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A consistent talmudic rule for division problems with multiple references

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Abstract We consider an extension of the classic division problem with claims, division problems with multiple references. We show that the theory of cooperative games is able to provide a single-valued allocation rule for this class of problems. Moreover, this rule can be related with the Talmud rule for the classic division problem, as in Aumann and Maschler (J. Econ. Theory 36, 195–213, 1985). Finally, we establish the consistency and other basic properties of the rule.

Keywords Division problems · Multiple references · Cooperative games · Talmud rule

Mathematics Subject Classification (2000) 91A06 · 91A12 · 91B32

1 Introduction

Aumann and Maschler (1985) proposed their well-known rule to divide an amount $E \in \mathbb{R}_{++}$ (the estate) of an infinitely divisible resource among a group of agents, N, having claims, $c_i \in \mathbb{R}_+$, on it, whose sum exceeds the estate. The rule assigns the

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amount $T_i(c, E)$ to each agent $i \in N$,

$$T_i(c, E) = \begin{cases} \min\{\frac{c_i}{2}, \lambda\} & \text{if } E \leq \sum_{i \in N} \frac{c_i}{2}, \\ c_i - \min\{\frac{c_i}{2}, \lambda\} & \text{otherwise,} \end{cases}$$

where $c = (c_i)_{i \in N}$ is the vector of claims and, in each case, $\lambda \in \mathbb{R}_+$ is chosen so as to achieve efficiency ($\sum_{i \in N} T_i(c, E) = E$).

This proposal rationalizes the recommendations made in the Talmud[1] for the *contested garment* and the *marriage contract* problems. Henceforth, the rule T was called the Talmud rule.

They also showed that the prenucleolus of a coalitional game associated with a claims problem, proposed by O'Neill (1982),[2] provides the division generated by the Talmud rule. A direct proof of this result is given in Benoît (1997).

The model can be reformulated to accommodate surplus sharing situations in which $E > \sum_{i \in N} c_i$. In this new setting, where the amount to divide is not necessarily below $\sum_{i \in N} c_i$, the vector containing the characteristics of the agents, c, will be called the vector of references.

The Talmud rule can be extended to this model by dividing equally the differences from the sum of the components of the vector of references when $E > \sum_{i \in N} c_i$ as

$$\bar{T}_i(c, E) = \begin{cases} \min\{\frac{c_i}{2}, \lambda\} & \text{if } E \leq \sum_{i \in N} \frac{c_i}{2}, \\ c_i - \min\{\frac{c_i}{2}, \lambda\} & \text{if } \sum_{i \in N} \frac{c_i}{2} < E \leq \sum_{i \in N} c_i, \\ c_i + \frac{E - \sum_{i \in N} c_i}{n} & \text{otherwise.} \end{cases}$$

The rule $\bar{T}$ still coincides with the prenucleolus of the corresponding coalitional game (see Serrano 1995).

In this paper we generalize this setting by considering an extension of the classic division problem to situations in which the characteristic of each agent is multidimensional, and therefore, several vectors of references have to be taken into account in the division. This kind of problems have been called in the literature *multi-issue allocation situations* and were introduced by Calleja et al. (2005). We will call these problems division problems with multiple references.[3]

In division problems with multiple references, two approaches are possible: In one of these approaches the budget is first allocated to the issues corresponding to each vector of references, and, in a second step, the amount assigned to each issue is divided among the agents; it is followed in Lorenzo-Freire et al. (2007, 2009), Moreno-Ternero (2009), Bergantiños and Lorenzo-Freire (2008), and Bergantiños et al. (2010). Another approach consists of directly providing a rule that assign an

[1]The Talmud is the collection of writings that constitute the basis for Jewish Law.

[2]The bankruptcy game proposed by O'Neill for the problem (c, E) is $(N, v_{(c,E)})$, where for each $S \subseteq N$, $v_{(c,E)}(S) = (E - c(N \setminus S))_+ = \max\{E - c(N \setminus S), 0\}$, where $c(N \setminus S) = \sum_{i \in N \setminus S} c_i$.

[3]This general terminology of "references" or "characteristics" is used in Ju et al. (2007) in order to subsume into their model, by choosing the meaning of variables appropriately, a number of existing and new allocations problems such as cost sharing, social choice with transferable utilities, income redistribution, bankruptcy with multiple types of assets. We borrow this terminology from them.

amount to each agent by taking into account simultaneously all the references. This is the approach followed in Calleja et al. (2005), González-Alcón et al. (2007), Ju et al. (2007), and also in the present paper.

Situations which can be represented by this model are, for instance, when University budgets are divided among different departments by taking into account different indicators such as number of students, research activity, teaching, etc. Once the quantity is assigned to the department, no constraint is imposed on what the final destination of the money should be. Other examples are: when the European Union distributes a budget between the members, by taking into account several characteristics of the countries, such as, total population, level of development, or total extension, or when the liquidation value of a bankrupt firm is divided among its creditors, and the different claims of each creditor are categorized by the type of assets.

The different references can also represent the assessments of rights or needs of the agents made by various experts, as in a situation in which funds are to be allocated to various research groups whose needs are assessed by different external evaluators.

This framework is also appropriate to address division problems where there is uncertainty about the references. For instance, if some creditors have to receive some assets from a firm at a future date, and the firm goes bankrupt before fulfilling its obligations, then, in order to divide the liquidation value of the firm, different future economic scenarios might be considered. The uncertainty about the claims of the creditors is included in the model by considering the values of the assets in different scenarios as references to allocate the estate.

Previous work on this model, Ju et al. (2007), considers a general framework with multi-dimensional characteristics and provides an axiomatic characterization of division rules which are non-manipulable by transfers among agents. The contributions of Pulido et al. (2002, 2008), can also be seen as particular cases of the model we are investigating, where only two vectors of references exist and one of them dominates the other.

The main objectives of this paper are to explore (i) whether the theory of cooperatives games is able to provide an allocation rule for the division problem with multiple references, and (ii) whether the rule can be related with the Talmud rule for the classic problem, as in Aumann and Maschler (1985).

For a division problem with multiple references, not only one, but several coalitional games can be associated with the problem by means of the procedure proposed by O'Neill (1982). In Hinojosa et al. (2005) this class of games is studied in a multi-scenario framework and a *generalized prenucleolus*, which unfortunately is not single-valued, is defined.

In this paper we explore a different approach to the division problem with multiple references. In Sect. 2 we describe the model. In Sect. 3 a single-valued rule for these problems, the prenucleolus of the coalitional game which assigns to each coalition the maximum value attained across the various references, is introduced. Section 4 is devoted to the analysis of the rule. We show that, in the two-agent case, the outcomes of the rule coincide with those of the extended Talmud rule, $\bar{T}$, applied to a division problem in which the agents claim the minimum of their references. We also show that, for more than two agents, the same is true if the amount to divide is not greater than the sum of the minimum claims. The behavior of the rule for the estates which are greater than the sum of the minima is also studied. In Sect. 5, we establish the consistency and other basic properties of the rule. Section 6 is devoted to conclusions, and the proofs are included in the appendix.

2 The model

Consider a set of potential agents $I \subset \mathbb{N}$, and a fixed finite set of issues[4] $M \subseteq \mathbb{N}$. Let $\mathcal{N}$ be the set of all non-empty finite subsets of I.

A *division problem with multiple references* is a pair $(C, E) \in \mathbb{R}_+^{N\times M} \times \mathbb{R}_{++}$, where $N \in \mathcal{N}$. We call matrix C the *matrix of references* and E is the estate to be allocated accordingly. By c_i^j we denote an element of matrix C. For each $i \in N$, the ith row of matrix C, $c_i \in \mathbb{R}^M$, represents the references of agent i with respect to the different issues. For each $j \in M$, the column $c^j \in \mathbb{R}^N$ represents the references of all the agents corresponding to the jth issue. We will also denote C as $(c_i)_{i\in N}$ or as $(c^j)^{j\in M}$. We assume that references are bounded from above, that is to say, there exists $q \in \mathbb{R}_{++}$ such that $c_i^j \le q$ for all $i \in N$, $j \in M$.

The class of all division problems with multiple references associated with the set of agents N and the set of issues M is denoted by $\mathcal{D}_N^M$, and the class of all division problems with multiple references where the set of issues is given by M is denoted by $\mathcal{D}^M$. Notice that $\mathcal{D}^M = \bigcup_{N\in\mathcal{N}} \mathcal{D}_N^M$.

A vector $x \in \mathbb{R}_+^N$, which satisfies the efficiency requirement, $\sum_{i\in N} x_i = E$, is called an *allocation* of the estate E. Let $X(E) \subseteq \mathbb{R}_+^N$ be the set of all the allocations of the estate E. A *division rule* over $\mathcal{D}_N^M$ is a function, R, that associates with each problem $(C, E) \in \mathcal{D}_N^M$ a unique allocation $R(C, E) \in X(E)$. We will also consider division rules over the class $\mathcal{D}^M$, which are functions that associate with each element of the class, a unique allocation of the estate.

3 The rule

For each division problem with multiple references, $(C, E) \in \mathcal{D}_N^M$, $|M|$ coalitional games, $(N, v_{(C,E)}^j)$, $j \in M$ can be defined by the procedure proposed by O'Neill. That is, for each $j \in M$ and for each $S \subseteq N$, $v_{(C,E)}^j(S) = (E - c^j(N \setminus S))_+ = \max\{E - c^j(N \setminus S), 0\}$, where $c^j(N \setminus S) = \sum_{i\in N\setminus S} c_i^j$.

Notice that the set of allocations of the estate, $X(E)$, is the set of vectors which accomplish efficiency in all these games. The proposal for a solution that we will discuss herein is based upon the differences between what the coalitions obtain with a certain allocation and their values in the coalitional games defined above. For each allocation, $x \in X(E)$, and each coalition, $S \subseteq N$, the $|M|$ *surplus functions* are $e_{v_{(C,E)}^j}(x, S) = v_{(C,E)}^j(S) - x(S)$, $j \in M$, where $x(S)$ denotes $\sum_{i\in S} x_i$. These func-

[4] We adopt this terminology, although the set M could also represent the set of different future scenarios, or the set of experts assessing the values that will be taken into account.

tions measure the dissatisfaction of coalition S at x with respect to all the issues, and plays a central role in the definition of our allocation rule.

The goal is to select allocations that are better in a lexicographic sense. If a unique vector of references is considered, a lexicographical order among the allocations can be defined, and a unique best outcome can be determined, the prenucleolus. For the case of several vectors of references, we will consider the maximum surplus across the issues as a measure of the dissatisfaction of coalition S at x, that is,

$$e_{(C,E)}(x, S) = \max_{j \in M}\left\{e_{v^j_{(C,E)}}(x, S)\right\} = \max_{j \in M}\left\{v^j_{(C,E)}(S)\right\} - x(S).$$

For each $x \in X(E)$ an $(2^N - 2)$-dimensional vector, $\pi_{(C,E)}(x)$, is constructed with the maximum surplus, $e_{(C,E)}(x, S)$, $S \subset N$, arranged in decreasing order. Vector $\pi_{(C,E)}(x)$ is a vector-valued measure of the performance of allocation x with respect to all the coalitions which takes into account all the issues.

We say that vector $\pi(x)$ is lexicographically better than vector $\pi(y)$, $\pi(x) <_{\text{lex}} \pi(y)$, if $\pi^k(x) < \pi^k(y)$ for the first component, k, in which vector $\pi(x)$ and vector $\pi(y)$ are different. This binary relation defines a complete order and therefore, a division rule can be defined in the class $\mathcal{D}^M_N$, by selecting, for each $(C, E) \in \mathcal{D}^M_N$, the allocation which minimizes lexicographically $\pi(x)$ from among all the allocations $x \in X(E)$.

Definition 3.1 The multiple-reference talmudic rule, *MT*, is for each $(C, E) \in \mathcal{D}^M_N$, $MT(C, E) = \arg \text{lex-min}_{x \in X(E)}\{\pi_{(C,E)}(x)\}$, where $\pi_{(C,E)}(x)$ is a vector whose components are the maximum surplus across issues, $e_{(C,E)}(x, S)$, $S \subset N$, arranged in decreasing order.

Note that for each $(C, E) \in \mathcal{D}^M_N$, the outcomes provided by the *MT* rule coincide with those obtained with the rpenucleolus of the coalitional game $(N, v^{\max}_{(C,E)})$, where $v^{\max}_{(C,E)}(S) = \max_{j \in M}\{v^j_{(C,E)}(S)\}$. This result follows from the fact that $e_{(C,E)}(x, S) = e_{v^{\max}_{(C,E)}}(x, S)$ for each $x \in X(E)$ and each $S \subset N$.

4 The path of awards of the rule

For each $(C, E) \in \mathcal{D}^M_N$, denote by $\underline{c}$ the vector whose components represent the minimum value from among the references of each agent, $\underline{c}_i = \min_{j \in M}\{c^j_i\}$, $i \in N$, and by $\underline{c}(N)$ the sum $\underline{c}(N) = \sum_{i=1}^n \underline{c}_i$.

In what follows, we will show that for values of the estate below the sum $\sum_{i=1}^n \underline{c}_i$, the *MT* rule behaves as the Talmud rule if the vector of minimum references, $\underline{c}$, is considered as the vector of claims. We will also analyze the path of awards of *MT* for values of the estate greater than the sum of the minimum references.

4.1 The value of the estate is below the sum of the minimum references

Without loss of generality, assume that $N = \{1, 2, \ldots, n\}$ and $\underline{c}_1 \leq \underline{c}_2 \leq \cdots \leq \underline{c}_n$. The result below establishes that, when the value of the estate does not exceed $\underline{c}(N)$,

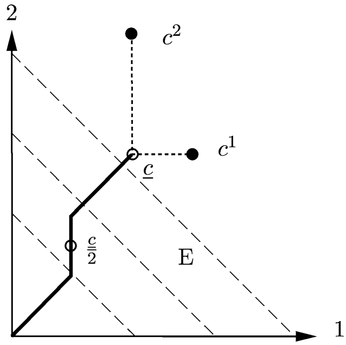

Fig. 1 Path of awards of *MT* in the two-agent case when $E < \underline{c}(N)$

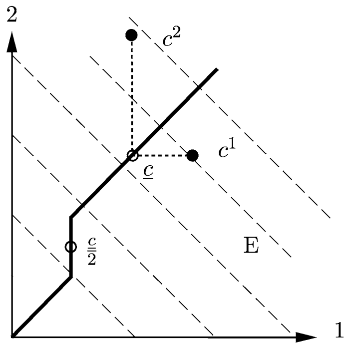

Fig. 2 Path of awards of *MT* in the two-agent case

then the *MT* rule provides the same vector of awards as the Talmud rule for a classic division problem with claims equal to $\underline{c}$. Figure 1 is an illustration for a two-agent case.

Theorem 4.1 *For each* $(C, E) \in \mathcal{D}_N^M$, *such that* $E \leq \underline{c}(N)$, *and each* $i \in N$,

$$MT_i(C, E) = T_i(\underline{c}, E) = \begin{cases} \min\{\frac{\underline{c}_i}{2}, \lambda\} & \text{if } E \leq \frac{\underline{c}(N)}{2}, \\ \underline{c}_i - \min\{\frac{\underline{c}_i}{2}, \lambda\} & \text{if } \frac{\underline{c}(N)}{2} < E \leq \underline{c}(N), \end{cases}$$

where $\lambda \in \mathbb{R}_+$ *is such that satisfies* $\sum_{i \in N} MT_i(C, E) = E$.

4.2 The value of the estate is above the sum of the minimum references

We first show that, in the two-agent case, when applying the *MT* rule, the difference $E - \underline{c}(N)$ is allocated equally between the agents (see Fig. 2). As a consequence, for the two-agent case, the outcomes obtained with this rule coincide with those obtained with the classic Talmud rule with claims $\underline{c}$.

Theorem 4.2 *For each* $(C, E) \in \mathcal{D}_N^M$ *with* $|N| = 2$ *and each* $i \in N$,

$$MT_i(C, E) = \begin{cases} \min\{\frac{c_i}{2}, \lambda\} & \textit{if } E \leq \frac{\underline{c}(N)}{2}, \\ \underline{c}_i - \min\{\frac{c_i}{2}, \lambda\} & \textit{if } \frac{\underline{c}(N)}{2} < E \leq \underline{c}(N), \\ \underline{c}_i + \frac{E - \underline{c}(N)}{|N|} & \textit{if } \underline{c}(N) < E, \end{cases}$$

where $\lambda \in \mathbb{R}_+$ *is such that satisfies* $\sum_{i \in N} MT_i(C, E) = E$.

Unfortunately, the result in Theorem 4.2 does not hold in the general case, as can be seen in the following example.

Example 4.3 Consider a division problem with three agents, $N = \{1, 2, 3\}$ and two references, $c^1 = (3, 7, 10)^t$ and $c^2 = (15, 9, 2)^t$.

When $E \leq \underline{c}(N) = 12$, the path of awards of *MT* (blue path) coincides with the path of awards of the Talmud rule with the reference $\underline{c} = (3, 7, 2)$. When $E \geq 12 = \underline{c}(N)$ the path of awards of *MT* is piecewise linear with four different slopes depending on the value of the estate. The following table shows how the *MT* rule allocates any additional unit of estate in the different intervals.

Allocation of an additional unit of estate			
	Agent 1	Agent 2	Agent 3
$12 = \underline{c}(N) < E \leq 16$	0.25	0.25	0.5
$16 < E \leq 28$	0.5	0	0.5
$28 < E \leq 32$	0.25	0.25	0.5
$32 < E < +\infty$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

In Fig. 3 we show the path of awards of the *MT* rule for this three-agent division problem, divided in two parts. The path when the amount to divide is not greater than the sum of the minimum references is shown on the left-hand side. On the right-hand side, the path for quantities greater than this sum is represented. In this second case, the coordinate axes are translated to the point $\underline{c} = (3, 7, 2)$.

The following result establishes that, in general, when $|N| \geq 3$ and $E > \underline{c}(N)$, the path of awards of *MT* is piecewise linear with the last piece of the path parallel to the line $x_1 = x_2 = \cdots = x_n$. That is, for each matrix of references, a value of the estate exists from which any additional amount is allocated equally to the agents.

Theorem 4.4 *MT has a continuous, piecewise linear path of awards. Moreover, there exists* $E(C) \in \mathbb{R}_{++}$ *such that, for each* $i \in N$, *and any* $A > 0$, $MT_i(C, E(C) + A) = MT_i(C, E(C)) + \frac{A}{n}$.

The proof of this result shows the hints to construct a procedure to obtain the allocations in any particular problem. We describe below an algorithm to compute the allocation provided by the *MT* rule in a general problem (C, E). This algorithm has been used to obtain the path of awards of *MT* in Example 4.3. As in the proof of Theo-

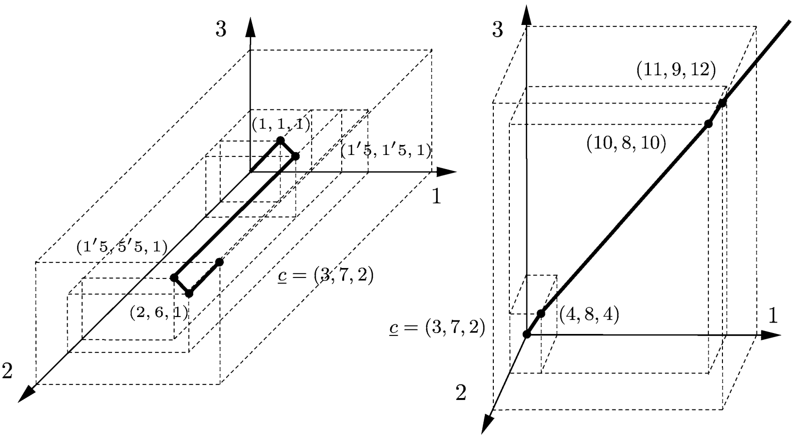

Fig. 3 Path of awards of *MT* in a three-agent example

rem 4.4, each estate $E \geq \underline{c}(N)$ will be denoted by E^x, where $x = MT(C, E^x)$; $E^* = \max\{\underline{c}(N), \max_{i \in N} \min_{j \in M} c^j(N \setminus \{i\})\}$; $F^x = \{i \in N \mid e(x, \{i\}) \geq e(x, \{j\})\ \forall j \in N\}$; α^x is an n-dimensional vector, satisfying $\sum_{i=1}^{n} \alpha_i^x = 1$, which represent the proportion of any infinitesimal increment of the state E^x which is assigned to each agent according to the rule *MT*; and A^x is the maximum possible increase of the state E^x for which any two coalitions with the same dissatisfaction level at x, maintain the same level of dissatisfaction.

Algorithm

IF $E \leq \underline{c}(N)$
    $x = T(\underline{c}, E)$
ELSE
    $i \leftarrow 0$
    $x_i \leftarrow \underline{c}$
    $E^{x_i} \leftarrow \underline{c}(N)$
    $F^{x_i} \leftarrow F^{\underline{c}}$
    Compute E^*
    WHILE $E^{x_i} < E^*$ or $F^{x_i} \neq N$
    DO
        Compute α^{x_i}, A^{x_i}, $y = x_i + A^{x_i}\alpha^{x_i}$ and $E^y = E^{x_i} + A^{x_i}$
        IF $E \leq E^y$
            $x = x_i + (E - E^{x_i})\alpha^{x_i}$
        ELSE
            $i \leftarrow i + 1$
            $x_i \leftarrow y$
            $E^{x_i} \leftarrow E^y$
        END IF
    END WHILE
    $x = x_i + (E - E^{x_i})\alpha^{x_i}$
END IF

5 Properties

Anonymity and neutrality are two basic properties that are satisfied by the *MT* rule on the class of problems with multiple references $\mathcal{D}_N^M$.

Let Π_N denote the set of permutations of the set of agents N. Let $\pi \in \Pi_N$, for $x \in \mathbb{R}^N$, denote $x_\pi \equiv (x_{\pi(i)})_{i \in N}$. For $C \in \mathbb{R}^{N \times M}$, denote by C_π the matrix whose kth row is $c_{\pi(k)}$ for $k \in N$. Anonymity states that the names of the agents are not relevant.

Anonymity: For each $(C, E) \in \mathcal{D}_N^M$ and each $\pi \in \Pi_N$, if $x = R(C, E)$, then $x_\pi = R(C_\pi, E)$.

Let Π^M denote the set of permutations of the set of issues M. For $C \in \mathbb{R}^{N \times M}$, denote by C^σ the matrix whose jth column is $c^{\sigma(j)}$ for all $j \in M$. Neutrality is a symmetry property with respect to the references. It states that the names of issues do not matter.

Neutrality: For each $(C, E) \in \mathcal{D}_N^M$ and each $\sigma \in \Pi^M$, $R(C, E) = R(C^\sigma, E)$.

Finally, the property below refers to division rules in the class where a fixed set of issues is considered, but the number of agents may vary. Denote by $C_{N'}$ the submatrix of C whose rows are the rows in C corresponding to the agents in N' (analogously $x_{N'}$).

Consistency: Given a finite set of issues, $M \subset \mathbb{N}$, for each $N \in \mathcal{N}$, each $(C, E) \in \mathcal{D}_N^M$, and each $N' \subset N$, if $x = R(C, E)$, then $x_{N'} = R(C_{N'}, x(N'))$.

Proposition 5.1 *MT satisfies anonymity, neutrality and consistency.*

6 Conclusions

We have addressed the extension of the classic division problem with claims to situations in which the agents involved are characterized by several parameters instead of by a single one.

For this class of problems we show that the theory of cooperative games is able to provide an allocation rule related to the classic Talmud rule, as is the case for single-dimensional problems. It is shown that for two-agents problems, the rule introduced generates the same results as the extended Talmud rule applied to a classic division problem obtained with the minimal reference of each agent. We also show that, for more than two agents, the same is true if the amount to divide is not greater than the sum of the components of the vector of minimal references. However, in the general case this result does not hold. Finally, we provide results about the behavior of the rule and its properties for the general case.

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Appendix

Proof Theorem 4.1 The proof is based on the proof provided by Benoît (1997) to show that the Talmud rule coincides with the prenucleolus of the corresponding bankruptcy game. In what follows, we will denote $T(\underline{c}, E)$ by $\bar{x}$. We are going to show that $\bar{x}$ coincides with the outcome provided by the prenucleolus of the coalitional game $(N, v^{\max}_{(C,E)})$ and therefore $\bar{x}$ is the allocation obtained from the rule *MT*, that is, $MT(C, E) = \bar{x}$. Our proof is based on several claims.

Claim 1 *For each $i, l \in N$, such that $i > l$, $\bar{x}_i \geq \bar{x}_l$ and $\underline{c}_i - \bar{x}_i \geq \underline{c}_l - \bar{x}_l$.*

Proof If $i > l$, then $\underline{c}_i \geq \underline{c}_l$, and since the awards provided by the Talmud rule, and the losses with respect to the claims obtained with the Talmud rule are order preserving, the result follows. □

In what follows, we will denote $v^{\max}$ instead of $v^{\max}_{(C,E)}$ and $e(x, S)$ instead of $e_{v^{\max}_{(C,E)}}(x, S)$ to simplify presentation.

Claim 2 *The surplus of coalition $S \subseteq N$ at $x \in X(E)$ in the coalitional game $(N, v^{\max})$ is*:

$$e(x, S) = \begin{cases} x(N \setminus S) - \min_{j \in M} c^j(N \setminus S) & \text{if } E - \min_{j \in M} c^j(N \setminus S) \geq 0, \\ -x(S) & \text{if } E - \min_{j \in M} c^j(N \setminus S) \leq 0. \end{cases}$$

Proof

$$\begin{aligned} e(x, S) &= v^{\max}(S) - x(S) = \max_{j \in M} v^j(S) - x(S) \\ &= \max_{j \in M} \big(E - c^j(N \setminus S)\big)_+ - x(S) \\ &= \Big(E - \min_{j \in M} c^j(N \setminus S)\Big)_+ - \big(E - x(N \setminus S)\big) \\ &= \begin{cases} x(N \setminus S) - \min_{j \in M} c^j(N \setminus S) & \text{if } E - \min_{j \in M} c^j(N \setminus S) \geq 0, \\ -x(S) & \text{if } E - \min_{j \in M} c^j(N \setminus S) \leq 0. \end{cases} \end{aligned}$$ □

Consider the following two cases:

Case (a) $E \leq \frac{c(N)}{2}$.

In this case, there exists $k \in N$, $0 \leq k \leq n - 1$, such that

$$\sum_{i=1}^{k} \frac{\underline{c}_i}{2} + (n-k)\frac{\underline{c}_k}{2} \leq E \leq \sum_{i=1}^{k} \frac{\underline{c}_i}{2} + (n-k)\frac{\underline{c}_{k+1}}{2},$$

where, when $k = 0$, $\underline{c}_0 = 0$ and $\sum_{i=1}^{0} \frac{\underline{c}_i}{2} = 0$.

Claim 3 *For each $l \in N$, $l \leq n-1$, and each $x \in X(E)$, $e(x, \{l\}) = -x_l$. Moreover, if $k \leq n-2$, then $e(x, \{n\}) = -x_n$.*

Proof $E - \min_{j \in M}\{c^j(N \setminus \{l\})\} \leq E - \underline{c}(N \setminus \{l\})$, since the sum of the minima is always less than or equal to the minimum of the sum. Therefore, since in this case, $E \leq \frac{\underline{c}(N)}{2}$, then $E - \min_{j \in M} c^j(N \setminus \{l\}) \leq E - \underline{c}(N \setminus \{l\}) \leq \frac{\underline{c}(N)}{2} - \underline{c}(N \setminus \{l\})$ holds, and this difference can be written as follows:

$$\frac{\underline{c}(N)}{2} - \underline{c}\big(N \setminus \{l\}\big) = \sum_{i=1}^{n-2}\left(\frac{\underline{c}_i}{2} - \underline{c}_i\right) + \left(\frac{\underline{c}_{n-1}}{2} + \frac{\underline{c}_n}{2} - \underline{c}_n\right) + (\underline{c}_l - \underline{c}_{n-1}) \leq 0.$$

Hence, by Claim 2, $e(x, \{l\}) = -x_l$. Moreover, if $k \leq n-2$, then $E \leq \sum_{i=1}^{n-2} \frac{\underline{c}_i}{2} + 2\frac{\underline{c}_{n-1}}{2}$ and therefore $E - \min_{j \in M} c^j(N \setminus \{n\}) \leq E - \underline{c}(N \setminus \{n\}) \leq \sum_{i=1}^{n-2} \frac{\underline{c}_i}{2} + 2\frac{\underline{c}_{n-1}}{2} - \underline{c}(N \setminus \{n\}) = \sum_{i=1}^{n-2}(\frac{\underline{c}_i}{2} - \underline{c}_i) \leq 0$. Thus, by Claim 2 again, $e(x, \{n\}) = -x_n$. □

Claim 4 *For each $l \in N$, $l \leq k$, and each $x \in X(E)$, it follows that $e(x, N \setminus \{l\}) = x_l - \underline{c}_l$.*

Proof Since $E \geq \sum_{i=1}^{k} \frac{\underline{c}_i}{2} + (n-k)\frac{\underline{c}_k}{2}$, then $E - \min_{j \in M} c^j(N \setminus (N \setminus \{l\})) = E - \underline{c}_l \geq \sum_{i=1}^{k} \frac{\underline{c}_i}{2} + (n-k)\frac{\underline{c}_k}{2} - \underline{c}_l \geq \sum_{i=1}^{k} \frac{\underline{c}_i}{2} + (n-k)\frac{\underline{c}_k}{2} - \underline{c}_k \geq \sum_{i=1}^{k-1} \frac{\underline{c}_i}{2} + (n-k-1)\frac{\underline{c}_k}{2} \geq 0$. Then by Claim 2, it follows that $e(x, N \setminus \{l\}) = x_l - \underline{c}_l$. □

Let $x \in X(E)$, $x \neq \bar{x}$, and let l, $1 \leq l \leq n-1$ be the lowest index in which x and $\bar{x}$ differ.

Claim 5 *For each coalition $S \subseteq N$ such that $e(\bar{x}, S) > -\bar{x}_l$, $e(\bar{x}, S) = e(x, S)$.*

Proof If $v^{\max}(S) = 0$, then, by Claim 2, $e(\bar{x}, S) = -\bar{x}(S)$ and, therefore, $-\bar{x}_l < e(\bar{x}, S) = -\bar{x}(S) \leq -\bar{x}_i$ for all $i \in S$. Thus, by Claim 1, $i < l$. This means that $e(\bar{x}, S) = e(x, S)$, since the first component for which x and $\bar{x}$ differ is component l.

If, on the other hand, $v^{\max}(S) > 0$, then, by Claim 2, $e(\bar{x}, S) = \bar{x}(N \setminus S) - \min_{j \in M} c^j(N \setminus S)$ and therefore the following chain of inequalities holds:

$$-\frac{\underline{c}_n}{2} \leq -\frac{\underline{c}_{n-1}}{2} \leq \cdots \leq -\frac{\underline{c}_{l+1}}{2} \leq -\frac{\underline{c}_l}{2} \leq -\bar{x}_l < e(\bar{x}, S)$$
$$= \bar{x}(N \setminus S) - \min_{j \in M} c^j(N \setminus S) \leq \frac{\underline{c}(N \setminus S)}{2} - \underline{c}(N \setminus S) = -\frac{\underline{c}(N \setminus S)}{2} \leq -\frac{\underline{c}_i}{2},$$

where the last inequality holds for all i which does not belong to S. Thus, by Claim 1, $i < l$. This means that $e(\bar{x}, S) = e(x, S)$, since the first component for which x and $\bar{x}$ differ is component l. □

Claim 6 *There exists $S \subseteq N$ such that $e(\bar{x}, S) = -\bar{x}_l$ and $e(x, S) > -\bar{x}_l$.*

Proof Suppose first that $l \leq k$. If $x_l < \bar{x}_l$, then, by Claim 3, $e(x, \{l\}) = -x_l > -\bar{x}_l = e(\bar{x}, \{l\})$. If, on the other hand, $x_l > \bar{x}_l$, then, by Claim 4, $e(x, N \setminus \{l\}) = x_l - \underline{c}_l > \bar{x}_l - \underline{c}_l = \frac{-\underline{c}_l}{2} = -\bar{x}_l = e(\bar{x}, N \setminus \{l\})$.

Suppose now that $l \geq k+1$. On the one hand, since $k \leq l-1 \leq n-2$, and by taking into account the result in Claim 3, $e(x, \{i\}) = -x_i$, for all $i \in N$ and all $x \in X(E)$, and on the other hand, since $\bar{x}$ is the outcome provided by the Talmud rule, $\bar{x}_i = \bar{x}_l$, for all $i \geq l$.

If $x_l < \bar{x}_l$, then $e(x, \{l\}) = -x_l > -\bar{x}_l = e(\bar{x}, \{l\})$. Otherwise, since $x_l > \bar{x}_l$, $x_i = \bar{x}_i$ for each $i < l$ and $x(N) = \bar{x}(N)$, then there exists $l^* > l$ such that $x_{l^*} < \bar{x}_{l^*}$ and therefore $e(x, \{l^*\}) = -x_{l^*} > -\bar{x}_{l^*} = -\bar{x}_l = e(\bar{x}, \{l^*\})$. □

Claims 5 and 6 permit us to conclude the proof in Case (a).

Case (b) $\frac{\underline{c}(N)}{2} \leq E \leq \underline{c}(N)$.

In this case, there exists $k \in N$, $n-1 \geq k \geq 0$, such that

$$\sum_{i=1}^{n} \underline{c}_i - \left(\sum_{i=1}^{k} \frac{\underline{c}_i}{2} + (n-k)\frac{\underline{c}_{k+1}}{2} \right) \leq E \leq \sum_{i=1}^{n} \underline{c}_i - \left(\sum_{i=1}^{k} \frac{\underline{c}_i}{2} + (n-k)\frac{\underline{c}_k}{2} \right),$$

where, when $k = 0$, $\underline{c}_0 = 0$ and $\sum_{i=1}^{0} \frac{\underline{c}_i}{2} = 0$.

Claim 7 *For each $l \in N$, $l \leq k$, and each $x \in X(E)$, $e(x, \{l\}) = -x_l$.*

Proof $E - \min_{j \in M} c^j(N \setminus \{l\}) \leq E - \underline{c}(N \setminus \{l\})$, since the sum of the minima is always less than or equal to the minimum of the sum. Thus, since in this case $E \leq \underline{c}(N) - (\sum_{i=1}^{k} \frac{\underline{c}_i}{2} + (n-k)\frac{\underline{c}_k}{2})$, then $E - \min_{j \in M} c^j(N \setminus \{l\}) \leq E - \underline{c}(N \setminus \{l\}) \leq -(\sum_{i=1}^{k} \frac{\underline{c}_i}{2} + (n-k)\frac{\underline{c}_k}{2}) + \underline{c}_l \leq -(\sum_{i=1}^{k} \frac{\underline{c}_i}{2} + (n-k)\frac{\underline{c}_k}{2}) + \frac{\underline{c}_l}{2} + \frac{\underline{c}_k}{2} = -(\sum_{i=1}^{k} \frac{\underline{c}_i}{2} - \frac{\underline{c}_l}{2} + (n-k-1)\frac{\underline{c}_k}{2}) \leq 0$. Then, by Claim 2, $e(x, \{l\}) = -x_l$. □

Claim 8 *For each $l \in N$, $l \leq n-1$, and each $x \in X(E)$, $e(x, N \setminus \{l\}) = x_l - \underline{c}_l$.*

Proof Since $E \geq \underline{c}(N) - (\sum_{i=1}^{k} \frac{\underline{c}_i}{2} + (n-k)\frac{\underline{c}_{k+1}}{2})$, then $E - \min_{j \in M} c^j(N \setminus (N \setminus \{l\})) = E - \underline{c}_l \geq \underline{c}(N) - (\sum_{i=1}^{k} \frac{\underline{c}_i}{2} + (n-k)\frac{\underline{c}_{k+1}}{2}) - \underline{c}_l = \sum_{i=1}^{k} \frac{\underline{c}_i}{2} + \sum_{i=k+1}^{n-2} (\underline{c}_i - \frac{\underline{c}_{k+1}}{2}) + (\underline{c}_{n-1} - \underline{c}_l) + (\underline{c}_n - \underline{c}_{k+1}) \geq 0$. Therefore, by Claim 2, it follows that $e(x, N \setminus \{l\}) = x_l - \underline{c}_l$. □

Consider $x \in X(E)$, $x \neq \bar{x}$, and let l, $1 \leq l \leq n-1$, be the lowest index in which x and $\bar{x}$ differ.

Claim 9 *For each coalition $S \subseteq N$ such that $e(\bar{x}, S) > \bar{x}_l - \underline{c}_l$, $e(\bar{x}, S) = e(x, S)$.*

Proof If $v^{\max}(S) = 0$, then, by Claim 2, $e(\bar{x}, S) = -\bar{x}(S)$ and therefore, $\bar{x}_l - \underline{c}_l < e(\bar{x}, S) = -\bar{x}(S) \leq -\bar{x}_i \leq \bar{x}_i - \underline{c}_i$ for all $i \in S$ (the last inequality holds because in Case 2 the losses of each agent from the minimum references are always less or equal than the allocation they obtain, that is, for each $i \in N$, $\bar{x}_i \geq \underline{c}_i - \bar{x}_i$). Thus, by Claim 1, $i > l$. This means that $e(\bar{x}, S) = e(x, S)$ because x and $\bar{x}$ does not differ until component l.

If, on the other hand, $v^{\max}(S) > 0$, then, by Claim 2, $\bar{x}_l - \underline{c}_l < e(\bar{x}, S) = x(N \setminus S) - \min_{j \in M} c^j(N \setminus S) \leq x(N \setminus S) - \underline{c}(N \setminus S) \leq \bar{x}_i - \underline{c}_i$ for all $i \notin S$. Thus, by Claim 1, if $i \geq l$, then $i \in S$. This means that $e(\bar{x}, S) = e(x, S)$ since the first component for which x and $\bar{x}$ differ is component l. □

Claim 10 *There exists $S \subseteq N$ such that $e(\bar{x}, S) = \bar{x}_l - \underline{c}_l$ and $e(x, S) > \bar{x}_l - \underline{c}_l$.*

Proof First, suppose that $l \leq k$. If $x_l < \bar{x}_l$, then, by Claim 7, $e(x, \{l\}) = -x_l > -\bar{x}_l = -\frac{\underline{c}_l}{2} = \frac{\underline{c}_l}{2} - \underline{c}_l = \bar{x}_l - \underline{c}_l = e(\bar{x}, \{l\})$. If, on the other hand, $x_l > \bar{x}_l$, then, by Claim 8, $e(x, N \setminus \{l\}) = x_l - \underline{c}_l > \bar{x}_l - \underline{c}_l = e(\bar{x}, N \setminus \{l\})$.

Suppose now that $l \geq k + 1$. Then, since $\bar{x}$ is the outcome provided by the Talmud rule, $\bar{x}_i - \underline{c}_i = \bar{x}_l - \underline{c}_l$, for all $i \geq l$.

If $x_l > \bar{x}_l$, then $e(x, N \setminus \{l\}) = x_l - \underline{c}_l > \bar{x}_l - \underline{c}_l = e(\bar{x}, N \setminus \{l\})$. Otherwise, since $x_l < \bar{x}_l$, $x_i = \bar{x}_i$ for each $i < l$ and $x(N) = \bar{x}(N)$, then there exists $l^* > l$ such that $x_{l^*} > \bar{x}_{l^*}$ and therefore $e(x, N \setminus \{l^*\}) = x_{l^*} - \underline{c}_{l^*} > \bar{x}_{l^*} - \underline{c}_{l^*} = \bar{x}_l - \underline{c}_l = e(\bar{x}, N \setminus \{l^*\})$. □

Claims 9 and 10 permit us to conclude the proof in Case (b). Therefore, $\bar{x}$ coincides with the prenucleolus of $(N, v^{\max})$ and, as a consequence, $\bar{x} = MT(C, E)$. □

Proof Theorem 4.2 In the two-agent case, $N = \{1, 2\}$, for each $i \in N$, $v^{\max}(\{i\}) = \max_{j \in M} v^j(\{i\}) = \max_{j \in M}(E - c^j_{N \setminus \{i\}})_+ = (E - \underline{c}_{N \setminus \{i\}})_+$, that is, $(N, v^{\max})$ coincides with the game $(N, v_{(\underline{c}, E)})$ and the prenucleolus of this game is $MT(C, E)$ (see Serrano 1995). □

Proof Theorem 4.4 Since the outcomes provided by MT coincide with those obtained from the prenucleolus of the coalitional game $(N, v^{\max})$, the path of awards of MT is continuous.

For the two-agent case, as a consequence of Theorems 4.1 and 4.2, the result follows and $E(C) = \underline{c}(N) - \underline{c}_1$ (recall that $\underline{c}_1 \leq \underline{c}_2$).

Suppose that $|N| \geq 3$. In what follows, we denote $\underline{c}(N)$ by $E^{\underline{c}}$, and each estate $E \geq E^{\underline{c}}$ will be denoted by E^x, where $x = MT(C, E^x)$. For each $E^{x'} = E^x + \varepsilon$ $(\varepsilon > 0)$, we write $x' = x + \varepsilon\alpha^x$, where α^x is an n-dimensional vector which satisfies $\sum_{i=1}^n \alpha_i^x = 1$. For each $S \subseteq N$, denote by α_S^x the sum $\alpha_S^x = \sum_{i \in S} \alpha_i^x$ and by α_{-S}^x the sum $\alpha_{-S}^x = \sum_{i \notin S} \alpha_i^x$. We also denote $\min_{i \in N}\{\alpha_i^x\}$ by $m(\alpha^x)$ and by I^{α^x} the set $I^{\alpha^x} = \{i \in N \mid \alpha_i^x = m(\alpha^x)\}$.

When the estate increases from E^x to $E^{x'} = E^x + \varepsilon$, the dissatisfaction of coalition S at x, $e(x, S) = v^{\max}(S) - x(S)$, changes to $e(x', S) = e(x, S) + \varepsilon\nabla^{\alpha^x}(S)$, where $\nabla^{\alpha^x}(S)$ can be written as shown in the following claim.

Claim 1 *For each* $E^x \in \mathbb{R}_+$, *such that* $E^{\underline{c}} \leq E^x < E^{x'} = E^x + \varepsilon$,

$$\varepsilon \nabla^{\alpha^x}(S) = e\big(x', S\big) - e(x, S) = \begin{cases} \varepsilon \alpha^x_{-S} & \text{if } E^x \geq \min_{j \in M} c^j(N \setminus S), \\ \varepsilon \alpha^x_{-S} - B & \text{if } E^x < \min_{j \in M} c^j(N \setminus S) < E^{x'}, \\ -\varepsilon \alpha^x_S & \text{if } E^{x'} \leq \min_{j \in M} c^j(N \setminus S), \end{cases}$$

where $B = \min_{j \in M}\{c^j(N \setminus S)\} - E^x$.

Moreover, if $E^x \geq E^* = \max\{E^{\underline{c}}, \max_{i \in N} \min_{j \in M} c^j(N \setminus \{i\})\}$, *then*:

1. *For each* $S \subset N$, $\varepsilon \nabla^{\alpha^x}(S) = \varepsilon \alpha^x_{-S}$.
2. *If* $m(\alpha^x) > 0$ *and* $i \in I^{\alpha^x}$, *then* $\nabla^{\alpha^x}(\{i\}) = \nabla^{\alpha^x}(\{j\})$ *for each* $j \in I^{\alpha^x}$ *and* $\nabla^{\alpha^x}(\{i\}) > \nabla^{\alpha^x}(S)$ *for each* $S \subset N$, $S \neq \{j\}$, $j \in I^{\alpha^x}$.
3. *If* $m(\alpha^x) = 0$, *then* $\nabla^{\alpha^x}(\{i\}) = \nabla^{\alpha^x}(S)$ *for each* $S \subset I^{\alpha^x}$, *and* $\nabla^{\alpha^x}(\{i\}) > \nabla^{\alpha^x}(S)$ *for each* $S \nsubseteq I^{\alpha^x}$.

Proof The difference of the dissatisfactions of coalition S at x and at x' is $e(x', S) - e(x, S) = (E^{x'} - \min_{j \in M} c^j(N \setminus S))_+ - x'(S) - (E^x - \min_{j \in M} c^j(N \setminus S))_+ + x(S)$. If $E^x \geq \min_{j \in M} c^j(N \setminus S)$, then $e(x', S) - e(x, S) = \varepsilon - \varepsilon \alpha^x_S = \varepsilon \alpha^x_{-S}$. If $E^{x'} \leq \min_{j \in M} c^j(N \setminus S)$, then $e(x', S) - e(x, S) = -\varepsilon \alpha^x_S$. Finally, if $E^x < \min_{j \in M} c^j(N \setminus S) < E^{x'}$, then $e(x', S) - e(x, S) = E^x + \varepsilon - \min_{j \in M} c^j(N \setminus S) - \varepsilon \alpha^x_S = \varepsilon \alpha^x_{-S} - (\min_{j \in M} c^j(N \setminus S) - E^x)$.

Moreover, if $E^x \geq E^*$, then

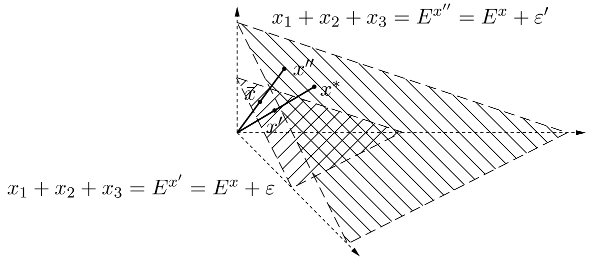
1. for each $S \subset N$, and each $i \in S$, since $E^x \geq \min_{j \in M} c^j(N \setminus \{i\}\} \geq \min_{j \in M} c^j(N \setminus S)$, then $\varepsilon \nabla^{\alpha^x}(S) = \varepsilon \alpha^x_{-S}$.
2. for each $i \in I^{\alpha^x}$, $\nabla^{\alpha^x}(\{i\}) = \alpha^x_{-i}$, since $m(\alpha^x) > 0$, if $j \notin I^{\alpha^x}$, then $\nabla^{\alpha^x}(\{i\}) = \alpha^x_{-i} > \alpha^x_{-j} = \nabla^{\alpha^x}(\{j\})$. Consider $S \subset N$ with more than one agent. If $i \in S$, then $\nabla^{\alpha^x}(\{i\}) = \alpha^x_{-i} > \alpha^x_{-S} = \nabla^{\alpha^x}(S)$. Otherwise, consider $j \in S$. Then, $\nabla^{\alpha^x}(\{i\}) \geq \nabla^{\alpha^x}(\{j\}) > \nabla^{\alpha^x}(S)$.
3. if $m(\alpha^x) = 0$, then, for each $i \in I^{\alpha^x}$, $\nabla^{\alpha^x}(\{i\}) = \alpha^x_{-i} = \alpha_N = \alpha_{-S} = \nabla^{\alpha^x}(S)$ for each $S \subset I^{\alpha^x}$, and $\nabla^{\alpha^x}(\{i\}) = \alpha^x_{-i} > \alpha_{-S} = \nabla^{\alpha^x}(S)$, for each $S \nsubseteq I^{\alpha^x}$. □

For each $E^x \geq E^{\underline{c}}$, consider the partition of the set of non-empty and proper coalitions, $\mathcal{P} = \{S \mid S \subset N\}$, into groups in which all the coalitions have the same level of dissatisfaction at x. For $l = 1, 2, \ldots, k^x$ $(1 \leq k^x \leq 2^n - 2)$, define:

$$\mathcal{S}^x_l = \left\{ S \subset N \mid e(x, S) \geq e(x, T) \text{ for all } T \subseteq \mathcal{P} \setminus \bigcup_{k=1}^{l-1} \mathcal{S}^x_k \right\},$$

where $\bigcup_{k=1}^{0} \mathcal{S}^x_k = \emptyset$. All coalitions in $\mathcal{S}^x_1$ have the same level of dissatisfaction and they are the groups with the maximum surplus at x. All the coalitions in $\mathcal{S}^x_2$ have the second maximum surplus at x, and so on. Denote by e^x_l, $l = 1, 2, \ldots, k^x$, the surplus of each coalition in each level, that is, if $S \in \mathcal{S}^x_l$, then $e^x_l = e(x, S)$ and, for each $l = 1, 2, \ldots, k^x - 1$, consider the function, L^x, that assigns to each coalition $S \subset N$ its level of dissatisfaction at x, that is, $L^x(S) = l$ if $S \in \mathcal{S}^x_l$.

Fig. 4 Illustration in the case of three agents



For each $E^x \geq E^{\underline{c}}$, denote by A^x the maximum increase of the estate, E^x, for which, if $0 < \varepsilon < \varepsilon' < A^x$ ($E^{x'} = E^x + \varepsilon$ and $E^{x''} = E^x + \varepsilon'$), then $k^{x'} = k^{x''}$ and for each $l = 1, 2, \ldots, k^{x'}$, $S_l^{x'} = S_l^{x''}$.

The following result shows that α^x is the same for each $\varepsilon < A^x$.

Claim 2 *For each* $E^x \geq E^{\underline{c}}$ *and each* $0 < \varepsilon < \varepsilon' < A^x$ ($E^{x'} = E^x + \varepsilon$ *and* $E^{x''} = E^x + \varepsilon'$), *if* $x' = x + \varepsilon\alpha^x$, *then* $x'' = x + \varepsilon'\alpha^x$.

Proof Suppose that $x'' = MT(C, E^{x''}) = x + \varepsilon'\beta^x \neq x^* = x + \varepsilon'\alpha^x$. Let $\bar{x}$ be $\bar{x} = x + \varepsilon\beta^x$ (see Fig. 4).

Since $x' = MT(C, E^{x'})$, there exists $l^* \leq k^{x'}$ and $T \in \mathcal{S}_{l^*}^{x'}$ such that, for each $l < l^*$ and each $S \in \mathcal{S}_l^{x'}$, $e(x', S) = e(\bar{x}, S)$, $e(x', T) < e(\bar{x}, T)$ and $e(\bar{x}, S) \leq e(\bar{x}, T)$ for each $S \in \bigcup_{l=l^*}^{k^{x'}} \mathcal{S}_l^{x'}$.

By using Claim 1, we are going to prove that if $e(x', S) = e(\bar{x}, S)$ (versus $e(x', S) < e(\bar{x}, S)$), then $e(x^*, S) = e(x'', S)$ (versus $e(x^*, S) < e(x'', S)$):

- If $E^x \geq \min_{j\in M} c^j(N \setminus S)$, then $e(x', S) = e(x, S) + \varepsilon\alpha_{-S}^x = (<)e(\bar{x}, S) = e(x, S) + \varepsilon\beta_{-S}^x$ and therefore, $\alpha_{-S}^x = (<)\beta_{-S}^x$. Thus, $e(x^*, S) = e(x, S) + \varepsilon'\alpha_{-S}^x = (<)e(x'', S) = e(x, S) + \varepsilon'\beta_{-S}^x$.
- If $E^x < \min_{j\in M} c^j(N \setminus S) < E^{x'}$, then $e(x', S) = e(x, S) + \varepsilon\alpha_{-S}^x - (\min_{j\in M} c^j(N \setminus S) - E^x) = (<)e(\bar{x}, S) = e(x, S) + \varepsilon\beta_{-S}^x - (\min_{j\in M} c^j(N \setminus S) - E^x)$ and $\alpha_{-S}^x = (<)\beta_{-S}^x$. Thus, $e(x^*, S) = e(x, S) + \varepsilon'\alpha_{-S}^x - (\min_{j\in M} c^j(N \setminus S) - E^x) = (<)e(x'', S) = e(x, S) + \varepsilon'\beta_{-S}^x - (\min_{j\in M} c^j(N \setminus S) - E^x)$.
- If $E^{x'} \leq \min_{j\in M} c^j(N \setminus S) \leq E^{x''}$, then $e(x', S) = e(x, S) - \varepsilon\alpha_S^x = (<)\, e(\bar{x}, S) = e(x, S) - \varepsilon\beta_S^x$ and $\alpha_S^x = (>)\beta_S^x$ or equivalently $\alpha_{-S}^x = (<)\beta_{-S}^x$. Therefore, $e(x^*, S) = e(x, S) + \varepsilon'\alpha_{-S}^x - (\min_{j\in M} c^j(N \setminus S) - E^x) = (<)e(x'', S) = e(x, S) + \varepsilon'\beta_{-S}^x - (\min_{j\in M} c^j(N \setminus S) - E^x)$.
- If $E^{x''} \leq \min_{j\in M} c^j(N \setminus S)$, then $e(x', S) = e(x, S) - \varepsilon\alpha_S^x = (<)e(\bar{x}, S) = e(x, S) - \varepsilon\beta_S^x$ and $-\beta_S^x = (>) - \alpha_S^x$. Therefore, $e(x^*, S) = e(x, S) - \varepsilon'\alpha_S^x = (<)e(x'', S) = e(x, S) - \varepsilon'\beta_S^x$.

Since $0 < \varepsilon < \varepsilon' < A^x$, then $k^{x'} = k^{x''}$ and $S_l^{x'} = S_l^{x''}$ for each $l = 1, 2, \ldots, k^{x'}$.

Thus, by taking into account the previous reasoning, we consider the above $l^* \leq k^{x''}$ and the above $T \in \mathcal{S}_{l^*}^{x''}$, and it follows that, for each $l < l^*$ and each $S \in \mathcal{S}_l^{x''}$, $e(x'', S) = e(x^*, S)$ and $e(x'', T) > e(x^*, T)$. This contradicts $x'' = MT(C, E^{x''})$. $\square$

Claim 3 *If there exists $E^x \geq E^*$ such that for each $i \in N$, $L^x(\{i\}) = 1$, holds, then $\alpha_i^x = \frac{1}{n}$ for each $i \in N$. Moreover, for each $E^{\bar{x}} > E^x$, $\alpha_i^{\bar{x}} = \frac{1}{n}$ for each $i \in N$.*

Proof Suppose there exists $\varepsilon > 0$, $\varepsilon < A^x$, such that $\alpha_i^x = \frac{1}{n}$ does not hold for all $i \in N$. Consider $x' = x + \varepsilon\alpha^x$ and x'', $x_i'' = x_i + \varepsilon\beta_i^x$, where $\beta_i^x = \frac{1}{n}$ for each $i \in N$. By Claim 1, $\mathcal{S}_1^{x''} = \{\{i\} \mid i \in N\}$ and therefore $\{\{i\} \mid i \in I^{\alpha^x}\} \subseteq \mathcal{S}_1^{x''}$. Moreover, since $\mathcal{S}_1^{x'} \subseteq \mathcal{S}_1^x$, for each $S \in \mathcal{S}_1^{x'}$ and each $i \in I^{\alpha^x}$, it follows that $\nabla^{\alpha^x}(S) \geq \nabla^{\alpha^x}(\{i\}) = \varepsilon(1 - m(\alpha^x)) > \varepsilon\frac{n-1}{n} = \nabla^{\beta^x}(\{i\})$. This contradicts the fact that $x' = MT(C, E^{x'})$ because $e_1^{x'} > e_1^{x''}$. Finally, for each $E^{\bar{x}} > E^x$, since $\mathcal{S}_1(E^{\bar{x}}) = \{\{i\} \mid i \in N\}$, the above reasoning can be applied to conclude that $\alpha_i^{\bar{x}} = \frac{1}{n}$ for each $i \in N$. □

Claim 4 *For each $E^x \geq E^*$, if $\alpha_i^x = \frac{1}{n}$ does not hold for all $i \in N$, then $\mathcal{I}^{\alpha^x} = \{\{i\} \mid i \in I^{\alpha^x},\ \{i\} \in \mathcal{S}_1^x\} = \emptyset$.*

Proof Suppose there exists $\varepsilon > 0$, $\varepsilon < A^x$, such that $\mathcal{I}^{\alpha^x} \neq \emptyset$. Consider $x' = x + \varepsilon\alpha^x$. Firstly, we assume $m(\alpha^x) \geq 0$. In this case, by Claim 1, $\mathcal{I}^{\alpha^x} \subseteq \mathcal{S}_1^{x'}$. Denote by δ the difference $\delta = \alpha_j^x - m(\alpha^x)$, where $\alpha_j^x = \min_{k \in N \setminus I^{\alpha^x}} \alpha_k^x$. Consider $x'' = x + \varepsilon\beta^x$, where $\beta_i^x = \alpha_i^x + \frac{\delta}{2|I^{\alpha^x}|}$, if $i \in I^{\alpha^x}$ and $\beta_i^x = \alpha_i^x - \frac{\delta}{2(n-|I^{\alpha^x}|)}$ otherwise. Notice that $I^{\alpha^x} = I^{\beta^x}$. By Claim 1, again, $\mathcal{I}^{\alpha^x} = \mathcal{I}^{\beta^x} \subseteq \mathcal{S}_1^{x''}$. This contradicts $x' = MT(C, E^{x'})$ since for each $i \in \mathcal{I}^{\alpha^x}$, $\nabla^{\alpha^x}(\{i\}) = \varepsilon(1 - \alpha_i^x) > \varepsilon(1 - \beta_i^x) = \nabla^{\beta^x}(\{i\})$, and therefore, $e_1^{x'} > e_1^{x''}$.

Now assume $m(\alpha^x) < 0$. Let $\{i\} \in \mathcal{I}^{\alpha^x}$, such that $\alpha_i^x = m(\alpha^x) < 0$. Then $\nabla^{\alpha^x}(\{i\}) = \varepsilon(1 - \alpha_i^x) > \varepsilon$ and, for each $S \in \mathcal{S}_1^{x'} \subseteq \mathcal{S}^x$, $\nabla^{\alpha^x}(S) \geq \nabla^{\alpha^x}(\{i\}) = \varepsilon(1 - \alpha_i^x) > \varepsilon$. This contradicts $x' = MT(C, E^{x'})$, since, by considering, for instance, $\beta_i^x = \frac{1}{n}$, for all $i \in N$ and $x'' = x + \varepsilon\beta^x$, none of the coalitions in $\mathcal{S}_1^{x''} \subseteq \mathcal{S}^x$ suffers an increment of her dissatisfaction greater than ε. □

Consider $x^{\underline{c}} = MT(C, E^{\underline{c}})$. By Claim 2, the path of awards of MT is linear from the estate $E^{\underline{c}}$ to the estate $E^{x^1} = E^{\underline{c}} + A^{\underline{c}}$, that is, $x = \underline{c} + \alpha^{\underline{c}}\varepsilon = MT(C, E^x)$, for each $E^x = E^{\underline{c}} + \varepsilon$, $\varepsilon \leq A^{\underline{c}}$. At $x^1 = \underline{c} + \alpha^{\underline{c}}A^{\underline{c}}$ the slope of the path of awards may change to α^{x^1}, and this new slope remains unchanged from estate E^{x^1} to estate E^{x^2} ($x = x^1 + \alpha^{x^1}\varepsilon = MT(C, E^x)$, for each $E^x = E^{x^1} + \varepsilon$, $\varepsilon \leq A^{x^1}$). At $x^2 = x^1 + \alpha^{x^1}Ax^1$ the path of awards of MT may change again and so on. Therefore, the path of awards of MT is piecewise linear.

Let $Q = \{x^1 = \underline{c}, x^2, x^3, \ldots\}$ be the set of allocations in the path of awards of MT, considered in Claim 2, at which the slope may change, and let $\bar{x}$ be the first of them such that $E^{\bar{x}} \geq E^*$. For each $x^j \in Q$, consider $F^{x^j} = \{i \in N \mid L^{x^j}(\{i\}) = 1\}$.

At $E^{\bar{x}}$, if for each $i \in N$, $L^{\bar{x}}(\{i\}) = 1$ holds, then, by Claim 3, any increase of the estate is divided equally among the agents and the result follows (and therefore there exists $E(C)$, $E(C) \leq E^{\bar{x}}$).

Otherwise, when $F^{\bar{x}} = \{i \in N \mid L^{\bar{x}}(\{i\}) = 1\} \neq N$, we are going to prove that the dissatisfaction of some agents $i \in N \setminus F^{\bar{x}}$ increases more than the dissatisfaction of some of the coalitions in $\mathcal{S}_l^{\bar{x}}$, where $l < L^{\bar{x}}(\{i\})$.

Case 1 There exists $i, j \in N$, $i \neq j$, such that $\alpha_i^{\bar{x}} \neq \alpha_j^{\bar{x}}$.

If $m(\alpha^{\bar{x}}) \geq 0$, by Claim 4, $I^{\alpha^{\bar{x}}} \cap F^{\bar{x}} = \emptyset$ and by Claim 1, $\nabla(\{i\}) > \nabla(S)$ for each $i \in I^{\alpha^{\bar{x}}}$ and each $S \in \mathcal{S}_1^{\bar{x}}$.

If, on the contrary, $m(\alpha^{\bar{x}}) < 0$, then there exists $S \subset N$, such that $L^{\bar{x}}(S) \leq L^{\bar{x}}(\{i\})$, for all $i \in I^{\alpha^{\bar{x}}}$, and $\alpha_S^{\bar{x}} \geq 0$, because, in other case, every coalition $S \in \mathcal{S}_1^{\bar{x}}$ would increment its dissatisfaction $\varepsilon(1 - \alpha_S^{\bar{x}}) > 1$. This contradicts $x' = MT(C, E^{x'})$ because by considering $\beta_i^x = \frac{1}{n}$, for all $i \in N$ and $x'' = x + \varepsilon\beta^x$, none of the coalitions in $\mathcal{S}_1^{x''} \subseteq \mathcal{S}^{\bar{x}}$ suffers an increment of her dissatisfaction greater than ε. Therefore, for each $i \in I^{\alpha^{\bar{x}}}$ and each of the above mentioned $S \subset N$, $\nabla(\{i\}) > \nabla(S)$.

Case 2 $\alpha_i^{\bar{x}} = \alpha_j^{\bar{x}} = \frac{1}{n}$, for each $i, j \in N$.

In this case, there exists a coalition $S \subset N$, with more than one agent, and $i \in N \setminus F^{\bar{x}}$, such that $L^{\bar{x}}(\{i\}) > L^{\bar{x}}(S) \geq 1$, since otherwise, for each $E^{x'} = E^{\bar{x}} + \varepsilon$, $\varepsilon < A^{\bar{x}}$, by considering $G^{\bar{x}} = \{i \in N \mid L^{\bar{x}}(\{i\}) = 2\}$, slightly increasing the proportion for agents in $F^{\bar{x}}$, appropriately decreasing the proportion for agents in $G^{\bar{x}}$, and fixing the proportion $\frac{1}{n}$ for agents in $N \setminus F^{\bar{x}} \cup G^{\bar{x}}$, we obtain a new allocation x'', different from $x' = \bar{x} + \frac{1}{n}\varepsilon$. This means a contradiction with $x' = MT(C, E)$. Therefore, by Claim 1, for the mentioned coalition $S \subset N$, with more than one agent, and that agent $i \in N \setminus F^{\bar{x}}$, $\nabla(\{i\}) > \nabla(S)$ holds.

Therefore, the level of dissatisfaction of the one-agent coalitions outside $F^{\bar{x}}$ approaches strictly the first level. Since the same reasoning described above can be applied to each $x^j \in Q$, $x^j \geq \bar{x}$, there eventually exists $x^k \in Q$ such that, for each $i \in N$, $L^{x_k}(\{i\}) = 1$ holds. Therefore, by Claim 3, any increase of the estate from E^{x^k} is divided equally among the agents. □

Proof Proposition 5.1 It is straightforward to see that MT satisfies anonymity and neutrality.

We will prove consistency: For $N \in \mathcal{N}$, $N' \subset N$, $M \subset \mathbb{N}$, and $(C, E) \in \mathcal{D}_N^M$, let $x = MT(C, E) = P(N, v_{(C,E)}^{\max})$. We will prove that $x_{N'} = MT(C_{N'}, x(N')) = P(N', v_{(C_{N'},x(N'))}^{\max})$.

Consider the reduction of game $(N, v_{(C,E)}^{\max})$, relative to N' and x,

$$r_{N'}^x\big(v_{(C,E)}^{\max}\big)\big(N'\big) \equiv v_{(C,E)}^{\max}(N) - x\big(N \setminus N'\big), \quad \text{and}$$

$$\text{for each } S \subset N', \quad r_{N'}^x\big(v_{(C,E)}^{\max}\big)(S) \equiv \max_{Q \subseteq N \setminus N'}\big(v_{(C,E)}^{\max}(S \cup Q) - x(Q)\big).$$

Since the prenucleolus is consistent with respect to this reduction (Sobolev 1975), it follows that $x_{N'} = P(N', r_{N'}^x(v_{(C,E)}^{\max}))$.

We are going to prove that $v_{(C_{N'},x(N'))}^{\max}(S) = r_{N'}^x(v_{(C,E)}^{\max})(S)$ for all $S \subseteq N'$.

It is clear that $v_{(C_{N'},x(N'))}^{\max}(N') = r_{N'}^x(v_{(C,E)}^{\max})(N') = x(N')$.

For each $S \subset N'$, on the one hand, $v^{\max}_{(C_{N'},x(N'))}(S) = \max_{j\in M}\{v^{j}_{(c^j_{N'},x(N'))}(S)\} = \max_{j\in M}(x(N') - c^j(N' \setminus S))_+$, and on the other hand,

$$\begin{aligned} r^x_{N'}\big(v^{\max}_{(C,E)}\big)(S) &= \max_{Q\subseteq N\setminus N'}\big(v^{\max}_{(C,E)}(S\cup Q) - x(Q)\big) \\ &= \max_{Q\subseteq N\setminus N'}\Big(\max_{j\in M}\big(x(N) - c^j\big(N\setminus(S\cup Q)\big)\big)_+ - x(Q)\Big) \\ &= \max_{j\in M}\Big(\max_{Q\subseteq N\setminus N'}\big(\big(x(N) - c^j\big(N\setminus(S\cup Q)\big)\big)_+ - x(Q)\big)\Big) \\ &= \max_{j\in M} r^x_{N'}\big(v^j\big)(S) = \max_{j\in M} v_{(c^j_{N'},x(N'))}(S) \\ &= \max_{j\in M}\big(x\big(N'\big) - c^j\big(N'\setminus S\big)\big)_+, \end{aligned}$$

and therefore, for each $S \subset N'$, $r^x_{N'}(v^{\max}_{(C,E)})(S) = v^{\max}_{(C_{N'},x(N'))}(S)$. □

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