Journal of Global Optimization (2019) 74:233–253
https://doi.org/10.1007/s10898-019-00766-1

Necessary and sufficient conditions for achieving global optimal solutions in multiobjective quadratic fractional optimization problems

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Received: 9 November 2017 / Accepted: 21 March 2019 / Published online: 1 April 2019

Abstract
If x^* is a local minimum solution, then there exists a ball of radius $r > 0$ such that $f(x) \geq f(x^*)$ for all $x \in B(x^*, r)$. The purpose of the current study is to identify the suitable $B(x^*, r)$ of the local optimal solution x^* for a particular multiobjective optimization problem. We provide a way to calculate the largest radius of the ball centered at local Pareto solution in which this solution is optimal. In this process, we present the necessary and sufficient conditions for achieving a global Pareto optimal solution. The results of this investigation might be useful to determine stopping criteria in the algorithms development.

Keywords Pareto optimality conditions · Multiobjective optimization · Quadratic fractional optimization problems

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1 Introduction

We consider the following multiobjective optimization problem (MOP):

$$\begin{array}{lll} \textbf{(MOP)} & \textbf{Minimize} & u(x) = (u_1(x), \ldots, u_m(x)) \\ & \text{subject to} & h_j(x) \leqq 0, \quad j \in J, \\ & & x \in \Omega, \end{array}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set, u_i and h_j are real-valued functions defined on Ω for each $i \in I \equiv \{1, \ldots, m\}$ and $j \in J \equiv \{1, \ldots, p\}$. Let S denotes the *set of feasible solutions* defined by $S = \{x \in \Omega \mid h_j(x) \leqq 0, \; j \in J\}$.

Let $\mathbb{R}_+$ be the set of nonnegative real numbers and x^T be the transpose of $x \in \mathbb{R}^n$. The inequality among vectors is to be understood in a component-wise sense: if $x, y \in \mathbb{R}^n$, then $x < y$ if and only if $x_i < y_i$; $x \leqq y$ if and only if $x_i \leqq y_i$; and $x \leq y$ if and only if $x \leqq y$ and $x \neq y$.

By considering Euclidean distance, let $B(x, r)$ be the *open ball* and $\bar{B}(x, r)$ be the *closed ball* of radius $r > 0$ centered at x. We denote by $\partial B(x, r)$ the *boundary* of $B(x, r)$. Similar to Chankong and Haimes [8], we present two optimality definitions for MOP.

Definition 1 A feasible solution x^* is said to be a *(locally) Pareto optimal solution* of MOP, if there does not exist another solution $(x \in B(x^*, r), r > 0)$ $x \in S$ such that $u(x) \leq u(x^*)$.

Definition 2 A feasible solution x^* is said to be a *(locally) weakly Pareto optimal solution* of MOP, if there does not exist another solution $(x \in B(x^*, r), r > 0)$ $x \in S$ such that $u(x) < u(x^*)$.

We observe that $x \in B(x^*, r)$ is not a restrictive hypothesis, since for an arbitrary layout neighborhood $N(x^*)$ of x^*, it is possible that $B(x^*, r) \subset N(x^*)$.

The literature presents different ways to characterize Pareto optimality conditions (see [8, 21] and references therein). This paper extends the results of the literature by proposing necessary and sufficient conditions for achieving a global Pareto optimal solution from a local solution. We calculate the largest radius of the ball centered at a local Pareto solution, in which this solution is optimal; then we verify the conditions for this solution to be globally optimal.

This paper is organized as follows: Sect. 2 presents the multiobjective quadratic fractional optimization problem. Our methodology is introduced in Sect. 3. Section 4 provides some optimality conditions. Conclusions are given in Sect. 5.

2 Quadratic fractional problem

In this paper, we consider the multiobjective quadratic fractional optimization problem (MQFP), where for each $i \in I$, $u_i(x) \equiv \frac{f_i(x)}{g_i(x)}$ in MOP, and f_i, g_i, are quadratic functions of n real variables, $g_i(x) > 0$ for all $x \in \Omega$.

Fractional optimization problems frequently arise in the decision-making applications, including portfolio selection, cutting and stock problems, and game theory, in the optimization of the ratio performance/cost, or profit/investment, or cost/time, and so on. Convexity and generalized convexity are used in the literature to achieve optimality conditions and main duality theorems for optimal solutions, including the scalar (single-objective) fractional optimization problem (FP) and multiobjective fractional optimization problem (MFP).

Jagannathan [11] has established optimality conditions by transforming the original problem FP to an associated non fractional problem. Liang et al. [19] obtained results that relate the primal-dual pair of FP. Craven [9] has presented several results for FP, and Stancu-Minasian [28] has provided a book containing various practical and theoretical aspects for FP. Emam [10] has studied a multiobjective integer quadratic programming problem under uncertainty. Liang et al. [20] have extended their approach [19] to MFP and examined the dual problem reported in [22]. Osuna-Gómez et al. [24] have used the parametric approaches of Jagannathan [11] and Bector et al. [6], and two classes of generalized convexity to establish weak Pareto optimality conditions and the main duality theorems for the differentiable MFP. Santos et al. [26] have expanded Osuna-Gómez's results to a more general, non-differentiable case of MFP.

We find few studies that involve quadratic functions at both the numerator and denominator in the ratio objective function. In most cases, the mixing of linear and quadratic functions is included. Schaible and Shi [27] and Benson [7] have considered a similar case of the scalar quadratic fractional optimization problem (QFP). Benson [7] has developed some theoretical properties and optimality conditions for a QFP consisting of the convex functions, and also presented an algorithm and its convergence properties. Jiao and Liu [12] have solved a QFP consisting of the quadratically constrained sum of quadratic ratios problem by using a range division and compression algorithm for global optimization.

Beato-Moreno et al. [4,5], Arévalo and Zapata [3], Konno and Inori [13], Rhode and Weber [25], Kornbluth and Steuer [16], and Korhonen and Yu [14,15] have addressed more similar approaches to MQFP. From an iterative computational test, Beato-Moreno et al. [4,5] have characterized the Pareto optimal solution for MQFP with linear and quadratic functions, and achieved some theoretical results by using the function linearization approach of Bector et al. [6]. Arévalo and Zapata [3], Konno and Inori [13], and Rhode and Weber [25] have analyzed the portfolio selection problem. Kornbluth and Steuer [16] have used a suitable Simplex method for MFP consisting of linear functions. Korhonen and Yu [14,15] have proposed an iterative computational method based on search directions and weighted sums for MQFP using linear and quadratic functions.

In their previous work, Oliveira et al. [23] have developed sufficient Pareto optimality conditions and duality results for a legitimate MQFP, where they have avoided the generalized convexities. Ammar [1,2] has studied the solutions of MQFP from a fuzzy random technique. Lachhwani [17,18] has provided a fuzzy goal programming approach for MQFP. Stancu-Minasian and Stancu [31] have presented a new class of generalized type-I univex functions and derived the weak, strong and converse duality theorems for MQFP. Stancu-Minasian [29,30] has listed a complete updated bibliography of fractional programming problems.

The approach taken in this paper is different from previous ones. Without generalized convexity assumptions on the objective functions, some necessary and sufficient conditions are given for a local Pareto optimal solution to be a global Pareto optimal solution. The central aspect of this contribution is to show how to calculate the largest radius of the ball centered at local Pareto optimal solution where this solution is globally optimal. These conditions might be useful to determine stopping criteria in the development of algorithms, including those based on quadratic approximations in more general problems.

3 Radius of efficiency

If $u(x) \leq u(z)$, we say that $x \in S$ *dominates* $z \in S$ in MOP. The Pareto (or globally efficient) optimal solution set is denoted by *Eff* (MOP), and the set of locally Pareto (or locally efficient) optimal solutions is denoted by *Leff* (MOP). We introduce the new concept of *radius of efficiency* for MOP, and then we extend this concept to MQFP.

3.1 Radius of efficiency for MOP

Definition 3 A feasible solution $x^* \in S$ is said to be λ-*efficient* or has *radius of efficiency* λ in MOP if $x^* \in$ *Leff* (MOP) and there does not exist another solution $x \in B(x^*, \lambda) \cap S$ that dominates x^*.

We note that if x^* is λ-efficient, then it is β-efficient, for all $\beta < \lambda$. If x^* is a globally optimal solution in S, we say that it is ∞-efficient.

If $x^* \in$ *Leff* (MOP), there exists $r > 0$ such that x^* is not dominated in $B(x^*, r) \cap S$. However, our goal is to calculate the largest radius $\lambda^* > 0$ of the ball $B(x^*, \lambda^*)$ such that x^* is not dominated in $B(x^*, \lambda^*) \cap S$. In particular, for λ^* sufficiently large we obtain a globally optimal solution in S. Thus, we consider a local Pareto optimal solution on a fixed and most appropriate ball. The value λ^* can be useful for solving MOP because if the radius of efficiency is known, the cost to find a new solution using a suitable search procedure can be estimated. Auxiliary techniques induced by the radius of efficiency can be used to conclude if a solution is globally optimal.

3.2 Radius of efficiency for MQFP

If $x^* \in$ *Leff* (MQFP), then the smallest value $\lambda > 0$, such that the inequality $\frac{f(x^*+\lambda d)}{g(x^*+\lambda d)} \leq \frac{f(x^*)}{g(x^*)}$ is valid for every unitary direction d, provides the maximum radius of efficiency of x^*. We use the following problem closely associated with MQFP, namely MQFP$_{x^*}$, where $u_i(x) \equiv f_i(x) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x)$, $x^* \in S$, and $f_i, g_i, i \in I$, are the quadratic functions of MQFP. Similar to Lemma 1.1 from [24], we consider Pareto optimal solutions to achieve next result.

Theorem 1 *$x^* \in$ Leff (MQFP) if and only if $x^* \in$ Leff (MQFP$_{x^*}$). In particular, there exists $r > 0$ such that x^* is locally Pareto optimal solution for MQFP in $B(x^*, r) \cap S$ if and only if it is locally Pareto optimal solution for MQFP$_{x^*}$ in $B(x^*, r) \cap S$.*

Proof Let x^* be locally Pareto optimal solution for MQFP in $B(x^*, r) \cap S$. Suppose that $x^* \notin$ *Leff* (MQFP$_{x^*}$), then there exists another solution $x \in B(x^*, r)$, satisfying $f(x) - \frac{f(x^*)}{g(x^*)} g(x) \leq f(x^*) - \frac{f(x^*)}{g(x^*)} g(x^*) = 0 \implies \frac{f(x)}{g(x)} \leq \frac{f(x^*)}{g(x^*)}$, which contradicts that $x^* \in$ *Leff* (MQFP). Therefore, $x^* \in$ *Leff* (MQFP$_{x^*}$). Now, let x^* be the locally Pareto optimal solution for MQFP$_{x^*}$ in $B(x^*, r) \cap S$. Suppose that $x^* \notin$ *Leff* (MQFP), then there exists another solution $x \in B(x^*, r)$, satisfying $\frac{f(x)}{g(x)} \leq \frac{f(x^*)}{g(x^*)} \implies f(x) - \frac{f(x^*)}{g(x^*)} g(x) \leq 0 = f(x^*) - \frac{f(x^*)}{g(x^*)} g(x^*)$, which contradicts that $x^* \in$ *Leff* (MQFP$_{x^*}$). Therefore, $x^* \in$ *Leff* (MQFP). □

From Theorem 1, we must choose which is more appropriate between MQFP and MQFP$_{x^*}$ to calculate the maximum radius of $B(x^*, r)$, and analyze the dominance of a local Pareto optimal solution in the set of feasible solutions.

4 Optimality conditions

First, we establish the optimality conditions for the unconstrained problem, and then we extend the results to the constrained problem with linear constraints.

4.1 Radius of efficiency in the unconstrained case

For each $i \in I$ and all $x \in \mathbb{R}^n$, we address $f_i(x) = x^T A_i x + a_i^T x + \bar{a}_i$ and $g_i(x) = x^T B_i x + b_i^T x + \bar{b}_i$, where $A_i, B_i \in \mathbb{R}^{n\times n}$ are matrices, and we assume that A_i is symmetric; B_i is symmetric and positive semidefinite; $a_i, b_i \in \mathbb{R}^n$ and $\bar{a}_i, \bar{b}_i \in \mathbb{R}$, with $\bar{b}_i > -({w^i}^T B_i w^i + b_i^T w^i)$, where w^i is the solution of the system $2B_i x + b_i = 0$; that is, w^i is the solution where the function $x^T B_i x + b_i^T x$ reaches its minimum and this choice ensures that $g_i(x) > 0$, $\forall x \in \mathbb{R}^n$. We do not consider cases where $2B_i x + b_i = 0$ has no solution.

Let $\nabla u(x)$ and $\nabla^2 u(x)$ be the gradient and the Hessian matrix of the function $u : \mathbb{R}^n \to \mathbb{R}$ at x, respectively. We denote by UMQFP and UMQFP$_{x^*}$ the unconstrained MQFP and MQFP$_{x^*}$, respectively. Further, we define some important sets and parameters for this work.

Definition 4 Let $d \in \mathbb{R}^n$ be a fixed and unitary direction, and $x^* \in \mathbb{R}^n$. Let

$$p_i(x) = x^T \left(A_i - \frac{f_i(x^*)}{g_i(x^*)} B_i \right) x + \left(a_i^T - \frac{f_i(x^*)}{g_i(x^*)} b_i^T \right) x, \quad i \in I, \tag{1}$$

be a quadratic function, and

$$I_0 = \left\{ i \in I \mid \left[d^T \nabla^2 p_i(x^*) d,\ \nabla p_i(x^*)^T d \right]^T \geq [0,0]^T \right\},$$

$$I_+ = \{ i \in I \mid d^T \nabla^2 p_i(x^*) d > 0 \text{ and } \nabla p_i(x^*)^T d < 0 \},$$

$$I_- = \{ i \in I \mid d^T \nabla^2 p_i(x^*) d < 0 \text{ and } \nabla p_i(x^*)^T d > 0 \},$$

$$\lambda_-^d = \max_{i \in I_-} \left\{ \frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d} \right\}, \qquad \lambda_+^d = \begin{cases} \min\limits_{i \in I_+} \left\{ \frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d} \right\}, & \text{if } I_+ \neq \emptyset \\ +\infty, & \text{if } I_+ = \emptyset, \end{cases}$$

$$\Lambda_-^d = [\lambda_-^d, \infty), \qquad \Lambda_+^d = \begin{cases} (0, \lambda_+^d], & \text{if } I_+ \neq \emptyset \\ (0, \lambda_+^d), & \text{if } I_+ = \emptyset, \end{cases} \qquad \Lambda^d = \Lambda_-^d \cap \Lambda_+^d.$$

Each function p_i defined in (1) is an objective function of MQFP$_{x^*}$ without the constant term $\bar{a}_i - \frac{f_i(x^*)}{g_i(x^*)} \bar{b}_i$, thus $p_i(x^*) = -[\bar{a}_i - \frac{f_i(x^*)}{g_i(x^*)} \bar{b}_i]$, $i \in I$. Taylor expansion around zero of the function $r_i(\lambda) = p_i(x^* + \lambda d)$, $i \in I$, $\lambda \in \mathbb{R}$, $\lambda \geqq 0$, is the quadratic function $r_i(\lambda) = p_i(x^*) + \lambda \nabla p_i(x^*)^T d + \frac{\lambda^2}{2} d^T \nabla^2 p_i(x^*) d$, where $p_i(x^*)$, $\nabla p_i(x^*)^T d$ and $\frac{1}{2} d^T \nabla^2 p_i(x^*) d$ are the constant, linear and quadratic terms, respectively.

For a fixed and unitary direction d, the set I_+ includes the index $i \in I$ whose function r_i decreases in value around $\lambda = 0$ but is convex; i.e., $d^T \nabla^2 p_i(x^*) d > 0$. The set I_- includes the index $i \in I$ whose function r_i increases in value around $\lambda = 0$ but is concave, i.e., $d^T \nabla^2 p_i(x^*) d < 0$ (see Fig. 5); and the set I_0 includes the index $i \in I$ whose function r_i does not decrease around $\lambda = 0$ (see Fig. 6).

Figures 6, 7 and 10 illustrate the positive roots λ_-^d and λ_+^d of the equation $r_i(\lambda) - p_i(x^*) = 0$. For the same direction d, the sets Λ_-^d, Λ_+^d and Λ^d are intervals of $\mathbb{R}_+ \backslash \{0\}$. λ_-^d is the left extreme of Λ_-^d, and λ_+^d is the right extreme of Λ_+^d. If $\lambda_-^d \leqq \lambda_+^d$ they are the end points of Λ^d, and if $\lambda_-^d > \lambda_+^d$ we have $\Lambda^d = \emptyset$ (see Fig. 8). Note that if $I_+ = \emptyset$ it is appropriate to choose $\lambda_+^d = +\infty$, because if $I_- \neq \emptyset$ we have $\Lambda^d = [\lambda_-^d, \infty) \cap (0, \infty) = [\lambda_-^d, \infty) \neq \emptyset$ (see Fig. 9).

Corollary 2 *Let $d \in \partial B(0,1)$. If $\Lambda^d \neq \emptyset$, then $p_i(x) \leqq p_i(x^*)$ for all $x = x^* + \lambda d$, $\lambda \in \Lambda^d$, and $i \in I_+ \cup I_-$.*

Proof By considering the Taylor expansion around zero of the functions $\bar{r}_i(\lambda) = f_i(x^* + \lambda d)$ and $\tilde{r}_i(\lambda) = g_i(x^* + \lambda d)$, $\lambda \in \Lambda^d$, $i \in I_+ \cup I_-$,

$$\bar{r}_i(\lambda) = f_i(x^*) + \lambda \nabla f_i(x^*)^T d + \lambda^2 d^T A_i d,$$
$$\tilde{r}_i(\lambda) = g_i(x^*) + \lambda \nabla g_i(x^*)^T d + \lambda^2 d^T B_i d,$$

we obtain

$$\frac{f_i(x^* + \lambda d)}{g_i(x^* + \lambda d)} \leqq \frac{f_i(x^*)}{g_i(x^*)} \iff \lambda d^T \left(\lambda A_i - \lambda \frac{f_i(x^*)}{g_i(x^*)} B_i \right) d \leqq \left(\lambda \frac{f_i(x^*)}{g_i(x^*)} \nabla g_i(x^*) - \lambda \nabla f_i(x^*) \right)^T d.$$

By using $p_i(x)$ from Eq. (1), we have

$$\begin{aligned} \lambda d^T \left(\lambda A_i - \lambda \frac{f_i(x^*)}{g_i(x^*)} B_i \right) d \; &\leqq \; \left(\lambda \frac{f_i(x^*)}{g_i(x^*)} \nabla g_i(x^*) - \lambda \nabla f_i(x^*) \right)^T d \\ &\iff \lambda \left(\nabla p_i(x^*)^T d + \frac{\lambda}{2} d^T \nabla^2 p_i(x^*) d \right) \leqq 0, \end{aligned} \tag{2}$$

where $\frac{1}{2}\nabla^2 p_i(x^*) = A_i - \frac{f_i(x^*)}{g_i(x^*)} B_i$ and $\nabla p_i(x^*) = \nabla f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \nabla g_i(x^*)$. Therefore, from (2)

$$\begin{aligned} p_i(x^* + \lambda d) - p_i(x^*) \; &= \; \lambda \left(\nabla p_i(x^*)^T d + \frac{\lambda}{2} d^T \nabla^2 p_i(x^*) d \right) \leqq 0 \\ &\iff r_i(\lambda) = p_i(x^* + \lambda d) \leqq p_i(x^*). \end{aligned}$$

□

Figures 5, 9 and 10 illustrate the conclusion of Corollary 2.

Condition 1 *Let $d \in \partial B(0,1)$, and suppose that $I_0 = \emptyset$, $I_+ \neq \emptyset$ and $I_- \neq \emptyset$. If $\lambda_-^d = \lambda_+^d = \frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d}$, for all $i \in I_+ \cup I_-$, then we define $\Lambda^d := \emptyset$.*

Condition 2 *Let $x^* \in Leff(UMQFP_{x^*})$ and $d \in \partial B(0,1)$. Then it does not simultaneously occur $\nabla p_i(x^*)^T d = d^T \nabla^2 p_i(x^*) d = 0$ for all $i \in I$.*

By imposing Conditions 1 and 2, we eliminate some rare cases that might occur with respect to the size of sets I_0, I_+ and I_-. For instance, if $|I_+| = |I_-| = 1$ and $\lambda_-^d = \lambda_+^d$ it is more suitable to use $\Lambda^d = \emptyset$ instead of using $\Lambda^d = \{\lambda_-^d\}$. Note that these conditions are useless for weakly Pareto optimal solutions.

Lemma 1 *Let $x^* \in Leff(UMQFP)$. Then x dominates x^* if and only if there exists $d \in \partial B(0,1)$ such that $I_0 = \emptyset$, $I_- \neq \emptyset$ and $\Lambda^d \neq \emptyset$, where $\lambda^* \in \Lambda^d$ and $x = x^* + \lambda^* d$.*

Proof ($\Rightarrow$) See Appendix A. ($\Leftarrow$) Suppose that $I_0 = \emptyset$, $I_- \neq \emptyset$ and $\Lambda^d \neq \emptyset$ in the direction d. If $I_+ = \emptyset$ then there exists $\lambda^* \in \Lambda^d$ (see Fig. 9), in which $x = x^* + \lambda^* d$ and $p(x) \leq p(x^*)$ have a solution. If $I_+ \neq \emptyset$ then either there exists a $\lambda^* \in \Lambda^d$, in which $x = x^* + \lambda^* d$ and $p(x) \leq p(x^*)$ have a solution, or $\lambda^d_- < \lambda^d_+$ (see Fig. 10) and there exists $\lambda^* \in \Lambda^d = [\lambda^d_-, \lambda^d_+]$, in which $x = x^* + \lambda^* d$ and $p(x) \leq p(x^*)$ have a solution. Thus, x dominates x^*. □

Theorem 3 *Let $x^* \in$ Leff (UMQFP). Then $x^* \in$ Eff (UMQFP) if and only if for all $d \in \partial B(0, 1)$, $I_0 \neq \emptyset$ or $\Lambda^d = \emptyset$ whenever $I_+ \neq \emptyset$.*

Proof ($\Rightarrow$) Suppose there exists $d \in \partial B(0, 1)$ where $I_+ \neq \emptyset$, $I_0 = \emptyset$ and $\Lambda^d \neq \emptyset$. As $x^* \in$ *Leff* (UMQFP), from Theorem 1, $x^* \in$ *Leff* (UMQFP$_{x^*}$) and $I_- \neq \emptyset$ in this direction. By Lemma 1, there exist $\lambda^* \in \Lambda^d$ and $x = x^* + \lambda^* d$ such that $p(x) \leq p(x^*)$ has a solution. By Eq. (1), $\frac{f(x)}{g(x)} \leq \frac{f(x^*)}{g(x^*)}$ has a solution. Which contradicts that $x^* \in$ *Eff* (UMQFP). Therefore, for all $d \in \partial B(0, 1)$, $I_0 \neq \emptyset$ or $\Lambda^d = \emptyset$ whenever $I_+ \neq \emptyset$.

($\Leftarrow$) By Theorem 1, $x^* \in$ *Leff* (UMQFP$_{x^*}$). Suppose that $I_0 \neq \emptyset$ for an arbitrary $d \in \partial B(0, 1)$. Then, by definition, there exists $i \in I$ for which $d^T \nabla^2 p_i(x^*) d > 0$ and $\nabla p_i(x^*)^T d \geqq 0$, or $d^T \nabla^2 p_i(x^*) d \geqq 0$ and $\nabla p_i(x^*)^T d > 0$, and then $r_i(\lambda)$ grows indefinitely for $\lambda > 0$, and $p(x) \leq p(x^*)$ does not have a solution for each $x = x^* + \lambda d$ near to x^* (see Fig. 11). Therefore, $x = x^* + \lambda d$ does not dominate x^*.

Now, suppose that $I_+ \neq \emptyset$ and $I_0 = \emptyset$, and $\Lambda^d = \emptyset$ in the direction d, then $I_- \neq \emptyset$. By Lemma 1, if $I_0 = \emptyset$, $I_+ \neq \emptyset$, $I_- \neq \emptyset$ and $\Lambda^d = \emptyset$, $p(x^* + \lambda d) \leq p(x^*)$ does not have a solution for $\lambda > 0$ (see Fig. 8). Therefore, $x = x^* + \lambda d$ does not dominate x^*. We conclude that if $d \in \partial B(0, 1)$ and $I_0 \neq \emptyset$, x^* is non-dominated; or whenever $I_+ \neq \emptyset$ and $\Lambda^d = \emptyset$, we again have x^* non-dominated. Thus $x^* \in$ *Eff* (UMQFP). □

We do not require any kind of generalized convexity to obtain the optimality conditions of Theorem 3. In the results that follow, we define $D = \{d \in \partial B(0, 1) \mid \Lambda^d \neq \emptyset\}$.

Corollary 4 *Let $x^* \in$ Leff (UMQFP) and $\beta = \inf_{d \in D} \{\lambda^d_-\}$. Then there does not exist another solution $x \in B(x^*, \beta)$ such that $\frac{f(x)}{g(x)} \leq \frac{f(x^*)}{g(x^*)}$.*

Proof It follows immediately from Theorem 3, if $d \in D$, then there exists $\lambda > 0$ in Λ^d such that another solution $x \in \mathbb{R}^n$ dominates x^* in this direction. However, the first solution x that dominates x^* along the direction d is $x = x^* + \lambda^d_- d$. By checking all directions in the set D, we conclude that the first solution x which dominates x^* is $x = x^* + \beta d$. Therefore x^* is β-efficient and there does not exist another solution $x \in B(x^*, \beta)$ such that $\frac{f(x)}{g(x)} \leq \frac{f(x^*)}{g(x^*)}$. □

Figure 10 illustrates a direction $d \in D$. There, we observe that $I_0 = \emptyset$, $I_+ = \{2, 3\}$, $I_- = \{1, 4\}$, $\Lambda^d \neq \emptyset$, and if $x = x^* + \lambda^d_- d$, $p(x) \leq p(x^*)$ has a solution. It is possible to rewrite Corollary 4 replacing the set D with the set $D' = \{d \in \partial B_n(0, 1) \mid I_0 = \emptyset\}$ (see the charts in Figs. 8 and 10).

Next, we present a desirable result which gives a lower bound for the radius of efficiency of a local Pareto optimality solution.

Corollary 5 *Let $x^* \in$ Leff (UMQFP) and $F(d) = \max_{i \in I_-} \{2 \nabla p_i(x^*)^T d\}$. Suppose there exists $\rho \in \mathbb{R}$, such that for all $d \in \partial B(0, 1)$, $F(d) \geqq \rho$. Then there does not exist another solution $x \in B(x^*, \frac{\rho}{-\gamma})$ such that $\frac{f(x)}{g(x)} \leq \frac{f(x^*)}{g(x^*)}$, where $\gamma < 0$, $\gamma = \min_{i \in I} \{\gamma_i\}$ and γ_i is the smallest negative eigenvalue of the matrix $\nabla^2 p_i(x^*)$, $i \in I$.*

Proof By Theorem 3 and the value of β in Corollary 4, if we search a solution $x = x^* + \lambda d$ that dominates x^*, we have to find a value $\lambda > 0$ that satisfies

$$\lambda \geqq \beta = \inf_{d \in D} \left\{ \max_{i \in I_-} \left\{ \frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d} \right\} \right\}.$$

From the symmetric and diagonalizable matrix $\nabla^2 p_i(x^*)$, $i \in I_-$, we obtain $0 > d^T \nabla^2 p_i(x^*) d \geq \gamma_i$, where γ_i is the smallest negative eigenvalue of the matrix $\nabla^2 p_i(x^*)$, thus

$$\begin{aligned}
\lambda \geqq \beta &= \inf_{d \in D} \left\{ \max_{i \in I_-} \left\{ \frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d} \right\} \right\} \\
&\geqq \inf_{d \in D} \left\{ \max_{i \in I_-} \left\{ \frac{-2\nabla p_i(x^*)^T d}{\gamma_i} \right\} \right\} \\
&\geqq \inf_{d \in D} \left\{ \max_{i \in I_-} \left\{ \frac{-2\nabla p_i(x^*)^T d}{\gamma} \right\} \right\} = \inf_{d \in D} \frac{F(d)}{-\gamma},
\end{aligned}$$

where $\gamma = \min_{i \in I} \{\gamma_i\}$. Since $F(d) \geqq \rho$, $\beta \geqq \frac{\rho}{-\gamma}$. If $\Lambda^d = \emptyset$ for all $d \in \partial B(0, 1)$, then x^* is ∞-efficient; but if there exists d such that $\Lambda^d \neq \emptyset$, then x^* is β-efficient. As $\infty > \beta \geqq \frac{\rho}{-\gamma}$, the theorem holds. □

Because $i \in I_-$, the changeover from the first to the second inequality of the Corollary 5 demonstration is valid, and it also provides $\rho > 0$. Therefore $\frac{\rho}{-\gamma} > 0$. In order to limit the search space, we present another beneficial result which gives an upper bound for the radius of efficiency of a local Pareto optimality solution. Let $\|\cdot\|$ be the Euclidean norm.

Theorem 6 *Let $x^* \in Leff\,(UMQFP)$ and $M = \min_{i \in I_+} \left\{ \frac{\|2\nabla p_i(x^*)\|}{\alpha} \right\}$. Suppose that for some $d \in D$ and for all $i \in I_+ \neq \emptyset$, $d^T \nabla^2 p_i(x^*) d \geqq \alpha > 0$. If there does not exist another solution $x \in B(x^*, M)$ such that $\frac{f(x)}{g(x)} \leq \frac{f(x^*)}{g(x^*)}$, then $x^* \in Eff\,(UMQFP)$.*

Proof Lemma 1 and Corollary 4 provide that if there exists a solution x that dominates x^* in the direction d, then $I_- \neq \emptyset$ and $\lambda_-^d \leqq \|x - x^*\| \leqq \lambda_+^d$. Let $P = \sup_{d \in D}\{\lambda_+^d\}$. By the hypotheses and $\|x - x^*\| \leqq P$, we also obtain

$$\begin{aligned}
P &= \sup_{d \in D} \left\{ \min_{i \in I_+} \left\{ \frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d} \right\} \right\} \\
&\leqq \sup_{d \in D} \left\{ \min_{i \in I_+} \left\{ \frac{-2\nabla p_i(x^*)^T d}{\alpha} \right\} \right\} \\
&\leqq \min_{i \in I_+} \left\{ \frac{\|2\nabla p_i(x^*)\|}{\alpha} \right\} = M.
\end{aligned}$$

Therefore, if there does not exist a solution that dominates x^* in $B(x^*, M)$, then there does not exist another solution in $\mathbb{R}^n$ with this property, and $x^* \in Eff$ (UMQFP). □

Because $i \in I_+$, the changeover from the first to the second inequality of the Theorem 6 demonstration is valid. It only has usefulness if $M < \infty$. Our results can be use to determine whether a locally Pareto optimal solution is dominated by some point within a limited subset

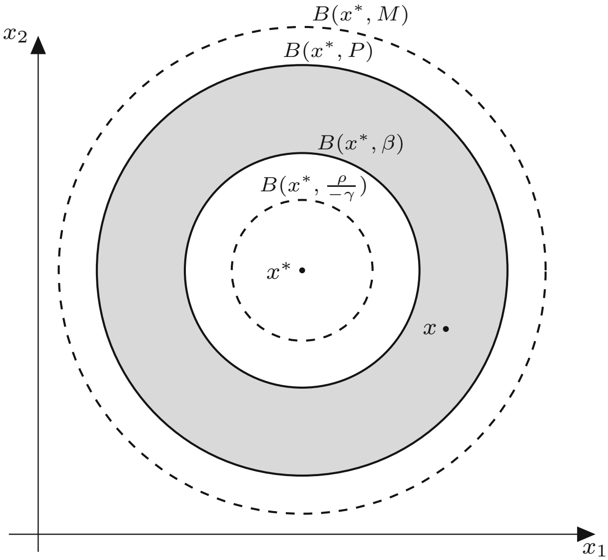

Fig. 1 Some interesting spherical regions of the solution $x^* \in$ *Leff* (UMQFP)

of $\mathbb{R}^n$. Note that Corollary 4 provides the maximum radius $\beta = \inf\limits_{d \in D} \left\{\lambda_-^d\right\}$ of $B(x^*, \beta)$ where x^* is not dominated. Corollary 5 provides a lower bound for β, i.e., a new $B(x^*, \frac{\rho}{-\gamma})$. On the other hand, Theorem 6 provides two balls centered at x^*, one of them with radius $P = \sup_{d \in D}\{\lambda_+^d\}$, and another with radius $M = \min\limits_{i \in I_+} \left\{ \frac{\|2\nabla p_i(x^*)\|}{\alpha} \right\}$ (upper bound of P). Therefore, we have four balls such that $B(x^*, \frac{\rho}{-\gamma}) \subseteq B(x^*, \beta) \subseteq B(x^*, P) \subseteq B(x^*, M)$.

The lower and upper bounds are more attractive computationally to calculate. A suitable search can be made in the subset $\bar{B}(x^*, P) \backslash B(x^*, \beta)$, or alternatively in the subset $\bar{B}(x^*, M) \backslash B(x^*, \frac{\rho}{-\gamma})$.

Figure 1 illustrates a particular case in $\mathbb{R}^2$. It shows four spheres centered at the solution x^*: in dashed lines and nearer to the center is the radius of $B(x^*, \frac{\rho}{-\gamma})$; in continuous lines and near the center is the radius of $B(x^*, \beta)$ and $B(x^*, P)$, respectively; the closed subset $\bar{B}(x^*, P) \backslash B(x^*, \beta)$ is the shaded area; finally, the radius of the biggest ball $B(x^*, M)$ is represented by the outer dashed lines. If there exists a solution x that dominates x^*, it must belong to subset $\bar{B}(x^*, P) \backslash B(x^*, \beta)$.

Auxiliary computational methods can be designed to further reduce the size of subset $\bar{B}(x^*, P) \backslash B(x^*, \beta)$. It is enough to observe that $x \in \bar{B}(x^*, P) \backslash B(x^*, \beta)$ if and only if there exists a unitary direction d such that $\lambda_-^d \leqq \lambda \leqq \lambda_+^d$ and $x = x^* + \lambda d$. Therefore, all those directions participating in subset $\bar{B}(x^*, P) \backslash B(x^*, \beta)$, but with $\Lambda^d = \emptyset$, can be excluded. Figure 2 illustrates this possibility, where there exist two shaded subsets of $\bar{B}(x^*, P) \backslash B(x^*, \beta)$, each containing a possible solution x. The shaded subsets represent only the directions d such that $\lambda_-^d \leqq \lambda \leqq \lambda_+^d$.

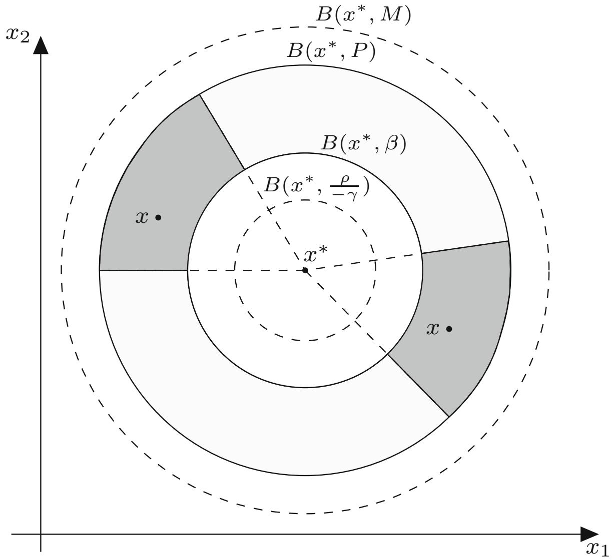

Fig. 2 Search space of x that dominates $x^* \in Leff$ (UMQFP)

4.2 Radius of efficiency in the constrained case

We extend the results achieved in Sect. 4.1 for the constrained case MQFP. Let S be the set of feasible solutions and $\mathrm{diam}(S) = \sup\{\|x - y\| : x, y \in S\}$. The radius of efficiency $\mathrm{diam}(S)$ is equivalent to the radius of efficiency ∞ in the UMQFP. If x^* is λ-efficient, then it is β-efficient, for all $\beta < \lambda \leqq \mathrm{diam}(S)$. If x^* is a globally Pareto optimal solution in S, we say that it is $\mathrm{diam}(S)$-efficient.

As a first contributed work we consider MQFP with the set of feasible solutions S defined as $S = \{x \in \mathbb{R}^n \mid Cx \leqq b,\ C \in \mathbb{R}^{p\times n}, b \in \mathbb{R}^p\}$. This choice makes it easy to calculate the distance from a solution $x^* \in S$ to a boundary S, which is indispensable in the calculus of radius of efficiency. However, we believe that the results extend for the more general constraint sets than linear inequalities.

The next two results extend naturally the existing relation between the associated problems MQFP and MQFP$_{x^*}$ shown in Sect. 4.1. Now we admit the existence of a particular ball of fixed radius for the solution x^* where it is not dominated, and we calculate this radius using our approach. In addition, we observe that the same fixed ball applies to both problems.

Theorem 7 *Let $x^* \in Leff$ (MQFP). x^* is λ-efficient for MQFP if and only if x^* is λ-efficient for MQFP$_{x^*}$.*

Proof To calculate the radius of efficiency of the solution $x^* \in Leff$ (MQFP$_{x^*}$), we must determine if there exist $\lambda > 0$ and a feasible unitary direction d such that $x^* + \lambda d \in S$ dominates x^*. In other words, we must find a pair $(\bar{\lambda}, \bar{d})$ which solves the following inequalities

$$f_i(x^* + \lambda d) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x^* + \lambda d) \leqq f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x^*) = 0, \text{ for all } i \in I.$$

This implies to solve Inequalities (2). Similarly, to calculate the radius of efficiency of the solution $x^* \in$ *Leff* (MQFP), we must find a pair $(\tilde{\lambda}, \tilde{d})$ which solves the following inequalities

$$\frac{f_i(x^*+\lambda d)}{g_i(x^*+\lambda d)} \leqq \frac{f_i(x^*)}{g_i(x^*)} \iff \lambda\left(\nabla p_i(x^*)^T d + \frac{\lambda}{2} d^T \nabla^2 p_i(x^*) d\right) \leqq 0, \text{ for all } i \in I.$$

This again implies the resolution of the Inequalities (2). Hence, if there exists a pair $(\bar{\lambda}, \bar{d})$ for the solution $x^* \in$ *Leff* (MQFP_{x^*}), and a pair $(\tilde{\lambda}, \tilde{d})$ for the same solution $x^* \in$ *Leff* (MQFP), we obtain $(\bar{\lambda}, \bar{d}) = (\tilde{\lambda}, \tilde{d})$, and x^* is λ-efficient for both problems, with $\lambda = \bar{\lambda} = \tilde{\lambda}$. □

Corollary 8 *$x^* \in$ Eff (MQFP) if and only if $x^* \in$ Eff $(MQFP_{x^*})$.*

Proof We replace the pair (λ, d) with the pair (∞, d) in Theorem 7. □

Let $K(x^*) = \{j \in J \mid C_j x^* = b_j\}$ be the active constraints set at the solution $x^* \in S$, where C_j is the jth row of the matrix C. Let $\widetilde{T}(x^*) = \{y \in \mathbb{R}^n \mid C_j y \leqq 0,\ \forall j \in K(x^*),\ \|y\| = 1\}$ be the tangent cone to S at the solution $x^* \in S$. $T(x^*) = \widetilde{T}(x^*) \cap \partial B(0, 1)$ is said to be the *feasible directions* set at the solution x^*.

Definition 5 Let $x^* \in S$, for each $d \in T(x^*)$ such that $x^* + \lambda_\ell^d d \in S$, we say that the real number $\lambda_\ell^d \in (0, +\infty)$ is the *bounding* for the radius of efficiency of x^* in the direction d. Analogously, to the Sect. 4.1, we define

$$\lambda_\ell^d = \min_{j \in J,\ C_j d > 0} \left\{\frac{b_j - C_j x^*}{C_j d}\right\},$$

$$\lambda_-^d = \max_{i \in I_-} \left\{\frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d}\right\},\quad \lambda_+^d = \begin{cases} \min\left\{\min\limits_{i \in I_+}\left\{\frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d}\right\}, \lambda_\ell^d\right\}, & \text{if } I_+ \neq \emptyset \\ \inf\left\{+\infty, \lambda_\ell^d\right\}, & \text{if } I_+ = \emptyset, \end{cases}$$

$$\Lambda_-^d = [\lambda_-^d, \infty), \qquad \Lambda_+^d = \begin{cases} (0, \lambda_+^d], & \text{if } I_+ \neq \emptyset \\ (0, \lambda_+^d), & \text{if } I_+ = \emptyset, \end{cases} \qquad \Lambda^d = \Lambda_-^d \cap \Lambda_+^d,$$

where $p_i, i \in I$, is the same function and I_0, I_+ and I_- are the same sets defined in Sect. 4.1.

By including the limit λ_ℓ^d to the radius of efficiency in direction d, Figs. 12, 13, 14 and 15 illustrate some examples to the sets and parameters in Definition 5. In Fig. 14, where there exists $\lambda > 0$, $\lambda_-^d \leqq \lambda \leqq \lambda_+^d < \lambda_\ell^d$, such that $x^* + \lambda d$ dominates x^*, the interval Λ^d is nonempty, and $p(x) \leq p(x^*)$ has solution. Also, in Fig. 15, if we move λ_ℓ^d towards the right of the λ-axis until $\lambda_-^d \leqq \lambda_\ell^d$, then λ_+^d becomes λ_ℓ^d, that is, we have $\lambda_-^d \leqq \lambda_+^d := \lambda_\ell^d$ and we obtain $\Lambda^d \neq \emptyset$.

Theorem 9 *Let $x^* \in$ Leff (MQFP). Then $x^* \in$ Eff (MQFP) if and only if for all $d \in T(x^*)$, $I_0 \neq \emptyset$ or $\Lambda^d = \emptyset$.*

Proof $(\Rightarrow)$ Identical to Theorem 3, by now considering the directions in $T(x^*)$ and the fact of S to be limited or not in those directions. $(\Leftarrow)$ See Appendix A. □

We observe that if $I_0 = \emptyset$, $I_+ = \emptyset$ and only $I_- \neq \emptyset$ in a direction $d \in T(x^*)$, it is possible to obtain $\Lambda^d = \emptyset$. However, in the unconstrained case always $\Lambda^d \neq \emptyset$ (see Fig. 9). Figure 15 illustrates the constrained case, where $\lambda_\ell^d < \lambda_-^d$, in which λ_+^d becomes λ_ℓ^d; that is, $\lambda_+^d := \lambda_\ell^d < \lambda_-^d$, so $\Lambda^d = \emptyset$. Since $\lambda_+^d = \inf\left\{+\infty, \lambda_\ell^d\right\}$, if for all $j \in I_-$, $\frac{-2\nabla p_j(x^*)^T d}{d^T \nabla^2 p_j(x^*) d} \leqq \lambda_\ell^d$, we return to the unconstrained case, because we would have $\lambda_-^d \leqq \lambda_+^d := \lambda_\ell^d$ and $\Lambda^d \neq \emptyset$. That is, there exists $\lambda \in \Lambda^d$ such that $x^* + \lambda d \in S$ dominates x^* (see Fig. 15, and move λ_ℓ^d towards the right of the λ-axis until $\lambda_-^d \leqq \lambda_\ell^d$).

Next results are extensions of Corollaries 4 and 5, and Theorem 6; we omit its proofs due to the similarity with the unconstrained case. To simplify the presentation, we define $\hat{D} = \left\{d \in T(x^*) \mid \Lambda^d \neq \emptyset\right\}$.

Corollary 10 *Let $x^* \in Leff$ (MQFP) and $\beta = \inf_{d\in\hat{D}} \left\{\lambda_-^d\right\}$. Then there does not exist another solution $x \in B(x^*, \beta) \cap S$ such that $\frac{f(x)}{g(x)} \leq \frac{f(x^*)}{g(x^*)}$.*

Corollary 11 *Let $x^* \in Leff$ (MQFP) and $F(d) = \max_{i\in I_-} \left\{2\nabla p_i(x^*)^T d\right\}$. Suppose there exists $\rho \in \mathbb{R}$, such that for all $d \in T(x^*)$, $F(d) \geqq \rho$. Then there does not exist another solution $x \in B(x^*, \frac{\rho}{-\gamma}) \cap S$ such that $\frac{f(x)}{g(x)} \leq \frac{f(x^*)}{g(x^*)}$, where $\gamma < 0$, $\gamma = \min_{i\in I} \{\gamma_i\}$ and γ_i is the smallest negative eigenvalue of the matrix $\nabla^2 p_i(x^*)$, $i \in I$.*

Theorem 12 *Let $x^* \in Leff$ (MQFP) and $M = \min\left\{\min_{i\in I_+} \frac{\|2\nabla p_i(x^*)\|}{\alpha}, \operatorname{diam}(S)\right\}$. Suppose that for some $d \in \hat{D}$ and for all $i \in I_+ \neq \emptyset$, $d^T \nabla^2 p_i(x^*) d \geqq \alpha > 0$. If there does not exist another solution $x \in B(x^*, M) \cap S$ such that $\frac{f(x)}{g(x)} \leq \frac{f(x^*)}{g(x^*)}$, then $x^* \in Eff$ (MQFP).*

We identify from Corollaries 10, 11 and Theorem 12 four particular subsets related to solution $x^* \in Leff$ (MQFP) that satisfying $(B(x^*, \frac{\rho}{-\gamma}) \cap S) \subseteq (B(x^*, \beta) \cap S) \subseteq (B(x^*, P) \cap S) \subseteq (B(x^*, M) \cap S)$, where $P = \sup_{d\in\hat{D}}\{\lambda_+^d\}$. If we have a suitable computational search method to find x that dominates the solution x^*, it is enough that this search is made in the subset $\left(\bar{B}(x^*, P) \cap S\right) \setminus (B(x^*, \beta) \cap S)$, or alternatively in the subset $\left(\bar{B}(x^*, M) \cap S\right) \setminus (B(x^*, \frac{\rho}{-\gamma}) \cap S)$.

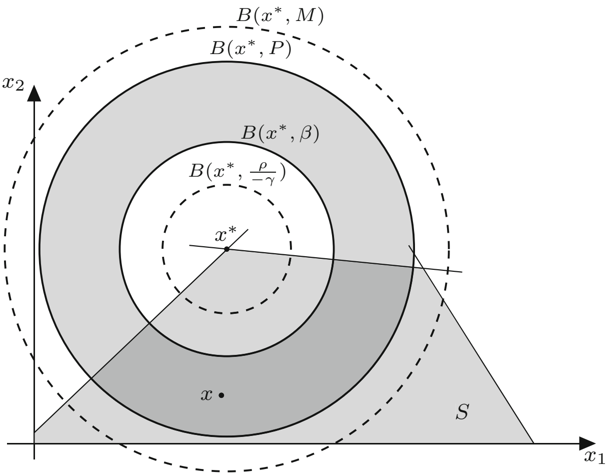

Fig. 3 Some interesting spherical regions of the solution $x^* \in Leff$ (MQFP)

Corollary 13 *Let $x^* \in Leff$ (MQFP). Suppose that the hypotheses of Corollaries 10 and 11, and Theorem 12 are satisfied. If there exists x that dominates x^* in MQFP, then* $x \in \left(\bar{B}(x^*, P) \cap S\right) \setminus (B(x^*, \beta) \cap S)$.

Figure 3 illustrates the search space from Corollary 13. It shows a solution $x^* \in Leff$ (MQFP) in the boundary of $S \subseteq \mathbb{R}^2$. In dashed lines, two subsets of interest are shown: $B(x^*, \frac{\rho}{-\gamma}) \cap S$ and $B(x^*, M) \cap S$. And in continuous lines, another two ones are shown: $B(x^*, \beta) \cap S$ and $B(x^*, P) \cap S$. If there exists x that dominates x^*, it must belong to subset $\left(\bar{B}(x^*, P) \cap S\right) \setminus (B(x^*, \beta) \cap S)$.

5 Conclusions

The present study was designed to develop necessary and sufficient conditions for a local Pareto optimal solution to be a global Pareto optimal solution. We focus our attention on multiobjective quadratic fractional optimization problems. We identify the spherical regions of a locally Pareto optimal solution; i.e., we show how to calculate the radius these spherical regions centered at it. If there exists another point that dominates this solution, it belongs to spherical regions. In this process, we also establish when the solution is globally optimal. The achieved results might be useful to determine stopping criteria in the development of algorithms, and new extensions can be established from these to more general multiobjective optimization problems.

Acknowledgements The authors are indebted to the anonymous reviewers for their helpful comments. This work was supported by the Coordination for the Improvement of Higher Level Personnel of Brazil (CAPES), the Fund to Support Teaching, Research and Extension of Unicamp (FAEPEX) (Grant No. 534/09), the Diretoria Executiva de Relações Internacionais (DERI)—Edital 62—Cátedras Ibero-Americanas (Grant No. 62/17), and the MINECO/ FEDER(UE) (Grant No. MTM2015-66185-P).

Appendix A: Proofs

Proof of Lemma 1 ($\Rightarrow$) If $x^* \in Leff$ (UMQFP) then $x^* \in Leff$ (UMQFP$_{x^*}$), and there exists $r > 0$ such that $f(x) - \frac{f(x^*)}{g(x^*)} g(x) \leq 0$ does not have a solution for each $x \in B(x^*, r)$, and by Eq. (1), $p(x) \leq p(x^*)$ does not have a solution for each $x \in B(x^*, r)$. Let $\bar{\lambda} = r$, then $p(x) \leq p(x^*)$ does not have a solution for each $x \in B(x^*, \bar{\lambda})$, and if $d \in \partial B(0, 1)$, $p(x) \leq p(x^*)$ does not have a solution for each $x = x^* + \lambda d$, $\|x - x^*\| \leqq \lambda$, and $\lambda \in (0, \bar{\lambda})$. Therefore, if x dominates x^*, $x \notin B(x^*, \bar{\lambda})$ and there exists $d \in \partial B(0, 1)$ such that $x = x^* + \lambda^* d$, and $\lambda^* \geqq \bar{\lambda}$.

Suppose that $I_0 \neq \emptyset$ in the direction $d \in \partial B(0, 1)$, there then exists $i \in I$ for which $d^T \nabla^2 p_i(x^*) d > 0$ and $\nabla p_i(x^*)^T d \geqq 0$ or $d^T \nabla^2 p_i(x^*) d \geqq 0$ and $\nabla p_i(x^*)^T d > 0$, and then $r_i(\lambda)$ grows indefinitely for $\lambda > 0$ (see Fig. 4). Therefore, I_0 has to be empty.

Suppose that $I_0 = \emptyset$ and $I_- = \emptyset$, then in the direction d, $I_+ \neq \emptyset$, $d^T \nabla^2 p_i(x^*) d > 0$ and $\nabla p_i(x^*)^T d < 0$ cannot occur; as well as $d^T \nabla^2 p_i(x^*) d \leqq 0$ and $\nabla p_i(x^*)^T d < 0$; or $d^T \nabla^2 p_i(x^*) d < 0$ and $\nabla p_i(x^*)^T d \leqq 0$ cannot occur (see Fig. 4). By Condition 2, we have $I_0 = \emptyset$ and $I_- \neq \emptyset$.

Now, suppose that $I_0 = \emptyset$ and $I_- \neq \emptyset$, and $\Lambda^d = \emptyset$ in the direction d. In this situation, $I_+ = \emptyset$ cannot occur; otherwise, we would have $\lambda_+^d = \infty$ and $\Lambda^d \neq \emptyset$. Then, if $I_0 = \emptyset$, $I_+ \neq \emptyset$, $I_- \neq \emptyset$ and $\Lambda^d = \emptyset$, either $\lambda_+^d < \lambda_-^d$, in which $p(x^* + \lambda d) \leq p(x^*)$ does not have a solution for $\lambda > 0$, or $\lambda_-^d = \lambda_+^d = \frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d}$, for all $i \in I_+ \cup I_-$, in which $p(x^* + \lambda d) \leq p(x^*)$ does not have a solution for $\lambda > 0$. We conclude that if x dominates x^*, then there exists $d \in \partial B(0, 1)$ such that $I_0 = \emptyset$, $I_- \neq \emptyset$ and $\Lambda^d \neq \emptyset$, where $\lambda^* \in \Lambda^d$ and $x = x^* + \lambda^* d$. □

Proof of Theorem 9 ($\Leftarrow$) By Theorem 7, $x^* \in$ *Leff* (MQFP$_{x^*}$). Let an arbitrary $d \in T(x^*)$ and suppose that $I_0 \neq \emptyset$ in the direction d. Then there exists $i \in I$ for which $r_i(\lambda)$ grows indefinitely for $\lambda > 0$ and $x = x^* + \lambda d$ does not dominate x^* in this direction (see Fig. 11).

If $I_0 = \emptyset$ and $\Lambda^d = \emptyset$, we verify two cases regarding the size of I_+. If $I_+ = \emptyset$, then only $I_- \neq \emptyset$. Since $\Lambda^d = \emptyset$ implies that S is limited ($\lambda_\ell^d < +\infty$) in this direction, and $\lambda_+^d := \lambda_\ell^d < \lambda_-^d$ (see Fig. 15). Therefore, x^* is not dominated in this direction. If $I_+ \neq \emptyset$ and x^* is a locally Pareto optimal solution, then, necessarily, $I_- \neq \emptyset$. We have $I_0 = \emptyset$, $I_+ \neq \emptyset$, $I_- \neq \emptyset$, $\Lambda^d = \emptyset$ and, as $\lambda_+^d := \lambda_\ell^d$ whenever $\lambda_\ell^d \leqq \lambda_+^d$, we return to the unconstrained case in Theorem 3 (see Figs. 12 and 13). Thus, x^* is not dominated in this direction.

We conclude that if $d \in T(x^*)$ and $I_0 \neq \emptyset$, we have x^* non-dominated, or if $\Lambda^d = \emptyset$, we again have x^* non-dominated, and $x^* \in$ *Eff* (MQFP). □

Appendix B: Illustrative figures

See Figs. 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 and 15.

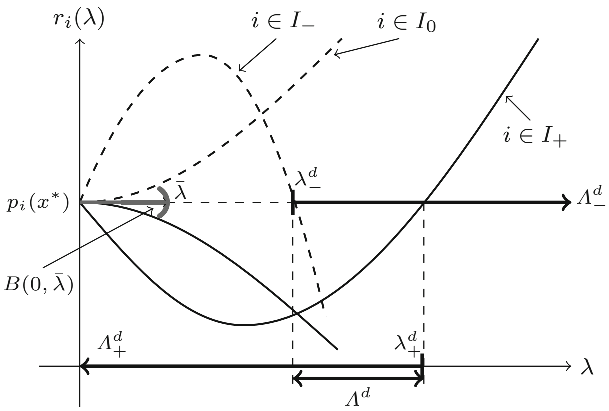

Fig. 4 Neighborhood of the solution x^* and the behavior of the function r_i

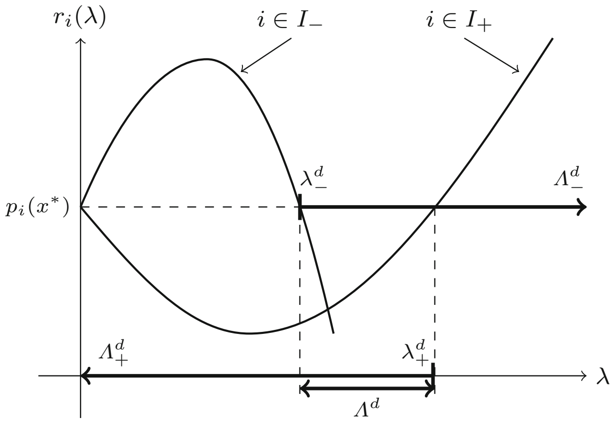

Fig. 5 Graphs of $r_i(\lambda)$ in which the index i belongs to the set I_+ or I_-

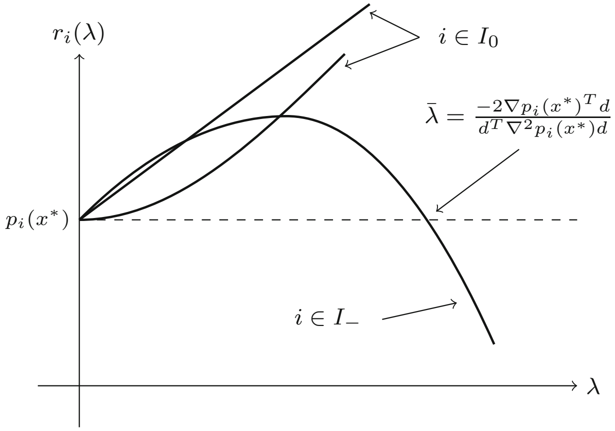

Fig. 6 Graphs of $r_i(\lambda)$ in which the index i belongs to the set I_0 or I_-, and $\bar{\lambda}$ is a root of the equation $r_i(\lambda) - p_i(x^*) = 0$

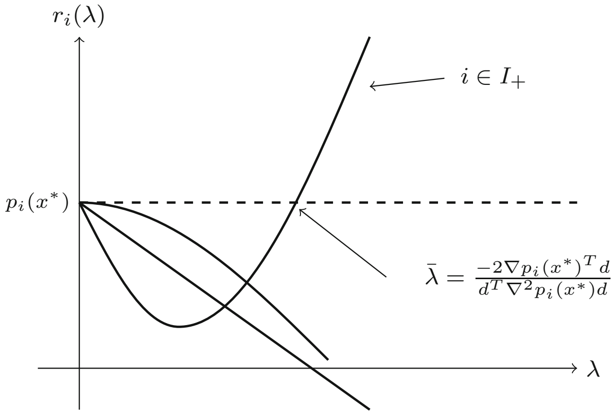

Fig. 7 Graphs of $r_i(\lambda)$ in which the index i belongs to the set I_+, and $\bar{\lambda}$ is a root of the equation $r_i(\lambda) - p_i(x^*) = 0$

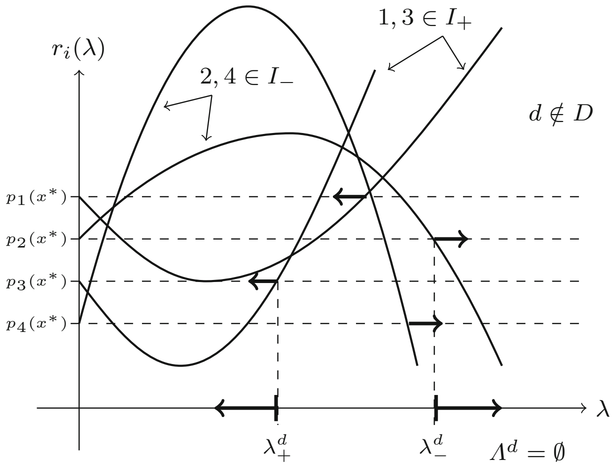

Fig. 8 The search for x that dominates $x^* \in Leff$ (UMQFP$_{x^*}$) in the search direction $d \notin D$, in which $I_0 = \emptyset$, $I_+ = \{1, 3\}$, $I_- = \{2, 4\}$ and $\Lambda^d = \emptyset$

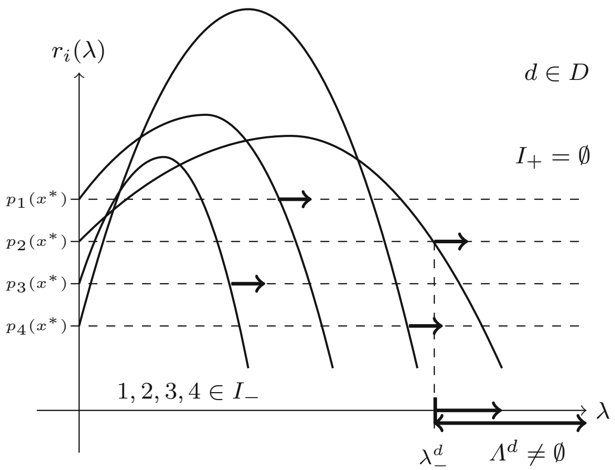

Fig. 9 The search for x that dominates $x^* \in Leff$ (UMQFP$_{x^*}$) in the search direction $d \in D$, in which $I_0 = \emptyset$, $I_+ = \emptyset$, $I_- = \{1, 2, 3, 4\}$ and $\Lambda^d \neq \emptyset$

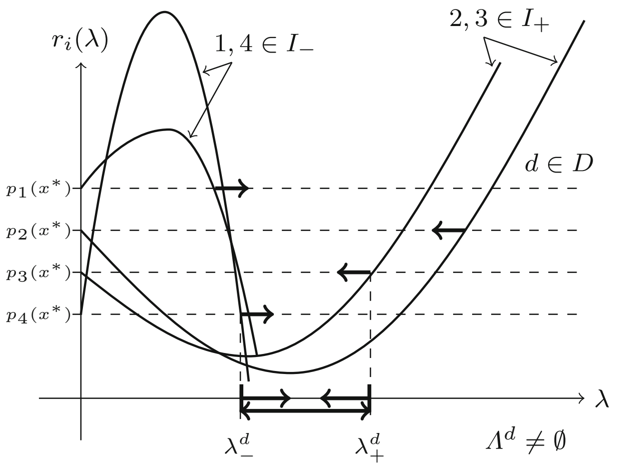

Fig. 10 The search for x that dominates $x^* \in$ *Leff* (UMQFP_{x^*}) in the search direction $d \in D$, in which $I_0 = \emptyset$, $I_+ = \{2, 3\}$, $I_- = \{1, 4\}$ and $\Lambda^d \neq \emptyset$

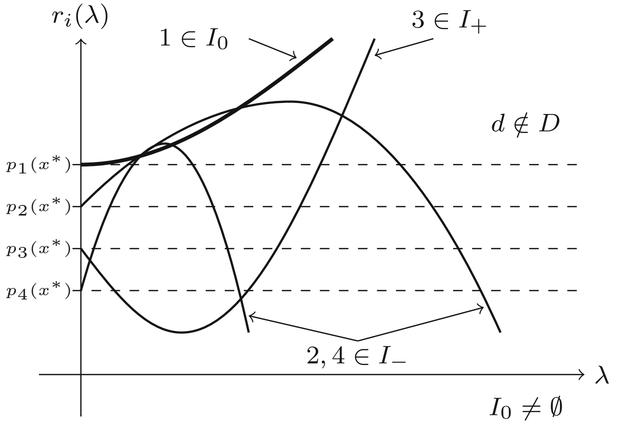

Fig. 11 The search for x that dominates $x^* \in$ *Leff* (UMQFP_{x^*}) in the search direction $d \notin D$, in which $I_0 = \{1\}$, $I_+ = \{3\}$ and $I_- = \{2, 4\}$

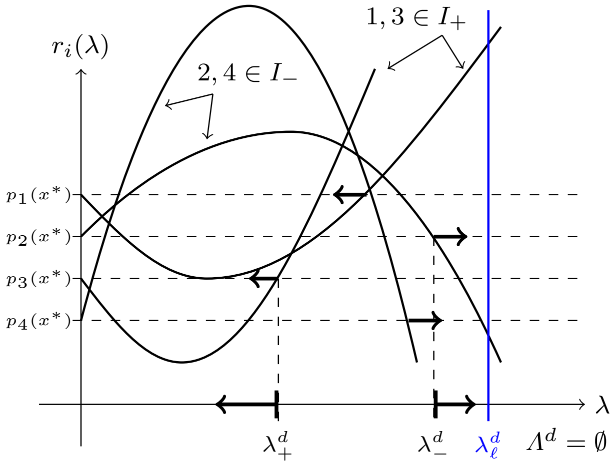

Fig. 12 The search for x that dominates $x^* \in \mathit{Leff}\ (\mathrm{MQFP}_{x^*})$ in the direction d, in which $I_0 = \emptyset$, $I_+ = \{1, 3\}$, $I_- = \{2, 4\}$, $\lambda_+^d < \lambda_-^d \leqq \lambda_\ell^d$ and $\Lambda^d = \emptyset$

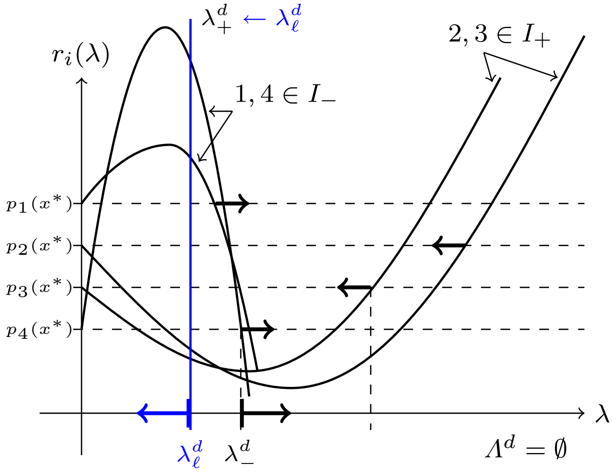

Fig. 13 The search for x that dominates $x^* \in \mathit{Leff}\ (\mathrm{MQFP}_{x^*})$ in the direction d, in which $I_0 = \emptyset$, $I_+ = \{2, 3\}$, $I_- = \{1, 4\}$, $\lambda_\ell^d < \lambda_-^d$, $\lambda_+^d \leftarrow \lambda_\ell^d$ and $\Lambda^d = \emptyset$

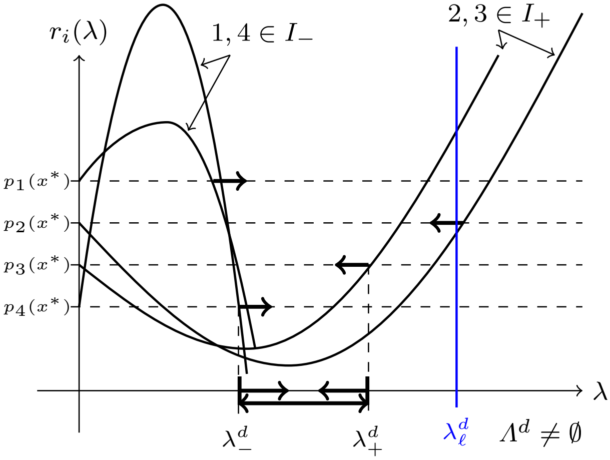

Fig. 14 The search for x that dominates $x^* \in Leff$ (MQFP_{x^*}) in the direction d, in which $I_0 = \emptyset$, $I_+ = \{2, 3\}$, $I_- = \{1, 4\}$, $\lambda_-^d \leqq \lambda_+^d < \lambda_\ell^d$ and $\Lambda^d \neq \emptyset$

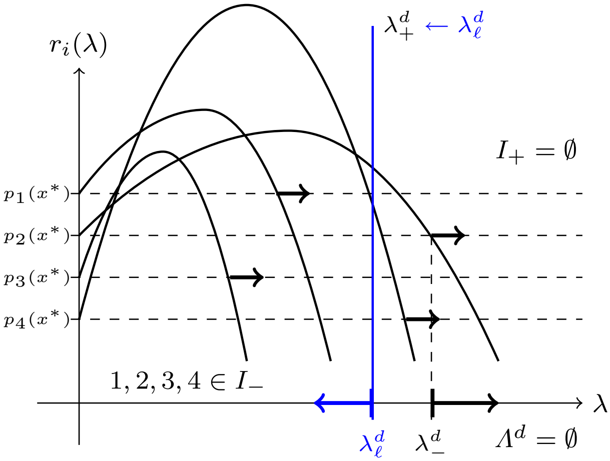

Fig. 15 The search for x that dominates $x^* \in Leff$ (MQFP_{x^*}) in the direction d, in which $I_0 = \emptyset$, $I_+ = \emptyset$, $I_- = \{1, 2, 3, 4\}$, $\lambda_\ell^d < \lambda_-^d$, $\lambda_+^d \leftarrow \lambda_\ell^d$ and $\Lambda^d = \emptyset$

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