REGULARITY OF GLOBAL ATTRACTORS AND EXPONENTIAL ATTRACTORS FOR 2D QUASI-GEOSTROPHIC EQUATIONS WITH FRACTIONAL DISSIPATION

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ABSTRACT. In this paper we investigate the regularity of global attractors and of exponential attractors for two dimensional quasi-geostrophic equations with fractional dissipation in $H^{2\alpha+s}(\mathbb{T}^2)$ with $\alpha > \frac{1}{2}$ and $s > 1$. We prove the existence of $(H^{2\alpha^-+s}(\mathbb{T}^2), H^{2\alpha+s}(\mathbb{T}^2))$-global attractor $\mathcal{A}$, that is, $\mathcal{A}$ is compact in $H^{2\alpha+s}(\mathbb{T}^2)$ and attracts all bounded subsets of $H^{2\alpha^-+s}(\mathbb{T}^2)$ with respect to the norm of $H^{2\alpha+s}(\mathbb{T}^2)$. The asymptotic compactness of solutions in $H^{2\alpha+s}(\mathbb{T}^2)$ is established by using commutator estimates for nonlinear terms, the spectral decomposition of solutions and new estimates of higher order derivatives. Furthermore, we show the existence of the exponential attractor in $H^{2\alpha+s}(\mathbb{T}^2)$, whose compactness, boundedness of the fractional dimension and exponential attractiveness for the bounded subset of $H^{2\alpha^-+s}(\mathbb{T}^2)$ are all in the topology of $H^{2\alpha+s}(\mathbb{T}^2)$.

1. **Introduction.** In this paper, we investigate the long-time behavior of solutions of the following 2D (surface) quasi-geostrophic equation with fractional dissipation defined on the 2D torus $\mathbb{T}^2 = [0,1]^2$:

$$\begin{cases} \dfrac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = F(x,\theta), \\ \theta(0) = \theta_0, \quad \displaystyle\int_{\mathbb{T}^2} \theta_0(x) dx = 0, \end{cases} \tag{1.1}$$

where $F(x,\theta)$ is a given external forcing term in the form

$$F(x,\theta) = g_1(x) f(\theta) + g_2(x), \tag{1.2}$$

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θ represents the potential temperature, $\kappa > 0$ is a diffusivity coefficient, $\alpha \in \left(\frac{1}{2}, 1\right]$ is a fractional exponent, $(-\Delta)^{\alpha}$ is the fractional Laplacian, and $u = (u_1, u_2)$ is the velocity field determined by θ through the relation:

$$u = \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1}\right), \quad \text{where} \quad (-\Delta)^{\frac{1}{2}}\psi = -\theta, \tag{1.3}$$

or, in a more explicit way,

$$u = (-R_2\theta, R_1\theta), \tag{1.4}$$

where $R_i, i = 1, 2$ are the Riesz transforms (see [25, p.299]). In the following, we will restrict ourselves to flows which have zero average on the torus, i.e.,

$$\int_{\mathbb{T}^2} \theta(t, x)dx = 0, \quad \forall t \geq 0.$$

Equations (1.1)-(1.3) are important models in geophysical fluid dynamics, especially for atmospheric and oceanic fluid. Indeed, the system describes the evolution of the temperature on the 2D boundary of a rapidly rotating half-space with small Rossby and Ekman numbers (see [28]).

Long time behavior of solutions to the 2D quasi-geostrophic equation with fractional dissipation has been studied in [3, 11, 22, 26, 27]. Using the framework of [31], the global weak attractor $\mathcal{A}$ in the space of weak solutions $W(f)$ was proved by Berselli for the sub-critical dissipative case $\alpha \in (\frac{1}{2}, 1)$. $\mathcal{A}$ attracts all bounded sets in the space of the generalized weak solutions $GW(f)$, see [3] for more details. With an improvement of the positivity lemma of [10] and a generalized maximum principle, Ju [22] established the existence of the global attractor $\mathcal{A}$ in $H^s(\mathbb{T}^2)$ for any $s > 2(1-\alpha)$ and $\alpha \in (\frac{1}{2}, 1]$. $\mathcal{A}$ attracts all bounded subsets of $H^s(\mathbb{T}^2)$ in the norm of $H^r(\mathbb{T}^2)$ for any $r \geq s > 2(1-\alpha)$, and for the case $\alpha \in (\frac{2}{3}, 1]$, $\mathcal{A}$ attracts all bounded subsets of $L^2(\mathbb{T}^2)$ in the norm of $H^s(\mathbb{T}^2)$ for any $s > 2(1-\alpha)$. More detailed results on decay characterization of solutions to 2D dissipative quasi-geostrophic equations were given by Niche and Schonbek [26, 27] for the homogeneous case (F=0). Wang and Tang [36] proved the existence of the global attractor in $L^p(\mathbb{R}^2)$ for 2D quasi-geostrophic equations with damping in the subcritical case $\alpha \in (\frac{1}{2}, 1]$. Dlotko, Kania and Sun [12] studied the existence of the global attractor $\mathcal{A}$ in $H^{s+2\alpha^-}(\mathbb{R}^2)$ for any $s > 1$ and $\alpha \in (\frac{1}{2}, 1]$. Very recently, the existence of the global attractor in $H^s(\mathbb{R}^2)$ for any $s > 2(1-\alpha)$ and $\alpha \in (\frac{1}{2}, 1]$ was proved by Farwig and Qian [15] for 2D quasi-geostrophic equations with a nonlocal damping. For the critical case $\alpha = \frac{1}{2}$, the existence of the global attractor in $H^{3/2}(\mathbb{T}^2)$ has been studied by Constantin, Coti Zelati, Kalita, Vicol and Tarfulea, see [7, 9, 11] for more details. However, to the best of our knowledge, there is no result available in the literature on the regularity of global attractors for 2D quasi-geostrophic equations. The first purpose of this work is to prove the regularity of global attractors of (1.1) in $H^{2\alpha+s}(\mathbb{T}^2)$ for any $s > 1$ and $\alpha \in (\frac{1}{2}, 1]$. More precisely, we will show that the global attractor $\mathcal{A}$ in $H^{2\alpha^-+s}(\mathbb{T}^2)$ is actually an $(H^{2\alpha^-+s}(\mathbb{T}^2), H^{2\alpha+s}(\mathbb{T}^2))$-global attractor in the sense that $\mathcal{A}$ is compact in $H^{2\alpha+s}(\mathbb{T}^2)$ and attracts all bounded subsets of $H^{2\alpha^-+s}(\mathbb{T}^2)$ with respect to the norm of $H^{2\alpha+s}(\mathbb{T}^2)$ (see Theorem 5.3).

The notion of exponential attractors, introduced by Eden, Foiaş, Nicolaenko and Temam [13], has been shown to be one of the most important concepts of limit sets in the theory of dynamical systems in infinite-dimensional spaces (see [2, 6, 30, 34]). The exponential attractor, as an intermediate object between the global attractor and the inertial manifold, satisfies some nice properties like the inertial manifold (e.g., finite fractal dimension, exponential attracting, stable with respect to some perturbations). However, contrarily to the global attractor, an exponential attractor is not necessarily unique, so that the concrete choice of an exponential attractor is in some sense artificial. The first technique to construct exponential attractors was developed in Hilbert spaces. This technique is based on the use of orthogonal projections (see [13]) and cannot be applied directly to dynamical systems defined in Banach spaces. A new method for constructing exponential attractors in Banach spaces was proposed by Efendiev, Miranville and Zelik in [14]. However, the existence of exponential attractors for 2D quasi-geostrophic equations is unsolved. The second purpose of this paper is to present some sufficient conditions for the construction of exponential attractors for autonomous dynamical systems on Banach spaces, and furthermore we apply our results to consider the existence of exponential attractors of (1.1) in $H^{2\alpha^-+s}(\mathbb{T}^2)$ and the regularity of exponential attractors of (1.1) with $g_2(x)$ instead of $F(x,\theta)$ in $H^{2\alpha+s}(\mathbb{T}^2)$ for any $s>1$ and $\alpha\in\left(\frac{1}{2},1\right]$.

There are several results on estimation of the fractal and Hausdorff dimensions of the global attractor for quasi-geostrophic equations. Wang [35] proved the existence of a compact, connected global attractor to the 3D baroclinic quasi-geostrophic equations of large scale atmosphere, and derived an upper bound of the Hausdorff and fractal dimensions of the global attractor. A precise upper bound of the fractal dimension of the global attractor for 2D quasi-geostrophic equations with fractional dissipation in $H^s(\Omega)$ for any $s\geq 2\alpha$ and $\alpha\in\left(\frac{1}{2},1\right]$ was obtained by Wang and Tang [37]. By using the fractional Lieb-Thirring inequality, estimates of the finite Hausdorff and fractal dimensions of the global attractor for 2D quasi-geostrophic equations with fractional dissipation were established by Farwig and Qian [15].

The paper is organized as follows. In the next section, we present some notation and recall the theory of global attractors for infinite dimensional dissipative dynamical systems and several preliminary results which will be used frequently. In Section 3, the global existence, uniqueness and regularity of solutions for problem (1.1) are established by using the theory of semilinear parabolic equations with sectorial operator. Section 4 is devote to a priori estimates which will yield the existence of bounded absorbing sets in $H^{2\alpha+s}(\mathbb{T}^2)$. In Section 5, we first establish the existence of the global attractor $\mathcal{A}$ in $H^{2\alpha^-+s}(\mathbb{T}^2)$ and then prove that $\mathcal{A}$ is indeed the $(H^{2\alpha^-+s}(\mathbb{T}^2),H^{2\alpha+s}(\mathbb{T}^2))$-global attractor. In Section 6, we study the existence of exponential attractors.

2. **Preliminaries.**

We first recall some notations and basic results from harmonic analysis. The fractional Laplacian $(-\Delta)^s$, with $s\in\mathbb{R}$ may be defined in this context as the Fourier multiplier with symbol $|k|^s$, i.e.,

$$(-\Delta)^s\varphi(x)=\sum_{k\in\mathbb{Z}^2_*}|k|^{2s}\widehat{\varphi}(k)\exp(ik\cdot x),$$

where $\mathbb{Z}^2_*=\mathbb{Z}^2\setminus\{0\}$,

$$\widehat{\varphi}(k)=\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}\varphi(x)e^{-ik\cdot x}dx \quad\text{and}\quad \varphi(x)=\sum_{k\in\mathbb{Z}^2_*}\widehat{\varphi}(k)\exp\left(ik\cdot x\right).$$

Notice that the eigenvalues of $\Lambda=(-\Delta)^{\frac{1}{2}}$ are given by $|k|$. Then we relabel them in increasing order as

$$0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_n\leq\cdots,$$

and denote the eigenfunction associated to λ_j by e_j. It is clear that $\{e_j\}_{j\geq 1}$ is an orthonormal basis of $L^2(\mathbb{T}^2)$, and the sets $\{\lambda_j\}_{j\geq 1}$ and $\{|k|\}_{k\in\mathbb{Z}^2_*}$ are equal.

As a consequence of the mean-free setting, for $s\in\mathbb{R}$ we define

$$\|\varphi\|_{H^s(\mathbb{T}^2)} = \|\Lambda^s\varphi\|_{L^2(\mathbb{T}^2)}$$

and $H^s(\mathbb{T}^2)$ denotes the Sobolev space of all f for which $\|f\|_{H^s(\mathbb{T}^2)}$ is finite. Moreover, for $s\in\mathbb{R}$ and $p\in[1,+\infty)$ we denote by $H^{s,p}(\mathbb{T}^2)$ the space of mean-free $L^p(\mathbb{T}^2)$ functions φ, which can be written as $\varphi=\Lambda^{-s}\psi$, with $\psi\in L^p(\mathbb{T}^2)$. The $H^{s,p}(\mathbb{T}^2)$ norm of φ is defined to be the $L^p(\mathbb{T}^2)$ norm of ψ, i.e.,

$$\|\varphi\|_{H^{s,p}(\mathbb{T}^2)} = \|\Lambda^s\varphi\|_{L^p(\mathbb{T}^2)}.$$

We recall the following important commutator and product estimates, cf. [23, 24].

Lemma 1 (Commutator and Product Estimates)**.** *Suppose that $\gamma>0$ and $p\in(1,+\infty)$. If $f,g\in\mathcal{S}(\mathbb{T}^2)$, the Schwartz class, then*

$$\begin{aligned}\|\Lambda^\gamma(fg)-f\Lambda^\gamma g\|_{L^p(\mathbb{T}^2)} \leq C(\gamma,p)\big(&\|\nabla f\|_{L^{p_1}(\mathbb{T}^2)}\|\Lambda^{\gamma-1}g\|_{L^{p_2}(\mathbb{T}^2)}\\ &+\|\Lambda^\gamma f\|_{L^{p_3}(\mathbb{T}^2)}\|g\|_{L^{p_4}(\mathbb{T}^2)}\big)\end{aligned} \tag{2.1}$$

and

$$\|\Lambda^\gamma(fg)\|_{L^p(\mathbb{T}^2)} \leq C(\gamma,p)\big(\|f\|_{L^{p_1}(\mathbb{T}^2)}\|\Lambda^\gamma g\|_{L^{p_2}(\mathbb{T}^2)}+\|\Lambda^\gamma f\|_{L^{p_3}(\mathbb{T}^2)}\|g\|_{L^{p_4}(\mathbb{T}^2)}\big) \tag{2.2}$$

with $p_2,\,p_3\in(1,+\infty)$ such that

$$\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}=\frac{1}{p}.$$

The following result can be obtained by the fact that the Riesz transforms commute with $(-\Delta)^l$ and the boundedness of the Riesz transforms in $L^p(\mathbb{T}^2)$, see [32, Chapter 3] for more details.

Lemma 2. *Let $1<p<\infty$ and $l\geq 0$. Then there exists a constant $C(l,p)$ such that*

$$\|(-\Delta)^l u\|_{L^p(\mathbb{T}^2)} \leq C(l,p)\|(-\Delta)^l\theta\|_{L^p(\mathbb{T}^2)}. \tag{2.3}$$

If $p=2$, the inequality (2.3) can be strengthened to

$$\|(-\Delta)^l u\|_{L^2(\mathbb{T}^2)} = \|(-\Delta)^l\theta\|_{L^2(\mathbb{T}^2)}. \tag{2.4}$$

Denote $A_{\gamma,\kappa}=\kappa(-\Delta)^\gamma$ where $\gamma>0$ and $\kappa>0$, and let

$$X^{\gamma,\kappa}=\big\{u\in L^2(\mathbb{T}^2): A_{\gamma,\kappa}u\in L^2(\mathbb{T}^2)\big\}$$

with norm $\|u\|_{X^{\gamma,\kappa}}=\|A_{\gamma,\kappa}u\|_{L^2(\mathbb{T}^2)}$. We recall the following well-known results for the semigroup generated by the positive operator $A_{\gamma,\kappa}$ (see [18, 38] for the similar results).

Proposition 3. *Let $0<\beta_1\leq\beta_2$ and $u\in X^{\beta_1,\kappa}$. Then there exists a constant $C_1=C_1(\beta_1,\beta_2,\gamma,\kappa)$ such that*

$$\|e^{-A_{\gamma,\kappa}t}u\|_{X^{\beta_2,\kappa}} \leq C_1 e^{-\frac{\kappa\lambda_1^{2\gamma}}{2}t}t^{-\frac{\beta_2-\beta_1}{\gamma}}\|u\|_{X^{\beta_1,\kappa}}, \quad t>0.$$

We recall the following improved positivity lemma, cf. [22, Lemma 3.3], which we use in the proof of Lemma 11.

Lemma 4 (Improve Positivity Lemma)**.** *Suppose $s \in [0,2]$ and θ, $\Lambda^s\theta \in L^p(\mathbb{T}^2)$, where $p \geq 2$. Then*

$$\int_{\mathbb{T}^2} |\theta|^{p-2}\theta\Lambda^s\theta dx \geq \frac{2}{p}\int_{\mathbb{T}^2}\left(\Lambda^{\frac{s}{2}}|\theta|^{\frac{p}{2}}\right)^2 dx.$$

The following variant of the classical Gronwall Lemma is due to Foiaş and Prodi [16]. See also [5, 8, 34].

Lemma 5 (Uniform Gronwall Lemma)**.** *Let g, h and y be non-negative locally integrable functions on $[t_0, +\infty)$ such that*

$$\frac{dy}{dt} \leq gy + h, \quad \forall t \geq t_0,$$

and

$$\int_t^{t+r} g(s)ds \leq a_1, \qquad \int_t^{t+r} h(s)ds \leq a_2, \qquad \int_t^{t+r} y(s)ds \leq a_3, \quad \forall t \geq t_0,$$

where $r > 0$ and a_1, a_2, a_3 are non-negative constants. Then

$$y(t+r) \leq \left(\frac{a_3}{r} + a_2\right)e^{a_1}, \quad \forall t \geq t_0.$$

In the sequel, C denotes an arbitrary positive constant, which may be different from line to line and even in the same line. For $r \in \mathbb{R}$, let r^- denotes the number strictly less than r but close to it.

2.1. **Semigroup and attractor.**

In this subsection, we recapitulate basic concepts and results on the bi-spaces global attractor and the exponential attractor. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|)$ be two Banach spaces such that the injection $Y \hookrightarrow X$ is continuous. The Hausdorff semidistance for nonempty subsets A and B of X is written as $dist_X(A,\, B)$ which is defined by

$$dist_X(A,\, B) = \sup\{d(a, B) : a \in A\},$$

where $d(a, B) = \inf\{\|a - b\|_X : b \in B\}$, and the similar notation will be used for subsets of Y.

Definition 6. Let X be a Banach space and $\{S(t)\}_{t\geq 0}$ be a family of operators on X. We say that $\{S(t)\}_{t\geq 0}$ is a continuous semigroup on X, if for all $t, s \in \mathbb{R}^+$, the following conditions are satisfied:

(i) $S(0) = I$ (the identity);
(ii) $S(t)S(s) = S(t+s)$;
(iii) $S(t)x$ is continuous in x and t.

Definition 7. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on X. Then $\{S(t)\}_{t\geq 0}$ is said to be asymptotically compact in X if for any bounded sequence $\{x_n\}_{n=1}^\infty \subset X$ and any sequence $t_n \to +\infty$, the sequence

$$\{S(t_n)x_n\}_{n=1}^\infty \text{ has a convergent subsequence in } X.$$

If, in addition, $S(t)$ maps X to Y for every $t \in \mathbb{R}^+$, and the sequence

$$\{S(t_n)x_n\}_{n=1}^\infty \text{ has a convergent subsequence in } Y$$

for any bounded sequence $\{x_n\}_{n=1}^\infty \subset X$ and any sequence $t_n \to +\infty$, then $\{S(t)\}_{t\geq 0}$ is said to be (X, Y)-asymptotically compact.

Definition 8. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on X. Then a set $\mathcal{A} \subset X$ is called a global attractor of $\{S(t)\}_{t\geq 0}$ in X if the following conditions (i)-(iii) are satisfied:

(i) $\mathcal{A}$ is compact in X;
(ii) $\mathcal{A}$ is invariant, i.e., $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$;
(iii) $\mathcal{A}$ attracts every bounded subset of X, that is, for any bounded set $B \subset X$,

$$\lim_{t\to+\infty} dist_X(S(t)B,\, \mathcal{A}) = 0.$$

If, in addition, the following conditions are satisfied:

(iv) $S(t)$ maps X to Y for every $t \in \mathbb{R}^+$;
(v) $\mathcal{A}$ is compact in Y;
(vi) $\mathcal{A}$ attracts every bounded subset of X in Y, that is, for any bounded set $B \subset X$,

$$\lim_{t\to+\infty} dist_Y(S(t)B,\, \mathcal{A}) = 0,$$

then $\mathcal{A}$ is called an (X, Y)-global attractor of $\{S(t)\}_{t\geq 0}$.

The following existence result for a global attractor for a semigroup can be found in [2, 5, 20, 29, 30, 33, 34, 40] (see also [19] for random case).

Proposition 9. *Suppose X and Y are Banach spaces with continuous injection $Y \hookrightarrow X$ such that all closed balls in Y are closed subsets of X. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on X. If $\{S(t)\}_{t\geq 0}$ is asymptotically compact in X and has a bounded absorbing set in X, then $\{S(t)\}_{t\geq 0}$ has a global attractor $\mathcal{A}$ in X.*

If, in addition, $S(t)$ maps X to Y for every $t \in \mathbb{R}^+$, and $\{S(t)\}_{t\geq 0}$ is (X, Y)-asymptotically compact, then $\mathcal{A}$ is also an (X, Y)-global attractor of $\{S(t)\}_{t\geq 0}$.

3. **Global existence and uniqueness of solutions.** In this section, we consider the global existence and uniqueness of solutions for problem (1.1) with initial data $\theta_0 \in H^{2\alpha^-+s}(\mathbb{T}^2)$, and then define a continuous semigroup.

Let $\theta_1,\, \theta_2 \in H^{2\alpha^-+s}(\mathbb{T}^2)$ and $u_1 = (-R_2\theta_1, R_1\theta_1)$, $u_2 = (-R_2\theta_2, R_1\theta_2)$. Note that $H^s(\mathbb{T}^2)$ is a Banach Algebra provided that $s > 1$ (see, e.g., [1, p.115]). Since Sobolev embeddings $H^{2\alpha^-+s}(\mathbb{T}^2) \subset H^{1+s}(\mathbb{T}^2) \subset H^s(\mathbb{T}^2)$ are valid for the subcritical case, by (2.3) we have

$$\begin{aligned}
\|u_1\cdot\nabla\theta_1 - u_2\cdot\nabla\theta_2\|_{H^s(\mathbb{T}^2)} &\leq \left\| R_2(\theta_1-\theta_2)\frac{\partial\theta_1}{\partial x_1} + R_2\theta_2\frac{\partial(\theta_1-\theta_2)}{\partial x_1}\right\|_{H^s(\mathbb{T}^2)} \\
&\quad + \left\| R_1(\theta_1-\theta_2)\frac{\partial\theta_1}{\partial x_2} + R_1\theta_2\frac{\partial(\theta_1-\theta_2)}{\partial x_2}\right\|_{H^s(\mathbb{T}^2)} \\
&\leq C\|\theta_1-\theta_2\|_{H^s(\mathbb{T}^2)}\|\theta_1\|_{H^{1+s}(\mathbb{T}^2)} + C\|\theta_2\|_{H^s(\mathbb{T}^2)}\|\theta_1-\theta_2\|_{H^{1+s}(\mathbb{T}^2)} \\
&\leq C\left(\|\theta_1\|_{H^{2\alpha^-+s}(\mathbb{T}^2)} + \|\theta_2\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}\right)\|\theta_1-\theta_2\|_{H^{2\alpha^-+s}(\mathbb{T}^2)},
\end{aligned} \tag{3.1}$$

which is the required local Lipschitz condition.

Following a standard approach in [6, p.55] and [21, Theorem 3.3.3] to semilinear parabolic equations, we obtain the local existence and uniqueness of solutions for problem (1.1) with initial data $\theta_0 \in H^{2\alpha^-+s}(\mathbb{T}^2)$.

Theorem 10. *Let $\alpha \in (\frac{1}{2}, 1]$, $\kappa > 0$ and $\theta_0 \in H^{2\alpha^-+s}(\mathbb{T}^2)$ with $s > 1$. Suppose further that $g_1, g_2 \in H^s(\mathbb{T}^2)$, and $f : H^{2\alpha^-+s}(\mathbb{T}^2) \to H^s(\mathbb{T}^2)$ is Lipschitz continuous on bounded subsets of $H^{2\alpha^-+s}(\mathbb{T}^2)$. Then there exists a unique solution θ to problem (1.1) such that*

$$\theta \in C((0,\tau); H^{2\alpha+s}(\mathbb{T}^2)) \cap C([0,\tau); H^{2\alpha^-+s}(\mathbb{T}^2)),\;\; \theta_t \in C((0,\tau); H^{2\gamma+s}(\mathbb{T}^2)),$$

where $\gamma < \alpha$ is arbitrary and $\tau > 0$ is the maximal time of existence. Moreover, θ is given by the formula

$$\theta(t) = e^{-A_{\alpha,\kappa}t}\theta_0 + \int_0^t e^{-A_{\alpha,\kappa}(t-s)}(F(x,\theta(s)) - u(s)\cdot\nabla\theta(s))ds, \quad \forall t \in [0,\tau), \tag{3.2}$$

where $e^{-A_{\alpha,\kappa}t}$ denotes the linear semigroup corresponding to the operator $A_{\alpha,\kappa} := \kappa(-\Delta)^\alpha$ in $H^s(\mathbb{T}^2)$, and $F(x,\theta)$ is given by (1.2).

Let $\theta \in H^{2\alpha^-+s}(\mathbb{T}^2)$ and $u = (-R_2\theta, R_1\theta)$. Similar to (3.1), we obtain

$$\|u\cdot\nabla\theta\|_{H^s(\mathbb{T}^2)} \le C\|\theta\|_{H^s(\mathbb{T}^2)}\|\theta\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}. \tag{3.3}$$

Further, we assume that $f : H^{2\alpha^-+s}(\mathbb{T}^2) \to H^s(\mathbb{T}^2)$ satisfies the sublinear growth restriction:

$$\|f(\theta)\|_{H^s(\mathbb{T}^2)} \le k_1\left(1 + \|\theta\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}\right), \quad \theta \in H^{2\alpha^-+s}(\mathbb{T}^2) \tag{3.4}$$

for some $k_1 > 0$. A priori estimate (4.47) below together with (3.3) and (3.4) will be used to guarantee that the local solution to problem (1.1) can be globally extended (see [6, p.71] for more details). On the other hand, following [21, Theorem 3.4.1], one can show that $\theta(t,\theta_0)$ is continuous with respect to θ_0 in $H^{2\alpha^-+s}(\mathbb{T}^2)$. Hence we now define a semigroup $S : \mathbb{R}^+ \times H^{2\alpha^-+s}(\mathbb{T}^2) \to H^{2\alpha^-+s}(\mathbb{T}^2)$ by

$$S(t)\theta_0 = \theta(t,\theta_0) \quad \text{for all } (t,\theta_0) \in \mathbb{R}^+ \times H^{2\alpha^-+s}(\mathbb{T}^2). \tag{3.5}$$

4. **Uniform estimates of solutions.** In this section, we derive uniform estimates on the solution of (1.1). Such estimates imply the existence of a bounded absorbing set, and also will be used to prove the asymptotic compactness of the semiflow associated with the equation.

We begin with uniform estimates of the solutions in $L^q(\mathbb{T}^2)$.

Lemma 11. *Let the conditions in Theorem 10 hold. Assume further that (3.4) holds and*

$$f(v)v \le k_2, \quad \forall v \in \mathbb{R} \tag{4.1}$$

for some $k_2 > 0$. Let $q \in [2,\infty)$. Then every solution $\theta(t,\theta_0)$ of (1.1) satisfies

$$\|\theta(t,\theta_0)\|^q_{L^q(\mathbb{T}^2)} \le \|\theta_0\|^q_{L^q(\mathbb{T}^2)}e^{-\kappa\lambda_1^{2\alpha}t} + C\|g_1\|^{\frac{q}{2}}_{L^{\frac{q}{2}}(\mathbb{T}^2)} + C\|g_2\|^q_{L^q(\mathbb{T}^2)}. \tag{4.2}$$

Proof. Multiplying (1.1) with $|\theta|^{q-2}\theta$ and then taking the inner product in $L^2(\mathbb{T}^2)$, we have

$$\begin{aligned}&\frac{1}{q}\frac{d}{dt}\int_{\mathbb{T}^2}|\theta|^q dx + \kappa\int_{\mathbb{T}^2}(-\Delta)^\alpha\theta|\theta|^{q-2}\theta dx + \int_{\mathbb{T}^2}u\cdot\nabla\theta|\theta|^{q-2}\theta dx\\ &\quad = \int_{\mathbb{T}^2}F(x,\theta)|\theta|^{q-2}\theta dx.\end{aligned} \tag{4.3}$$

For the second term, by Lemma 4 we obtain

$$\kappa\int_{\mathbb{T}^2}(-\Delta)^\alpha\theta|\theta|^{q-2}\theta dx \ge \frac{2\kappa}{q}\int_{\mathbb{T}^2}\left(\Lambda^\alpha|\theta|^{\frac{q}{2}}\right)^2 dx \ge \frac{2\kappa}{q}\lambda_1^{2\alpha}\int_{\mathbb{T}^2}|\theta|^q dx. \tag{4.4}$$

Due to $\nabla\cdot u = 0$, by using integration by parts we have

$$\int_{\mathbb{T}^2}u\cdot\nabla\theta|\theta|^{q-2}\theta dx = 0. \tag{4.5}$$

Since $f(\theta)\theta \le k_2$, we deduce from Young's inequality that for the case $q > 2$,

$$\begin{aligned}\int_{\mathbb{T}^2} F(x,\theta)|\theta|^{q-2}\theta dx &= \int_{\mathbb{T}^2} (g_1(x)f(\theta) + g_2(x))|\theta|^{q-2}\theta dx \\ &\le k_2 \int_{\mathbb{T}^2} g_1(x)|\theta|^{q-2} dx + \int_{\mathbb{T}^2} g_2(x)|\theta|^{q-2}\theta dx \\ &\le \frac{\kappa}{q}\lambda_1^{2\alpha} \int_{\mathbb{T}^2} |\theta|^q dx + C\int_{\mathbb{T}^2} |g_1(x)|^{\frac{q}{2}} dx + C\int_{\mathbb{T}^2} |g_2(x)|^q dx.\end{aligned} \tag{4.6}$$

For the case $q = 2$,

$$\begin{aligned}\int_{\mathbb{T}^2} F(x,\theta)\theta dx &\le k_2 \int_{\mathbb{T}^2} g_1(x) dx + \int_{\mathbb{T}^2} g_2(x)\theta dx \\ &\le \frac{\kappa}{q}\lambda_1^{2\alpha} \int_{\mathbb{T}^2} |\theta|^2 dx + k_2\int_{\mathbb{T}^2} |g_1(x)| dx + C\int_{\mathbb{T}^2} |g_2(x)|^2 dx.\end{aligned} \tag{4.7}$$

Inserting the above estimates into (4.3) gives

$$\frac{d}{dt}\int_{\mathbb{T}^2} |\theta|^q dx + \kappa\lambda_1^{2\alpha}\int_{\mathbb{T}^2} |\theta|^q dx \le C\int_{\mathbb{T}^2} |g_1(x)|^{\frac{q}{2}} dx + C\int_{\mathbb{T}^2} |g_2(x)|^q dx. \tag{4.8}$$

Then the desired result (4.2) follows from the Gronwall inequality. □

We next derive uniform estimates for θ in $H^{\alpha+s}(\mathbb{T}^2)$ and for $\partial_t\theta$ in $H^s(\mathbb{T}^2)$.

Lemma 12. *Let the conditions in Lemma 11 hold. Then, every solution $\theta(\cdot)$ of problem (1.1) satisfies for all $t \ge 0$,*

$$\begin{aligned}\|\theta(t)\|^2_{H^{\alpha+s}(\mathbb{T}^2)} &\le e^{-\kappa\lambda_1^{2\alpha}t}\|\theta_0\|^2_{H^{\alpha+s}(\mathbb{T}^2)} + C\|g_2\|^2_{H^s(\mathbb{T}^2)} + C\|g_1\|^2_{H^s(\mathbb{T}^2)} \\ &\quad + Cte^{-\kappa\lambda_1^{2\alpha}t}\|\theta_0\|^2_{L^2(\mathbb{T}^2)}\|g_1\|^{\frac{2}{1-\eta'}}_{H^s(\mathbb{T}^2)} + CG_2\|g_1\|^{\frac{2}{1-\eta'}}_{H^s(\mathbb{T}^2)} \\ &\quad + Ce^{-\kappa\lambda_1^{2\alpha}t}\|\theta_0\|^2_{L^2(\mathbb{T}^2)}\|\theta_0\|^{\frac{2}{1-\eta}}_{L^{p_2}(\mathbb{T}^2)} + CG_1te^{-\kappa\lambda_1^{2\alpha}t}\|\theta_0\|^2_{L^2(\mathbb{T}^2)} \\ &\quad + CG_2e^{-\kappa\lambda_1^{2\alpha}t}\|\theta_0\|^{\frac{2}{1-\eta}}_{L^{p_2}(\mathbb{T}^2)} + CG_1G_2,\end{aligned} \tag{4.9}$$

$$\begin{aligned}\kappa\int_t^{t+1} \|\theta(r)\|^2_{H^{2\alpha+s}(\mathbb{T}^2)} dr &\le e^{-\kappa\lambda_1^{2\alpha}t}\|\theta_0\|^2_{H^{\alpha+s}(\mathbb{T}^2)} + C\|g_2\|^2_{H^s(\mathbb{T}^2)} + C\|g_1\|^2_{H^s(\mathbb{T}^2)} \\ &+ Cte^{-\kappa\lambda_1^{2\alpha}t}\|\theta_0\|^2_{L^2(\mathbb{T}^2)}\|g_1\|^{\frac{2}{1-\eta'}}_{H^s(\mathbb{T}^2)} + CG_2\|g_1\|^{\frac{2}{1-\eta'}}_{H^s(\mathbb{T}^2)} \\ &+ Ce^{-\kappa\lambda_1^{2\alpha}t}\|\theta_0\|^2_{L^2(\mathbb{T}^2)}\|\theta_0\|^{\frac{2}{1-\eta}}_{L^{p_2}(\mathbb{T}^2)} + CG_1(t+1)e^{-\kappa\lambda_1^{2\alpha}t}\|\theta_0\|^2_{L^2(\mathbb{T}^2)} \\ &+ CG_2e^{-\kappa\lambda_1^{2\alpha}t}\|\theta_0\|^{\frac{2}{1-\eta}}_{L^{p_2}(\mathbb{T}^2)} + CG_1G_2,\end{aligned} \tag{4.10}$$

where $p_2 = \frac{2}{2\alpha^- - 1}$, $\eta' = \frac{2\alpha^- + s}{2\alpha + s}$, $\eta \in \left[\frac{s+1}{2\alpha+s}, 1\right)$ is a constant,

$$G_1 = \|g_1\|^{\frac{1}{1-\eta}}_{L^{\frac{p_2}{2}}(\mathbb{T}^2)} + \|g_2\|^{\frac{2}{1-\eta}}_{L^{p_2}(\mathbb{T}^2)} \quad \text{and} \quad G_2 = \|g_1\|_{L^1(\mathbb{T}^2)} + \|g_2\|^2_{L^2(\mathbb{T}^2)}.$$

Proof. Multiplying (1.1) with $(-\Delta)^{\alpha+s}\theta$ and then integrating over $\mathbb{T}^2$, we find that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} [(-\Delta)^{\frac{\alpha+s}{2}}\theta]^2 dx + \kappa\int_{\mathbb{T}^2} [(-\Delta)^{\frac{2\alpha+s}{2}}\theta]^2 dx \\ &= \int_{\mathbb{T}^2} F(x,\theta)(-\Delta)^{\alpha+s}\theta dx - \int_{\mathbb{T}^2} (u\cdot\nabla\theta)(-\Delta)^{\alpha+s}\theta dx.\end{aligned} \tag{4.11}$$

We proceed to estimate the last two terms in (4.11). First, Hölder's and Young's inequalities imply that

$$\begin{aligned}\left|\int_{\mathbb{T}^2} g_2(x)(-\Delta)^{\alpha+s}\theta dx\right| &\le \|(-\Delta)^{\frac{s}{2}} g_2\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{2\alpha+s}{2}}\theta\|_{L^2(\mathbb{T}^2)} \\ &\le \frac{\kappa}{8}\|\theta\|^2_{H^{2\alpha+s}(\mathbb{T}^2)} + C\|g_2\|^2_{H^s(\mathbb{T}^2)}.\end{aligned} \tag{4.12}$$

By Lemma 1, Hölder's and Young's inequalities, $g_1 \in H^s(\mathbb{T}^2)$ and (3.4), we have

$$\begin{aligned}&\left|\int_{\mathbb{T}^2} g_1(x)f(\theta)(-\Delta)^{\alpha+s}\theta dx\right| \\ &\le \|(-\Delta)^{\frac{s}{2}}(g_1 f(\theta))\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{2\alpha+s}{2}}\theta\|_{L^2(\mathbb{T}^2)} \\ &\le C\left(\|g_1\|_{L^\infty(\mathbb{T}^2)}\|(-\Delta)^{\frac{s}{2}} f(\theta)\|_{L^2(\mathbb{T}^2)} + \|(-\Delta)^{\frac{s}{2}} g_1\|_{L^2(\mathbb{T}^2)}\|f(\theta)\|_{L^\infty(\mathbb{T}^2)}\right)\|\theta\|_{H^{2\alpha+s}(\mathbb{T}^2)} \\ &\le C\|g_1\|_{H^s(\mathbb{T}^2)}\left(1+\|\theta\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}\right)\|\theta\|_{H^{2\alpha+s}(\mathbb{T}^2)} \\ &\le C\|g_1\|_{H^s(\mathbb{T}^2)}\left(1+\|\theta\|^{\eta'}_{H^{2\alpha+s}(\mathbb{T}^2)}\|\theta\|^{1-\eta'}_{L^2(\mathbb{T}^2)}\right)\|\theta\|_{H^{2\alpha+s}(\mathbb{T}^2)} \\ &\le \frac{\kappa}{8}\|\theta\|^2_{H^{2\alpha+s}(\mathbb{T}^2)} + C\|g_1\|^2_{H^s(\mathbb{T}^2)} + C\|g_1\|^{\frac{2}{1-\eta'}}_{H^s(\mathbb{T}^2)}\|\theta\|^2_{L^2(\mathbb{T}^2)},\end{aligned} \tag{4.13}$$

where we have used the Sobolev embedding $H^s(\mathbb{T}^2) \subset L^\infty(\mathbb{T}^2)$ for $s > 1$ and the following Gagliardo-Nirenberg inequality:

$$\|\theta\|_{H^{2\alpha^-+s}(\mathbb{T}^2)} \le C\|\theta\|^{\eta'}_{H^{2\alpha+s}(\mathbb{T}^2)}\|\theta\|^{1-\eta'}_{L^2(\mathbb{T}^2)},$$

where $\eta' := \frac{2\alpha^-+s}{2\alpha+s}$. Next, we analyze the last term in (4.11). By Hölder's inequality, we have

$$\left|\int_{\mathbb{T}^2}(u\cdot\nabla\theta)(-\Delta)^{\alpha+s}\theta dx\right| \le \|(-\Delta)^{\frac{2\alpha+s}{2}}\theta\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s}{2}}(u\cdot\nabla\theta)\|_{L^2(\mathbb{T}^2)}. \tag{4.14}$$

Note that $\nabla\cdot u = 0$. Then by making use of Lemmas 1 and 2, we obtain

$$\begin{aligned}\|(-\Delta)^{\frac{s}{2}}(u\cdot\nabla\theta)\|_{L^2(\mathbb{T}^2)} &= \|(-\Delta)^{\frac{s+1}{2}}(u\theta)\|_{L^2(\mathbb{T}^2)} \\ &\le C\|(-\Delta)^{\frac{s+1}{2}}u\|_{L^{p_1}(\mathbb{T}^2)}\|\theta\|_{L^{p_2}(\mathbb{T}^2)} + C\|u\|_{L^{p_2}(\mathbb{T}^2)}\|(-\Delta)^{\frac{s+1}{2}}\theta\|_{L^{p_1}(\mathbb{T}^2)} \\ &\le C\|(-\Delta)^{\frac{s+1}{2}}\theta\|_{L^{p_1}(\mathbb{T}^2)}\|\theta\|_{L^{p_2}(\mathbb{T}^2)},\end{aligned} \tag{4.15}$$

where

$$p_1 := \frac{1}{1-\alpha^-}, \quad p_2 := \frac{2}{2\alpha^- - 1}.$$

To deal with the term $\|(-\Delta)^{\frac{s+1}{2}}\theta\|_{L^{p_1}(\mathbb{T}^2)}$, we use the following Gagliardo-Nirenberg inequality:

$$\|(-\Delta)^{\frac{s+1}{2}}\theta\|_{L^{p_1}(\mathbb{T}^2)} \le C\|(-\Delta)^{\frac{2\alpha+s}{2}}\theta\|^{\eta}_{L^2(\mathbb{T}^2)}\|\theta\|^{1-\eta}_{L^2(\mathbb{T}^2)}, \tag{4.16}$$

where $\eta \in \left[\frac{s+1}{2\alpha+s}, 1\right)$. Then it follows from (4.14)-(4.16) and Young's inequality that

$$\begin{aligned}-\int_{\mathbb{T}^2}(u\cdot\nabla\theta)(-\Delta)^{\alpha+s}\theta dx &\le C\|(-\Delta)^{\frac{2\alpha+s}{2}}\theta\|^{\eta+1}_{L^2(\mathbb{T}^2)}\|\theta\|^{1-\eta}_{L^2(\mathbb{T}^2)}\|\theta\|_{L^{p_2}(\mathbb{T}^2)} \\ &\le \frac{\kappa}{4}\|(-\Delta)^{\frac{2\alpha+s}{2}}\theta\|^2_{L^2(\mathbb{T}^2)} + C\|\theta\|^2_{L^2(\mathbb{T}^2)}\|\theta\|^{\frac{2}{1-\eta}}_{L^{p_2}(\mathbb{T}^2)}.\end{aligned} \tag{4.17}$$

Inserting (4.12)-(4.13) and (4.17) in (4.11), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|(-\Delta)^{\frac{\alpha+s}{2}}\theta\|^2_{L^2(\mathbb{T}^2)}+\kappa\|(-\Delta)^{\frac{2\alpha+s}{2}}\theta\|^2_{L^2(\mathbb{T}^2)}\\
&\leq C\|\theta\|^2_{L^2(\mathbb{T}^2)}\|\theta\|^{\frac{2}{1-\eta}}_{L^{p_2}(\mathbb{T}^2)}+C\|g_2\|^2_{H^s(\mathbb{T}^2)}+C\|g_1\|^2_{H^s(\mathbb{T}^2)}+C\|g_1\|^{\frac{2}{1-\eta'}}_{H^s(\mathbb{T}^2)}\|\theta\|^2_{L^2(\mathbb{T}^2)}.
\end{aligned}
\tag{4.18}
$$

Applying Gronwall's inequality to (4.18) and using

$$
\kappa\|(-\Delta)^{\frac{2\alpha+s}{2}}\theta\|^2_{L^2(\mathbb{T}^2)}\geq\kappa\lambda_1^{2\alpha}\|(-\Delta)^{\frac{\alpha+s}{2}}\theta\|^2_{L^2(\mathbb{T}^2)},
$$

we find that for all $t\geq 0$,

$$
\begin{aligned}
\|\theta(t)\|^2_{H^{\alpha+s}(\mathbb{T}^2)}\leq{}& e^{-\kappa\lambda_1^{2\alpha}t}\|\theta_0\|^2_{H^{\alpha+s}(\mathbb{T}^2)}+C\|g_2\|^2_{H^s(\mathbb{T}^2)}+C\|g_1\|^2_{H^s(\mathbb{T}^2)}\\
&+C\|g_1\|^{\frac{2}{1-\eta'}}_{H^s(\mathbb{T}^2)}\int_0^t e^{-\kappa\lambda_1^{2\alpha}(t-r)}\|\theta(r)\|^2_{L^2(\mathbb{T}^2)}dr\\
&+C\int_0^t e^{-\kappa\lambda_1^{2\alpha}(t-r)}\|\theta(r)\|^2_{L^2(\mathbb{T}^2)}\|\theta(r)\|^{\frac{2}{1-\eta}}_{L^{p_2}(\mathbb{T}^2)}dr.
\end{aligned}
\tag{4.19}
$$

Furthermore, integrating (4.18) from t to $t+1$, we deduce from (4.19) that for all $t\geq 0$,

$$
\begin{aligned}
\kappa\int_t^{t+1}\|\theta(r)\|^2_{H^{2\alpha+s}(\mathbb{T}^2)}dr\leq{}& e^{-\kappa\lambda_1^{2\alpha}t}\|\theta_0\|^2_{H^{\alpha+s}(\mathbb{T}^2)}+C\|g_2\|^2_{H^s(\mathbb{T}^2)}+C\|g_1\|^2_{H^s(\mathbb{T}^2)}\\
&+C\|g_1\|^{\frac{2}{1-\eta'}}_{H^s(\mathbb{T}^2)}\int_0^t e^{-\kappa\lambda_1^{2\alpha}(t-r)}\|\theta(r)\|^2_{L^2(\mathbb{T}^2)}dr\\
&+C\int_0^t e^{-\kappa\lambda_1^{2\alpha}(t-r)}\|\theta(r)\|^2_{L^2(\mathbb{T}^2)}\|\theta(r)\|^{\frac{2}{1-\eta}}_{L^{p_2}(\mathbb{T}^2)}dr\\
&+C\|g_1\|^{\frac{2}{1-\eta'}}_{H^s(\mathbb{T}^2)}\int_t^{t+1}\|\theta(r)\|^2_{L^2(\mathbb{T}^2)}dr\\
&+C\int_t^{t+1}\|\theta(r)\|^2_{L^2(\mathbb{T}^2)}\|\theta(r)\|^{\frac{2}{1-\eta}}_{L^{p_2}(\mathbb{T}^2)}dr.
\end{aligned}
\tag{4.20}
$$

Inserting (4.2) with $q=2$ or $q=p_2$ into (4.19) and (4.20), the assertions of the lemma follow. □

Lemma 13. *Let the conditions in Lemma 11 hold. Then for any bounded set $B\subset H^{2\alpha^-+s}(\mathbb{T}^2)$, there exists $T_0=T_0(B)>0$ such that any solution $\theta(t,\theta_0)$ of problem (1.1) with $\theta_0\in B$ satisfies*

$$
\int_t^{t+1}\|\partial_t\theta\|^2_{H^s(\mathbb{T}^2)}d\tau\leq C,\quad \forall t\geq T_0. \tag{4.21}
$$

Proof. Taking the inner product of (1.1) with $(-\Delta)^s\partial_t\theta$ in $L^2(\mathbb{T}^2)$, we find that

$$
\begin{aligned}
&\|(-\Delta)^{\frac{s}{2}}(\partial_t\theta)\|^2_{L^2(\mathbb{T}^2)}+\frac{\kappa}{2}\frac{d}{dt}\|(-\Delta)^{\frac{s+\alpha}{2}}\theta\|^2_{L^2(\mathbb{T}^2)}\\
&=\int_{\mathbb{T}^2}F(x,\theta)(-\Delta)^s(\partial_t\theta)dx-\int_{\mathbb{T}^2}(u\cdot\nabla\theta)(-\Delta)^s(\partial_t\theta)dx\\
&=\int_{\mathbb{T}^2}g_1(x)f(\theta)(-\Delta)^s(\partial_t\theta)dx+\int_{\mathbb{T}^2}g_2(x)(-\Delta)^s(\partial_t\theta)dx\\
&\quad-\int_{\mathbb{T}^2}(u\cdot\nabla\theta)(-\Delta)^s(\partial_t\theta)dx\\
&:=I_1+I_2+I_3.
\end{aligned}
\tag{4.22}
$$

By Lemma 1, Hölder's and Young's inequalities, $g_1\in H^s(\mathbb{T}^2)$ and (3.4), I_1 is bounded by

$$
\begin{aligned}
I_1&\le\|(-\Delta)^{\frac{s}{2}}(g_1(x)f(\theta))\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s}{2}}(\partial_t\theta)\|_{L^2(\mathbb{T}^2)}\\
&\le C\left(\|g_1\|_{L^\infty(\mathbb{T}^2)}\|(-\Delta)^{\frac{s}{2}}f(\theta)\|_{L^2(\mathbb{T}^2)}\right.\\
&\quad\left.+\|(-\Delta)^{\frac{s}{2}}g_1\|_{L^2(\mathbb{T}^2)}\|f(\theta)\|_{L^\infty(\mathbb{T}^2)}\right)\|(-\Delta)^{\frac{s}{2}}(\partial_t\theta)\|_{L^2(\mathbb{T}^2)}\\
&\le C\|g_1\|_{H^s(\mathbb{T}^2)}\left(1+\|\theta\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}\right)\|\partial_t\theta\|_{H^s(\mathbb{T}^2)}\\
&\le\frac{1}{8}\|\partial_t\theta\|^2_{H^s(\mathbb{T}^2)}+C\|g_1\|^2_{H^s(\mathbb{T}^2)}+C\|g_1\|^2_{H^s(\mathbb{T}^2)}\|\theta\|^2_{H^{2\alpha+s}(\mathbb{T}^2)},
\end{aligned}
\tag{4.23}
$$

where we have used the Sobolev embedding $H^{2\alpha+s}(\mathbb{T}^2)\subset H^{2\alpha^-+s}(\mathbb{T}^2)$ in the last inequality. For I_2, applying Hölder's and Young's inequalities again, we have

$$
I_2\le\|(-\Delta)^{\frac{s}{2}}g_2\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s}{2}}(\partial_t\theta)\|_{L^2(\mathbb{T}^2)}\le\frac{1}{8}\|\partial_t\theta\|^2_{H^s(\mathbb{T}^2)}+C\|g_2\|^2_{H^s(\mathbb{T}^2)}. \tag{4.24}
$$

Using (4.15), Hölder's and Young's inequalities, and the Sobolev embedding $H^{2\alpha+s}(\mathbb{T}^2)\subset H^{s+1,p_1}(\mathbb{T}^2)$, we deduce that

$$
\begin{aligned}
I_3&\le\|(-\Delta)^{\frac{s}{2}}(u\cdot\nabla\theta)\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s}{2}}(\partial_t\theta)\|_{L^2(\mathbb{T}^2)}\\
&\le C\|\theta\|_{H^{s+1,p_1}(\mathbb{T}^2)}\|\theta\|_{L^{p_2}(\mathbb{T}^2)}\|\partial_t\theta\|_{H^s(\mathbb{T}^2)}\\
&\le\frac{1}{4}\|\partial_t\theta\|^2_{H^s(\mathbb{T}^2)}+C\|\theta\|^2_{H^{2\alpha+s}(\mathbb{T}^2)}\|\theta\|^2_{L^{p_2}(\mathbb{T}^2)},
\end{aligned}
\tag{4.25}
$$

where $p_1=\frac{1}{1-\alpha^-}$ and $p_2=\frac{2}{2\alpha^--1}$ are given in (4.15). Inserting (4.23)-(4.25) into (4.22) yields

$$
\begin{aligned}
&\|\partial_t\theta\|^2_{H^s(\mathbb{T}^2)}+\kappa\frac{d}{dt}\|\theta\|^2_{H^{\alpha+s}(\mathbb{T}^2)}\\
&\le C\|g_1\|^2_{H^s(\mathbb{T}^2)}+C\|g_2\|^2_{H^s(\mathbb{T}^2)}+C\left(\|g_1\|^2_{H^s(\mathbb{T}^2)}+\|\theta\|^2_{L^{p_2}(\mathbb{T}^2)}\right)\|\theta\|^2_{H^{2\alpha+s}(\mathbb{T}^2)}.
\end{aligned}
\tag{4.26}
$$

Integrating (4.26) on $[t,t+1]$, in view of g_1, $g_2\in H^s(\mathbb{T}^2)$, (4.2) and (4.9)-(4.10), we obtain that for any given bounded set $B\subset H^{2\alpha^-+s}(\mathbb{T}^2)$, there exists $T_0=T_0(B)>0$ such that any solution $\theta(t,\theta_0)$ of problem (1.1) with $\theta_0\in B$ satisfies

$$
\int_t^{t+1}\|\partial_t\theta\|^2_{H^s(\mathbb{T}^2)}d\tau\le C,\quad\forall t\ge T_0,
$$

which implies the assertion of the lemma. □

Lemma 14. *Let the conditions in Lemma 11 hold. Also, assume that*

$$\|f(\theta)\|_{L^2(\mathbb{T}^2)} \leq k_3\left(1+\|\theta\|_{H^{2\alpha}(\mathbb{T}^2)}\right), \quad \theta \in H^{2\alpha}(\mathbb{T}^2), \tag{4.27}$$

and $f \in C^1(\mathbb{R})$ *satisfying*

$$\|f'(\theta)\|_{H^s(\mathbb{T}^2)} \leq k_4\left(1+\|\theta\|_{H^{\alpha+s}(\mathbb{T}^2)}\right), \quad \theta \in H^{\alpha+s}(\mathbb{T}^2) \tag{4.28}$$

for some $k_3, k_4 > 0$. *Then, for any bounded set* $B \subset H^{2\alpha^- + s}(\mathbb{T}^2)$, *there exists* $T_1 = T_1(B) > 0$ *such that any solution* $\theta(t, \theta_0)$ *of problem* (1.1) *with* $\theta_0 \in B$ *satisfies*

$$\|\partial_t \theta(t)\|^2_{H^s(\mathbb{T}^2)} \leq C, \quad \forall t \geq T_1. \tag{4.29}$$

Proof. By differentiating (1.1) in time and writing $w = \partial_t \theta$, we have

$$\partial_t w + u \cdot \nabla w + \kappa(-\Delta)^\alpha w = -u_t \cdot \nabla\theta + g_1(x) f'(\theta) w. \tag{4.30}$$

Multiplying (4.30) by $(-\Delta)^s w$ and then integrating over $\mathbb{T}^2$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}[(-\Delta)^{\frac{s}{2}} w]^2 dx + \kappa \int_{\mathbb{T}^2}[(-\Delta)^{\frac{s+\alpha}{2}} w]^2 dx \\ &= \int_{\mathbb{T}^2} g_1(x) f'(\theta) w (-\Delta)^s w dx - \int_{\mathbb{T}^2}(u\cdot\nabla w)(-\Delta)^s w dx - \int_{\mathbb{T}^2}(u_t \cdot \nabla\theta)(-\Delta)^s w dx. \end{aligned} \tag{4.31}$$

Now we estimate the last three terms in (4.31). First, by Hölder's inequality and Lemma 1, we deduce that for some $0 < \frac{2}{p_3} < p_4 < \alpha$,

$$\begin{aligned} &\left|\int_{\mathbb{T}^2} g_1(x) f'(\theta) w (-\Delta)^s w dx\right| \\ &\leq \|(-\Delta)^{\frac{s-\alpha}{2}}(g_1 f'(\theta) w)\|_{L^2(\mathbb{T}^2)} \|(-\Delta)^{\frac{s+\alpha}{2}} w\|_{L^2(\mathbb{T}^2)} \\ &\leq C\|g_1 f'(\theta)\|_{L^{p_3}(\mathbb{T}^2)} \|w\|_{H^{s-\alpha, \frac{2p_3}{p_3-2}}(\mathbb{T}^2)} \|w\|_{H^{s+\alpha}(\mathbb{T}^2)} \\ &\quad + C\|g_1 f'(\theta)\|_{H^{s-\alpha, \frac{2}{1-\alpha+p_4}}(\mathbb{T}^2)} \|w\|_{L^{\frac{2}{\alpha-p_4}}(\mathbb{T}^2)} \|w\|_{H^{s+\alpha}(\mathbb{T}^2)} \\ &\leq C\|g_1 f'(\theta)\|_{L^{p_3}(\mathbb{T}^2)} \|w\|^{\eta_1+1}_{H^{s+\alpha}(\mathbb{T}^2)} \|w\|^{1-\eta_1}_{L^2(\mathbb{T}^2)} \\ &\quad + C\|g_1 f'(\theta)\|_{H^{s-\alpha, \frac{2}{1-\alpha+p_4}}(\mathbb{T}^2)} \|w\|^{\eta_2+1}_{H^{s+\alpha}(\mathbb{T}^2)} \|w\|^{1-\eta_2}_{L^2(\mathbb{T}^2)}. \end{aligned} \tag{4.32}$$

where we have used the following Gagliardo-Nirenberg inequalities:

$$\|w\|_{H^{s-\alpha, \frac{2p_3}{p_3-2}}(\mathbb{T}^2)} \leq C\|w\|^{\eta_1}_{H^{s+\alpha}(\mathbb{T}^2)} \|w\|^{1-\eta_1}_{L^2(\mathbb{T}^2)},$$

and

$$\|w\|_{L^{\frac{2}{\alpha-p_4}}(\mathbb{T}^2)} \leq C\|w\|^{\eta_2}_{H^{s+\alpha}(\mathbb{T}^2)} \|w\|^{1-\eta_2}_{L^2(\mathbb{T}^2)},$$

where $\eta_1 \in \left[\frac{s-\alpha}{s+\alpha}, 1\right)$ and $\eta_2 \in (0,1)$. Using Lemma 1 again, (4.28), $g_1 \in H^s(\mathbb{T}^2)$ and the Sobolev embeddings $H^s(\mathbb{T}^2) \subset L^{p_3}(\mathbb{T}^2)$ and $H^s(\mathbb{T}^2) \subset H^{s-\alpha, \frac{2}{1-\alpha+p_4-2/p_3}}(\mathbb{T}^2)$ for $s > 1$, we have

$$\begin{aligned} \|g_1 f'(\theta)\|_{H^{s-\alpha, \frac{2}{1-\alpha+p_4}}(\mathbb{T}^2)} \leq\ & C\Big(\|f'(\theta)\|_{L^{p_3}(\mathbb{T}^2)} \|g_1\|_{H^{s-\alpha, \frac{2}{1-\alpha+p_4-2/p_3}}(\mathbb{T}^2)} \\ & + \|g_1\|_{L^{p_3}(\mathbb{T}^2)} \|f'(\theta)\|_{H^{s-\alpha, \frac{2}{1-\alpha+p_4-2/p_3}}(\mathbb{T}^2)}\Big) \\ \leq\ & C\|f'(\theta)\|_{H^s(\mathbb{T}^2)} \|g_1\|_{H^s(\mathbb{T}^2)} \end{aligned}$$

$$\leq \quad C\left(1+\|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)}\right)\|g_1\|_{H^s(\mathbb{T}^2)}. \tag{4.33}$$

Noticing that $H^s(\mathbb{T}^2)\subset L^\infty(\mathbb{T}^2)$ for $s>1$ and $g_1\in H^s(\mathbb{T}^2)$, hence

$$\|g_1 f'(\theta)\|_{L^{p_3}(\mathbb{T}^2)} \leq C\|g_1\|_{L^\infty(\mathbb{T}^2)}\|f'(\theta)\|_{H^s(\mathbb{T}^2)} \leq C\left(1+\|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)}\right)\|g_1\|_{H^s(\mathbb{T}^2)}, \tag{4.34}$$

where we have used the Sobolev embedding $H^s(\mathbb{T}^2)\subset L^{p_3}(\mathbb{T}^2)$ for $s>1$. Inserting (4.33) and (4.34) into (4.32), we obtain from Young's inequality that

$$\begin{aligned}
&\left|\int_{\mathbb{T}^2} g_1(x)f'(\theta)w(-\Delta)^s w dx\right| \\
&\leq C\left(1+\|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)}\right)\|g_1\|_{H^s(\mathbb{T}^2)}\left(\|w\|^{\eta_1+1}_{H^{s+\alpha}(\mathbb{T}^2)}\|w\|^{1-\eta_1}_{L^2(\mathbb{T}^2)}+\|w\|^{\eta_2+1}_{H^{s+\alpha}(\mathbb{T}^2)}\|w\|^{1-\eta_2}_{L^2(\mathbb{T}^2)}\right) \\
&\leq \frac{\kappa}{4}\|w\|^2_{H^{s+\alpha}(\mathbb{T}^2)}+C\|w\|^2_{L^2(\mathbb{T}^2)}\left(1+\|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)}\right)^{\frac{2}{1-\eta_1}}\|g_1\|^{\frac{2}{1-\eta_1}}_{H^s(\mathbb{T}^2)} \\
&\quad + C\|w\|^2_{L^2(\mathbb{T}^2)}\left(1+\|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)}\right)^{\frac{2}{1-\eta_2}}\|g_1\|^{\frac{2}{1-\eta_2}}_{H^s(\mathbb{T}^2)}.
\end{aligned} \tag{4.35}$$

Next, we consider the nonlinear terms involving $u_t\cdot\nabla\theta$ and $u\cdot\nabla w$ on the right hand side of (4.31). Indeed, it suffices to analyze the term involving $u_t\cdot\nabla\theta$, since the other term satisfies similar estimates. Notice that $\nabla\cdot u_t=0$. Then by Hölder's inequality and Lemmas 1-2, we deduce that

$$\begin{aligned}
&\left|\int_{\mathbb{T}^2}(u_t\cdot\nabla\theta)(-\Delta)^s w dx\right| \\
&\leq \|(-\Delta)^{\frac{s+\alpha}{2}}w\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\alpha}{2}}(u_t\cdot\nabla\theta)\|_{L^2(\mathbb{T}^2)} \\
&\leq \|(-\Delta)^{\frac{s+\alpha}{2}}w\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\alpha+1}{2}}(u_t\theta)\|_{L^2(\mathbb{T}^2)} \\
&\leq C\|(-\Delta)^{\frac{s+\alpha}{2}}w\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\alpha+1}{2}}u_t\|_{L^{p_1}(\mathbb{T}^2)}\|\theta\|_{L^{p_2}(\mathbb{T}^2)} \\
&\quad + C\|(-\Delta)^{\frac{s+\alpha}{2}}w\|_{L^2(\mathbb{T}^2)}\|u_t\|_{L^{p_2}(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\alpha+1}{2}}\theta\|_{L^{p_1}(\mathbb{T}^2)} \\
&\leq C\|(-\Delta)^{\frac{s+\alpha}{2}}w\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\alpha+1}{2}}w\|_{L^{p_1}(\mathbb{T}^2)}\|\theta\|_{L^{p_2}(\mathbb{T}^2)} \\
&\quad + C\|(-\Delta)^{\frac{s+\alpha}{2}}w\|_{L^2(\mathbb{T}^2)}\|w\|_{L^{p_2}(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\alpha+1}{2}}\theta\|_{L^{p_1}(\mathbb{T}^2)},
\end{aligned} \tag{4.36}$$

where $p_1=\frac{1}{1-\alpha^-}$ and $p_2=\frac{2}{2\alpha^--1}$ are given in (4.15). Using the following Gagliardo-Nirenberg inequalities:

$$\|(-\Delta)^{\frac{s-\alpha+1}{2}}w\|_{L^{p_1}(\mathbb{T}^2)} \leq C\|(-\Delta)^{\frac{s+\alpha}{2}}w\|^{\eta_3}_{L^2(\mathbb{T}^2)}\|w\|^{1-\eta_3}_{L^2(\mathbb{T}^2)}, \tag{4.37}$$

and

$$\|w\|_{L^{p_2}(\mathbb{T}^2)} \leq C\|(-\Delta)^{\frac{s+\alpha}{2}}w\|^{\eta_4}_{L^2(\mathbb{T}^2)}\|w\|^{1-\eta_4}_{L^2(\mathbb{T}^2)}, \tag{4.38}$$

where $\eta_3\in\left[\frac{s-\alpha+1}{s+\alpha},1\right)$ and $\eta_4\in(0,1)$, and inserting (4.37) and (4.38) into (4.36), we obtain from Young's inequality that

$$\begin{aligned}
&\left|\int_{\mathbb{T}^2}(u_t\cdot\nabla\theta)(-\Delta)^s w dx\right| \\
&\leq C\|(-\Delta)^{\frac{s+\alpha}{2}}w\|^{1+\eta_3}_{L^2(\mathbb{T}^2)}\|w\|^{1-\eta_3}_{L^2(\mathbb{T}^2)}\|\theta\|_{L^{p_2}(\mathbb{T}^2)} \\
&\quad + C\|(-\Delta)^{\frac{s+\alpha}{2}}w\|^{1+\eta_4}_{L^2(\mathbb{T}^2)}\|w\|^{1-\eta_4}_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s+\alpha}{2}}\theta\|^{\eta_3}_{L^2(\mathbb{T}^2)}\|\theta\|^{1-\eta_3}_{L^2(\mathbb{T}^2)} \\
&\leq \frac{\kappa}{8}\|w\|^2_{H^{s+\alpha}(\mathbb{T}^2)}+C\|w\|^2_{L^2(\mathbb{T}^2)}\|\theta\|^{\frac{2}{1-\eta_3}}_{L^{p_2}(\mathbb{T}^2)}+C\|w\|^2_{L^2(\mathbb{T}^2)}\|\theta\|^{\frac{2}{1-\eta_4}}_{H^{s+\alpha}(\mathbb{T}^2)},
\end{aligned} \tag{4.39}$$

where we have used the Sobolev embedding $H^{s+\alpha}(\mathbb{T}^2) \subset L^2(\mathbb{T}^2)$ in the last inequality. In a similar way as above, we have

$$\begin{aligned} \left| \int_{\mathbb{T}^2} (u \cdot \nabla w)(-\Delta)^s w dx \right| \leq & \frac{\kappa}{8} \|w\|_{H^{s+\alpha}(\mathbb{T}^2)}^2 + C\|w\|_{L^2(\mathbb{T}^2)}^2 \|\theta\|_{L^{p_2}(\mathbb{T}^2)}^{\frac{2}{1-\eta_3}} \\ & + C\|w\|_{L^2(\mathbb{T}^2)}^2 \|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)}^{\frac{2}{1-\eta_4}}. \end{aligned} \tag{4.40}$$

Inserting (4.35) and (4.39)-(4.40) into (4.31) gives

$$\begin{aligned} & \frac{d}{dt}\|w\|_{H^s(\mathbb{T}^2)}^2 + \kappa\|w\|_{H^{s+\alpha}(\mathbb{T}^2)}^2 \\ & \leq C\|w\|_{L^2(\mathbb{T}^2)}^2 \left(1 + \|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)}\right)^{\frac{2}{1-\eta_1}} \|g_1\|_{H^s(\mathbb{T}^2)}^{\frac{2}{1-\eta_1}} \\ & \quad + C\|w\|_{L^2(\mathbb{T}^2)}^2 \left(1 + \|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)}\right)^{\frac{2}{1-\eta_2}} \|g_1\|_{H^s(\mathbb{T}^2)}^{\frac{2}{1-\eta_2}} + C\|w\|_{L^2(\mathbb{T}^2)}^2 \|\theta\|_{L^{p_2}(\mathbb{T}^2)}^{\frac{2}{1-\eta_3}} \\ & \quad + C\|w\|_{L^2(\mathbb{T}^2)}^2 \|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)}^{\frac{2}{1-\eta_4}}. \end{aligned} \tag{4.41}$$

On the other hand, arguing as in (4.13) and (4.15)-(4.16), in view of (4.27) and the Sobolev embeddings $H^{s+\alpha}(\mathbb{T}^2) \subset H^{2\alpha}(\mathbb{T}^2)$, $H^{s+\alpha}(\mathbb{T}^2) \subset H^{1,p_1}(\mathbb{T}^2)$ for $s > 1$ and $\alpha \in (\frac{1}{2}, 1]$, we obtain

$$\begin{aligned} \|g_1 f(\theta)\|_{L^2(\mathbb{T}^2)} & \leq \|g_1\|_{L^\infty(\mathbb{T}^2)} \|f(\theta)\|_{L^2(\mathbb{T}^2)} \\ & \leq C\|g_1\|_{H^s(\mathbb{T}^2)} \left(1 + \|\theta\|_{H^{2\alpha}(\mathbb{T}^2)}\right) \\ & \leq C\|g_1\|_{H^s(\mathbb{T}^2)} \left(1 + \|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)}\right), \end{aligned} \tag{4.42}$$

$$\begin{aligned} \|u \cdot \nabla\theta\|_{L^2(\mathbb{T}^2)} & = \|(-\Delta)^{\frac{1}{2}}(u\theta)\|_{L^2(\mathbb{T}^2)} \\ & \leq C\|(-\Delta)^{\frac{1}{2}} u\|_{L^{p_1}(\mathbb{T}^2)} \|\theta\|_{L^{p_2}(\mathbb{T}^2)} + C\|u\|_{L^{p_2}(\mathbb{T}^2)} \|(-\Delta)^{\frac{1}{2}}\theta\|_{L^{p_1}(\mathbb{T}^2)} \\ & \leq C\|(-\Delta)^{\frac{1}{2}}\theta\|_{L^{p_1}(\mathbb{T}^2)} \|\theta\|_{L^{p_2}(\mathbb{T}^2)} \\ & \leq C\|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)} \|\theta\|_{L^{p_2}(\mathbb{T}^2)}, \end{aligned} \tag{4.43}$$

where $p_1 = \frac{1}{1-\alpha^-}$ and $p_2 = \frac{2}{2\alpha^- - 1}$ are given in (4.15). By (4.42)-(4.43) and g_1, $g_2 \in H^s(\mathbb{T}^2)$, it follows from (1.1) that

$$\begin{aligned} \|w\|_{L^2(\mathbb{T}^2)}^2 & = \|\partial_t\theta\|_{L^2(\mathbb{T}^2)}^2 \\ & \leq C\|u \cdot \nabla\theta\|_{L^2(\mathbb{T}^2)}^2 + C\|(-\Delta)^\alpha\theta\|_{L^2(\mathbb{T}^2)}^2 + \|g_1 f(\theta)\|_{L^2(\mathbb{T}^2)}^2 + C\|g_2\|_{L^2(\mathbb{T}^2)}^2 \\ & \leq C\|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)}^2 \|\theta\|_{L^{p_2}(\mathbb{T}^2)}^2 + C\|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)}^2 \\ & \quad + C\|g_1\|_{H^s(\mathbb{T}^2)}^2 \left(1 + \|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)}^2\right) + C\|g_2\|_{L^2(\mathbb{T}^2)}^2. \end{aligned}$$

This together with (4.2) and (4.9) implies that for any given bounded set $B \subset H^{s+2\alpha^-}(\mathbb{T}^2)$, there exists a $T_1' = T_1'(B) > 0$ such that any solution $\theta(t, \theta_0)$ of problem (1.1) with $\theta_0 \in B$ satisfies

$$\|\theta(t)\|_{H^{\alpha+s}(\mathbb{T}^2)}^2 \leq C, \quad \forall t \geq T_1', \tag{4.44}$$

$$\|\partial_t\theta(t)\|_{L^2(\mathbb{T}^2)}^2 \leq C, \quad \forall t \geq T_1'. \tag{4.45}$$

Combining (4.44)-(4.45) and g_1, $g_2 \in H^s(\mathbb{T}^2)$, we obtain from (4.41) that

$$\frac{d}{dt}\|w\|_{H^s(\mathbb{T}^2)}^2 + \kappa\|w\|_{H^{s+\alpha}(\mathbb{T}^2)}^2 \leq C, \quad \forall t \geq T_1'. \tag{4.46}$$

Finally, applying the uniform Gronwall lemma and (4.21), the assertion of the lemma follows. □

Finally, we establish uniform estimates for θ in $H^{2\alpha+s}(\mathbb{T}^2)$.

Lemma 15. *Let conditions in Lemma 14 hold. Then for any bounded set $B \subset H^{2\alpha^-+s}(\mathbb{T}^2)$, there exists $T_2 = T_2(B) > 0$ such that any solution $\theta(t,\theta_0)$ of problem (1.1) with $\theta_0 \in B$ satisfies*

$$\|\theta(t)\|^2_{H^{2\alpha+s}(\mathbb{T}^2)} \leq C, \quad \forall t \geq T_2. \tag{4.47}$$

Proof. Applying the operator $(-\Delta)^{\frac{s}{2}}$ to (1.1), in view of (1.2), we find that

$$\begin{aligned}\|(-\Delta)^{\frac{2\alpha+s}{2}}\theta\|^2_{L^2(\mathbb{T}^2)} &\leq C\|(-\Delta)^{\frac{s}{2}}(\partial_t\theta)\|^2_{L^2(\mathbb{T}^2)} + C\|(-\Delta)^{\frac{s}{2}}g_2\|^2_{L^2(\mathbb{T}^2)}\\ &\quad + C\|(-\Delta)^{\frac{s}{2}}(u\cdot\nabla\theta)\|^2_{L^2(\mathbb{T}^2)} + C\|(-\Delta)^{\frac{s}{2}}(g_1 f(\theta))\|^2_{L^2(\mathbb{T}^2)}\\ &:= C\|(-\Delta)^{\frac{s}{2}}(\partial_t\theta)\|^2_{L^2(\mathbb{T}^2)} + C\|g_2\|^2_{H^s(\mathbb{T}^2)} + I_4 + I_5.\end{aligned} \tag{4.48}$$

By (4.15)-(4.16) and Young's inequality, I_4 is bounded by

$$\begin{aligned} I_4 &\leq C\|(-\Delta)^{\frac{s+1}{2}}\theta\|^2_{L^{p_1}(\mathbb{T}^2)}\|\theta\|^2_{L^{p_2}(\mathbb{T}^2)}\\ &\leq C\|(-\Delta)^{\frac{2\alpha+s}{2}}\theta\|^{2\eta}_{L^2(\mathbb{T}^2)}\|\theta\|^{2-2\eta}_{L^2(\mathbb{T}^2)}\|\theta\|^2_{L^{p_2}(\mathbb{T}^2)}\\ &\leq \frac{1}{4}\|(-\Delta)^{\frac{2\alpha+s}{2}}\theta\|^2_{L^2(\mathbb{T}^2)} + C\|\theta\|^2_{L^2(\mathbb{T}^2)}\|\theta\|^{\frac{2}{1-\eta}}_{L^{p_2}(\mathbb{T}^2)},\end{aligned} \tag{4.49}$$

where $\eta \in \left[\frac{s+1}{2\alpha+s}, 1\right)$, $p_1 = \frac{1}{1-\alpha^-}$ and $p_2 = \frac{2}{2\alpha^- -1}$ are given in (4.15) and (4.16). Arguing as in (4.13), we have

$$\begin{aligned} I_5 &\leq C\|g_1\|^2_{H^s(\mathbb{T}^2)}\left(1 + \|\theta\|^2_{H^{2\alpha^-+s}(\mathbb{T}^2)}\right)\\ &\leq C\|g_1\|^2_{H^s(\mathbb{T}^2)}\left(1 + \|\theta\|^{2\eta'}_{H^{2\alpha+s}(\mathbb{T}^2)}\|\theta\|^{2-2\eta'}_{L^2(\mathbb{T}^2)}\right)\\ &\leq \frac{1}{4}\|(-\Delta)^{\frac{2\alpha+s}{2}}\theta\|^2_{L^2(\mathbb{T}^2)} + C\|g_1\|^{\frac{2}{1-\eta'}}_{H^s(\mathbb{T}^2)}\|\theta\|^2_{L^2(\mathbb{T}^2)} + C\|g_1\|^2_{H^s(\mathbb{T}^2)}.\end{aligned} \tag{4.50}$$

Inserting (4.49) and (4.50) into (4.48) gives

$$\begin{aligned}\|\theta\|^2_{H^{2\alpha+s}(\mathbb{T}^2)} &\leq C\|\partial_t\theta\|^2_{H^s(\mathbb{T}^2)} + C\|\theta\|^2_{L^2(\mathbb{T}^2)}\|\theta\|^{\frac{2}{1-\eta}}_{L^{p_2}(\mathbb{T}^2)} + C\|g_1\|^{\frac{2}{1-\eta'}}_{H^s(\mathbb{T}^2)}\|\theta\|^2_{L^2(\mathbb{T}^2)}\\ &\quad + C\|g_1\|^2_{H^s(\mathbb{T}^2)} + C\|g_2\|^2_{H^s(\mathbb{T}^2)}.\end{aligned} \tag{4.51}$$

Using g_1, $g_2 \in H^s(\mathbb{T}^2)$, (4.2) and (4.29), the assertion of the lemma follows from (4.51). □

5. **Existence and regularity of global attractors.** In this section, we prove the existence of an $(H^{2\alpha^-+s}(\mathbb{T}^2), H^{2\alpha^-+s}(\mathbb{T}^2))$-global attractor and an $(H^{2\alpha^-+s}(\mathbb{T}^2), H^{2\alpha+s}(\mathbb{T}^2))$-global attractor for the semigroup associated with the 2D quasi-geostrophic equation (1.1).

Theorem 16. *Assume that the conditions of Lemma 14 hold. Then the semigroup $\{S(t)\}_{t\geq 0}$ associated with problem (1.1) has an $(H^{2\alpha^-+s}(\mathbb{T}^2), H^{2\alpha^-+s}(\mathbb{T}^2))$-global attractor.*

Proof. Lemma 15 implies that $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $H^{2\alpha+s}(\mathbb{T}^2)$, and is asymptotically compact in $H^{2\alpha^-+s}(\mathbb{T}^2)$. Hence the existence of an $(H^{2\alpha^-+s}(\mathbb{T}^2), H^{2\alpha^-+s}(\mathbb{T}^2))$-global attractor for $\{S(t)\}_{t\geq 0}$ follows from Proposition 9. $\square$

In order to obtain the existence of the $(H^{2\alpha^-+s}(\mathbb{T}^2), H^{2\alpha+s}(\mathbb{T}^2))$-global attractor, we need the following auxiliary lemma.

Lemma 17. *Assume that the conditions of Lemma 14 hold. Then for any bounded set $B \subset H^{2\alpha^-+s}(\mathbb{T}^2)$, there exist $T_3 = T_3(B) > 0$ and $N_0 \geq 1$ such that any solution $\theta(t, \theta_0)$ of problem (1.1) with $\theta_0 \in B$ satisfies*

$$\|(I - P_m)\partial_t\theta(t)\|^2_{H^s(\mathbb{T}^2)} \leq \varepsilon, \quad \forall t \geq T_3,\, m \geq N_0, \tag{5.1}$$

where $P_m : L^2(\mathbb{T}^2) \to H_m$ is the projection operator and H_m is the space spanned by $\{e_j\}_{j=1}^m$.

Proof. Let $w_2 = (I - P_m)w$. Multiplying (4.30) by $(\Delta)^s w_2$ and then integrating over $\mathbb{T}^2$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}[(-\Delta)^{\frac{s}{2}}w_2]^2dx + \kappa\int_{\mathbb{T}^2}[(-\Delta)^{\frac{s+\alpha}{2}}w_2]^2dx\\ &= \int_{\mathbb{T}^2} g_1(x)f'(\theta)w(-\Delta)^s w_2dx - \int_{\mathbb{T}^2}(u\cdot\nabla w)(-\Delta)^s w_2dx - \int_{\mathbb{T}^2}(u_t\cdot\nabla\theta)(-\Delta)^s w_2dx.\end{aligned} \tag{5.2}$$

For the first term on the right-hand side of (5.2), by similar arguments as in (4.32)-(4.34) we obtain

$$\begin{aligned}&\left|\int_{\mathbb{T}^2} g_1(x)f'(\theta)w(-\Delta)^s w_2dx\right|\\ &\leq \|(-\Delta)^{\frac{s-\alpha}{2}}(g_1f'(\theta)w)\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s+\alpha}{2}}w_2\|_{L^2(\mathbb{T}^2)}\\ &\leq C\|g_1f'(\theta)\|_{L^{p_3}(\mathbb{T}^2)}\|w\|_{H^{s-\alpha,\frac{2p_3}{p_3-2}}(\mathbb{T}^2)}\|w_2\|_{H^{s+\alpha}(\mathbb{T}^2)}\\ &\quad + C\|g_1f'(\theta)\|_{H^{s-\alpha,\frac{2}{1-\alpha+p_4}}(\mathbb{T}^2)}\|w\|_{L^{\frac{2}{\alpha-p_4}}(\mathbb{T}^2)}\|w_2\|_{H^{s+\alpha}(\mathbb{T}^2)}\\ &\leq C\left(1+\|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)}\right)\|g_1\|_{H^s(\mathbb{T}^2)}\|w\|_{H^s(\mathbb{T}^2)}\|w_2\|_{H^{s+\alpha}(\mathbb{T}^2)}\\ &\leq \frac{\kappa}{4}\|w_2\|^2_{H^{s+\alpha}(\mathbb{T}^2)} + C\left(1+\|\theta\|^2_{H^{s+\alpha}(\mathbb{T}^2)}\right)\|g_1\|^2_{H^s(\mathbb{T}^2)}\|w\|^2_{H^s(\mathbb{T}^2)},\end{aligned} \tag{5.3}$$

where p_3 and p_4 are given in (4.32), and we have used Young's inequality and the Sobolev embeddings $H^s(\mathbb{T}^2) \subset H^{s-\alpha,\frac{2p_3}{p_3-2}}(\mathbb{T}^2)$ and $H^s(\mathbb{T}^2) \subset L^{\frac{2}{\alpha-p_4}}(\mathbb{T}^2)$ for $s > 1$. For the last term in (5.2), by Lemmas 1-2, Hölder's and Young's inequalities, we find that

$$
\begin{aligned}
&\left|\int_{\mathbb{T}^2}(u_t\cdot\nabla\theta)(-\Delta)^s w_2 dx\right| \\
&\le \|(-\Delta)^{\frac{s+\alpha}{2}} w_2\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\alpha}{2}}(u_t\cdot\nabla\theta)\|_{L^2(\mathbb{T}^2)} \\
&\le C\|(-\Delta)^{\frac{s+\alpha}{2}} w_2\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\alpha}{2}} u_t\|_{L^{p_5}(\mathbb{T}^2)}\|\nabla\theta\|_{L^{p_6}(\mathbb{T}^2)} \\
&\quad + C\|(-\Delta)^{\frac{s+\alpha}{2}} w_2\|_{L^2(\mathbb{T}^2)}\|u_t\|_{L^{p_2}(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\alpha+1}{2}}\theta\|_{L^{p_1}(\mathbb{T}^2)} \\
&\le C\|(-\Delta)^{\frac{s+\alpha}{2}} w_2\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\alpha}{2}} w\|_{L^{p_5}(\mathbb{T}^2)}\|\nabla\theta\|_{L^{p_6}(\mathbb{T}^2)} \\
&\quad + C\|(-\Delta)^{\frac{s+\alpha}{2}} w_2\|_{L^2(\mathbb{T}^2)}\|w\|_{L^{p_2}(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\alpha+1}{2}}\theta\|_{L^{p_1}(\mathbb{T}^2)} \\
&\le C\|w_2\|_{H^{s+\alpha}(\mathbb{T}^2)}\|w\|_{H^s(\mathbb{T}^2)}\|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)} \\
&\le \frac{\kappa}{8}\|w_2\|^2_{H^{\alpha+s}(\mathbb{T}^2)} + C\|w\|^2_{H^s(\mathbb{T}^2)}\|\theta\|^2_{H^{s+\alpha}(\mathbb{T}^2)},
\end{aligned} \tag{5.4}
$$

where

$$
p_1 = \frac{1}{1-\alpha^-},\quad p_2 = \frac{2}{2\alpha^- - 1},\quad p_5 = \frac{2}{1-\alpha^-},\quad p_6 = \frac{2}{\alpha^-},
$$

and we have used the Sobolev embeddings $H^s(\mathbb{T}^2) \subset H^{s-\alpha,p_5}(\mathbb{T}^2)$, $H^s(\mathbb{T}^2) \subset L^{p_2}(\mathbb{T}^2)$, $H^{s+\alpha}(\mathbb{T}^2) \subset H^{1,p_6}(\mathbb{T}^2)$ and $H^{s+\alpha}(\mathbb{T}^2) \subset H^{s-\alpha+1,p_1}(\mathbb{T}^2)$ in the last inequality. For the second term on the right-hand side of (5.2), noticing that $(-\Delta)^{\frac{s}{2}}$ and ∇ are commutable, and

$$
\langle u\cdot\nabla((-\Delta)^{\frac{s}{2}} w_2), (-\Delta)^{\frac{s}{2}} w_2\rangle = 0,
$$

hence we have

$$
\begin{aligned}
-\int_{\mathbb{T}^2}(u\cdot\nabla w)(-\Delta)^s w_2 dx &= -\langle(-\Delta)^{\frac{s}{2}}(u\cdot\nabla w_2) - u\cdot\nabla((-\Delta)^{\frac{s}{2}} w_2), (-\Delta)^{\frac{s}{2}} w_2\rangle \\
&\quad - \langle(-\Delta)^{\frac{s}{2}}(u\cdot\nabla w_1), (-\Delta)^{\frac{s}{2}} w_2\rangle \\
&= -\langle(-\Delta)^{\frac{s}{2}}(u\cdot\nabla w_2) - u\cdot((-\Delta)^{\frac{s}{2}}\nabla w_2), (-\Delta)^{\frac{s}{2}} w_2\rangle \\
&\quad - \langle(-\Delta)^{\frac{s}{2}}(u\cdot\nabla w_1), (-\Delta)^{\frac{s}{2}} w_2\rangle \\
&:= I_6 + I_7,
\end{aligned} \tag{5.5}
$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\mathbb{T}^2)$ and $w_1 = P_m w$. Then using Lemmas 1-2 and Hölder's inequality, we obtain

$$
\begin{aligned}
|I_6| &\le C\|w_2\|_{H^s(\mathbb{T}^2)}\Big(\|\nabla u\|_{L^{p_6}(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-1}{2}}\nabla w_2\|_{L^{p_5}(\mathbb{T}^2)} \\
&\quad + \|(-\Delta)^{\frac{s}{2}} u\|_{L^{p_6}(\mathbb{T}^2)}\|\nabla w_2\|_{L^{p_5}(\mathbb{T}^2)}\Big) \\
&\le C\|w_2\|_{H^s(\mathbb{T}^2)}\Big(\|\nabla\theta\|_{L^{p_6}(\mathbb{T}^2)}\|w_2\|_{H^{s,p_5}(\mathbb{T}^2)} + \|\theta\|_{H^{s,p_6}(\mathbb{T}^2)}\|\nabla w_2\|_{L^{p_5}(\mathbb{T}^2)}\Big) \\
&\le C\|w\|_{H^s(\mathbb{T}^2)}\|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)}\|w_2\|_{H^{s+\alpha}(\mathbb{T}^2)},
\end{aligned} \tag{5.6}
$$

where $p_5 = \frac{2}{1-\alpha^-}$, $p_6 = \frac{2}{\alpha^-}$, and we have used the Sobolev embeddings $H^{s+\alpha}(\mathbb{T}^2) \subset H^{s,p_6}(\mathbb{T}^2)$ and $H^{s+\alpha}(\mathbb{T}^2) \subset H^{s,p_5}(\mathbb{T}^2)$. By similar arguments as in (4.36), using the equivalence property of norm in the finite dimensional case, I_7 is bounded by

$$\begin{aligned}|I_7| &\le \|(-\Delta)^{\frac{s}{2}} w_2\|_{L^2(\mathbb{T}^2)} \|(-\Delta)^{\frac{s}{2}} (u\cdot\nabla w_1)\|_{L^2(\mathbb{T}^2)} \\ &\le C\|(-\Delta)^{\frac{s}{2}} w_2\|_{L^2(\mathbb{T}^2)} \|(-\Delta)^{\frac{s+1}{2}} (uw_1)\|_{L^2(\mathbb{T}^2)} \\ &\le C\|(-\Delta)^{\frac{s}{2}} w_2\|_{L^2(\mathbb{T}^2)} \|(-\Delta)^{\frac{s+1}{2}} u\|_{L^{p_1}(\mathbb{T}^2)} \|w_1\|_{L^{p_2}(\mathbb{T}^2)} \\ &\quad + C\|(-\Delta)^{\frac{s}{2}} w_2\|_{L^2(\mathbb{T}^2)} \|u\|_{L^{p_2}(\mathbb{T}^2)} \|(-\Delta)^{\frac{s+1}{2}} w_1\|_{L^{p_1}(\mathbb{T}^2)} \\ &\le C\|(-\Delta)^{\frac{s}{2}} w_2\|_{L^2(\mathbb{T}^2)} \|(-\Delta)^{\frac{s+1}{2}} \theta\|_{L^{p_1}(\mathbb{T}^2)} \|w_1\|_{L^{p_2}(\mathbb{T}^2)} \\ &\quad + C\|(-\Delta)^{\frac{s}{2}} w_2\|_{L^2(\mathbb{T}^2)} \|\theta\|_{L^{p_2}(\mathbb{T}^2)} \|(-\Delta)^{\frac{s+1}{2}} w_1\|_{L^{p_1}(\mathbb{T}^2)} \\ &\le C\|w_2\|_{H^{s+\alpha}(\mathbb{T}^2)} \|\theta\|_{H^{2\alpha+s}(\mathbb{T}^2)} \|w\|_{H^s(\mathbb{T}^2)},\end{aligned} \tag{5.7}$$

where $p_1 = \frac{1}{1-\alpha^-}$ and $p_2 = \frac{2}{2\alpha^- - 1}$ are given in (4.15), and we have used the Sobolev embeddings $H^{\alpha+s}(\mathbb{T}^2) \subset H^s(\mathbb{T}^2)$, $H^{2\alpha+s}(\mathbb{T}^2) \subset H^{s+1,p_1}(\mathbb{T}^2)$ and $H^{2\alpha+s}(\mathbb{T}^2) \subset L^{p_2}(\mathbb{T}^2)$. Inserting (5.6) and (5.7) into (5.5) gives

$$\begin{aligned}\left|\int_{\mathbb{T}^2} (u\cdot\nabla w)(-\Delta)^s w_2 dx\right| &\le C\|w_2\|_{H^{s+\alpha}(\mathbb{T}^2)} \|\theta\|_{H^{2\alpha+s}(\mathbb{T}^2)} \|w\|_{H^s(\mathbb{T}^2)} \\ &\le \frac{\kappa}{8}\|w_2\|^2_{H^{s+\alpha}(\mathbb{T}^2)} + C\|w\|^2_{H^s(\mathbb{T}^2)} \|\theta\|^2_{H^{2\alpha+s}(\mathbb{T}^2)},\end{aligned} \tag{5.8}$$

due to Young's inequality and the Sobolev embedding $H^{2\alpha+s}(\mathbb{T}^2) \subset H^{s+\alpha}(\mathbb{T}^2)$. Combining (5.2)-(5.4) and (5.8), we have

$$\begin{aligned}&\frac{d}{dt}\|w_2\|^2_{H^s(\mathbb{T}^2)} + \kappa\|w_2\|^2_{H^{s+\alpha}(\mathbb{T}^2)} \\ &\le C\left(1+\|\theta\|^2_{H^{s+\alpha}(\mathbb{T}^2)}\right)\|g_1\|^2_{H^s(\mathbb{T}^2)}\|w\|^2_{H^s(\mathbb{T}^2)} + C\|w\|^2_{H^s(\mathbb{T}^2)}\|\theta\|^2_{H^{2\alpha+s}(\mathbb{T}^2)}.\end{aligned} \tag{5.9}$$

From (4.9), (4.29) and (4.47) we see that for any given bounded set $B \subset H^{s+2\alpha^-}(\mathbb{T}^2)$, there exists a $T_3' = T_3'(B) > 0$ such that any solution $\theta(t,\theta_0)$ of problem (1.1) with $\theta_0 \in B$ satisfies

$$\|\theta(t)\|^2_{H^{s+\alpha}(\mathbb{T}^2)} \le C, \quad \forall t \ge T_3', \tag{5.10}$$

$$\|\theta(t)\|^2_{H^{2\alpha+s}(\mathbb{T}^2)} \le C, \quad \forall t \ge T_3', \tag{5.11}$$

$$\|\partial_t\theta(t)\|^2_{H^s(\mathbb{T}^2)} \le C, \quad \forall t \ge T_3'. \tag{5.12}$$

Then (5.9)-(5.12) and $g_1 \in H^s(\mathbb{T}^2)$ imply that

$$\frac{d}{dt}\|w_2\|^2_{H^s(\mathbb{T}^2)} + \kappa\lambda_{m+1}^{2\alpha}\|w_2\|^2_{H^s(\mathbb{T}^2)} \le C, \quad \forall t \ge T_3'.$$

Finally, by Gronwall's inequality we obtain (5.1), and thus the proof of this lemma is completed. □

We are now ready to prove the main result of this section.

Theorem 18. *Assume that the conditions of Lemma 14 hold. Then the semigroup* $\{S(t)\}_{t\ge0}$ *associated with problem* (1.1) *has an* $(H^{2\alpha^-+s}(\mathbb{T}^2), H^{2\alpha+s}(\mathbb{T}^2))$*-global attractor* $\mathcal{A}$.

Proof. Thanks to Theorem 16 and Proposition 9, now it only remains to show that the semigroup $\{S(t)\}_{t\ge0}$ is $(H^{2\alpha^-+s}(\mathbb{T}^2), H^{2\alpha+s}(\mathbb{T}^2))$-asymptotically compact.

Let a bounded set $B \subset H^{2\alpha^-+s}(\mathbb{T}^2)$, sequences $\{t_n\}_{n=1}^\infty$ with $t_n \to +\infty$ and $\{\theta_0^n\}_{n=1}^\infty \subset B$ are given arbitrarily. We will show that the sequence $\{S(t_n)\theta_0^n\}_{n=1}^\infty$ has a convergent subsequence in $H^{2\alpha+s}(\mathbb{T}^2)$. Notice that Lemma 15 implies that

$\{S(t_n)\theta_0^n\}_{n=1}^\infty$ is bounded in $H^{2\alpha+s}(\mathbb{T}^2)$. By the compactness of embedding $H^{2\alpha+s}(\mathbb{T}^2)\subset H^{2\alpha^-+s}(\mathbb{T}^2)$, it follows that there is $\xi\in H^{2\alpha^-+s}(\mathbb{T}^2)$ such that, up to a subsequence,

$$S(t_n)\theta_0^n\to\xi \text{ strongly in } H^{2\alpha^-+s}(\mathbb{T}^2). \tag{5.13}$$

This together with (3.1) implies that

$$\{u^n(t_n)\cdot\nabla\theta^n(t_n)\}_{n=1}^\infty \text{ is a Cauchy sequence in } H^s(\mathbb{T}^2). \tag{5.14}$$

Since $f: H^{2\alpha^-+s}(\mathbb{T}^2)\to H^s(\mathbb{T}^2)$ is Lipschitz continuous on bounded subset of $H^{2\alpha^-+s}(\mathbb{T}^2)$, in view of (1.2) and (5.13), we have that

$$\{F(x,\theta^n(t_n))\}_{n=1}^\infty \text{ is a Cauchy sequence in } H^s(\mathbb{T}^2). \tag{5.15}$$

Observe that

$$\begin{aligned}&\frac{\partial\theta^n(t_n)}{\partial t_n}-\frac{\partial\theta^{n'}(t_{n'})}{\partial t_{n'}}+u^n(t_n)\cdot\nabla\theta^n(t_n)-u^{n'}(t_{n'})\cdot\nabla\theta^{n'}(t_{n'})\\&\quad+\kappa(-\Delta)^\alpha(\theta^n(t_n)-\theta^{n'}(t_{n'}))=F(x,\theta^n(t_n))-F(x,\theta^{n'}(t_{n'})).\end{aligned} \tag{5.16}$$

Taking the inner product of (5.16) with $(-\Delta)^{\alpha+s}(\theta^n(t_n)-\theta^{n'}(t_{n'}))$, we obtain

$$\begin{aligned}&\kappa\|\theta^n(t_n)-\theta^{n'}(t_{n'})\|^2_{H^{2\alpha+s}(\mathbb{T}^2)}\\&=-\left\langle(-\Delta)^{\frac{s}{2}}\left(\frac{\partial\theta^n(t_n)}{\partial t_n}-\frac{\partial\theta^{n'}(t_{n'})}{\partial t_{n'}}\right),(-\Delta)^{\frac{2\alpha+s}{2}}\left(\theta^n(t_n)-\theta^{n'}(t_{n'})\right)\right\rangle\\&\quad-\left\langle(-\Delta)^{\frac{s}{2}}\left(u^n(t_n)\cdot\nabla\theta^n(t_n)-u^{n'}(t_{n'})\cdot\nabla\theta^{n'}(t_{n'})\right),\right.\\&\quad\left.(-\Delta)^{\frac{2\alpha+s}{2}}\left(\theta^n(t_n)-\theta^{n'}(t_{n'})\right)\right\rangle\\&\quad+\left\langle(-\Delta)^{\frac{s}{2}}\left(F(x,\theta^n(t_n))-F(x,\theta^{n'}(t_{n'}))\right),(-\Delta)^{\frac{2\alpha+s}{2}}\left(\theta^n(t_n)-\theta^{n'}(t_{n'})\right)\right\rangle\\&\le\frac{\kappa}{2}\|\theta^n(t_n)-\theta^{n'}(t_{n'})\|^2_{H^{2\alpha+s}(\mathbb{T}^2)}+C\left\|\frac{\partial\theta^n(t_n)}{\partial t_n}-\frac{\partial\theta^{n'}(t_{n'})}{\partial t_{n'}}\right\|^2_{H^s(\mathbb{T}^2)}\\&\quad+C\|u^n(t_n)\cdot\nabla\theta^n(t_n)-u^{n'}(t_{n'})\cdot\nabla\theta^{n'}(t_{n'})\|^2_{H^s(\mathbb{T}^2)}\\&\quad+\|F(x,\theta^n(t_n))-F(x,\theta^{n'}(t_{n'}))\|^2_{H^s(\mathbb{T}^2)},\end{aligned} \tag{5.17}$$

thanks to Young's and Hölder's inequalities. Lemma 17 implies that for every $\varepsilon>0$, there exist $N_0'>0$ and $m_0\ge1$ such that

$$\left\|(I-P_{m_0})\frac{\partial\theta^n(t_n)}{\partial t_n}\right\|_{H^s(\mathbb{T}^2)}+\left\|(I-P_{m_0})\frac{\partial\theta^{n'}(t_{n'})}{\partial t_{n'}}\right\|_{H^s(\mathbb{T}^2)}\le\varepsilon,\quad\forall n,\,n'\ge N_0'. \tag{5.18}$$

On the other hand, by (4.29) we find that the sequence $\left\{P_{m_0}\frac{\partial\theta^n(t_n)}{\partial t_n}\right\}_{n=1}^\infty$ is bounded in $H^s(\mathbb{T}^2)$, and thus $\left\{P_{m_0}\frac{\partial\theta^n(t_n)}{\partial t_n}\right\}_{n=1}^\infty$ is precompact in $H^s(\mathbb{T}^2)$. This together with (5.18) shows that $\left\{\frac{\partial\theta^n(t_n)}{\partial t_n}\right\}_{n=1}^\infty$ has a finite open covering of balls with radii less than ε in $H^s(\mathbb{T}^2)$. This implies that, up to a subsequence,

$$\left\{\frac{\partial\theta^n(t_n)}{\partial t_n}\right\}_{n=1}^\infty \text{ is a Cauchy sequence in } H^s(\mathbb{T}^2). \tag{5.19}$$

Combining (5.14)-(5.15), (5.17) and (5.19), we have that, up to a subsequence,

$$\{\theta^n(t_n)\}_{n=1}^{\infty} \text{ is a Cauchy sequence in } H^{2\alpha+s}(\mathbb{T}^2),$$

and thus $\{S(t)\}_{t\geq 0}$ is $(H^{2\alpha^-+s}(\mathbb{T}^2), H^{2\alpha+s}(\mathbb{T}^2))$-asymptotically compact. The proof of this theorem is completed. □

6. **Exponential attractors.** In this section, we are concerned with the construction of exponential attractors. First, we recall the definition of exponential attractors and the fractal dimension for a general set.

Definition 19. Let A be a compact subset of a Banach space X. Then the fractal dimension $\dim_X^f(A)$ of A is defined by

$$\dim_X^f(A) = \limsup_{\varepsilon\to 0} \frac{\log \widehat{N}(A,\varepsilon)}{\log \frac{1}{\varepsilon}},$$

where $\varepsilon > 0$ and $\widehat{N}(A,\varepsilon)$ is the minimal number of closed balls in X having a radius ε which cover the set A.

In particular, when $\dim_X^f(A) < \infty$, A is said to have a finite fractal dimension.

Definition 20. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a Banach space X, and let $\mathcal{A} \subset X$ be a global attractor of $\{S(t)\}_{t\geq 0}$ in X. Then a set $\mathcal{M} \subset X$ is called an exponential attractor for $\{S(t)\}_{t\geq 0}$ in X if the following properties hold:

(i) $\mathcal{M}$ is a compact subset of X such that $\mathcal{A} \subset \mathcal{M} \subset X$;
(ii) $\mathcal{M}$ is positively invariant, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$ for all $t \geq 0$;
(iii) $\mathcal{M}$ has finite fractal dimension in X;
(iv) $\mathcal{M}$ attracts exponentially every bounded subset of X, i.e., there exists an exponent $\sigma > 0$ such that for any bounded set $B \subset X$,

$$dist_X(S(t)B, \mathcal{M}) \leq C_B e^{-\sigma t}, \quad t \geq \widehat{t}_B$$

with two positive constants C_B and $\widehat{t}_B$ depending on B.

For the convenience of applications, we reformulate the abstract result on the construction of exponential attractors [4, 14, 17, 39] for $\{S(t)\}_{t\geq 0}$ under slightly modified conditions.

Theorem 21. *Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a Banach space X, and let $\mathcal{A} \subset X$ be a global attractor of $\{S(t)\}_{t\geq 0}$ in X. Assume that there exists a closed bounded set $\mathfrak{X} \subset X$ satisfying:*

(i) it is positively invariant, i.e., $S(t)\mathfrak{X} \subset \mathfrak{X}$ for all $t \geq 0$;
(ii) it is an absorbing set of $\{S(t)\}_{t\geq 0}$, i.e., for any bounded set $B \subset X$, there exists a time $t_B > 0$ such that $S(t)B \subset \mathfrak{X}$ for all $t \geq t_B$;
(iii) there exist $0 < \gamma_1 \leq 1$, $0 < \gamma_2 \leq 1$, $t^* > 0$ and $L = L(t^*) > 0$ such that for any $t_1, t_2 \in [t^*, 2t^*]$ and $u, v \in \mathfrak{X}$,

$$\|S(t_1)u - S(t_2)v\|_X \leq L\left(|t_1 - t_2|^{\gamma_1} + \|u - v\|_X^{\gamma_2}\right);$$

(iv) there exist a positive constant $\delta \in \left[0, \frac{1}{2}\right)$ and a N-dimensional subspace X_N of X such that the bounded projection $P_N : X \to X_N$ satisfies that for any $u, v \in \mathfrak{X}$,

$$\|(I - P_N)(S(t^*)u - S(t^*)v)\|_X \leq \delta \|u - v\|_X^{\gamma_2}.$$

Then the semigroup $\{S(t)\}_{t\geq 0}$ has an exponential attractor $\mathcal{E}$ satisfying the following properties:

(1) $\mathcal{E}$ contains a global attractor $\mathcal{A}$ of $\{S(t)\}_{t\geq 0}$;
(2) $\mathcal{E}$ is a compact subset of X with finite fractal dimension

$$\dim_X^f(\mathcal{E}) \leq \frac{1}{\gamma_0}\left(-\frac{\log \widehat{N}_\mu}{\log a_\mu}+1\right);$$

(3) $\mathcal{E}$ is positively invariant, i.e., $S(t)\mathcal{E}\subset\mathcal{E}$ for all $t\geq 0$;
(4) $dist_X(S(t)\mathfrak{X},\,\mathcal{E})\leq L(R^{\gamma_2})^{\gamma_2}a_\mu^{-2\gamma_2}e^{-\left(\frac{\gamma_2}{t^*}\log a_\mu^{-1}\right)t}$ for every $t\geq t^*$.

Here $\gamma_0=\min\{\gamma_1,\gamma_2\}$, R is the diameter of $\mathfrak{X}$, $0<a_\mu<1$ is the exponent given by $a_\mu=2(\delta+\mu L)$ with $0<\mu<\frac{1-2\delta}{2L}$, and $\widehat{N}_\mu$ is the minimal number of closed balls of X with radius μ which cover the closed unit ball of X_N, $\mathscr{B}_N(0;1)=\{x\in X_N:\|x\|_X\leq 1\}$ centered at 0.

Proof. Following a similar procedure to the method given in [17, 39], we can construct the exponential attractor $\mathcal{E}$ for $\{S(t)\}_{t\geq 0}$.

(1) Let $S^n=S(nt^*)$ for $n\in\mathbb{N}$. We will first construct an exponential attractor $\mathcal{E}^*$ for the discrete dynamical system $\{S^n\}_{n\geq 0}$.

Let μ be any exponent such that $0<\mu<\frac{1-2\delta}{2L}$, and let $a_\mu=2(\delta+\mu L)$. Then it is clear that $0<a_\mu<1$. The closed unit ball, $\mathscr{B}_N(0;1)=\{x\in X_N:\|x\|_X\leq 1\}$ centered at 0, is a finite N-dimensional compact ball in X. Therefore, $\mathscr{B}_N(0;1)$ can be covered by finite closed balls of X with radius μ. Denote by $\widehat{N}_\mu$ the minimal number of balls of X with radius μ which cover $\mathscr{B}_N(0;1)$. Following the arguments in [39, Theorem 6.12] step by step, we obtain that for $n=0,1,2,\cdots$, there exists a finite covering of $S^n\mathfrak{X}$ such that

$$S^n\mathfrak{X}\subset\bigcup_{i=1}^{\widehat{N}_\mu^n}\overline{\mathscr{B}}(W_{n,i};R^{\gamma_2}a_\mu^n),$$

with centers $W_{n,i}\in S^n\mathfrak{X}$, $1\leq i\leq\widehat{N}_\mu^n$, where R is the diameter of $\mathfrak{X}$.

Let $\mathscr{P}=\{W_{n,i}:0\leq n<\infty,\,1\leq i\leq\widehat{N}_\mu^n\}$ and $\mathcal{E}^*=\overline{\bigcup_{n=0}^\infty S^n\mathscr{P}}$. Then $\mathcal{E}^*$ is an exponential attractor for the discrete dynamical system $\{S^n\}_{n\geq 0}$. More precisely, $\mathcal{E}^*$ contains a global attractor $\mathcal{A}^*$ of $\{S^n\}_{n\geq 0}$, the fractal dimension of $\mathcal{E}^*$ is

$$\dim_X^f(\mathcal{E}^*)\leq-\frac{\log\widehat{N}_\mu}{\log a_\mu}, \tag{6.1}$$

$S^n\mathcal{E}^*\subset\mathcal{E}^*$ for each n and

$$dist_X(S^n\mathfrak{X},\,\mathcal{E}^*)\leq R^{\gamma_2}a_\mu^n\quad\text{for all } n\geq 0. \tag{6.2}$$

(2) Now we consider the continuous case. To this aim, we define

$$\mathcal{E}=\bigcup_{t\in[t^*,2t^*]}S(t)\mathcal{E}^*.$$

Thanks to the condition (iii), it is obvious that $\mathcal{E}$ is compact. Then it follows from (6.1) and the condition (iii) that

$$\dim_X^f(\mathcal{E})\leq\frac{1}{\gamma_0}\left(1+\dim_X^f(\mathcal{E}^*)\right)\leq\frac{1}{\gamma_0}\left(1-\frac{\log\widehat{N}_\mu}{\log a_\mu}\right),$$

where $\gamma_0=\min\{\gamma_1,\gamma_2\}$. Moreover, by the positive invariance for the discrete exponential attractor, we have $S(t)\mathcal{E}\subset\mathcal{E}$ for any $t\geq 0$.

For $t \geq t^*$, writing $t = nt^* + \tau$ with $n \geq 1$ and $\tau \in [0, t^*]$, by the condition (iii) we obtain that

$$dist_X(S(t)\mathfrak{X},\, \mathcal{E}) \leq dist_X(S(t)\mathfrak{X},\, S(t^*)\mathcal{E}^*) \leq L(dist_X(S(t-t^*)\mathfrak{X},\, \mathcal{E}^*))^{\gamma_2}. \tag{6.3}$$

Furthermore, using the condition (i) and (6.2), we have

$$\begin{aligned} dist_X(S(t-t^*)\mathfrak{X},\, \mathcal{E}^*) &\leq dist_X(S((n-1)t^*)S(\tau)\mathfrak{X},\, \mathcal{E}^*) \\ &\leq dist_X(S^{n-1}\mathfrak{X},\, \mathcal{E}^*) \\ &\leq R^{\gamma_2} a_\mu^{n-1} = R^{\gamma_2} a_\mu^{-1} a_\mu^{\frac{t-\tau}{t^*}} \\ &\leq R^{\gamma_2} a_\mu^{-2} e^{-\left(\frac{1}{t^*}\log a_\mu^{-1}\right)t}. \end{aligned}$$

This and (6.3) imply that for all $t \geq t^*$,

$$dist_X(S(t)\mathfrak{X},\, \mathcal{E}) \leq L(R^{\gamma_2})^{\gamma_2} a_\mu^{-2\gamma_2} e^{-\left(\frac{\gamma_2}{t^*}\log a_\mu^{-1}\right)t}. \tag{6.4}$$

By the condition (ii), we conclude that for the global attractor $\mathcal{A}$ of $\{S(t)\}_{t\geq 0}$, there exists a time $t_{\mathcal{A}} > 0$ such that $S(t)\mathcal{A} \subset \mathfrak{X}$ for all $t \geq t_{\mathcal{A}}$. This and the invariance of $\mathcal{A}$ ensure that

$$dist_X(\mathcal{A},\, \mathcal{E}) = dist_X(S(t)\mathcal{A},\, \mathcal{E}) \leq dist_X(S((t-t_{\mathcal{A}})\mathfrak{X},\, \mathcal{E})).$$

Thanks to (6.4), we see that $\mathcal{A} \subset \mathcal{E}$.

The proof is complete. □

In order to construct an exponential attractor $\mathcal{E}$ for the semigroup $\{S(t)\}_{t\geq 0}$ associated with (1.1) in $H^{2\alpha+s}(\mathbb{T}^2)$ with $\alpha > \frac{1}{2}$ and $s > 1$, now it suffices to show that there exists a closed bounded set $\mathfrak{X} \subset H^{2\alpha+s}(\mathbb{T}^2)$ having the properties (i)-(iv) in Theorem 21.

Lemma 15 implies that $\{S(t)\}_{t\geq 0}$ has a closed bounded absorbing set $\mathfrak{B}$ in $H^{2\alpha+s}(\mathbb{T}^2)$, and there exists a time $t_{\mathfrak{B}} > 0$ such that $S(t)\mathfrak{B} \subset \mathfrak{B}$ for every $t \geq t_{\mathfrak{B}}$. We define

$$\mathfrak{X} = \overline{\bigcup_{t\geq t_{\mathfrak{B}}} S(t)\mathfrak{B}}. \tag{6.5}$$

Then it follows that $\mathfrak{X}$ is a closed bounded set in $H^{2\alpha+s}(\mathbb{T}^2)$, and the conditions (i) and (ii) in Theorem 21 are fulfilled.

In order to verify the Hölder continuity property, we need the following lemma on estimates of $\partial_t\theta$.

Lemma 22. *Assume that the conditions of Lemma 14 hold. Then for any fixed* $\widetilde{\eta}$ *with* $0 < \widetilde{\eta} \leq 2\alpha - 2\alpha^-$, *there exists* $T_0^* = T_0^*(\mathfrak{X}) > 0$ *such that any solution* $\theta(t, \theta_0)$ *of problem* (1.1) *with* $\theta_0 \in \mathfrak{X}$ *satisfies for any* $t_1,\, t_2 \geq T_0^*$,

$$\int_{t_1}^{t_2} \|\partial_t\theta\|^2_{H^{2\alpha+s-\widetilde{\eta}}(\mathbb{T}^2)} d\tau \leq C|t_2 - t_1| + C. \tag{6.6}$$

Proof. Multiplying (4.30) by $(-\Delta)^{\alpha+s-\widetilde{\eta}}w$ and then integrating over $\mathbb{T}^2$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} [(-\Delta)^{\frac{\alpha+s-\widetilde{\eta}}{2}} w]^2 dx + \kappa \int_{\mathbb{T}^2} [(-\Delta)^{\frac{s-\widetilde{\eta}+2\alpha}{2}} w]^2 dx \\ &= \int_{\mathbb{T}^2} g_1(x) f'(\theta) w (-\Delta)^{\alpha+s-\widetilde{\eta}} w dx - \int_{\mathbb{T}^2} (u\cdot\nabla w)(-\Delta)^{\alpha+s-\widetilde{\eta}} w dx \\ &\quad - \int_{\mathbb{T}^2} (u_t \cdot \nabla\theta)(-\Delta)^{\alpha+s-\widetilde{\eta}} w dx. \end{aligned} \tag{6.7}$$

For the first term on the right-hand side of (6.7), by similar arguments as in (4.32)-(4.34) we have

$$
\begin{aligned}
&\left|\int_{\mathbb{T}^2} g_1(x)f'(\theta)w(-\Delta)^{\alpha+s-\widetilde{\eta}}wdx\right| \\
&\le \|(-\Delta)^{\frac{s-\widetilde{\eta}}{2}}(g_1f'(\theta)w)\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\widetilde{\eta}+2\alpha}{2}}w\|_{L^2(\mathbb{T}^2)} \\
&\le C\|g_1f'(\theta)\|_{L^{p_3'}(\mathbb{T}^2)}\|w\|_{H^{s-\widetilde{\eta},\frac{2p_3'}{p_3'-2}}(\mathbb{T}^2)}\|w\|_{H^{s-\widetilde{\eta}+2\alpha}(\mathbb{T}^2)} \\
&\quad + C\|g_1f'(\theta)\|_{H^{s-\widetilde{\eta},\frac{2}{1-\widetilde{\eta}+p_4'}}(\mathbb{T}^2)}\|w\|_{L^{\frac{2}{\widetilde{\eta}-p_4'}}(\mathbb{T}^2)}\|w\|_{H^{s-\widetilde{\eta}+2\alpha}(\mathbb{T}^2)} \\
&\le C\left(1+\|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)}\right)\|g_1\|_{H^s(\mathbb{T}^2)}\|w\|_{H^s(\mathbb{T}^2)}\|w\|_{H^{s-\widetilde{\eta}+2\alpha}(\mathbb{T}^2)} \\
&\quad + C\Bigg(\|f'(\theta)\|_{L^{p_3'}(\mathbb{T}^2)}\|g_1\|_{H^{s-\widetilde{\eta},\frac{2}{1-\widetilde{\eta}+p_4'-2/p_3'}}(\mathbb{T}^2)} \\
&\quad + \|g_1\|_{L^{p_3'}(\mathbb{T}^2)}\|f'(\theta)\|_{H^{s-\widetilde{\eta},\frac{2}{1-\widetilde{\eta}+p_4'-2/p_3'}}(\mathbb{T}^2)}\Bigg)\|w\|_{H^s(\mathbb{T}^2)}\|w\|_{H^{s-\widetilde{\eta}+2\alpha}(\mathbb{T}^2)} \\
&\le C\left(1+\|\theta\|_{H^{s+\alpha}(\mathbb{T}^2)}\right)\|g_1\|_{H^s(\mathbb{T}^2)}\|w\|_{H^s(\mathbb{T}^2)}\|w\|_{H^{s-\widetilde{\eta}+2\alpha}(\mathbb{T}^2)} \\
&\le \frac{\kappa}{4}\|w\|^2_{H^{s-\widetilde{\eta}+2\alpha}(\mathbb{T}^2)} + C\left(1+\|\theta\|^2_{H^{s+\alpha}(\mathbb{T}^2)}\right)\|g_1\|^2_{H^s(\mathbb{T}^2)}\|w\|^2_{H^s(\mathbb{T}^2)}
\end{aligned} \tag{6.8}
$$

for some $0 < \frac{2}{p_3'} < p_4' < \widetilde{\eta}$, and we have used Lemma 1, Young's inequality and the Sobolev embeddings $H^s(\mathbb{T}^2) \subset H^{s-\widetilde{\eta},\frac{2p_3'}{p_3'-2}}(\mathbb{T}^2)$, $H^s(\mathbb{T}^2) \subset L^\infty(\mathbb{T}^2)$, $H^s(\mathbb{T}^2) \subset L^{\frac{2}{\widetilde{\eta}-p_4'}}(\mathbb{T}^2)$, $H^s(\mathbb{T}^2) \subset L^{p_3'}(\mathbb{T}^2)$ and $H^s(\mathbb{T}^2) \subset H^{s-\widetilde{\eta},\frac{2}{1-\widetilde{\eta}+p_4'-2/p_3'}}(\mathbb{T}^2)$ for $s > 1$. For the last term in (6.7), using Lemmas 1-2, Hölder's and Young's inequalities, we deduce that

$$
\begin{aligned}
&\left|\int_{\mathbb{T}^2}(u_t\cdot\nabla\theta)(-\Delta)^{\alpha+s-\widetilde{\eta}}wdx\right| \\
&\le \|(-\Delta)^{\frac{s-\widetilde{\eta}+2\alpha}{2}}w\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\widetilde{\eta}}{2}}(u_t\cdot\nabla\theta)\|_{L^2(\mathbb{T}^2)} \\
&\le C\|(-\Delta)^{\frac{s-\widetilde{\eta}+2\alpha}{2}}w\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\widetilde{\eta}}{2}}u_t\|_{L^{p_5'}(\mathbb{T}^2)}\|\nabla\theta\|_{L^{p_6'}(\mathbb{T}^2)} \\
&\quad + C\|(-\Delta)^{\frac{s-\widetilde{\eta}+2\alpha}{2}}w\|_{L^2(\mathbb{T}^2)}\|u_t\|_{L^{p_2'}(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\widetilde{\eta}+1}{2}}\theta\|_{L^{p_1'}(\mathbb{T}^2)} \\
&\le C\|(-\Delta)^{\frac{s-\widetilde{\eta}+2\alpha}{2}}w\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\widetilde{\eta}}{2}}w\|_{L^{p_5'}(\mathbb{T}^2)}\|\nabla\theta\|_{L^{p_6'}(\mathbb{T}^2)} \\
&\quad + C\|(-\Delta)^{\frac{s-\widetilde{\eta}+2\alpha}{2}}w\|_{L^2(\mathbb{T}^2)}\|w\|_{L^{p_2'}(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\widetilde{\eta}+1}{2}}\theta\|_{L^{p_1'}(\mathbb{T}^2)} \\
&\le C\|w\|_{H^{s-\widetilde{\eta}+2\alpha}(\mathbb{T}^2)}\|w\|_{H^s(\mathbb{T}^2)}\|\theta\|_{H^{s+2\alpha}(\mathbb{T}^2)} \\
&\le \frac{\kappa}{8}\|w\|^2_{H^{s-\widetilde{\eta}+2\alpha}(\mathbb{T}^2)} + C\|w\|^2_{H^s(\mathbb{T}^2)}\|\theta\|^2_{H^{s+2\alpha}(\mathbb{T}^2)},
\end{aligned} \tag{6.9}
$$

where

$$
p_1' = \frac{1}{1-\alpha}, \quad p_2' = \frac{2}{2\alpha-1}, \quad p_5' = \frac{2}{1-\widetilde{\eta}}, \quad p_6' = \frac{2}{\widetilde{\eta}},
$$

and we have used the Sobolev embeddings $H^s(\mathbb{T}^2) \subset H^{s-\widetilde{\eta},p_5'}(\mathbb{T}^2)$, $H^{s+2\alpha}(\mathbb{T}^2) \subset H^{1,p_6'}(\mathbb{T}^2)$, $H^s(\mathbb{T}^2) \subset L^{p_2'}(\mathbb{T}^2)$, $H^{s+2\alpha}(\mathbb{T}^2) \subset H^{s-\widetilde{\eta}+1,p_1'}(\mathbb{T}^2)$ in the last inequality. Arguing as in (4.36)-(4.39), the second term on the right-hand side of (6.7) is bounded by

$$
\begin{aligned}
&\left|\int_{\mathbb{T}^2}(u\cdot\nabla w)(-\Delta)^{\alpha+s-\widetilde{\eta}}w dx\right| \\
&\le \|(-\Delta)^{\frac{s-\widetilde{\eta}+2\alpha}{2}}w\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\widetilde{\eta}}{2}}(u\cdot\nabla w)\|_{L^2(\mathbb{T}^2)} \\
&\le \|(-\Delta)^{\frac{s-\widetilde{\eta}+2\alpha}{2}}w\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\widetilde{\eta}+1}{2}}(uw)\|_{L^2(\mathbb{T}^2)} \\
&\le C\|(-\Delta)^{\frac{s-\widetilde{\eta}+2\alpha}{2}}w\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\widetilde{\eta}+1}{2}}u\|_{L^{p_1}(\mathbb{T}^2)}\|w\|_{L^{p_2}(\mathbb{T}^2)} \\
&\quad + C\|(-\Delta)^{\frac{s-\widetilde{\eta}+2\alpha}{2}}w\|_{L^2(\mathbb{T}^2)}\|u\|_{L^{p_2}(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\widetilde{\eta}+1}{2}}w\|_{L^{p_1}(\mathbb{T}^2)} \\
&\le C\|(-\Delta)^{\frac{s-\widetilde{\eta}+2\alpha}{2}}w\|_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\widetilde{\eta}+1}{2}}\theta\|_{L^{p_1}(\mathbb{T}^2)}\|w\|_{L^{p_2}(\mathbb{T}^2)} \\
&\quad + C\|(-\Delta)^{\frac{s-\widetilde{\eta}+2\alpha}{2}}w\|_{L^2(\mathbb{T}^2)}\|\theta\|_{L^{p_2}(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\widetilde{\eta}+1}{2}}w\|_{L^{p_1}(\mathbb{T}^2)} \\
&\le C\|(-\Delta)^{\frac{s-\widetilde{\eta}+2\alpha}{2}}w\|^{1+\eta_4'}_{L^2(\mathbb{T}^2)}\|w\|^{1-\eta_4'}_{L^2(\mathbb{T}^2)}\|(-\Delta)^{\frac{s-\widetilde{\eta}+1}{2}}\theta\|_{L^{p_1}(\mathbb{T}^2)} \\
&\quad + C\|(-\Delta)^{\frac{s-\widetilde{\eta}+2\alpha}{2}}w\|^{1+\eta_3'}_{L^2(\mathbb{T}^2)}\|w\|^{1-\eta_3'}_{L^2(\mathbb{T}^2)}\|\theta\|_{L^{p_2}(\mathbb{T}^2)} \\
&\le \frac{\kappa}{8}\|w\|^2_{H^{s-\widetilde{\eta}+2\alpha}(\mathbb{T}^2)} + C\|w\|^2_{L^2(\mathbb{T}^2)}\|\theta\|^{\frac{2}{1-\eta_4'}}_{H^{s+2\alpha}(\mathbb{T}^2)} + C\|w\|^2_{L^2(\mathbb{T}^2)}\|\theta\|^{\frac{2}{1-\eta_3'}}_{H^{s+2\alpha}(\mathbb{T}^2)}.
\end{aligned} \tag{6.10}
$$

Here $p_1=\frac{1}{1-\alpha^-}$, $p_2=\frac{2}{2\alpha^- -1}$, and we have used the Sobolev embeddings $H^{s+2\alpha}(\mathbb{T}^2)\subset H^{s-\widetilde{\eta}+1,p_1}(\mathbb{T}^2)$, $H^{s+2\alpha}(\mathbb{T}^2)\subset L^{p_2}(\mathbb{T}^2)$, and the following Gagliardo-Nirenberg inequalities:

$$
\|(-\Delta)^{\frac{s-\widetilde{\eta}+1}{2}}w\|_{L^{p_1}(\mathbb{T}^2)}\le C\|(-\Delta)^{\frac{s-\widetilde{\eta}+2\alpha}{2}}w\|^{\eta_3'}_{L^2(\mathbb{T}^2)}\|w\|^{1-\eta_3'}_{L^2(\mathbb{T}^2)},
$$

$$
\|w\|_{L^{p_2}(\mathbb{T}^2)}\le C\|(-\Delta)^{\frac{s-\widetilde{\eta}+2\alpha}{2}}w\|^{\eta_4'}_{L^2(\mathbb{T}^2)}\|w\|^{1-\eta_4'}_{L^2(\mathbb{T}^2)},
$$

where $\eta_3'\in\left[\frac{s-\widetilde{\eta}+1}{s-\widetilde{\eta}+2\alpha},1\right)$ and $\eta_4'\in(0,1)$.

Inserting (6.8)-(6.10) into (6.7) gives

$$
\begin{aligned}
&\frac{d}{dt}\|w\|^2_{H^{\alpha+s-\widetilde{\eta}}(\mathbb{T}^2)}+\kappa\|w\|^2_{H^{s-\widetilde{\eta}+2\alpha}(\mathbb{T}^2)} \\
&\le C\left(1+\|\theta\|^2_{H^{s+\alpha}(\mathbb{T}^2)}\right)\|g_1\|^2_{H^s(\mathbb{T}^2)}\|w\|^2_{H^s(\mathbb{T}^2)}+C\|w\|^2_{H^s(\mathbb{T}^2)}\|\theta\|^2_{H^{s+2\alpha}(\mathbb{T}^2)} \\
&\quad + C\|w\|^2_{L^2(\mathbb{T}^2)}\|\theta\|^{\frac{2}{1-\eta_4'}}_{H^{s+2\alpha}(\mathbb{T}^2)}+C\|w\|^2_{L^2(\mathbb{T}^2)}\|\theta\|^{\frac{2}{1-\eta_3'}}_{H^{s+2\alpha}(\mathbb{T}^2)}.
\end{aligned} \tag{6.11}
$$

Recall that $\theta_0\in\mathfrak{X}$ and $\mathfrak{X}$ is a closed bounded set in $H^{2\alpha+s}(\mathbb{T}^2)$ having the positively invariant property. In view of Lemma 14, we deduce that there exists $T_0^{**}=T_0^{**}(\mathfrak{X})>0$ such that any solution $\theta(t,\theta_0)$ of problem (1.1) with $\theta_0\in\mathfrak{X}$ satisfies

$$
\|\partial_t\theta(t)\|^2_{H^s(\mathbb{T}^2)}\le C,\quad \forall t\ge T_0^{**},
$$

$$
\|\partial_t\theta(t)\|^2_{L^2(\mathbb{T}^2)}\le C,\quad \forall t\ge T_0^{**},
$$

and by (4.46), we have that for any t_1, $t_2\ge T_0^{**}$,

$$
\int_{t_1}^{t_2}\|\partial_t\theta\|^2_{H^{s+\alpha}(\mathbb{T}^2)}d\tau\le C|t_2-t_1|+C.
$$

Using $g_1 \in H^s(\mathbb{T}^2)$ and the Sobolev embedding $H^{s+2\alpha}(\mathbb{T}^2) \subset H^{s+\alpha}(\mathbb{T}^2)$, it follows from (6.11) that

$$\frac{d}{dt}\|w\|^2_{H^{\alpha+s-\widetilde{\eta}}(\mathbb{T}^2)} + \kappa\|w\|^2_{H^{s-\widetilde{\eta}+2\alpha}(\mathbb{T}^2)} \le C, \quad \forall t \ge T_0^{**}. \tag{6.12}$$

Applying the uniform Gronwall lemma to (6.12) results in

$$\|\partial_t\theta(t)\|^2_{H^{\alpha+s-\widetilde{\eta}}(\mathbb{T}^2)} \le C, \quad \forall t \ge T_0^{**} + 1. \tag{6.13}$$

Integrating the differential inequality (6.12), then the assertion of the lemma follows immediately from (6.13). □

The Lipschitz continuity of the solutions with respect to the initial data is presented in the following lemma.

Lemma 23. *Assume that the conditions of Lemma 14 hold. Let θ and ξ be the solutions of problem (1.1) with the initial data θ_0 and $\xi_0 \in \mathfrak{X}$, respectively. Then, for any fixed $\widetilde{\eta}$ with $0 < \widetilde{\eta} < 2\alpha - 2\alpha^-$,*

$$\begin{aligned}
\|\theta(t)-\xi(t)\|_{H^{2\alpha^-+s}(\mathbb{T}^2)} &\le C\|\theta_0-\xi_0\|_{H^{2\alpha^-+s}(\mathbb{T}^2)} e^{\frac{C}{\nu}t} e^{-\frac{\kappa\lambda_1^{2\alpha}}{4}t},\\
\|\theta(t)-\xi(t)\|_{H^{2\alpha+s-\widetilde{\eta}}(\mathbb{T}^2)} &\le Ce^{-\frac{\kappa\lambda_1^{2\alpha}}{2}t} t^{-\frac{\alpha-\alpha^- - \frac{\widetilde{\eta}}{2}}{\alpha}} \|\theta_0-\xi_0\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}\\
&\quad + Ce^{-\frac{\kappa\lambda_1^{2\alpha}}{4}t} e^{\frac{C}{\nu}t}\|\theta_0-\xi_0\|_{H^{2\alpha^-+s}(\mathbb{T}^2)},
\end{aligned} \tag{6.14}$$

where ν is a constant with $\nu > \frac{\alpha}{\alpha-\alpha^-}$.

Proof. Thanks to (3.2), we have

$$\begin{aligned}
\theta(t)-\xi(t) = e^{-A_{\alpha,\kappa}t}(\theta_0-\xi_0) + \int_0^t e^{-A_{\alpha,\kappa}(t-r)}\Big(&(F(x,\theta(r)) - F(x,\xi(r)))\\
&- (u(r)\cdot\nabla\theta(r) - v(r)\cdot\nabla\xi(r))\Big)dr,
\end{aligned} \tag{6.15}$$

where $u = (-R_2\theta, R_1\theta)$ and $v = (-R_2\xi, R_1\xi)$. Denote

$$A_{\alpha^-+\frac{s}{2},\kappa} = \kappa(-\Delta)^{\alpha^-+\frac{s}{2}}, \quad A_{\frac{s}{2},\kappa} = \kappa(-\Delta)^{\frac{s}{2}}.$$

By Proposition 3 and (6.15), we deduce that

$$\begin{aligned}
&\left\|A_{\alpha^-+\frac{s}{2},\kappa}(\theta(t)-\xi(t))\right\|_{L^2(\mathbb{T}^2)}\\
&\le \left\|A_{\alpha^-+\frac{s}{2},\kappa}e^{-A_{\alpha,\kappa}t}(\theta_0-\xi_0))\right\|_{L^2(\mathbb{T}^2)}\\
&\quad + \int_0^t \left\|A_{\alpha^-+\frac{s}{2},\kappa}e^{-A_{\alpha,\kappa}(t-r)}(g_1(x)(f(\theta(r)) - f(\xi(r))))\right\|_{L^2(\mathbb{T}^2)} dr\\
&\quad + \int_0^t \left\|A_{\alpha^-+\frac{s}{2},\kappa}e^{-A_{\alpha,\kappa}(t-r)}(u(r)\cdot\nabla\theta(r) - v(r)\cdot\nabla\xi(r))\right\|_{L^2(\mathbb{T}^2)} dr\\
&\le Ce^{-\frac{\kappa\lambda_1^{2\alpha}}{2}t}\left\|A_{\alpha^-+\frac{s}{2},\kappa}(\theta_0-\xi_0)\right\|_{L^2(\mathbb{T}^2)}\\
&\quad + C\int_0^t e^{-\frac{\kappa\lambda_1^{2\alpha}}{2}(t-r)}(t-r)^{-\frac{\alpha^-}{\alpha}}\left\|A_{\frac{s}{2},\kappa}(g_1(x)(f(\theta(r)) - f(\xi(r))))\right\|_{L^2(\mathbb{T}^2)} dr\\
&\quad + C\int_0^t e^{-\frac{\kappa\lambda_1^{2\alpha}}{2}(t-r)}(t-r)^{-\frac{\alpha^-}{\alpha}}\left\|A_{\frac{s}{2},\kappa}(u(r)\cdot\nabla\theta(r) - v(r)\cdot\nabla\xi(r))\right\|_{L^2(\mathbb{T}^2)} dr.
\end{aligned} \tag{6.16}$$

Since $g_1 \in H^s(\mathbb{T}^2)$ and $f : H^{2\alpha^-+s}(\mathbb{T}^2) \to H^s(\mathbb{T}^2)$ is Lipschitz continuous on bounded subsets of $H^{2\alpha^-+s}(\mathbb{T}^2)$, by using Lemma 1 and the Sobolev embedding $H^s(\mathbb{T}^2) \subset L^\infty(\mathbb{T}^2)$ for $s > 1$, we have

$$\begin{aligned}
&C\int_0^t e^{-\frac{\kappa\lambda_1^{2\alpha}}{2}(t-r)}(t-r)^{-\frac{\alpha^-}{\alpha}}\left\|A_{\frac{s}{2},\kappa}\left(g_1(x)(f(\theta(r))-f(\xi(r)))\right)\right\|_{L^2(\mathbb{T}^2)}dr\\
&\leq C\int_0^t e^{-\frac{\kappa\lambda_1^{2\alpha}}{2}(t-r)}(t-r)^{-\frac{\alpha^-}{\alpha}}\Big(\|g_1\|_{L^\infty(\mathbb{T}^2)}\|(-\Delta)^{\frac{s}{2}}(f(\theta(r))-f(\xi(r)))\|_{L^2(\mathbb{T}^2)}\\
&\quad+\|(-\Delta)^{\frac{s}{2}}g_1\|_{L^2(\mathbb{T}^2)}\|(f(\theta(r))-f(\xi(r)))\|_{L^\infty(\mathbb{T}^2)}\Big)dr\\
&\leq C\|g_1\|_{H^s(\mathbb{T}^2)}\int_0^t e^{-\frac{\kappa\lambda_1^{2\alpha}}{2}(t-r)}(t-r)^{-\frac{\alpha^-}{\alpha}}\|(f(\theta(r))-f(\xi(r)))\|_{H^s(\mathbb{T}^2)}dr\\
&\leq C\int_0^t e^{-\frac{\kappa\lambda_1^{2\alpha}}{2}(t-r)}(t-r)^{-\frac{\alpha^-}{\alpha}}\|\theta(r)-\xi(r)\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}dr.
\end{aligned}\tag{6.17}$$

In view of (3.1) and $\theta(r)$, $\xi(r) \in \mathfrak{X}$ for all $r \geq 0$, we find that

$$\begin{aligned}
&C\int_0^t e^{-\frac{\kappa\lambda_1^{2\alpha}}{2}(t-r)}(t-r)^{-\frac{\alpha^-}{\alpha}}\left\|A_{\frac{s}{2},\kappa}(u(r)\cdot\nabla\theta(r)-v(r)\cdot\nabla\xi(r))\right\|_{L^2(\mathbb{T}^2)}dr\\
&\leq C\int_0^t e^{-\frac{\kappa\lambda_1^{2\alpha}}{2}(t-r)}(t-r)^{-\frac{\alpha^-}{\alpha}}\Big(\|\theta(r)\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}\\
&\quad+\|\xi(r)\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}\Big)\|\theta(r)-\xi(r)\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}dr\\
&\leq C\int_0^t e^{-\frac{\kappa\lambda_1^{2\alpha}}{2}(t-r)}(t-r)^{-\frac{\alpha^-}{\alpha}}\|\theta(r)-\xi(r)\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}dr.
\end{aligned}\tag{6.18}$$

Inserting (6.17)-(6.18) into (6.16) result in

$$\begin{aligned}
\|\theta(t)-\xi(t)\|_{H^{2\alpha^-+s}(\mathbb{T}^2)} &\leq Ce^{-\frac{\kappa\lambda_1^{2\alpha}}{2}t}\|\theta_0-\xi_0\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}\\
&\quad+C\int_0^t e^{-\frac{\kappa\lambda_1^{2\alpha}}{2}(t-r)}(t-r)^{-\frac{\alpha^-}{\alpha}}\|\theta(r)-\xi(r)\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}dr\\
&\leq Ce^{-\frac{\kappa\lambda_1^{2\alpha}}{2}t}\|\theta_0-\xi_0\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}\\
&\quad+C\left(\int_0^t e^{-\frac{\kappa\lambda_1^{2\alpha}\nu}{4(\nu-1)}(t-r)}(t-r)^{-\frac{\alpha^-\nu}{\alpha(\nu-1)}}dr\right)^{\frac{\nu-1}{\nu}}\\
&\quad\times\left(\int_0^t e^{-\frac{\kappa\lambda_1^{2\alpha}\nu}{4}(t-r)}\|\theta(r)-\xi(r)\|^\nu_{H^{2\alpha^-+s}(\mathbb{T}^2)}dr\right)^{\frac{1}{\nu}}\\
&\leq Ce^{-\frac{\kappa\lambda_1^{2\alpha}}{2}t}\|\theta_0-\xi_0\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}\\
&\quad+C\left(\int_0^t e^{-\frac{\kappa\lambda_1^{2\alpha}\nu}{4}(t-r)}\|\theta(r)-\xi(r)\|^\nu_{H^{2\alpha^-+s}(\mathbb{T}^2)}dr\right)^{\frac{1}{\nu}},
\end{aligned}\tag{6.19}$$

where $\nu > \frac{\alpha}{\alpha-\alpha^-}$ and we have used the property of Gamma function in the last inequality. Therefore,

$$\begin{aligned}
&e^{\frac{\kappa\lambda_1^{2\alpha}\nu}{4}t}\|\theta(t)-\xi(t)\|^{\nu}_{H^{2\alpha^-+s}(\mathbb{T}^2)}\\
&\quad\le C\|\theta_0-\xi_0\|^{\nu}_{H^{2\alpha^-+s}(\mathbb{T}^2)}+C\int_0^t e^{\frac{\kappa\lambda_1^{2\alpha}\nu}{4}r}\|\theta(r)-\xi(r)\|^{\nu}_{H^{2\alpha^-+s}(\mathbb{T}^2)}dr.
\end{aligned}$$

Applying Gronwall's lemma,

$$\|\theta(t)-\xi(t)\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}\le C\|\theta_0-\xi_0\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}e^{\frac{C}{\nu}t}e^{-\frac{\kappa\lambda_1^{2\alpha}}{4}t}. \tag{6.20}$$

For any fixed $\widetilde{\eta}$ with $0<\widetilde{\eta}<2\alpha-2\alpha^-$, denote $A_{\alpha+\frac{s-\widetilde{\eta}}{2},\kappa}=\kappa(-\Delta)^{\alpha+\frac{s-\widetilde{\eta}}{2}}$, by similar arguments as in (6.16)-(6.18), we have

$$\begin{aligned}
&t^{\frac{\alpha-\alpha^--\frac{\widetilde{\eta}}{2}}{\alpha}}\left\|A_{\alpha+\frac{s-\widetilde{\eta}}{2},\kappa}(\theta(t)-\xi(t))\right\|_{L^2(\mathbb{T}^2)}\\
&\le Ce^{-\frac{\kappa\lambda_1^{2\alpha}}{2}t}\left\|A_{\alpha^-+\frac{s}{2},\kappa}(\theta_0-\xi_0))\right\|_{L^2(\mathbb{T}^2)}\\
&\quad+Ct^{\frac{\alpha-\alpha^--\frac{\widetilde{\eta}}{2}}{\alpha}}\int_0^t e^{-\frac{\kappa\lambda_1^{2\alpha}}{2}(t-r)}(t-r)^{-\frac{\alpha-\frac{\widetilde{\eta}}{2}}{\alpha}}\\
&\quad\times\left\|A_{\frac{s}{2},\kappa}(g_1(x)(f(\theta(r))-f(\xi(r))))\right\|_{L^2(\mathbb{T}^2)}dr\\
&\quad+Ct^{\frac{\alpha-\alpha^--\frac{\widetilde{\eta}}{2}}{\alpha}}\int_0^t e^{-\frac{\kappa\lambda_1^{2\alpha}}{2}(t-r)}(t-r)^{-\frac{\alpha-\frac{\widetilde{\eta}}{2}}{\alpha}}\\
&\quad\times\left\|A_{\frac{s}{2},\kappa}(u(r)\cdot\nabla\theta(r)-v(r)\cdot\nabla\xi(r))\right\|_{L^2(\mathbb{T}^2)}dr\\
&\le Ce^{-\frac{\kappa\lambda_1^{2\alpha}}{2}t}\|\theta_0-\xi_0\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}\\
&\quad+Ct^{\frac{\alpha-\alpha^--\frac{\widetilde{\eta}}{2}}{\alpha}}\int_0^t e^{-\frac{\kappa\lambda_1^{2\alpha}}{2}(t-r)}(t-r)^{-\frac{\alpha-\frac{\widetilde{\eta}}{2}}{\alpha}}\|\theta(r)-\xi(r))\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}dr.
\end{aligned}\tag{6.21}$$

Inserting (6.20) into (6.21) gives

$$\begin{aligned}
&t^{\frac{\alpha-\alpha^--\frac{\widetilde{\eta}}{2}}{\alpha}}\|\theta(t)-\xi(t)\|_{H^{2\alpha+s-\widetilde{\eta}}(\mathbb{T}^2)}\\
&\le Ce^{-\frac{\kappa\lambda_1^{2\alpha}}{2}t}\|\theta_0-\xi_0\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}+Ct^{\frac{\alpha-\alpha^--\frac{\widetilde{\eta}}{2}}{\alpha}}e^{-\frac{\kappa\lambda_1^{2\alpha}}{4}t}e^{\frac{C}{\nu}t}\|\theta_0-\xi_0\|_{H^{2\alpha^-+s}(\mathbb{T}^2)},
\end{aligned}\tag{6.22}$$

where the property of Gamma function is also used. Thus the assertion of the lemma follows immediately from (6.20) and (6.22). □

Now we are ready to state and prove the main results of this section.

Theorem 24. *Assume that the conditions of Lemma 14 hold. Then the semigroup $\{S(t)\}_{t\ge 0}$ associated with problem (1.1) possesses an exponential attractor $\mathcal{E}$ in $H^{2\alpha^-+s}(\mathbb{T}^2)$, which is bounded in $H^{2\alpha+s}(\mathbb{T}^2)$.*

Proof. Recall that $\mathfrak{X}$ given in (6.5) is a closed bounded positively invariant set in $H^{2\alpha+s}(\mathbb{T}^2)$, and for any bounded set $B\subset H^{2\alpha^-+s}(\mathbb{T}^2)$, there exists $t_B>0$ such that

$$S(t)B\subset\mathfrak{X},\quad \forall t\ge t_B. \tag{6.23}$$

In order to apply Theorem 21, now it only remains to verify the conditions (iii) and (iv) in Theorem 21. For any θ_0, $\xi_0 \in \mathfrak{X}$, let $\theta(t) = S(t)\theta_0$ and $\xi(t) = S(t)\xi_0$. Notice that

$$\frac{\partial \theta}{\partial t} - \frac{\partial \xi}{\partial t} + u \cdot \nabla\theta - v \cdot \nabla\xi + \kappa(-\Delta)^{\alpha}(\theta - \xi) = F(x,\theta) - F(x,\xi), \tag{6.24}$$

where $u = (-R_2\theta, R_1\theta)$ and $v = (-R_2\xi, R_1\xi)$. Taking the inner product of (6.24) with $(-\Delta)^{\alpha+s}(\theta-\xi)$, we find that for any fixed $\widetilde{\eta}$ with $0 < \widetilde{\eta} < \min\left\{2\alpha - 2\alpha^-, \frac{\alpha}{2}\right\}$,

$$\begin{aligned}
&\kappa\|\theta - \xi\|^2_{H^{2\alpha+s}(\mathbb{T}^2)} \\
&\quad= -\left\langle (-\Delta)^{\frac{s+\widetilde{\eta}}{2}}\left(\frac{\partial \theta}{\partial t} - \frac{\partial \xi}{\partial t}\right), (-\Delta)^{\frac{2\alpha+s-\widetilde{\eta}}{2}}(\theta - \xi)\right\rangle \\
&\qquad - \left\langle (-\Delta)^{\frac{s+\widetilde{\eta}}{2}}\left(u \cdot \nabla\theta - v \cdot \nabla\xi\right), (-\Delta)^{\frac{2\alpha+s-\widetilde{\eta}}{2}}(\theta - \xi)\right\rangle \\
&\qquad + \left\langle (-\Delta)^{\frac{s}{2}}\left(F(x,\theta) - F(x,\xi)\right), (-\Delta)^{\frac{2\alpha+s}{2}}(\theta - \xi)\right\rangle \\
&\quad\le \|\theta - \xi\|_{H^{2\alpha+s-\widetilde{\eta}}(\mathbb{T}^2)}\left(\left\|\frac{\partial \theta}{\partial t} - \frac{\partial \xi}{\partial t}\right\|_{H^{s+\widetilde{\eta}}(\mathbb{T}^2)} + \|u \cdot \nabla\theta - v \cdot \nabla\xi\|_{H^{s+\widetilde{\eta}}(\mathbb{T}^2)}\right) \\
&\qquad + \|\theta - \xi\|_{H^{2\alpha+s}(\mathbb{T}^2)}\|F(x,\theta) - F(x,\xi)\|_{H^{s}(\mathbb{T}^2)},
\end{aligned} \tag{6.25}$$

thanks to Hölder's inequality. Arguing as in (3.1), in view of the Sobolev embeddings $H^{2\alpha+s}(\mathbb{T}^2) \subset H^{1+s+\widetilde{\eta}}(\mathbb{T}^2)$ and $H^{2\alpha+s}(\mathbb{T}^2) \subset H^{s+\widetilde{\eta}}(\mathbb{T}^2)$ and $\theta(t)$, $\xi(t) \in \mathfrak{X}$ for all $t \ge 0$, we have

$$\begin{aligned}
&\|u \cdot \nabla\theta - v \cdot \nabla\xi\|_{H^{s+\widetilde{\eta}}(\mathbb{T}^2)} \\
&\le C\|\theta - \xi\|_{H^{s+\widetilde{\eta}}(\mathbb{T}^2)}\|\theta\|_{H^{1+s+\widetilde{\eta}}(\mathbb{T}^2)} + C\|\xi\|_{H^{s+\widetilde{\eta}}(\mathbb{T}^2)}\|\theta - \xi\|_{H^{1+s+\widetilde{\eta}}(\mathbb{T}^2)} \\
&\le C\|\theta - \xi\|_{H^{2\alpha+s}(\mathbb{T}^2)}\left(\|\theta\|_{H^{2\alpha+s}(\mathbb{T}^2)} + \|\xi\|_{H^{2\alpha+s}(\mathbb{T}^2)}\right) \\
&\le C\left(\|\theta\|_{H^{2\alpha+s}(\mathbb{T}^2)} + \|\xi\|_{H^{2\alpha+s}(\mathbb{T}^2)}\right) \le C.
\end{aligned} \tag{6.26}$$

Since $f : H^{2\alpha^-+s}(\mathbb{T}^2) \to H^s(\mathbb{T}^2)$ is Lipschitz continuous on bounded subsets of $H^{2\alpha^-+s}(\mathbb{T}^2)$, by using Lemma 1, (1.2), $g_1 \in H^s(\mathbb{T}^2)$ and the Sobolev embedding $H^s(\mathbb{T}^2) \subset L^\infty(\mathbb{T}^2)$ for $s > 1$, we obtain that

$$\begin{aligned}
&\|F(x,\theta) - F(x,\xi)\|_{H^s(\mathbb{T}^2)} \\
&= \|g_1(x)(f(\theta) - f(\xi))\|_{H^s(\mathbb{T}^2)} \\
&\le C\left(\|g_1\|_{L^\infty(\mathbb{T}^2)}\|f(\theta) - f(\xi)\|_{H^s(\mathbb{T}^2)} + \|g_1\|_{H^s(\mathbb{T}^2)}\|f(\theta) - f(\xi)\|_{L^\infty(\mathbb{T}^2)}\right) \\
&\le C\|g_1\|_{H^s(\mathbb{T}^2)}\|f(\theta) - f(\xi)\|_{H^s(\mathbb{T}^2)} \\
&\le C\|\theta - \xi\|_{H^{2\alpha^-+s}(\mathbb{T}^2)}.
\end{aligned} \tag{6.27}$$

Inserting (6.26) and (6.27) into (6.25), in view of (6.13), the Sobolev embedding $H^{\alpha+s-\widetilde{\eta}}(\mathbb{T}^2) \subset H^{s+\widetilde{\eta}}(\mathbb{T}^2)$ and $\theta(t)$, $\xi(t) \in \mathfrak{X}$ for all $t \ge 0$, we conclude that there exists $t_0^* > 0$ such that for any $t \ge t_0^*$,

$$\begin{aligned}
&\|\theta(t) - \xi(t)\|^2_{H^{2\alpha+s}(\mathbb{T}^2)} \\
&\le C\|\theta(t) - \xi(t)\|_{H^{2\alpha+s-\widetilde{\eta}}(\mathbb{T}^2)} + C\|\theta(t) - \xi(t)\|_{H^{2\alpha^-+s}(\mathbb{T}^2)} \\
&\le Ce^{-\frac{\kappa\lambda_1^{2\alpha}}{2}t}t^{-\frac{\alpha-\alpha^- - \frac{\widetilde{\eta}}{2}}{\alpha}}\|\theta_0 - \xi_0\|_{H^{2\alpha^-+s}(\mathbb{T}^2)} + Ce^{-\frac{\kappa\lambda_1^{2\alpha}}{4}t}e^{\frac{C}{\nu}t}\|\theta_0 - \xi_0\|_{H^{2\alpha^-+s}(\mathbb{T}^2)},
\end{aligned} \tag{6.28}$$

thanks to Lemma 23. Hence, it follows from Lemma 22 that there exists $t^* \geq t_0^*$ such that for all $t_1,\, t_2 \in [t^*, 2t^*]$ with $t_1 \leq t_2$,

$$
\begin{aligned}
&\|S(t_1)\theta_0 - S(t_2)\xi_0\|_{H^{2\alpha^-+s}(\mathbb{T}^2)} \\
&\leq \|S(t_1)\theta_0 - S(t_2)\theta_0\|_{H^{2\alpha^-+s}(\mathbb{T}^2)} + \|S(t_2)\theta_0 - S(t_2)\xi_0\|_{H^{2\alpha^-+s}(\mathbb{T}^2)} \\
&\leq \int_{t_1}^{t_2} \|\partial_t\theta\|_{H^{2\alpha^-+s}(\mathbb{T}^2)} d\tau + \|\theta(t_2) - \xi(t_2)\|_{H^{2\alpha^-+s}(\mathbb{T}^2)} \\
&\leq |t_1 - t_2|^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \|\partial_t\theta\|^2_{H^{2\alpha^-+s}(\mathbb{T}^2)} d\tau \right)^{\frac{1}{2}} + C\|\theta(t_2) - \xi(t_2)\|_{H^{2\alpha+s}(\mathbb{T}^2)} \\
&\leq C|t_1 - t_2|^{\frac{1}{2}} + C\|\theta_0 - \xi_0\|^{\frac{1}{2}}_{H^{2\alpha^-+s}(\mathbb{T}^2)},
\end{aligned}
\tag{6.29}
$$

where we have used Hölder's inequality and (6.28). This implies that the condition (iii) in Theorem 21 holds true. Finally, by (6.28) we obtain that there exists $N \geq 1$ such that

$$
\begin{aligned}
&\|(I - P_N)(S(t^*)\theta_0 - S(t^*)\xi_0)\|_{H^{2\alpha^-+s}(\mathbb{T}^2)} \\
&\leq \frac{1}{\lambda_{N+1}^{2(\alpha-\alpha^-)}} \|(I - P_N)(S(t^*)\theta_0 - S(t^*)\xi_0)\|_{H^{2\alpha+s}(\mathbb{T}^2)} \\
&\leq \frac{1}{\lambda_{N+1}^{2(\alpha-\alpha^-)}} \|\theta(t^*) - \xi(t^*)\|_{H^{2\alpha+s}(\mathbb{T}^2)} \\
&\leq \frac{C}{\lambda_{N+1}^{2(\alpha-\alpha^-)}} \|\theta_0 - \xi_0\|^{\frac{1}{2}}_{H^{2\alpha^-+s}(\mathbb{T}^2)} < \frac{1}{4}\|\theta_0 - \xi_0\|^{\frac{1}{2}}_{H^{2\alpha^-+s}(\mathbb{T}^2)},
\end{aligned}
\tag{6.30}
$$

where $P_N : L^2(\mathbb{T}^2) \to H_N$ is the projection operator and H_N is the space spanned by $\{e_j\}_{j=1}^N$. Therefore, the condition (iv) in Theorem 21 holds true and consequently the assertion of this theorem follows immediately from Theorem 21. □

Theorem 25. *Let $\alpha \in \left[\frac{1}{2}, 1\right)$, $\kappa > 0$ and $g_2 \in H^s(\mathbb{T}^2)$ with $s > 1$. Then the semigroup $\{S(t)\}_{t\geq 0}$ associated with problem (1.1) with $g_2(x)$ instead of $F(x,\theta)$ has an exponential attractor $\mathcal{E}$, whose compactness, boundedness of the fractional dimension and exponential attractiveness for the bounded subset B of $H^{2\alpha^-+s}(\mathbb{T}^2)$ are all in the topology of $H^{2\alpha+s}(\mathbb{T}^2)$.*

Proof. By similar arguments as in (6.25), in view of (6.26), (6.13), the Sobolev embedding $H^{\alpha+s-\widetilde{\eta}}(\mathbb{T}^2) \subset H^{s+\widetilde{\eta}}(\mathbb{T}^2)$ and $\theta(t) \in \mathfrak{X}$ for all $t \geq 0$, we deduce that for any fixed $\widetilde{\eta}$ with $0 < \widetilde{\eta} < \min\left\{2\alpha - 2\alpha^-, \frac{\alpha}{2}\right\}$, there exists $t_1^* > t_0^*$ such that for any $t_1,\, t_2 \geq t_1^*$ with $t_1 \leq t_2$,

$$
\begin{aligned}
&\kappa\|\theta(t_1) - \theta(t_2)\|^2_{H^{2\alpha+s}(\mathbb{T}^2)} \\
&= -\left\langle (-\Delta)^{\frac{s+\widetilde{\eta}}{2}} \left(\frac{\partial\theta(t_1)}{\partial t_1} - \frac{\partial\theta(t_2)}{\partial t_2} \right), (-\Delta)^{\frac{2\alpha+s-\widetilde{\eta}}{2}} (\theta(t_1) - \theta(t_2)) \right\rangle \\
&\quad - \left\langle (-\Delta)^{\frac{s+\widetilde{\eta}}{2}} (u(t_1)\cdot\nabla\theta(t_1) - u(t_2)\cdot\nabla\theta(t_2)), (-\Delta)^{\frac{2\alpha+s-\widetilde{\eta}}{2}} (\theta(t_1) - \theta(t_2)) \right\rangle \\
&\leq C\|\theta(t_1) - \theta(t_2)\|_{H^{2\alpha+s-\widetilde{\eta}}(\mathbb{T}^2)}.
\end{aligned}
\tag{6.31}
$$

Combining (6.6), (6.28) and (6.31) together, we find that there exists $t_2^* > t_1^*$ such that for any $t_1,\, t_2 \in [t_2^*, 2t_2^*]$ with $t_1 \le t_2$,

$$\begin{aligned}
&\|S(t_1)\theta_0 - S(t_2)\xi_0\|_{H^{2\alpha+s}(\mathbb{T}^2)}\\
&\le \|S(t_1)\theta_0 - S(t_2)\theta_0\|_{H^{2\alpha+s}(\mathbb{T}^2)} + \|S(t_2)\theta_0 - S(t_2)\xi_0\|_{H^{2\alpha+s}(\mathbb{T}^2)}\\
&\le C\|S(t_1)\theta_0 - S(t_2)\theta_0\|^{\frac{1}{2}}_{H^{2\alpha+s-\widetilde{\eta}}(\mathbb{T}^2)} + C\|\theta_0-\xi_0\|^{\frac{1}{2}}_{H^{2\alpha^-+s}(\mathbb{T}^2)}\\
&\le C\left(\int_{t_1}^{t_2}\|\partial_t\theta\|_{H^{2\alpha+s-\widetilde{\eta}}(\mathbb{T}^2)}d\tau\right)^{\frac{1}{2}} + C\|\theta_0-\xi_0\|^{\frac{1}{2}}_{H^{2\alpha^-+s}(\mathbb{T}^2)}\\
&\le C|t_1-t_2|^{\frac{1}{4}}\left(\int_{t_1}^{t_2}\|\partial_t\theta\|^2_{H^{2\alpha+s-\widetilde{\eta}}(\mathbb{T}^2)}d\tau\right)^{\frac{1}{4}} + C\|\theta_0-\xi_0\|^{\frac{1}{2}}_{H^{2\alpha^-+s}(\mathbb{T}^2)}\\
&\le C|t_1-t_2|^{\frac{1}{4}} + C\|\theta_0-\xi_0\|^{\frac{1}{2}}_{H^{2\alpha+s}(\mathbb{T}^2)}.
\end{aligned}\tag{6.32}$$

On the other hand, multiplying (6.24) by $(\Delta)^{\alpha+s}(I-P_{N'})(\theta-\xi)$ and then integrating over $\mathbb{T}^2$, in view of (6.26), (6.13), the Sobolev embedding $H^{\alpha+s-\widetilde{\eta}}(\mathbb{T}^2) \subset H^{s+\widetilde{\eta}}(\mathbb{T}^2)$ and $\theta(t),\, \xi(t) \in \mathfrak{X}$ for all $t \ge 0$, we obtain that

$$\begin{aligned}
&\kappa\|(I-P_{N'})\left(\theta(t_2^*)-\xi(t_2^*)\right)\|^2_{H^{2\alpha+s}(\mathbb{T}^2)}\\
&= -\left\langle(-\Delta)^{\frac{s+\widetilde{\eta}}{2}}\left(\frac{\partial\theta(t_2^*)}{\partial t_2^*} - \frac{\partial\xi(t_2^*)}{\partial t_2^*}\right), (-\Delta)^{\frac{2\alpha+s-\widetilde{\eta}}{2}}(I-P_{N'})\left(\theta(t_2^*)-\xi(t_2^*)\right)\right\rangle\\
&\quad - \Big\langle(-\Delta)^{\frac{s+\widetilde{\eta}}{2}}\left(u(t_2^*)\cdot\nabla\theta(t_2^*) - v(t_2^*)\cdot\nabla\xi(t_2^*)\right),\\
&\qquad (-\Delta)^{\frac{2\alpha+s-\widetilde{\eta}}{2}}(I-P_{N'})\left(\theta(t_2^*)-\xi(t_2^*)\right)\Big\rangle\\
&\le C\|(I-P_{N'})\left(\theta(t_2^*)-\xi(t_2^*)\right)\|_{H^{2\alpha+s-\widetilde{\eta}}(\mathbb{T}^2)},
\end{aligned}\tag{6.33}$$

where $0 < \widetilde{\eta} < \min\left\{2\alpha - 2\alpha^-, \frac{\alpha}{2}\right\}$ is given in (6.31), $P_{N'} : L^2(\mathbb{T}^2) \to H_{N'}$ is the projection operator and $H_{N'}$ is the space spanned by $\{e_j\}_{j=1}^{N'}$. Hence (6.28) and (6.33) ensure that for N' sufficiently large,

$$\begin{aligned}
&\|(I-P_{N'})(S(t_2^*)\theta_0 - S(t_2^*)\xi_0)\|_{H^{2\alpha+s}(\mathbb{T}^2)}\\
&\le C\|(I-P_{N'})(S(t_2^*)\theta_0 - S(t_2^*)\xi_0)\|_{H^{2\alpha+s-\widetilde{\eta}}(\mathbb{T}^2)}\\
&\le \frac{C}{\lambda_{N'+1}^{\widetilde{\eta}}}\|(I-P_{N'})(S(t_2^*)\theta_0 - S(t_2^*)\xi_0)\|_{H^{2\alpha+s}(\mathbb{T}^2)}\\
&\le \frac{C}{\lambda_{N'+1}^{\widetilde{\eta}}}\|\theta(t_2^*)-\xi(t_2^*)\|_{H^{2\alpha+s}(\mathbb{T}^2)}\\
&\le \frac{C}{\lambda_{N'+1}^{\widetilde{\eta}}}\|\theta_0-\xi_0\|^{\frac{1}{2}}_{H^{2\alpha^-+s}(\mathbb{T}^2)} < \frac{1}{4}\|\theta_0-\xi_0\|^{\frac{1}{2}}_{H^{2\alpha+s}(\mathbb{T}^2)}.
\end{aligned}\tag{6.34}$$

Since $\mathfrak{X}$ given in (6.5) is a closed bounded positively invariant set in $H^{2\alpha+s}(\mathbb{T}^2)$, and (6.23) holds true for $\mathfrak{X}$, by using (6.32) and (6.34), the assertion of this theorem follows immediately from Theorem 21. □

Remark 26. In fact, in addition to the hypotheses in Lemma 14, if we also assume that for any fixed $\widetilde{\eta}$ with $0 < \widetilde{\eta} < 2\alpha - 2\alpha^-$, $g_1 \in H^{s+\widetilde{\eta}}(\mathbb{T}^2)$ and $f : H^{2\alpha^-+s+\widetilde{\eta}}(\mathbb{T}^2) \to H^{s+\widetilde{\eta}}(\mathbb{T}^2)$ is Lipschitz continuous on bounded subsets of $H^{2\alpha^-+s+\widetilde{\eta}}(\mathbb{T}^2)$, then by similar arguments as in (6.27), we deduce that

$$\|F(x,\theta)-F(x,\xi)\|_{H^{s+\tilde{\eta}}(\mathbb{T}^2)} \leq C\|\theta-\xi\|_{H^{2\alpha^-+s+\tilde{\eta}}(\mathbb{T}^2)}. \tag{6.35}$$

Arguing as in the proof of Theorem 25, by (6.35) we obtain that the semigroup $\{S(t)\}_{t\geq 0}$ associated with problem (1.1) has an exponential attractor $\mathcal{E}$, whose compactness, boundedness of the fractional dimension and exponential attractiveness for the bounded subset B of $H^{2\alpha^-+s}(\mathbb{T}^2)$ are all in the topology of $H^{2\alpha+s}(\mathbb{T}^2)$.

7. **Summary.** In this work we studied the regularity of global attractors for the surface quasi-geostrophic equations with fractional dissipation in the subcritical case. We proved the existence of the global attractor that is compact in $H^{2\alpha+s}(\mathbb{T}^2)$ and attracts all bounded subsets of $H^{2\alpha^-+s}(\mathbb{T}^2)$ with respect to the norm of $H^{2\alpha+s}(\mathbb{T}^2)$. It is worth mentioning that, similarly, the results in this work can be extended to a bounded domain $\Omega\subset\mathbb{R}^2$ with smooth boundary. Furthermore, if we can show that the $H^{2\alpha+s}$-norm of solutions is arbitrary small uniformly on the exterior domains $\mathbb{R}^2\setminus\Omega_K$, where $\Omega_K=\{x\in\mathbb{R}^2: |x|\leq K\}$ for $K>0$, then we can also obtain the regularity of global attractors in the unbounded domain case. Here we mainly want to show how to study the regularity of global attractors for the surface quasi-geostrophic equations with fractional dissipation, the basic idea can be more easily obtained for readers by considering the periodic domain $\mathbb{T}^2$. When proving the asymptotic compactness in $H^{2\alpha+s}(\mathbb{T}^2)$ for problem (1.1), the dissipative term $(-\Delta)^\alpha$, $1/2<\alpha\leq 1$, and the nonlinear term $u\cdot\nabla\theta$ give much more trouble than for reaction-diffusion systems. In addition, the uniform estimates in $H^{2\alpha+s}(\mathbb{T}^2)$ cannot be obtained immediately, since problem (1.1) is treated in the base space $H^s(\mathbb{T}^2)$. For the external forcing term $F(x,\theta)$, it is necessary to use product estimates for $g_1(x)f(\theta)$ and composition estimates for $f(\theta)$. Another highlight of the work is that we present some sufficient conditions for the construction of exponential attractors for autonomous dynamical systems on Banach space, which can be used to establish the existence of exponential attractors of problem (1.1) in $H^{2\alpha^-+s}(\mathbb{T}^2)$ and furthermore the regularity of the exponential attractor $\mathcal{E}$ of problem (1.1) with $g_2(x)$ instead of $F(x,\theta)$ in $H^{2\alpha+s}(\mathbb{T}^2)$ for $s>1$ and $\alpha\in\left(\frac{1}{2},1\right]$. More precisely, $\mathcal{E}$ is compact in $H^{2\alpha+s}(\mathbb{T}^2)$, an upper bound of the fractal dimension of $\mathcal{E}$ is given in the topology of $H^{2\alpha+s}(\mathbb{T}^2)$, and $\mathcal{E}$ attracts exponentially all bounded subsets of $H^{2\alpha^-+s}(\mathbb{T}^2)$ with respect to the norm of $H^{2\alpha+s}(\mathbb{T}^2)$.

REFERENCES

[1] Adams, R.A., Sobolev Spaces, Pure and Applied Mathematics, vol. 65, Academic Press, New York-London, 1975.

[2] Babin, A.V., Vishik, M.I., Attractors of Evolution Equations, Studies in Mathematics and its Applications, vol. 25, North-Holland Publishing Co., Amsterdam, 1992.

[3] Berselli, L.C., Vanishing viscosity limit and long-time behavior for 2D quasi-geostrophic equations, Indiana Univ. Math. J. **51** (2002), 905-930.

[4] Berti, V., Gatti, S., Parabolic-hyperbolic time-dependent Ginzburg-Landau-Maxwell equations, Quart. Appl. Math. **64** (2006), 617-639.

[5] Cholewa, J.W., Dlotko, T., Bi-spaces global attractors in abstract parabolic equations, Evolution equations, Banach Center Publ., Polish Acad. Sci. Inst. Math., Warsaw **60** (2003), 13-26.

[6] Cholewa, J.W., Dlotko, T., Global Attractors in Abstract Parabolic Problems, London Mathematical Society Lecture Note Series, vol. 278, Cambridge University Press, Cambridge, 2000.

[7] Constantin, P., Coti Zelati, M., Vicol, V., Uniformly attracting limit sets for the critically dissipative SQG equation, Nonlinearity **29** (2016), 298-318.
[8] Constantin, P., Foiaş, C., Navier-Stokes Equations, Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.
[9] Constantin, P., Tarfulea, A., Vicol, V., Long time dynamics of forced critical SQG, Comm. Math. Phys. **335** (2015), 93-141.
[10] Córdoba, A., Córdoba, D., A maximum principle applied to quasi-geostrophic equations, Comm. Math. Phys. **249** (2004), 511-528.
[11] Coti Zelati, M., Kalita, P., Smooth attractors for weak solutions of the SQG equation with critical dissipation, Discrete Contin. Dyn. Syst. Ser. B **22** (2017), 1857-1873.
[12] Dlotko, T., Kania, M.B., Sun, C., Quasi-geostrophic equation in $\mathbb{R}^2$, J. Differential Equations **259** (2015), 531-561.
[13] Eden, A., Foias, C., Nicolaenko, B., Temam, R., Exponential Attractors for Dissipative Evolution Equation, Research in Applied Mathematics, vol. 37, Masson, Paris; John Wiley & Sons, Ltd., Chichester, 1994.
[14] Efendiev, M., Miranville, A., Zelik, S., Exponential attractors for a nonlinear reaction-diffusion system in $\mathbb{R}^3$, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), 713-718.
[15] Farwig, R., Qian, C., Asymptotic behavior for the quasi-geostrophic equations with fractional dissipation in $\mathbb{R}^2$, J. Differential Equations **266** (2019), 6525-6579.
[16] Foiaş, C., Prodi, G., Sur le comportement global des solutions non-stationnaires des équations de Navier-Stokes en dimension 2, Rend. Sem. Mat. Univ. Padova **39** (1967), 1-34.
[17] Gatti, S., Grasselli, M., Miranville, A., Pata, V., A construction of a robust family of exponential attractors, Proc. Amer. Math. Soc. **134** (2006), 117-127.
[18] Giga, Y., Miyakawa, T., Solutions in L_r of the Navier-Stokes initial value problem, Arch. Rational Mech. Anal. **89** (1985), 267-281.
[19] Gu, A., Li, D., Wang, B., Yang, H., Regularity of random attractors for fractional stochastic reaction-diffusion equations on $\mathbb{R}^n$, J. Differential Equations **264** (2018), 7094-7137.
[20] Hale, J.K., Asymptotic Behavior of Dissipative Systems, Mathematical Surveys and Monographs, vol. 25, American Mathematical Society, Providence, RI, 1988.
[21] Henry, D., Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, vol. 840, Springer-Verlag, Berlin-New York, 1981.
[22] Ju, N., The maximum principle and the global attractor for the dissipative 2D quasi-geostrophic equations, Comm. Math. Phys. **255** (2005), 161-181.
[23] Kato, T., Ponce, G., Commutator estimates and the Euler and Navier-Stokes equations, Comm. Pure Appl. Math. **41** (1988), 891-907.
[24] Kenig, C.E., Ponce, G., Vega, L., Well-posedness of the initial value problem for the Korteweg-de Vries equation, J. Amer. Math. Soc. **4** (1991), 323-347.
[25] Martínez Carracedo, C., Sanz Alix, M., The Theory of Fractional Powers of Operators, North-Holland Mathematics Studies, vol. 187, North-Holland Publishing Co., Amsterdam, 2001.
[26] Niche, C.J., Schonbek, M.E., Decay of weak solutions to the 2D dissipative quasi-geostrophic equation, Comm. Math. Phys. **276** (2007), 93-115.
[27] Niche, C.J., Schonbek, M.E., Decay characterization of solutions to dissipative equations, J. Lond. Math. Soc. **91** (2015), 573-595.
[28] Pedlosky, J., Geophysical Fluid Dynamics, Springer, Berlin, 1982.
[29] Raugel, G., Global Attractors in Partial Differential Equations, Handbook of dynamical systems, vol. 2, 885-982, North-Holland, Amsterdam, 2002.
[30] Robinson, J.C., Infinite-Dimensional Dynamical Systems, An introduction to dissipative parabolic PDEs and the theory of global attractors, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001.
[31] Sell, G.R., Global attractors for the three-dimensional Navier-Stokes equations, J. Dynam. Differential Equations **8** (1996), 1-33.
[32] Stein, E.M., Singular Integrals and Differentiability Properties of Functions, Princeton Mathematical Series, Princeton University Press, Princeton, N.J., 1970.
[33] Sun, C., Zhong, C., Attractors for the semilinear reaction-diffusion equation with distribution derivatives in unbounded domains, Nonlinear Anal. **63** (2005), 49-65.
[34] Temam, R., Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Applied Mathematical Sciences, vol. 68, Springer-Verlag, New York, 1997.

[35] Wang, S., Attractors for the 3D baroclinic quasi-geostrophic equations of large-scale atmosphere, J. Math. Anal. Appl. **165** (1992), 266-283.

[36] Wang, M., Tang, Y., Long time dynamics of 2D quasi-geostrophic equations with damping in L^p, J. Math. Anal. Appl. **412** (2014), 866-877.

[37] Wang, M., Tang, Y., On dimension of the global attractor for 2D quasi-geostrophic equations, Nonlinear Anal. Real World Appl. **14** (2013), 1887-1895.

[38] Wu, J., Dissipative quasi-geostrophic equations with L^p data, Electron. J. Differential Equations **2001** (2001), 1-13.

[39] Yagi, A., Abstract Parabolic Evolution Equations and their Applications, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010.

[40] Zhong, C., Yang, M., Sun, C., The existence of global attractors for the norm-to-weak continuous semigroup and application to the nonlinear reaction-diffusion equations, J. Differential Equations **223** (2006), 367-399.

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