# New sufficient conditions for global asymptotic stability of a kind of nonlinear neutral differential equations

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**Abstract**

This paper addresses the stability study for nonlinear neutral differential equations. Thanks to a new technique based on the fixed point theory, we find some new sufficient conditions ensuring the global asymptotic stability of the solution. In this work we extend and improve some related results presented in recent works of literature. Two examples are exhibited to show the effectiveness and advantage of the proved results.
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## 1 Introduction

It is well-known that the theory of neutral functional differential equations has attracted many types of research due to its wide and great applications in many fields of mathematical science and engineering such as neural networks, population dynamics, control theory, and many other phenomena. For appropriate literature we can refer to the books [17]-[20]. A neutral delay differential equation is a kind of delay differential equation where the delay argument occurs in the highest order derivative of the state, which can be used to describe many real-world phenomena that arise in the areas for example, lossless transmission lines, theory of automatic control and others, we refer the reader to references

Brayton [10], Hale [18], Kuang [19], Kolmanovskii and Myshkis, [20], and the sources related.

Up today, significant progress has been made in the qualitative theory (e.g. oscillation theory, periodicity, stability, the existence of a periodic solution, asymptotic behavior, boundedness, instability, and so on) of neutral delay differential equations. For more details, we can refer to ( [1]-[5] and [10]-[16],[23]-[31] ), and the references of these sources. One of the most qualitative concepts in mathematic theory is determining the stability of a given model. The theory of stability was created at the end of the 19th century by Lyapunov. This method is now known as the Lyapunov's direct method or Lyapunov function.

For decades, Lyapunov developed a method for determining stability in many areas of differential equations without solving the equations themselves. This theory has been proven significantly its effectiveness over a century and it has been achieved wide applications in various fields of physics and mathematical sciences. Unfortunately, when we try to carry over the principles of the Lyapunov stability theory to special problems, we face a large number of difficulties and it appears that new methods are needed to overcome those obstacles (see [5]-[7]). Luckily, Burton and many authors have used fixed point theory as an alternative to studying the stability of deterministic or stochastic systems, where some of these problems in Lyapunov functions have been solved. In the current study, we use this method to address a kind of nonlinear neutral differential equations (see [8], [10], [15], [21]).

In [16], Jin and Luo studied the asymptotic stability in the space $C^0$ of the scalar nonlinear neutral differential equation of the form

$$u'(t) = -a(t)u(t) + c(t)u'(t-\tau(t)) - b(t)\,u(t-\tau(t)), t \geq 0. \tag{1.1}$$

The work of Jin and Luo in [16] requires that the delay $\tau$ is twice differentiable, and $\tau'(t) \neq 1$ for $t \geq 0$ and $c$ is differentiable. However, there are many interesting examples where these conditions are not satisfied. It is our purpose in this paper to remove these restrictive conditions by studying the global stability in the space $C^1$.

As it is known, there are a few papers [1]- [3] and [22],[28] have discussed the global asymptotic stability of solutions of neutral differential equations in $C^1$. For example, Liu and Yang in [22] was the first to establish necessary and sufficient conditions for the asymptotic stability in $C^1$ for the equation

$$u'(t) = -a(t)u(t) + c(t)u'(t-\tau_1(t)) + Q(t, u(t), u(t-\tau_2(t))), \tag{1.2}$$

where $Q$ is a Lipschitz continuous function in $u$. Liu and Yang have been able, in their work, to avoid the derivative of the coefficient $c$ and they also do not need that the delay $\tau$ is twice differentiable, and $\tau'(t) \neq 1$ for $t \geq 0$. Otherwise, a good contribution of their work was obtained to relax the conditions on the coefficient $c$ and the delay $3c4$.

Recently, by the same method of Liu et Yang [22], Ardjouni and Djoudi [1] have adressed general form than (1.2) as follows

$$\begin{aligned} u'(t) &= -a(t)u(t) + g(t, u'(t-\tau_1(t)), u'(t-\tau_2(t)), ..., u'(t-\tau_n(t)) \\ &\quad + f(t, u(t-\tau_1(t)), u(t-\tau_2(t)), ..., u(t-\tau_n(t)), \end{aligned} \tag{1.3}$$

where $f(t, u_1, ..., u_n)$,$g(t, u_1, ..., u_n)$ are continuous and satisfy Lipschitz condition in $u_1, ..., u_n$, respectively. However, it still remains unexplored the case in which one considers all terms of the equation (1.3) are nonlinear, what is the main reason for the analysis we will perform in the current paper.

In 2020 Zaid *et al.* [31] obtained stability results in $C^0$ about the zero solution of the standard form of the totally nonlinear delay differential equation

$$u'(t) = -\sum_{i=1}^{N} a_i(t, u_t) u(t) + f(t, u_t), t \geq t_0. \tag{1.4}$$

In the case $N = 1$, Eq. (1.4) reduces to the one in [13]. With the previous motivation, in this paper, we extend the results in [31] to the totally nonlinear neutral differential equation represented in (2.1), (see below). More precisely, we will study the stability in the space $C^1$ (as described in more details below) which is a stronger concept of stability than the usual one in $C^0$. The study of stability in $C^1$ is therefore much richer than the classical stability in $C^0$. By applying the fixed point theory, we will state new and more applicable stability criteria in $C^1$. The sufficient conditions obtained are quite practicable and we will no longer need the delay to be twice differentiable or coefficients are differentiable, which is required in some previous relevant works [3],[4],[12], [16],[30]. It is this new feature that makes the asymptotic behavior in $C^1$ more important and more useful as well. Our work extends and improves the results in [1],[13],[16],[22],[31]. In addition, two examples are given to test the feasibility and advantage of the proved results.

## 2 Notations and preliminaries

Let $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^-$ denote $(-\infty, +\infty)$,$[0, +\infty)$, $(-\infty, 0]$ respectively.

In the current paper, we aim to discuss the asympotic stabilty in $C^1$ for standard form of neutral differential equations as follows,

$$u'(t) = -\sum_{i=1}^{N} a_i(t, u_t) u(t) + g(t, u'_t) + f(t, u_t), t \geq t_0, \tag{2.1}$$

where $f, g \in C(\mathbb{R}^+ \times B, \mathbb{R})$ and $a_i \in C(\mathbb{R}^+ \times B, \mathbb{R}), \left(i = \overline{1, N}\right)$, with,

$$B = \left\{\phi \in C\left(\mathbb{R}^-, \mathbb{R}\right) : \phi \text{ bounded}\right\},$$

with the norm $\|\phi\|_\circ := \sup_{\theta \in (-\infty, 0]} |\phi(\theta)|$. Define also

$$C_L = \{\xi \in C : \|\xi\|_\circ \leq L\} \text{ and } C^1_{L'} = \left\{\xi \in C^1 : \left\|\xi'\right\|_\circ \leq L'\right\}.$$

If $u \in C^1(\mathbb{R},\mathbb{R})$ is bounded and for $t \geq 0$ is a fixed number, we let $u_t, u_t' \in C$ be defined by

$$u_t(\theta) = u(t+\theta) \text{ and } u_t'(\theta) = u'(t+\theta) \text{ for } \theta \in \mathbb{R}^-. \tag{2.2}$$

We define

$$\|x\|^{[s,t]} := \sup_{\xi \in [s,t]} |x(\xi)|,$$

for a function $x : \mathbb{R} \to \mathbb{R}$.

Before starting the main result of this paper, we impose the following assumptions:

**(A1)** there exists a constant $L > 0$ and a function $b_1 \in C(\mathbb{R},\mathbb{R}^+)$ such that, for all $\phi, \psi \in C_L$ and for all $t \geq 0$,

$$|f(t,\phi) - f(t,\psi)| \leq |b_1(t)| \, \|\phi - \psi\|_\circ . \tag{2.3}$$

**(A2)** there exists a constant $L' > 0$ and a function $b_2 \in C(\mathbb{R},\mathbb{R}^+)$ such that, for all $\phi, \psi \in C_{L'}^1$ and for all $t \geq 0$,

$$\left|g\left(t,\phi'\right) - g\left(t,\psi'\right)\right| \leq |b_2(t)| \left\|\phi' - \psi'\right\|_\circ \tag{2.4}$$

**(A3)** $\forall \varepsilon > 0$ and $t_1 \geq 0$, there exists a $t_2 > t_1$ such that $[t \geq t_2, u_t \in C_L]$, imply

$$|f(t,u_t)| \leq |b_1(t)| \left(\varepsilon + \|u\|^{[t_1,t]}\right). \tag{2.5}$$

**(A4)** $\forall \varepsilon > 0$ and $t_1 \geq 0$, there exists a $t_3 > t_1$ such that $\left[t \geq t_3, u' \in C_{L'}^1\right]$, imply

$$|g(t,u_t')| \leq |b_2(t)| \left(\varepsilon + \|u'\|^{[t_1,t]}\right). \tag{2.6}$$

**(A5)** there exists $\alpha_1, \alpha_2 \in C(\mathbb{R},\mathbb{R})$, ($\alpha_2$ is bounded) such that

$$\alpha_1(t) \leq \sum_{i=1}^{N} a_i(t,u_t) \leq \alpha_2(t).$$

**(A6)** Assume furthermore that,

$$f(t,0) = g(t,0) = 0 \text{ for all } t \geq t_0 . \tag{2.7}$$

which guarantees that (2.1) possesses a trivial solution $u(t) = 0$.

For each $t_0 \in [0,\infty)$, denote $C_{t_0}^1 = C^1(]-\infty,t_0],\mathbb{R})$ with the norm defined by

$$|u|_{t_0} := \max_{t \in (-\infty,t_0]} \{|u(t)|, |u'(t)|\},$$

for $u \in C_{t_0}^1 = C^1((-\infty,t_0],\mathbb{R})$. In addition, denote $\Phi_{t_0}$, where

$$\Phi_{t_0} = \left\{\varphi \in C_{t_0}^1 : \varphi_-'(t_0) = -\sum_{i=1}^{N} a_i\left(t_0,\varphi_{t_0}\right)\varphi(t_0) + g\left(t_0,\varphi_{t_0}'\right) + f\left(t_0,\varphi_{t_0}\right)\right\}.$$

For each $t_0 \in [0, \infty)$, we choose initial functions for equation (2.1) of the type $\varphi \in \Phi_{t_0}$.

The definitions of stability in $C^1$, as well as the necessary notation for our study are borrowed from the paper [22], but the nonlinearities in our model and the fact that we are considering a neutral term, make our study nontrivial and meaningful.

We now recall some basic informations.

**Definition 1.1.** For each initial value $(t_0, \varphi) \in [0, \infty) \times \Phi_{t_0}$, $u$ is called a solution of (2.1) with $(t_0, \varphi)$ if $u \in C^1((-\infty, +\infty), \mathbb{R})$ satisfies equation (2.1) for all most $t \geq t_0$ and $u = \varphi$ for $t \leq t_0$. Such a solution will be denoted by $u(t) = u(t, t_0, \varphi)$.

We now prepare definitions in order to prove asymptotic stability in $C^1$ for (2.1).

**Definition 1.2.** i) The trivial solution of (2.1) is:

i) stable in $C^1$, if for any $\varepsilon > 0$ and $t \geq t_0$, there is a scalar $\delta = \delta(\varepsilon, t_0) > 0$, such that for any initial function $\varphi \in \Phi_{t_0}$ satisfying $|\varphi|_{t_0} < \delta$, we have for the corresponding solution that

$$\max_{s \in (-\infty, t]} \{|u(s, t_0, \varphi)|, |u'(s, t_0, \varphi)|\} < \varepsilon \text{ for } t \geq t_0.$$

ii) asymptotically stable in $C^1$, if $u(t)$ is stable in $C^1$, and for any initial function $\varphi \in \Phi_{t_0}$ we have for the corresponding solution that

$$\lim_{t \to \infty} u(t, t_0, \varphi) = \lim_{t \to \infty} u'(t, t_0, \varphi) = 0.$$

At the light in definition 1.1. For the problem of the initial value of the equation (2.1), sensible conditions are imposed.

Since our model (2.1) involves nonlinear term $\sum_{i=1}^{N} a_i(t, u_t) u(t)$, so that it is more complex and different than those of the above literature [1], [13], [16],[22], [31] which also implies some difficulties in mathematical analysis. That means we study how the asymptotic behavior property in $C^1$ will be when (1.4) is added to the perturbed nonlinear neutral term $g(t, u'_t)$. Motivated by the previously cited literature related to fixed point approch [5], [11], [12], [13], [15], [15]. As the main tool, it used Banach's fixed point to obtain some new sufficient conditions ensuring the global asymptotic stability results in $C^1$ to Eq. (2.1). Finally, two examples are given to illustrate the real interest and importance of the proposed results.

# 3 Stability by contraction mapping

It is well known that studying the stability of an equation by Banach's fixed point method based on three essential points: a complete metric space, a variation of parameters, and the formulation of an appropriate contraction mapping.

The advantage of this method is that the fixed point argument leads, the existence, uniqueness, boundedness, and stability of the equation, all at once. Up till now, no work has concerned the equation (2.1) to establish sufficient conditions for the global asymptotic behavior in $C^1$. Let us begin to explore this world of stability.

In this section we shall discuss the asymptotic stability in $C^1$ for equation (2.1).

**Theorem 3.1**. Assume hypotheses *(A1)–(A6) hold, and for any* $t \geq t_0$, *if there exists* $\eta \in (0, \frac{1}{2})$ *such that,*

$$\liminf_{t\to\infty} \int_{t_0}^{t} \alpha_1(s)\, ds > -\infty, \tag{3.1}$$

*and*

$$\int_{t_0}^{t} e^{-\int_s^t \alpha_1(z)dz} \left(|b_1(s)| + |b_2(s)|\right) ds \leq \eta, \tag{3.2}$$

$$|\alpha_2(t)| \int_{t_0}^{t} e^{-\int_s^t \alpha_1(z)dz} \left(|b_1(s)| + |b_2(s)|\right) ds + \left(|b_1(t)| + |b_2(t)|\right) \leq \eta, \tag{3.3}$$

*and*

$$\int_0^t \alpha_1(s)ds \to \infty \text{ as } t \to \infty. \tag{3.4}$$

*Then the equation* (2.1) *has a unique trivial solution. Moreover, it is asymptotic stable in* $C^1$.

**Proof**. First, suppose that $\int_0^t \alpha_1(s)ds \to \infty$ as $t \to \infty$. For each $t_0 \in [0, \infty)$, let $\varphi \in C((-\infty, t_0], \mathbb{R})$ be a fixed initial function. We define $S$ as the following space

$$S = \left\{u \in C^1(\mathbb{R}, \mathbb{R}) : \lim_{t\to\infty} u(t) = \lim_{t\to\infty} u'(t) = 0\right\},$$

with the metric defined by

$$\|u\| := \max_{t\in\mathbb{R}} \{|u(t)|, |u'(t)|\}.$$

Then $S$ is a complete metric space.

Next, we define for any $\varphi \in \Phi_{t_0}$,

$$D_\varphi^l = \left\{u \in S : u_{t_0} = \varphi \text{ and } \max_{t\geq t_0} \{\|u_t\|_\circ, \|u_t'\|_\circ\} \leq l\right\},$$

Obviously, we know $D_\varphi^l$ is a closed convex and bounded subset of $S$, where $l = \max\{L, L'\}$.

We can use the variation of parameter formula for writing the equation (2.1) as an integral equation suitable for Banach's fixed point theorem. The application $\mathcal{P}$ is given in literature without details of the proof. It can be deduced from [22]. Hence, we omit it.

Define the application $\mathcal{P}(u):\mathbb{R}\to\mathbb{R}$ with $(\mathcal{P}u)(t)=\varphi(t)$ for $t\in(-\infty,t_0]$, and

$$(\mathcal{P}u)(t)=e^{-\int_{t_0}^{t}\sum\limits_{i=1}^{N}a_i(s,u_s)ds}\varphi(t_0)+\int_{t_0}^{t}e^{-\int_{s}^{t}\sum\limits_{i=1}^{N}a_i(z,u_z)dz}\left[g\left(s,u_s'\right)+f\left(s,u_s\right)\right]ds, \tag{3.5}$$

for $t\geq t_0$. It is not difficult to see that $\mathcal{P}(u):\mathbb{R}\to\mathbb{R}$ is continuous.

Initially, we show that, $\mathcal{P}:D_\varphi^l\to D_\varphi^l$. In view of (3.5), we can derive,

$$\begin{aligned}(\mathcal{P}u)'(t) &= -\varphi(t_0)\sum_{i=1}^{N}a_i(t,u_t)\,e^{-\int_{t_0}^{t}\sum\limits_{i=1}^{N}a_i(s,u_s)ds}+g\left(t,u_t'\right)+f\left(t,u_t\right)\\ &\quad-\sum_{i=1}^{N}a_i(t,u_t)\int_{t_0}^{t}e^{-\int_{s}^{t}\sum\limits_{i=1}^{N}a_i(z,u_z)dz}\left[g\left(s,u_s'\right)+f\left(s,u_s\right)\right]ds\\ &= -\sum_{i=1}^{N}a_i(t,u_t)(\mathcal{P}u)(t)+g\left(t,u_t'\right)+f\left(t,u_t\right),\end{aligned} \tag{3.6}$$

for $t\geq t_0$.

By the definition of $\Phi_{t_0}$, (3.6) yields

$$(\mathcal{P}u)_+'(t_0)=-\sum_{i=1}^{N}a_i(t_0,u_{t_0})\varphi(t_0)+g\left(t_0,u_{t_0}'\right)+f\left(t_0,u_{t_0}\right)=\varphi_-'(t_0).$$

Hence, $\mathcal{P}u\in C^1(\mathbb{R})$ for $u\in D_\varphi^l$.

Next, we verify that $\max\limits_{t\geq t_0}\left\{\left\|(\mathcal{P}u)_t'\right\|_\circ,\left\|(\mathcal{P}u)_t\right\|_\circ\right\}<l$. Let

$$A=\sup_{t\geq t_0}\left\{\left|\alpha_2(t)\right|\right\}\text{ and }K=\sup_{t\geq t_0}e^{-\int_{t_0}^{t}\alpha_1(s)ds}.$$

By conditions (3.4),(3.1), $K,A\in[0,\infty)$. For a given small bounded initial function $\varphi$ with $|\varphi|_{t_0}<\delta_0$, where $\delta_0>0$ satisfies

$$\delta_0<l\min\left\{1,\frac{1-\eta}{K},\frac{1-2\eta}{KA}\right\}. \tag{3.7}$$

Let $u\in D_\varphi^l$, then $\max\limits_{t\geq t_0}\left\{\|u_t'\|_\circ,\|u_t\|_\circ\right\}\leq l$. Du the conditions $(2.3),(2.4),(3.7),$ and $(3.2)$, we can get

$$\begin{aligned}\left|(\mathcal{P}u)(t)\right| &\leq \left|\varphi(t_0)\right|e^{-\int_{t_0}^{t}\alpha_1(s)ds}+\int_{t_0}^{t}e^{-\int_{s}^{t}\alpha_1(z)dz}\left|b_2(s)\right|\|u_s'\|_\circ ds\\ &\quad+\int_{t_0}^{t}e^{-\int_{s}^{t}\alpha_1(z)dz}\left|b_1(s)\right|\|u_s\|_\circ ds\\ &\leq K\delta_0+\eta l<l.\end{aligned}$$

Now, (3.6) and $(2.3), (2.4), (2.7)$ and $(3.2), (3.3), (3.7)$ imply that

$$\begin{aligned}
\left|(\mathcal{P}u)'(t)\right| &\leq |\varphi(t_0)| \sum_{i=1}^{N} |a_i(t, u_t)| e^{-\int_{t_0}^{t} \sum_{i=1}^{N} a_i(s,u_s)ds} + |g(t, u_t')| + |f(t, u_t)| \\
&\quad + \sum_{i=1}^{N} |a_i(t, u_t)| \int_{t_0}^{t} e^{-\int_{s}^{t} \sum_{i=1}^{N} a_i(z,u_z)dz} \left[|g(s, u_s')| + |f(s, u_s)|\right] ds \\
&\leq KA\delta_0 + |g(t, u_t') - g(t, 0)| + |f(t, u_t) - f(t, 0)| \\
&\quad + l\left\{|\alpha_2(t)| \int_{t_0}^{t} e^{-\int_{s}^{t} \alpha_1(z)dz} \left(|b_1(s)| + |b_2(s)|\right) ds\right\} \\
&\leq KA\delta_0 + l\left(|b_1(t)| + |b_2(t)|\right) + \eta l \\
&\leq KA\delta_0 + 2\eta l < l,
\end{aligned}$$

by the choice of $\delta_0$. This implies, $\max_{t \geq t_0} \left\{|(\mathcal{P}u)(t)|, \left|(\mathcal{P}u)'(t)\right|\right\} < l$. We will now show $(\mathcal{P}u)(t)$ approaches zero as $t \to \infty$.

Du the condition (3.4), we have

$$\lim_{t \to \infty} e^{-\int_{t_0}^{t} \alpha_1(z)dz} = 0.$$

Therefore, it is obvious that the first term of $(\mathcal{P}u)(t)$ tends to zero as $t \to \infty$ because of condition (3.4). Next, we will show that the last term of $(\mathcal{P}u)(t)$ tends to zero too. Since $\lim_{t \to \infty} u(t) = \lim_{t \to \infty} u'(t) = 0$, we can find $T_1 > t_0$ such that $\forall t \geq T_1$, $\max\{|u(t)|, |u'(t)|\} < \varepsilon$, and the fact $u \in D_{\varphi}^{l}$ implies that $\forall\ t \geq t_0$, $\max\{\|u_t\|_{\circ}, \|u_t'\|_{\circ}\} < l$. It is therefore follows from (2.5) and (2.6) that we can find $t_2 > T_1$ such that

$$|f(t, u_t)| \leq |b_1(t)| \left(\varepsilon + \|u\|^{[T_1, t]}\right),$$

and

$$|g(t, u_t')| \leq |b_2(t)| \left(\varepsilon + \|u'\|^{[T_1, t]}\right),$$

for $t \geq t_2$.

Hence for $t \geq t_2$, we have

$$
\begin{aligned}
& \left| \int_{t_0}^{t} e^{-\int_s^t \sum_{i=1}^{N} a_i(z,u_z)dz} \left[ g\left(s,u'_s\right) + f\left(s,u_s\right) \right] ds \right| \\
\leq & \int_{t_0}^{t_2} e^{-\int_s^t \alpha_1(z)dz} \left| g\left(s,u'_s\right) + f\left(s,u_s\right) \right| ds \\
& + \int_{t_2}^{t} e^{-\int_s^t \alpha_1(z)dz} \left| g\left(s,u'_s\right) + f\left(s,u_s\right) \right| ds \\
\leq & \int_{t_0}^{t_2} e^{-\int_s^t \alpha_1(z)dz} \left[ \left|b_1(s)\right| + \left|b_2(s)\right| \right] \max_{s \geq t_0} \left\{ \left\|u'_s\right\|_\circ, \left\|u_s\right\|_\circ \right\} ds \\
& + \int_{t_2}^{t} e^{-\int_s^t \alpha_1(z)dz} \left|b_1(s)\right| \left( \varepsilon + \left\|u'\right\|^{[T_1,s]} \right) ds \\
& + \int_{t_2}^{t} e^{-\int_s^t \alpha_1(z)dz} \left|b_2(s)\right| \left( \varepsilon + \left\|u\right\|^{[T_1,s]} \right) ds,
\end{aligned}
$$

since $\max\left\{ \left\|u\right\|^{[T_1,t]}, \left\|u'\right\|^{[T_1,t]} \right\} \leq \varepsilon$ for $t \geq t_2$. Then,

$$
\begin{aligned}
\leq & \int_{t_0}^{t_2} e^{-\int_s^t \alpha_1(z)dz} \left[ \left|b_1(s)\right| + \left|b_2(s)\right| \right] \max_{s \geq t_0} \left\{ \left|u'_s\right|, \left|u_s\right| \right\} ds \\
& +2\varepsilon \int_{t_2}^{t} e^{-\int_s^t \alpha_1(z)dz} \left[ \left|b_1(s)\right| + \left|b_2(s)\right| \right] ds \\
\leq & \; l \int_{t_0}^{t_2} e^{-\int_s^{t_2} \alpha_1(z)dz} e^{-\int_{t_2}^{t} \alpha_1(z)dz} \left[ \left|b_1(s)\right| + \left|b_2(s)\right| \right] ds + 2\eta\varepsilon.
\end{aligned}
$$

By using condition (3.4), we can find $T \geq t_2$ such that for $t \geq T$, we get

$$
l e^{-\int_T^t \alpha_1(z)dz} \int_{t_0}^{t_2} e^{-\int_s^T \alpha_1(z)dz} \left[ \left|b_1(s)\right| + \left|b_2(s)\right| \right] ds \leq \varepsilon.
$$

This yields $\lim\limits_{t \to \infty} \left(\mathcal{P}u\right)(t) = 0$ for $u \in D_\varphi^l$.

Moreover, for each $u \in D_\varphi^l$, $\lim\limits_{t \to \infty} u(t) = \lim\limits_{t \to \infty} u'(t) = 0$, given $\varepsilon > 0$ there exists $T_2 > t_0$ such that $\forall t \geq T_2$, $\max\left\{ \left|u(t)\right|, \left|u'(t)\right| \right\} < \varepsilon$ . By conditions (2.5),(2.6) we can find a $T' > T_2$ such that, for $t \geq T'$, we have

$$
\left| g\left(t,u'_t\right) \right| \leq \left|b_1(t)\right| \left( \varepsilon + \left\|u'\right\|^{[T_2,t]} \right),
$$

and

$$
\left| f\left(t,u_t\right) \right| \leq \left|b_2(t)\right| \left( \varepsilon + \left\|u\right\|^{[T_2,t]} \right).
$$

For $t \geq T'$, we have from $(3.6)$,

$$\begin{aligned}
\left|(\mathcal{P}u)'(t)\right| &\leq \sum_{i=1}^{N} |a_i(t,u_t)| \, |(\mathcal{P}u)(t)| + |g(t,u_t')| + |f(t,u_t)| \\
&\leq \sum_{i=1}^{N} |a_i(t,u_t)| \, |(\mathcal{P}u)(t)| + |b_1(t)| \left(\varepsilon + \|u'\|^{[T_2,t]}\right) \\
&\quad + |b_2(t)| \left(\varepsilon + \|u\|^{[T_2,t]}\right) \\
&\leq |\alpha_2(t)| \, |(\mathcal{P}u)(t)| + 2\eta\varepsilon
\end{aligned}$$

This, together with $(3.1) - (3.3)$, leads to $\lim_{t\to\infty} (\mathcal{P}u)'(t) = 0$ for $u \in D_\varphi^l$. Therefore, $\mathcal{P}u \in D_\varphi^l$ for $u \in D_\varphi^l$, *i.e.* $\mathcal{P} : D_\varphi^l \to D_\varphi^l$.

We now show that $\mathcal{P} : D_\varphi^l \to D_\varphi^l$ is contractive. To this end, suppose that $u, y \in D_\varphi^l$, by the conditions $(2.3), (2.4)$ $(3.2), (3.3), (3.6)$, that for $t \geq t_0$,

$$\begin{aligned}
& |(\mathcal{P}u)(t) - (\mathcal{P}y)(t)| \\
\leq\ & \int_{t_0}^{t} e^{-\int_s^t \sum_{i=1}^{N} a_i(z,u_z)dz} \left[|g(s,u_s') - g(s,y_s')| + |f(s,u_s) - f(s,y_s)|\right] ds \\
\leq\ & \int_{t_0}^{t} e^{-\int_s^t \alpha_1(z)dz} |b_1(s)| \, \|u_s' - y_s'\|_\circ \, ds + \int_{t_0}^{t} e^{-\int_s^t \alpha_1(z)dz} |b_2(s)| \, \|u_s - y_s\|_\circ \, ds \\
\leq\ & \int_{t_0}^{t} e^{-\int_s^t \alpha_1(z)dz} \left[|b_1(s)| + |b_2(s)|\right] \max_{s\geq t_0} \left\{\|u_s - y_s\|_\circ, \|u_s' - y_s'\|_\circ\right\} ds \\
\leq\ & \eta \|u - y\|. \qquad (3.8)
\end{aligned}$$

In addition,

$$\begin{aligned}
& \left|(\mathcal{P}u)'(t) - (\mathcal{P}y)'(t)\right| \\
\leq\ & |\alpha_2(t)| \, |(\mathcal{P}u)(t) - (\mathcal{P}y)(t)| + |g(t,u_t') - g(t,y_t')| \\
& + |f(t,u_t) - f(t,y_t)| \\
\leq\ & \|u - y\| \left\{ |\alpha_2(t)| \int_{t_0}^{t} e^{-\int_s^t \alpha_1(z)dz} \left[|b_1(s)| + |b_2(s)|\right] ds + |b_1(t)| + |b_2(t)| \right\} \\
\leq\ & \eta \|u - y\|. \qquad (3.9)
\end{aligned}$$

From $(3.8)$ and $(3.9)$, as $0 < \eta < \dfrac{1}{2}$, $\mathcal{P} : D_\varphi^l \to D_\varphi^l$ is a contraction mapping and hence there exists a unique fixed point $u$ in $D_\varphi^l$ which means $u$ is a solution of $(2.1)$ through $(t_0, \varphi)$, bounded by $l$ and $\lim_{t\to\infty} u(t) = \lim_{t\to\infty} u'(t) = 0$ as $t \to \infty$.

The following step represents another way in which we can establish the stability of $(2.1)$ by using as the main tool Banach's fixed point method. For comprehensive works done on the stability of some particular cases of the equation mentioned, the readers can refer to the papers of Raffoul [5] and Burton [11]. Let $\varepsilon > 0$ be given, by proceeding now a different way than before, that is, replacing $l$ by $\varepsilon$ in $D^l_\varphi$, ( we obtain the existence of a sufficiently small $\delta > 0$ so that (3.7) satisfies with $\delta_0 = \delta$ ) such that for $|\varphi| < \delta$ leads that the unique solution $u$ of (2.1) with $u_{t_0} = \varphi$ on $(-\infty, t_0]$ satisfies $\max\limits_{t\geq t_0} \{|u(t)|, |u'(t)|\} < \varepsilon$. Meanwhile, $\lim\limits_{t\to\infty} u(t, t_0, \varphi) = \lim\limits_{t\to\infty} u'(t, t_0, \varphi) = 0$. We can therefore conclude that the trivial solution of (2.1) is asymptotic stable in $C^1$.

In the end, we proceed to show the asymptotic stability in $C^1$ of the trivial solution to equation (2.1). For all $\varepsilon > 0$, let $\delta > 0$ such that

$$\delta < \varepsilon \min\left\{1, \frac{1-\eta}{K}, \frac{1-\eta}{KA}\right\}.$$

If $u(t) = u(t, t_0, \varphi)$ is a solution of equation (2.1) with $|\varphi|_{t_0} < \delta$, then $u(t) = (\mathcal{P}u)(t)$ on $[t_0, \infty)$. We claim that $\|u\| < \varepsilon$. Otherwise, there would exist $t^* > t_0$ such that

$$\max\{|u(t^*, t_0, \varphi)|, |u'(t^*, t_0, \varphi)|\} = \varepsilon,$$

and

$$\max\{|u(t, t_0, \varphi)|, |u'(t, t_0, \varphi)|\} < \varepsilon,$$

for $t \leq t^*$, if $|u(t^*, t_0, \varphi)| = \varepsilon$, then it follows from (3.5) and (2.3), (2.4), (3.2) that

$$\begin{aligned}
&|u(t^*, t_0, \varphi)| \\
= &\left|\varphi(t_0) e^{-\int_{t_0}^{t^*} \sum_{i=1}^{N} a_i(s, u_s) ds} + \int_{t_0}^{t^*} e^{-\int_{t_0}^{t^*} \sum_{i=1}^{N} a_i(z, u_z) dz} \left[g(s, u'_s) + f(s, u_s)\right] ds\right| \\
\leq &|\varphi(t_0)| e^{-\int_{t_0}^{t^*} \alpha_1(z) dz} \\
&+ \int_{t_0}^{t^*} e^{-\int_{s}^{t^*} \alpha_1(z) dz} \left[|g(s, u'_s) - g(s, 0)| + |f(s, u_s) - f(s, 0)|\right] ds \\
\leq &\delta_0 e^{-\int_{t_0}^{t^*} \alpha_1(z) dz} + \int_{t_0}^{t^*} e^{-\int_{s}^{t^*} \alpha_1(z) dz} \left[|b_1(s)| \|u'_s\|_\circ + |b_2(s)| \|u_s\|_\circ\right] ds \\
\leq &K\delta + \eta\varepsilon < \varepsilon,
\end{aligned}$$

which contradicts $|u(t^*, t_0, \varphi)| = \varepsilon$.
If $|u'(t^*, t_0, \varphi)| = \varepsilon$, it then follows from (3.6),(2.3), (2.4), (3.3) that

$$\begin{aligned}
|u'(t^*, t_0, \varphi)| \leq &|\varphi(t_0)| |\alpha_2(t^*)| e^{-\int_{t_0}^{t^*} \sum_{i=1}^{N} a_i(s, u_s) ds} + |g(t^*, u'_{t^*})| + |f(t^*, u_{t^*})| \\
&+ |\alpha_2(t^*)| \int_{t_0}^{t^*} e^{-\int_{s}^{t^*} \alpha_1(z) dz} \left(|g(s, u'_s)| + |f(s, u_s)|\right) ds \\
\leq &KA\delta + \varepsilon |\alpha_2(t^*)| \int_{t_0}^{t^*} e^{-\int_{s}^{t^*} \alpha_1(z) dz} \left(|b_1(s)| + |b_2(s)|\right) ds \\
&+ |b_1(t^*)| + |b_2(t^*)| \\
\leq &KA\delta + \eta\varepsilon < \varepsilon,
\end{aligned}$$

which contradicts $|u'(t^*, t_0, \varphi)| = \varepsilon$ too. Thus, $\max\{|u(t)|, |u'(t)|\} < \varepsilon$ for all $t \geq t_0$, and the zero solution of equation (2.1) is stable in $C^1$. Combining with the fact that

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} u'(t) = 0.$$

Thus, the zero solution of (2.1) is asymptotic stable in $C^1$ if (3.4) holds.

**Theorem 3.2**. *Suppose that conditions* (2.3) − (2.7) *and* (3.1) − (3.3) *hold for* (2.1). *If the trivial solution of* (2.1) *is globally asymptotic stable in* $C^1$, *then*

$$\lim_{t\to\infty} \int_0^t \alpha_2(s)\, ds = +\infty. \tag{3.10}$$

**Proof.** Conversely, suppose the condition (3.10) fails. Then (3.1) implies that $\liminf_{t\to\infty} \int_0^t \alpha_2(s)\, ds > -\infty$, and we find a sequence $\{t_n\} \subset [0, \infty)$, $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \int_0^{t_n} \alpha_2(s)\, ds = F, \text{ for each } F \in \mathbb{R}^+.$$

We may also select a constant $q \in \mathbb{R}^+$ such that

$$-q \leq \int_0^{t_n} \alpha_2(s)\, ds \leq +q, n = 1, 2, ...$$

Set

$$\overset{0}{K} = \sup_{t\geq t_0} e^{-\int_{t_0}^t \alpha_1(s) ds} \text{ and } \overset{0}{A} = \sup_{t\geq t_0} \{|\alpha_2(t)|\}, J = \liminf_{t\to\infty} \int_0^t \alpha_1(s)\, ds, \ .$$

Hence, it therfore follows from (3.1) that $J \in \mathbb{R}$, $\overset{0}{K}, \overset{0}{A} \in \mathbb{R}^+$.

Since (3.10) fails, then $\int_0^t \alpha_1(s)\, ds$ tends $+\infty$ as $t \to \infty$ fails too. By (3.1), for the sequence $\{t_n\}$ defined above, one can select a constant $J \in \mathbb{R}^+$ such that

$$-J \leq \int_0^{t_n} \alpha_1(s)\, ds \leq +J, n = 1, 2, ... \tag{3.11}$$

Denote

$$I_n = \int_0^{t_n} e^{\int_0^s \alpha_1(z)dz} (|b_1(s)| + |b_2(s)|)\, ds, n = 1, 2, ...$$

But, in view of condition (3.2) we have

$$I_n = \int_0^{t_n} e^{\int_0^s \alpha_1(z)dz} (|b_1(s)| + |b_2(s)|)\, ds \leq \eta$$

From (3.11), it then follows that

$$\begin{aligned} I_n &= e^{\int_0^{t_n} \alpha_1(z)dz} \int_0^{t_n} e^{\int_0^s \alpha_1(z)dz} (|b_1(s)| + |b_2(s)|)\, ds \\ &\leq \eta e^{\int_0^{t_n} \alpha_1(z)dz} < e^J. \end{aligned}$$

Therefore the sequence $\{I_n\}$ is bounded. Thus, the sequence $\{I_n\}$ has a convergent subsequence. Without loss of generality, we can assume that

$$\lim_{n\to\infty}\int_0^{t_n} e^{\int_0^s \alpha_1(z)dz}\left(|b_1(s)|+|b_2(s)|\right)ds=\mu,\ \text{for some } \mu\in\mathbb{R}^+.$$

Let $m$ be an integer such that

$$\int_{t_m}^{t_n} e^{\int_0^s \alpha_1(z)dz}\left(|b_1(s)|+|b_2(s)|\right)ds<\frac{1-\eta}{4Be^{2q}\left(e^{-J}+1\right)}, \tag{3.12}$$

and

$$e^{-\int_{t_m}^{t_n}\alpha_1(z)dz}>\frac{1}{2},\ e^{-\int_0^{t_n}\alpha_1(z)dz}<e^{-J}+1,\ e^{\int_0^{t_m}\alpha_1(z)dz}<e^{J}+1, \tag{3.13}$$

for all $n>m$, where

$$B=\max\left\{\overset{0}{K}\left(e^J+1\right),\overset{0}{K}\overset{0}{A}\left(e^J+1\right),1\right\}.$$

For any $\delta_0>0$, we consider $u(t)=u(t,t_m,\varphi)$ the solution of (2.1) with $|\varphi|_{t_m}<\delta_0$ and $|\varphi(t_m)|>\dfrac{\delta_0}{2}$ for $t<t_m$. It therfore follows from (3.5), (3.6), (3.13) and (3.1) – (3.3), that for $t\in[t_m,\infty)$,

$$\begin{aligned}
|u(t)| &\le \delta_0 e^{-\int_{t_m}^{t}\alpha_1(s)ds}+\int_{t_m}^{t} e^{-\int_s^t\alpha_1(z)dz}\left[|g(s,u_s')|+|f(s,u_s)|\right]ds\\
&\le \overset{0}{K}\left(e^J+1\right)\delta_0+\|u\|_{t_m}\int_{t_m}^{t} e^{-\int_s^t\alpha_1(z)dz}\left[|b_1(s)|+|b_2(s)|\right]ds\\
&\le B\delta_0+\eta\|u\|_{t_m},
\end{aligned}$$

and

$$\begin{aligned}
|u'(t)| &\le |u(t_m)|\,|\alpha_2(t)|\,e^{-\int_{t_m}^{t}\alpha_1(s)ds}+|g(t,u_t')|+|f(t,u_t)|\\
&\quad+|\alpha_2(t)|\int_{t_m}^{t} e^{-\int_s^t\alpha_1(z)dz}\left(|g(s,u_s')|+|f(s,u_s)|\right)ds\\
&\le \overset{0}{K}\overset{0}{A}\left(e^J+1\right)\delta_0\\
&\quad+\|u\|_{t_m}\left\{|\alpha_2(t)|\int_{t_m}^{t} e^{-\int_s^t\alpha_1(z)dz}\left[|b_1(s)|+|b_2(s)|\right]ds+\left[|b_1(t)|+|b_2(t)|\right]\right\}\\
&\le B\delta_0+\eta\|u\|_{t_m}.
\end{aligned}$$

Hence, $\|u\|_{t_m}\le B\delta_0+\eta\|u\|_{t_m}$, thus we have

$$\|u\|_{t_m}\le\frac{B}{1-\eta}\delta_0,\ \text{for all } t\ge t_m. \tag{3.14}$$

It then follows from $(3.5), (3.12)-(3.14)$ and $(2.3), (2.4), (2.7)$ that for any $n > m$.

$$\begin{aligned} |u(t_n)| &\geq |\varphi(t_m)| e^{-\int_{t_m}^{t_n} \alpha_2(s)ds} - \left| \int_{t_m}^{t_n} e^{-\int_s^{t_n} \sum_{i=1}^{N} a_i(z,u_z)dz} [g(s,u_s') + f(s,u_s)]\, ds \right| \\ &\geq \delta_0 e^{-\int_{t_m}^{t_n} \alpha_2(s)ds} - \int_{t_m}^{t_n} e^{-\int_s^{t_n} \sum_{i=1}^{N} a_i(z,u_z)dz} |g(s,u_s') + f(s,u_s)|\, ds \\ &\geq \delta_0 e^{-\int_{t_m}^{t_n} \alpha_2(s)ds} - \|u\|_{t_m} e^{-\int_0^{t_n} \alpha_1(z)dz} \int_{t_m}^{t_n} e^{\int_0^s \alpha_1(z)dz} [|b_1(s)| + |b_2(s)|]\, ds \\ &\geq \delta_0 e^{-\int_{t_m}^{t_n} \alpha_2(s)ds} - \|u\|_{t_m} e^{-\int_0^{t_n} \alpha_1(z)dz} \int_{t_m}^{t_n} e^{\int_0^s \alpha_1(z)dz} [|b_1(s)| + |b_2(s)|]\, ds. \end{aligned}$$

But

$$\begin{aligned} e^{-\int_{t_m}^{t_n} \alpha_2(z)ds} &= e^{\int_{t_n}^{0} \alpha_2(z)dz} e^{\int_0^{t_m} \alpha_2(z)dz} \\ &= e^{-\int_0^{t_n} \alpha_2(z)dz} e^{\int_0^{t_m} \alpha_2(z)dz} \geq e^{-2q}, \end{aligned}$$

and $e^{-\int_0^{t_n} \alpha_1(z)dz} \leq e^{-J} + 1$, which implies

$$|u(t_n)| \geq \frac{1}{2}\delta_0 e^{-2q} - \frac{\delta_0 B}{1-\eta}\left(e^{-J}+1\right) \frac{1-\eta}{4Be^{2q}(e^{-J}+1)} = \frac{1}{2}\delta_0 e^{-2q}. \qquad (3.15)$$

The facts that $\lim_{n\to\infty} t_n = \infty$ and the trivial solution of (2.1) is asymptotic stable in $C^1$ implies $\lim_{n\to\infty} u(t, t_n, \varphi) = \lim_{n\to\infty} u'(t, t_n, \varphi) = 0$, which is in contradiction with $(3.15)$. The proof of Theorem 3.1 is completed.

**Corollary 3.1.** *Assume that (A1)–(A6) hold, and for any $t \geq t_0$, if there is an $\eta \in (0, \frac{1}{2})$ such that*

$$\liminf_{t\to\infty} \int_{t_0}^{t} \alpha_1(s)\, ds > -\infty,$$

*and*

$$\int_{t_0}^{t} e^{-\int_s^t \alpha_1(z)dz} (|b_1(s)| + |b_2(s)|)\, ds \leq \eta,$$

*Then zero the solution equation* (2.1) *shows asymptotic stable in $C^0$ if*

$$\int_{t_0}^{t} \alpha_1(s)\, ds \to \infty \text{ as } t \to \infty.$$

For equation (2.1), we also have

**Corollary 3.2.** *Suppose that (A1)–(A6) and $(3.1), (3.2)$ hold. If the trivial solution of (2.1) is asymptotic stable in $C^0$, then we get*

$$\int_{t_0}^{t} \alpha_2(s)\, ds \to \infty \text{ as } t \to \infty.$$

**Remark 3.1.** According to Ziad *et al.* [31]. Corollary 3.1 and Corollary 3.2 are natural generalizations of Theorem 3.1 and Theorem 3 2, respectively. In fact, when $g(t, u'_t) = 0$ our conditions reduce to those of Ziad *et al.* [31].

Now we consider the standard form of tottaly nonlinear neutral differential equations

$$u'(t) = -h\left(t, u(t)\right) + g\left(t, u'_t\right) + f\left(t, u_t\right), t \geq t_0. \tag{3.16}$$

Similar to equation (2.1), if we assume that

**(A7)** $h\left(t, 0\right)$ and there exist $\alpha_1, \alpha_2 \in C\left(\mathbb{R}, \mathbb{R}\right)$ such that

$$\alpha_1\left(t\right) \leq \frac{\partial h\left(t, u\right)}{\partial u} \leq \alpha_2\left(t\right),$$

then we can get the following theorem.

**Theorem 3.3.** *Suppose that (A1)-(A7), and* $(3.1)-(3.4)$ *hold, then the trivial solution of* (3.16) *is asymptotic stable in* $C^1$.

**Proof.** For any $h \in C^1$, since $h(t, 0) = 0$, it is clear to see that

$$h(t, u) = \left[\int_0^1 \frac{\partial h\left(t, su\right)}{\partial u} ds\right] u.$$

If we set $\sum\limits_{i=1}^{N} a_i\left(t, u_t\right) = \int_0^1 \frac{\partial h\left(t, su\right)}{\partial u} ds$, then we can rewrite (2.1) as (3.16) with

$$\alpha_1\left(t\right) \leq \sum_{i=1}^{N} a_i\left(t, u_t\right) \leq \alpha_2\left(t\right).$$

Then the claim is true thanks to Theorem 3.1.

In addition, we get another result for equation (3.16) as follows.

**Theorem 3.4.** *If conditions (A1)-(A6), and* $(3.1)-(3.3)$ *are satisied, then the zero solution of* (3.16) *with a small initial function is asymptotic stable in* $C^1$. *If the zero solution of* (3.16) *is globally asymptotic stable in* $C^1$, *then*

$$\int_0^t \alpha_2\left(t\right) \to \infty \text{ as } t \to \infty,$$

*holds.*

Choosing $N = 1$ and $a_1\left(t, u_t\right) = a(t)$ in Theorem 3.1, we have the following result.

**Corollary 3.3.** *Assume that (A1)–(A6) hold, and for any* $t \geq t_0$, *and there exists a constant* $\eta \in (0, \frac{1}{2})$ *such that,*

$$\liminf_{t \to \infty} \int_{t_0}^{t} a\left(s\right) ds > -\infty, \tag{3.17}$$

*and*

$$\int_{t_0}^{t} e^{-\int_s^t a(z)dz} \left(|b_1(s)| + |b_2(s)|\right) ds \leq \eta, \tag{3.18}$$

$$|a\,(t)| \int_{t_0}^{t} e^{-\int_s^t a(z)dz} \left(|b_1(s)| + |b_2(s)|\right) ds + \left(|b_1(t)| + |b_2(t)|\right) \leq \eta. \tag{3.19}$$

*Then equation* (2.1) *has a unique trivial solution and, it is an asymptotic stable in* $C^1$ *if only if*

$$\int_{t_0}^{t} a\,(s)\,ds \to \infty \text{ as } t \to \infty.$$

**Remark 3.2.** Theorem 3.1 remains true if the conditions (3.2), (3.3) are fulfilled for all $t \geq t_\sigma$ for some $t_\sigma \in \mathbb{R}^+$.

# 4 Remarks and illustrative Examples

Let us discuss two examples for ullistration.

**Example 4.1.** Given a nonlinear neutral differential equation

$$u'(t) = -a\,(t, u(t-\tau\,(t)))\,u(t) + f\,(t, u(t-\tau\,(t))) + g\,(t, u'(t-\tau\,(t)))\,, \tag{4.1}$$

$t \geq 0$, where

$$a\,(t, u(t-\tau\,(t))) = \frac{1}{1+t}\left(1 + \frac{|\sin t|}{1+u^2\,(t-\tau\,(t))}\right),$$

and

$$\begin{aligned} g\,(t, u'(t-\tau\,(t))) &= \frac{0.1}{1+t}\sin\frac{u'(t-\tau\,(t))}{10}, \\ f\,(t, u(t-\tau\,(t))) &= 0.4\ln\left(1 + \frac{|u\,(t-\tau\,(t))|}{10\,(1+t)}\right). \end{aligned}$$

One can take $\alpha_1(t) = \dfrac{1}{1+t}$ and $\alpha_2(t) = \dfrac{2\,|\sin t|}{1+t}$, then

$$\alpha_1(t) \leq a\,(t, u_t) \leq \alpha_2(t).$$

It is easy to check

$$|\alpha_2\,(t)| < 2,\ \forall t \in [0, \infty)\,,\ \int_0^t \alpha_1(s)ds \to \infty \text{ as } t \to \infty.$$

By straightforward computations, we can check that conditions (2.2) and (2.3) in Theorem 3.1 hold true, where $\tau \in C(\mathbb{R}^+, \mathbb{R}^+)$, and $\delta \in C(\mathbb{R}^+, \mathbb{R}^+)$ with

$$t - \tau\,(t) \to \infty \text{ and } t - \delta\,(t) \to \infty \text{ as } t \to \infty. \tag{4.2}$$

Assume that $b_1(t) = \dfrac{0.1}{2(1+t)}$ and $b_2(t) = \dfrac{0.5}{10(1+t)}$. Then (2.3), (2.4) hold. Also assume that $\eta = 1/3$, then for $t \in [0, \infty)$

$$\begin{aligned} &\int_0^t e^{-\int_s^t \alpha_1(z)dz} \left(|b_1(s)| + |b_2(s)|\right) ds \\ &\quad \leq \int_0^t e^{-\int_s^t \dfrac{1}{1+z}dz} \left(\frac{1}{10(1+s)}\right) ds = \frac{1}{10} \leq \eta, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &|\alpha_2(t)| \int_0^t e^{-\int_s^t \alpha_1(z)dz} \left(|b_1(s)| + |b_2(s)|\right) ds + \left(|b_1(t)| + |b_2(t)|\right) \\ &\leq 2\int_0^t e^{-\int_s^t \dfrac{1}{1+z}dz} \left(\frac{1}{10(1+s)}\right) ds + \frac{1}{10(1+t)} \\ &= \frac{3}{10} \leq \eta. \end{aligned} \tag{4.4}$$

Hence, all the conditions in Theorem 3.1 are verified. Therefore, the zero solution of equation (4.1) is asymptotic stable in $C^1$.

**Example 4.2.** Consider the following equation in the form of (2.1),

$$\begin{aligned} u'(t) \quad = \quad & -\sum_{i=1}^{2} a_i\left(t, u(t-\tau(t))\right) u(t) + f\left(t, u(t-\tau_1(t)), u(t-\tau_2(t))\right) \\ & + g\left(t, u'(t-\tau_1(t)), u'(t-\tau_2(t))\right). \end{aligned} \tag{4.5}$$

By taking $a_1(t,u) = \dfrac{0.5e^t}{1+e^t}\left(1 + \dfrac{|\cos t|}{\left(1+e^{-u^2}\right)}\right)$, $a_2(t,u)) = \dfrac{0.5e^t}{1+e^t}\left(1 + \dfrac{|\sin(u)|}{2}\right)$, $\tau \in C(\mathbb{R}^+, \mathbb{R}^+)$, and $\tau_i \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfy

$$t - \tau_i(t) \to \infty \text{ as } t \to \infty, i = 1, 2. \tag{4.6}$$

By calculation, we have

$$\alpha_1(t) := \frac{e^t}{1+e^t} \leq \sum_{i=1}^{2} a_i\left(t, u(t-\tau(t))\right) \leq \frac{1.75e^t}{1+e^t} =: \alpha_2(t),$$

and it is straightforward to check that

$$|\alpha_2(t)| < 1.75, \ \forall t \in [0, \infty), \text{ and } \int_0^t \alpha_1(s)ds \to \infty \text{ as } t \to \infty.$$

Let $f(t, u_1, u_2) = \ln\left(1 + \dfrac{5(|u_1| + |u_2|)}{100(1+e^{-t})}\right)$, $g(t, u_1, u_2) = 0.1 \sin\left(\dfrac{u_1}{5(1+e^{-t})}\right) + 0.12 \sin\left(\dfrac{u_2}{4(1+e^{-t})}\right)$, then we obtain

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq |b_1(t)| |u_1 - v_1| + |b_2(t)| |u_2 - v_2|,$$

$$|g(t,u_1,u_2)-g(t,v_1,v_2)| \leq |c_1(t)|\,|u_1-v_1| + |c_2(t)|\,|u_2-v_2|,$$

where

$$b_1(t) = b_2(t) = \frac{5}{100\left(1+e^{-t}\right)},$$

and

$$c_1(t) = \frac{0.02}{1+e^{-t}},\, c_2(t) = \frac{0.03}{1+e^{-t}}.$$

Then (A1)-(A6) hold. In addition, let $\eta = 4/9$, then for $t \in [0,\infty)$,

$$\begin{aligned}
&\int_0^t e^{-\int_s^t \alpha_1(z)dz} \sum_{j=1}^{2} |b_j(s)| + |c_j(s)|\, ds \\
&\quad < \int_0^t e^{-\int_s^t \frac{e^z}{1+e^z}dz} \left[\frac{e^s}{10\left(1+e^s\right)} + \frac{0.05e^s}{\left(1+e^s\right)}\right] ds < 0.15 < \eta, \quad (4.7)
\end{aligned}$$

and

$$\begin{aligned}
&|\alpha_2(t)| \int_0^t e^{-\int_s^t \alpha_1(z)dz} \sum_{j=1}^{2} \left(|b_j(s)| + |c_j(s)|\right) ds + \sum_{j=1}^{2} \left(|b_j(t)| + |c_j(t)|\right) \\
&< 1.75 \times \int_0^t e^{-\int_s^t \frac{e^z}{1+e^z}dz} \left[\frac{e^s}{10\left(1+e^s\right)} + \frac{0.05e^s}{\left(1+e^s\right)}\right] ds \\
&+ \frac{e^t}{10\left(1+e^t\right)} + \frac{0.05e^t}{\left(1+e^t\right)} \\
&< 1.75 \times 0.15 + \frac{e^t}{10\left(1+e^t\right)} + \frac{0.05e^t}{\left(1+e^t\right)} \\
&< 1.75 \times 0.15 + 0.15 = 0.413 \leq \eta. \quad (4.8)
\end{aligned}$$

Hence, (3.2) and (3.3) hold. According to Theorem 3.1, the zero solution of Eq. (4.5) is globally asymptotically stable in $C^1$.

**Remark 4.1.** Theorem 3.1 includes and generalizes the result of Ardjouni and Djoudi [1]. In fact, when we chose $N = 1$ and $a_1(t,u_t) = a(t)$ ($a$ is bounded), $g(t,u_t') = g(t,u'(t-\tau_1(t)), u'(t-\tau_2(t), ..., u'(t-\tau_n(t)))$ and $f(t,u_t) = f(t,u(t-\tau_1(t)), u(t-\tau_2(t), ..., u(t-\tau_n(t)))$, our conditions reduce to that of Ardjouni and Djoudi [1, Theorem 2.1].

**Remark 4.2.** It has been noted in [27] that a fading memory condition such as $(2.5)$, $(2.6)$ or $(4.6)$ is necessary for the asymptotic behavior of a general neutral differential equation. This means that the equation representing a physical system must remember its past, but the memory must fade over time.

**Conclusion 1** ***Conclusion:*** *In this work, a standard tottally nonlinear neutral differential equations have been studied. Based on Banach's fixed point* *theorem, some new sufficient conditions to guarantee that the trivial solution of equation* (2.1) *is globally asymptotic stable in* $C^1$ *have been achieved. The main contribution of this paper is confirming the importance and advantage of using the fixed point theory. The derived stability criteria are easily to apply in practice and do not need the differentiability of the delays or coefficients, which are required in [16]. Moreover, we can easily see Theorem 3.1 and Corollaries cited above are independent of some restrictive conditions in reference [16]. Up now, the results derived here have not been published in the corresponding literature. An illustrative examples were given to show the efficiency of the results introduced. Hence, in futur, we would like to extend the application of this precious approach to more complex delay models such as the equations with damped stochastic perturbations and so on. We will leave this open problem for future research subjects.*

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