THE CONTINUITY, REGULARITY AND POLYNOMIAL STABILITY OF MILD SOLUTIONS FOR STOCHASTIC 2D-STOKES EQUATIONS WITH UNBOUNDED DELAY DRIVEN BY TEMPERED FRACTIONAL GAUSSIAN NOISE

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We consider stochastic 2D-Stokes equations with unbounded delay in fractional power spaces and moments of order $p \geq 2$ driven by a tempered fractional Brownian motion (TFBM) $B^{\sigma,\lambda}(t)$ with $-1/2 < \sigma < 0$ and $\lambda > 0$. First, the global existence and uniqueness of mild solutions are established by using a new technical lemma for stochastic integrals with respect to TFBM in the sense of p-th moment. Moreover, based on the relations between the stochastic integrals with respect to TFBM and fractional Brownian motion, we show the continuity of mild solutions in the case of $\lambda \to 0$, $\sigma \in (-1/2, 0)$ or $\lambda > 0$, $\sigma \to \sigma_0 \in (-1/2, 0)$. In particular, we obtain p-th moment Hölder regularity in time and p-th polynomial stability of mild solutions. This paper can be regarded as a first step to study the challenging model: stochastic 2D-Navier-Stokes equations with unbounded delay driven by tempered fractional Gaussian noise.

Keywords: Stochastic Stokes equation, Tempered fractional Brownian motion, Unbounded delay, Continuity with respect to parameters, Hölder regularity, Polynomial stability.

1. Introduction

Tempered fractional Brownian motion (TFBM) [40], defined by exponentially

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tempering the power law kernel in the moving average representation of a fractional Brownian motion (FBM), denotes a family of Gaussian processes with continuous sample paths that are indexed by tempered parameter λ and Hurst parameter H ($H = 1/2 - \sigma$). This extra parameter λ controls the deviation from a FBM's power law spectrum at low frequencies. Different with the long range dependence of fractional Gaussian noise (FGN), tempered FGN exhibits semi-long range dependence, i.e., the increments in TFBM decays essentially like a power law over fine/moderate scales (fractional or scale invariant behavior), however quasi-exponentially over large scales. Tempered FGN has been successfully applied in wind speed modeling. Tempered fractional processes have attracted much attention in recent years due to a wide range of applications such as in the physics and modeling of transient anomalous diffusion [12,32,47,49,53], geophysical flows [10,42,43] and finance [15,22,31,60].

In spite of the fast growth of the literature on tempered fractional processes, there has been little mention of stochastic differential equations driven by tempered fractional Gaussian noise even in the nondelay case. Very recently, we proved the existence, uniqueness and exponential stability of mild solutions for stochastic delay evolution equations driven by tempered fractional Gaussian noise in mean square [54].

Navier-Stokes equations have been extensively studied over the last century, since they are crucial for fluid mechanics and turbulence. Due to the importance of considering some delay terms in the models, stochastic Navier-Stokes equations with delay have attracted increasing attention in recent years; see [9,30] for Brownian motion and [50] for Lévy process. However, there are some difficulties to study delay Navier-Stokes equations driven by tempered fractional Gaussian noise even in the fractional noise case. Since Stokes equations provide a first approximation of the more general Navier-Stokes equations in situations where the flow is nearly steady, slow and has small velocity gradients, in this paper, we investigate the following stochastic 2D-Stokes equation with unbounded delay in the sense of p moment ($p \geq 2$):

$$\begin{cases} du(t) = \Delta u(t)dt - \delta u dt + \nabla p dt + F(t, u_t)dt + G(t, u_t)dB^{\sigma,\lambda}(t) & \text{in } \mathbb{R}^2, \ t > 0, \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^2, \quad t > 0, \\ u(t, x) = \varphi(t, x), & \text{in } \mathbb{R}^2, \quad t \in (-\infty, 0]. \end{cases}$$

For convenience, let us rewrite it in an abstract form

$$\begin{cases} du(t) = -Au(t)dt + f(t, u_t)dt + g(t, u_t)dB^{\sigma,\lambda}(t), \qquad t > 0, \\ u(t) = \varphi(t), \qquad t \in (-\infty, 0], \end{cases} \tag{1.1}$$

where $A = -P\Delta + \delta PI = -\Delta P + \delta PI$, $f(t, u_t) = PF(t, u_t)$, $g(t, u_t) = PG(t, u_t)$, φ is the initial data, $B^{\sigma,\lambda}(t)$ is a tempered fractional Brownian motion with $-1/2 < \sigma < 0$ and $\lambda > 0$ over a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Here $\delta > 0$, P is the Helmholtz-Leray projector and A is the Stokes operator.

Our purposes in this current work are in four aspects:

(i) To prove global existence, uniqueness and Hölder regularity of mild solutions to (1.1) in fractional power spaces and moments of order $p \geq 2$;

(ii) To prove that the mild solution $u^{\sigma,\lambda}$ of (1.1) converges to the mild solution $u^{\sigma,0}$ of (1.1) but with FBM $B^{\sigma,0}$ instead of TFBM $B^{\sigma,\lambda}$ as $\lambda \to 0$, and to present the continuity of the mild solution $u^{\sigma,\lambda}$ of (1.1) with respect to the parameter $\sigma \in (-1/2, 0)$ in the sense of p moment;
(iii) To prove the p-th polynomial (as well as exponential) stability of global mild solutions to (1.1) in the phase space

$$\mathcal{C}^{p,\zeta}(H^\gamma) = \Big\{\psi \in C\big(-\infty, 0; L^p(\Omega; H^\gamma)\big) : \lim_{\theta \to -\infty} e^{\zeta\theta}\psi(\theta) \text{ exists in } L^p(\Omega; H^\gamma)\Big\},$$

where $p \geq 2$, $\zeta > 0$ and the Banach space H^γ given in Section 2;
(iv) At light that the conditions imposed for (iii) do not allow to consider the case of variable delay within that formulation, we use the Banach fixed point theorem and complicated analysis, to prove the global existence and p-th polynomial stability of mild solutions to (1.1) in the particular (but still interesting) case of proportional delay, when g becomes independent of the state variable, where the phase space is

$$\mathcal{C}^{p}(H^\gamma) = \Big\{\psi \in C\big(-\infty, 0; L^p(\Omega; H^\gamma)\big) : \lim_{\theta \to -\infty} \psi(\theta) \text{ exists in } L^p(\Omega; H^\gamma)\Big\}.$$

Regularity of solutions for stochastic partial differential equations driven by space-time white noise has been extensively developed over the last one and a half decades (see, e.g. [1,5,8,17,29,45]). However, the study on the regularity of the solutions of stochastic equations in an infinite-dimensional space with a fractional Brownian motion has been relatively limited. Regularity of the solutions for stochastic semilinear equations with an additive fractional Gaussian noise, the formal derivative of a fractional Brownian motion, has been considered in [33,48,52,56]. It is worthy mentioning that our Hölder regularity results for the mild solutions are established for stochastic delay 2D-Stokes equations with multiplicative nonlinear tempered fractional Gaussian noise in fractional power spaces and moments of order $p \geq 2$.

In recent years, stability of stochastic ordinary and stochastic partial differential equations, providing relevant information on the long time behavior of the solutions of such equations, has received much attention (see, e.g., [28,34,35,37,46,51,55,58,61]). Hölder continuous paths approach has been used in [13,14,25] to study the exponential stability of a ordinary or partial differential equation driven by fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. Based on the generalized Itô formula and representation of fractional Brownian motion, the exponential stability has been obtained in [57] for a class of stochastic differential equations driven by additive fractional noise with Hurst parameter $H \in (1/2, 1)$. Exponential stability for impulsive stochastic differential equations has been considered in [2,3,18]. Almost sure exponential stability has been studied in [27,59] for stochastic scalar non-autonomous linear stochastic differential delay equation and Black-Scholes model driven by fractional Brownian motion with Hurst index $> \frac{1}{2}$. Up to date, we do not know any published work on polynomial stability of stochastic differential equations driven by fractional Brownian motion. In the current work, two different methods are used to analyze the p-th polynomial stability of stochastic 2D-Stokes equations with infinite delay (distributed delay or unbounded variable delay) and proportional delay (which is a particular case of variable delay) driven by TFBM.

The paper is organized as follows. In Section 2 we recall some preliminary definitions and results regarding TFBM, while in Section 3 the global existence and uniqueness of mild solutions to Eq. (1.1) are considered. Section 4 is devoted to the relationship between mild solutions of Eq. (1.1) driven by TFBM and FBM. In Section 5, we establish the continuity of mild solutions of Eq. (1.1) with respect to the Hurst parameter $H \in (1/2, 1)$ where $H = 1/2 - \sigma$. Hölder regularity in time for mild solutions to Eq. (1.1) is proved in Section 6, while Section 7 is devoted to providing a first stability result for the case of proportional delay. This requires a new method to analyze the global existence and p-th polynomial stability of mild solutions to Eq. (1.1), and it also needs to consider an additive tempered fractional Gaussian noise. Finally, in Section 8, we consider a different phase space and provide not only polynomial but also exponential stability results by imposing different assumptions.

2. Preliminaries

Let X be a Banach space with the norm $\|\cdot\|_X$. We denote by $C(a,b;X)$ the Banach space of all continuous X-valued functions on $[a,b]$ equipped with the sup norm. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space equipped with some filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition, i.e., the filtration is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. For $2 \leq p < \infty$, the collection of all strongly-measurable, L^p integrable X-valued random variable, denoted by $L^p(\Omega;X) = L^p(\Omega,\mathcal{F},\mathbb{P};X)$, is a Banach space equipped with the norm $\|u(\cdot)\|_{L^p(\Omega;X)} = \big(E\|u(\cdot)\|_X^p\big)^{\frac{1}{p}}$. We denote by $C\big(a,b;L^p(\Omega;X)\big) = C\big(a,b;L^p(\Omega,\mathcal{F},\mathbb{P};X)\big)$ the Banach space of all continuous functions from $[a,b]$ into $L^p(\Omega;X)$ equipped with the sup norm $\|u(t)\|_{C(a,b;L^p(\Omega;X))} = \big(\sup_{t\in[a,b]} E\|u(t)\|_X^p\big)^{\frac{1}{p}}$. As usual, let $u \vee v$ denote the maximum of $u, v \in \mathbb{R}$, and $u \wedge v$ their minimum. In the sequel $\mathbb{C}$ denotes an arbitrary positive constant, which may be different from line to line and even in the same line. If we want to emphasize the dependence of $\mathbb{C}$ on some variable x, we denote it by $\mathbb{C}(x)$.

We now recall the definitions of tempered fractional Brownian motion and fractional Brownian motion as well as the Wiener integrals with respect to them; for more details, we refer to [7,36,39,41].

Let $\{B(t)\}_{t\in\mathbb{R}}$ be a two-sided one-dimensional Brownian motion, which is a process with stationary independent increments such that $B(t)$ has a Gaussian distribution with mean zero and variance $|t|$ for all $t \in \mathbb{R}$.

Definition 2.1. For any $\sigma < \frac{1}{2}$ and $\lambda > 0$, a tempered fractional Brownian motion (TFBM) is defined by the following integral:

$$B^{\sigma,\lambda}(t) = \int_{-\infty}^{\infty} \big[e^{-\lambda(t-x)_+}(t-x)_+^{-\sigma} - e^{-\lambda(-x)_+}(-x)_+^{-\sigma}\big]dB(x), \tag{2.1}$$

where $(x)_+ = xI_{(x>0)}$, $0^0 = 0$ and λ is called tempered parameter.

In particular, when $\lambda = 0$ and $\sigma < -\frac{1}{2}$, TFBM (2.1) does not exist, since the integrand is not in $L^2(\mathbb{R})$. However, TFBM with $\lambda > 0$ and $\sigma < -\frac{1}{2}$ is well-defined, because the exponential tempering keeps the integrand in $L^2(\mathbb{R})$. When $\sigma < -\frac{1}{2}$ and $\lambda > 0$, or when $\sigma = 0$ and $\lambda > 0$, TFBM (2.1) is a continuous semimartingale, so the classical Itô stochastic calculus is applicable to TFBM in these cases. TFBM is neither a semimartingale nor a Markov process in the remaining case when $\sigma \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$ and $\lambda > 0$.

When $-\frac{1}{2} < \sigma < \frac{1}{2}$ and $\lambda = 0$, TFBM (2.1) reduces to a fractional Brownian motion (FBM) $\{B^{\sigma,0}(t)\}_{t\in\mathbb{R}}$, a self-similar Gaussian stochastic process with Hurst scaling index $H = \frac{1}{2} - \sigma$. For the normalized case, we have

Definition 2.2. For $-\frac{1}{2} < \sigma < \frac{1}{2}$ and $\lambda = 0$, a normalized fractional Brownian motion with $H = \frac{1}{2} - \sigma$ is defined by

$$B^H(t) = C_H \int_{-\infty}^{\infty} \left[(t-x)_+^{-\sigma} - (-x)_+^{-\sigma}\right] dB(x), \tag{2.2}$$

where $C_H = \frac{\left(2H \sin \pi H \Gamma(2H)\right)^{\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}$. Here $\Gamma(\cdot)$ is Euler's gamma function.

Thanks to Proposition 2.3 in [41], it follows that TFBM $\{B^{\sigma,\lambda}(t)\}_{t\in\mathbb{R}}$, with $\sigma < \frac{1}{2}$ and $\lambda > 0$, is a Gaussian stochastic process with mean $E[B^{\sigma,\lambda}(t)] = 0$ for all $t \in \mathbb{R}$, and covariance

$$E[B^{\sigma,\lambda}(t)B^{\sigma,\lambda}(s)] = \frac{1}{2}\left[C_t^2|t|^{2H} + C_s^2|s|^{2H} - C_{t-s}^2|t-s|^{2H}\right] \tag{2.3}$$

for any $s, t \in \mathbb{R}$, where $H = \frac{1}{2} - \sigma$, and

$$C_t^2 = \frac{2\Gamma(2H)}{(2\lambda|t|)^{2H}} - \frac{2\Gamma(H+\frac{1}{2})}{\sqrt{\pi}} \frac{1}{(2\lambda|t|)^H} K_H(\lambda|t|), \quad t \neq 0, \tag{2.4}$$

in which $K_H(\cdot)$ is the modified Bessel function of the second kind, and $C_0^2 = 0$. It is clear that $B^{\sigma,\lambda}(0) = 0$.

For the normalized FBM $\{B^H(t)\}_{t\in\mathbb{R}}$ with $H \in (0,1)$, it is well known that it is a Gaussian stochastic process having the properties $B^H(0) = 0$, $E[B^H(t)] = 0$ for all $t \in \mathbb{R}$, and

$$E[B^H(t)B^H(s)] = \frac{1}{2}\left[|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right], \quad t, s \in \mathbb{R}. \tag{2.5}$$

In order to consider the stochastic integrals with respect to TFBM and FBM, we now present the definitions of fractional integral and tempered fractional integral.

Definition 2.3. Let $\alpha > 0$ and $T > 0$. For any $f \in L^p(0,T)$ (where $1 \leq p < \infty$) and for any $a, b \in [0,T]$ with $b > a$, the left and right Riemann-Liouville fractional integral on (a,b) are defined by

$${}_a\mathbb{I}_t^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds$$

and

$$ {}_t\mathbb{I}_b^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds, $$

respectively, where $\Gamma(\cdot)$ is mentioned in Definition 2.2.

Definition 2.4. Let $\alpha > 0$, $\lambda > 0$ and $T > 0$. For any $f \in L^p(0,T)$ (where $1 \le p < \infty$) and for any $a, b \in [0,T]$ with $b > a$, the left and right Riemann-Liouville tempered fractional integral on (a,b) are defined by

$$ {}_a\mathbb{I}_t^{\alpha,\lambda} f(t) := e^{-\lambda t} {}_a\mathbb{I}_t^{\alpha}[e^{\lambda t} f(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} f(s) ds $$

and

$$ {}_t\mathbb{I}_b^{\alpha,\lambda} f(t) := e^{\lambda t} {}_t\mathbb{I}_b^{\alpha}[e^{-\lambda t} f(t)] = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} e^{-\lambda(s-t)} f(s) ds, $$

respectively.

Definition 2.5. For any $-\frac{1}{2} < \sigma < 0$, $\lambda > 0$, and for any $a, b \in [0,T]$ with $b > a$, we define

$$ \int_a^b f(t) dB^{\sigma,\lambda}(t) := \Gamma(k+1) \int_a^b \big({}_t\mathbb{I}_b^{k,\lambda} f(t) - \lambda {}_t\mathbb{I}_b^{k+1,\lambda} f(t) \big) dB(t) $$

for any $f \in \mathcal{A}_1 := \big\{ f \in L^2(a,b) : \int_a^b \big| {}_t\mathbb{I}_b^{k,\lambda} f(t) - \lambda {}_t\mathbb{I}_b^{k+1,\lambda} f(t) \big|^2 dt < \infty \big\}$. Here $k = -\sigma$, and $\mathcal{A}_1$ is a linear space with inner product $\langle f, g \rangle_{\mathcal{A}_1} := \langle F, G \rangle_{L^2(a,b)}$ where

$$ F(t) = \Gamma(k+1)\big({}_t\mathbb{I}_b^{k,\lambda} f(t) - \lambda {}_t\mathbb{I}_b^{k+1,\lambda} f(t) \big), $$
$$ G(t) = \Gamma(k+1)\big({}_t\mathbb{I}_b^{k,\lambda} g(t) - \lambda {}_t\mathbb{I}_b^{k+1,\lambda} g(t) \big). $$

Definition 2.6. For any $H \in (\frac{1}{2}, 1)$ and $a, b \in [0,T]$ with $b > a$, we define

$$ \int_a^b f(t) dB^H(t) := C_H \Gamma(H + \frac{1}{2}) \int_a^b {}_t\mathbb{I}_b^{H-\frac{1}{2}} f(t) dB(t), $$

for any $f \in \mathcal{A}_0 := \big\{ f \in L^2(a,b) : \int_a^b |{}_t\mathbb{I}_b^{H-\frac{1}{2}} f(t)|^2 dt < \infty \big\}$. Here C_H is given in Definition 2.2 and $\mathcal{A}_0$ is a linear space with inner product $\langle f, g \rangle_{\mathcal{A}_0} := \langle F_0, G_0 \rangle_{L^2(a,b)}$ where

$$ F_0(t) = C_H \Gamma(H + \frac{1}{2}) {}_t\mathbb{I}_b^{H-\frac{1}{2}} f(t), \qquad G_0(t) = C_H \Gamma(H + \frac{1}{2}) {}_t\mathbb{I}_b^{H-\frac{1}{2}} g(t). $$

For the stochastic integrals with respect to Brownian motion, FBM and TFBM, we have the following properties; for the particular case of $p = 2$ see, e.g., [7,11,19,54].

Lemma 2.1. *If $\phi : [0,T] \times \Omega \to L^2$ is a progressively measurable function satisfying $E\big(\int_0^T \|\phi(s)\|_{L^2}^2 ds \big)^{\frac{p}{2}} < \infty$, then for any $t \in [0,T]$,*

$$ E\Big\| \int_0^t \phi(s) dB(s) \Big\|_{L^2}^p \le C_p E\Big(\int_0^t \|\phi(s)\|_{L^2}^2 ds \Big)^{\frac{p}{2}}, \tag{2.6} $$

where $C_p > 0$ and $p \geq 2$.

Lemma 2.2. *Let $-\frac{1}{2} < \sigma < 0$, $\lambda > 0$, $p \geq 2$. If $\phi : [0,T]\times\Omega \to L^2$ is a progressively measurable function satisfying $E\big(\int_0^T \|\phi(s)\|_{L^2}^2 ds\big)^{\frac{p}{2}} < \infty$, then for any $t \in [0,T]$,*

$$E\Big\|\int_0^t \phi(s)dB^{\sigma,\lambda}(s)\Big\|_{L^2}^p \leq C_p(N_t)^{\frac{p}{2}} E\Big(\int_0^t \|\phi(s)\|_{L^2}^2 ds\Big)^{\frac{p}{2}}, \tag{2.7}$$

where C_p is given in Lemma 2.1,

$$N_t = (2H-1)t^{2H-1}\beta(2-2H, H-\frac{1}{2}) + 4\lambda^2 t^{2H+1}\frac{\beta(2-2H, H+\frac{1}{2})}{2H-1},$$

$H = \frac{1}{2} - \sigma$ and $\beta(\cdot,\cdot)$ is the beta function.

Proof. To prove (2.7), we first need to show that $\big({}_s\mathbb{I}_t^{-\sigma,\lambda}\phi(s) - \lambda_s\mathbb{I}_t^{1-\sigma,\lambda}\phi(s)\big)$ is progressively measurable. Let $\tilde{\phi}(t)$ be an elementary process with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ defined by

$$\tilde{\phi}(t) = \sum_{j=0}^{k-1}\tilde{\phi}_j \mathbb{1}_{(t_j,t_{j+1})}(t), \quad j = 0,1,\ldots,k-1, \tag{2.8}$$

where $0 = t_0 < t_1 < \cdots < t_k = t$, and for each index j the random variable $\tilde{\phi}_j$ is measurable relative to $\mathcal{F}_{t_j}$. Hence the elementary process $\tilde{\phi}(t)$ is progressively measurable. Then we obtain that for $\tilde{\phi}$,

$$\begin{aligned}
&{}_s\mathbb{I}_t^{-\sigma,\lambda}\big(\sum_{j=0}^{k-1}\tilde{\phi}_j\mathbb{1}_{(t_j,t_{j+1})}(s)\big) - \lambda_s\mathbb{I}_t^{1-\sigma,\lambda}\big(\sum_{j=0}^{k-1}\tilde{\phi}_j\mathbb{1}_{(t_j,t_{j+1})}(s)\big)\\
&= \frac{1}{\Gamma(-\sigma)}\int_s^t (u-s)^{-\sigma-1}e^{-\lambda(u-s)}\big(\sum_{j=0}^{k-1}\tilde{\phi}_j\mathbb{1}_{(t_j,t_{j+1})}(u)\big)du\\
&\quad - \frac{\lambda}{\Gamma(1-\sigma)}\int_s^t (u-s)^{-\sigma}e^{-\lambda(u-s)}\big(\sum_{j=0}^{k-1}\tilde{\phi}_j\mathbb{1}_{(t_j,t_{j+1})}(u)\big)du.
\end{aligned}$$

For $u \in (0,t_1)$ we have

$$\begin{aligned}
&\frac{\tilde{\phi}_0}{\Gamma(-\sigma)}\int_s^{t_1}(u-s)^{-\sigma-1}e^{-\lambda(u-s)}du - \frac{\lambda\tilde{\phi}_0}{\Gamma(1-\sigma)}\int_s^{t_1}(u-s)^{-\sigma}e^{-\lambda(u-s)}du\\
&= \frac{\tilde{\phi}_0}{\Gamma(1-\sigma)}\int_s^{t_1}e^{-\lambda(u-s)}d(u-s)^{-\sigma} + \frac{\tilde{\phi}_0}{\Gamma(1-\sigma)}\int_s^{t_1}(u-s)^{-\sigma}de^{-\lambda(u-s)}\\
&= \frac{\tilde{\phi}_0}{\Gamma(1-\sigma)}e^{-\lambda(t_1-s)}(t_1-s)^{-\sigma}.
\end{aligned}$$

If $u \in (t_j, t_{j+1})$, then we find that

$$\begin{aligned}
&\frac{\tilde{\phi}_j}{\Gamma(-\sigma)}\int_{s\vee t_j}^{t_{j+1}}(u-s)^{-\sigma-1}e^{-\lambda(u-s)}du - \frac{\lambda\tilde{\phi}_j}{\Gamma(1-\sigma)}\int_{s\vee t_j}^{t_{j+1}}(u-s)^{-\sigma}e^{-\lambda(u-s)}du\\
&= \frac{\tilde{\phi}_j}{\Gamma(1-\sigma)}\big(e^{-\lambda(t_{j+1}-s)}(t_{j+1}-s)^{-\sigma} - e^{-\lambda(s\vee t_j-s)}(s\vee t_j-s)^{-\sigma}\big).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&{}_s\mathbb{I}_t^{-\sigma,\lambda}\big(\sum_{j=0}^{k-1}\tilde{\phi}_j\mathbb{1}_{(t_j,t_{j+1})}(s)\big)-\lambda_s\mathbb{I}_t^{1-\sigma,\lambda}\big(\sum_{j=0}^{k-1}\tilde{\phi}_j\mathbb{1}_{(t_j,t_{j+1})}(s)\big)\\
&=\sum_{j=1}^{k-1}\frac{\tilde{\phi}_j}{\Gamma(1-\sigma)}\big(e^{-\lambda(t_{j+1}-s)}(t_{j+1}-s)^{-\sigma}-e^{-\lambda(s\vee t_j-s)}(s\vee t_j-s)^{-\sigma}\big)\\
&+\tilde{\phi}_0\frac{e^{-\lambda(t_1-s)}(t_1-s)^{-\sigma}}{\Gamma(1-\sigma)}.
\end{aligned}\tag{2.9}$$

It follows from (2.8) and (2.9) that mappings

$$\omega\to{}_s\mathbb{I}_t^{-\sigma,\lambda}\tilde{\phi}(s)-\lambda_s\mathbb{I}_t^{1-\sigma,\lambda}\tilde{\phi}(s)\quad\text{and}\quad s\to{}_s\mathbb{I}_t^{-\sigma,\lambda}\tilde{\phi}(s)-\lambda_s\mathbb{I}_t^{1-\sigma,\lambda}\tilde{\phi}(s),$$

are $\mathcal{F}_t$-measurable for each $\omega\in\Omega$ and continuous with respect to s, respectively. This implies that the mapping

$$(s,\omega)\to{}_s\mathbb{I}_t^{-\sigma,\lambda}\tilde{\phi}(s)-\lambda_s\mathbb{I}_t^{1-\sigma,\lambda}\tilde{\phi}(s),\quad 0\le s\le t,\ \omega\in\Omega,$$

on the product space $[0,t]\times\Omega$ is $\mathcal{B}([0,t])\times\mathcal{F}_t$-measurable. Then ${}_s\mathbb{I}_t^{-\sigma,\lambda}\tilde{\phi}(s)-\lambda_s\mathbb{I}_t^{1-\sigma,\lambda}\tilde{\phi}(s)$ is progressively measurable. Notice that $E\big(\int_0^T\|\phi(s)\|_{L^2}^2ds\big)^{\frac{p}{2}}<\infty$ and thus, for a sequence of elementary processes denoted by $\{\tilde{\phi}_n\}$,

$$E\big(\int_0^T\|\phi(s)-\tilde{\phi}_n(s)\|_{L^2}^2ds\big)^{\frac{p}{2}}\to 0\quad\text{as } n\to\infty.\tag{2.10}$$

On the other hand, according to Lemmas 2.2 and 3.6 in [41], we have

$$\begin{aligned}
&E\Big(\int_0^T\big\|{}_s\mathbb{I}_T^{-\sigma,\lambda}(\phi(s)-\tilde{\phi}_n(s))-\lambda_s\mathbb{I}_T^{1-\sigma,\lambda}(\phi(s)-\tilde{\phi}_n(s))\big\|_{L^2}^2ds\Big)^{\frac{p}{2}}\\
&\le E\Big(\int_0^T\big({}_s\mathbb{I}_T^{-\sigma,\lambda}\|\phi(s)-\tilde{\phi}_n(s)\|_{L^2}-\lambda_s\mathbb{I}_T^{1-\sigma,\lambda}\|\phi(s)-\tilde{\phi}_n(s)\|_{L^2}\big)^2ds\Big)^{\frac{p}{2}}\\
&\le E\big(\int_0^T\|\phi(s)-\tilde{\phi}_n(s)\|_{L^2}^2ds\big)^{\frac{p}{2}}\to 0\quad\text{as } n\to\infty.
\end{aligned}$$

Since the fact that limits of progressively measurable processes are progressively measurable, we conclude that $\big({}_s\mathbb{I}_t^{-\sigma,\lambda}\phi(s)-\lambda_s\mathbb{I}_t^{1-\sigma,\lambda}\phi(s)\big)$ is progressively measurable.

Now we are ready to prove (2.7). By using Lemma 2.1, Definitions 2.4 and 2.5, we deduce that

$$
\begin{aligned}
&E\Big\|\int_0^t \phi(s)dB^{\sigma,\lambda}(s)\Big\|_{L^2}^p\\
&=\big(\Gamma(1-\sigma)\big)^p E\Big\|\int_0^t \big({}_s\mathbb{I}_t^{-\sigma,\lambda}\phi(s)-\lambda_s\mathbb{I}_t^{1-\sigma,\lambda}\phi(s)\big)dB(s)\Big\|_{L^2}^p\\
&\le \big(\Gamma(1-\sigma)\big)^p C_p E\Big(\int_0^t \big\|{}_s\mathbb{I}_t^{-\sigma,\lambda}\phi(s)-\lambda_s\mathbb{I}_t^{1-\sigma,\lambda}\phi(s)\big\|_{L^2}^2 ds\Big)^{\frac{p}{2}}\\
&\le C_p 2^{\frac{p}{2}} E\Big(\int_0^t \sigma^2\Big(\int_s^t (u-s)^{-\sigma-1}e^{-\lambda(u-s)}\|\phi(u)\|_{L^2}du\Big)^2\\
&\quad+\lambda^2\Big(\int_s^t (x-s)^{-\sigma}e^{-\lambda(x-s)}\|\phi(x)\|_{L^2}dx\Big)^2 ds\Big)^{\frac{p}{2}}\\
&=C_p 2^{\frac{p}{2}} E\Big(\sigma^2\int_0^t\int_s^t\int_s^t \|\phi(u)\|_{L^2}\|\phi(r)\|_{L^2}(u-s)^{-\sigma-1}(r-s)^{-\sigma-1}\\
&\quad\times e^{-\lambda(u-s)}e^{-\lambda(r-s)}dudrds\\
&\quad+\lambda^2\int_0^t\int_s^t\int_s^t \|\phi(x)\|_{L^2}\|\phi(y)\|_{L^2}(y-s)^{-\sigma}(x-s)^{-\sigma}\\
&\quad\times e^{-\lambda(y-s)}e^{-\lambda(x-s)}dxdyds\Big)^{\frac{p}{2}}\\
&\le C_p 2^{\frac{p}{2}} E\Big(\sigma^2\int_0^t\int_0^t\int_0^{u\wedge r} \|\phi(u)\|_{L^2}\|\phi(r)\|_{L^2}(u-s)^{-\sigma-1}(r-s)^{-\sigma-1}dsdudr\\
&\quad+\lambda^2\int_0^t\int_0^t\int_0^{x\wedge y} \|\phi(x)\|_{L^2}\|\phi(y)\|_{L^2}(y-s)^{-\sigma}(x-s)^{-\sigma}dsdxdy\Big)^{\frac{p}{2}}\\
&\le C_p E\Big(2\sigma^2\int_0^t\int_0^t \|\phi(r)\|_{L^2}^2|r-u|^{-2\sigma-1}\beta(1+2\sigma,-\sigma)dudr\\
&\quad+2\lambda^2t^2\int_0^t\int_0^t \|\phi(y)\|_{L^2}^2|y-x|^{-2\sigma-1}\beta(1+2\sigma,1-\sigma)dxdy\Big)^{\frac{p}{2}}\\
&\le C_p\Big((2H-1)t^{2H-1}\beta(2-2H,H-\frac{1}{2})+4\lambda^2t^{2H+1}\frac{\beta(2-2H,H+\frac{1}{2})}{2H-1}\Big)^{\frac{p}{2}}\\
&\quad\times E\Big(\int_0^t \|\phi(s)\|_{L^2}^2ds\Big)^{\frac{p}{2}},
\end{aligned}
$$

where we have used the following inequalities (see [54]):

$$\int_0^{u\wedge r}(u-s)^{-\sigma-1}(r-s)^{-\sigma-1}ds\le |r-u|^{-2\sigma-1}\beta(1+2\sigma,-\sigma) \tag{2.11}$$

and

$$\int_0^{x\wedge y}(x-s)^{-\sigma}(y-s)^{-\sigma}ds\le (x\vee y)^2|x-y|^{-2\sigma-1}\beta(1+2\sigma,1-\sigma) \tag{2.12}$$

for any $-\frac{1}{2}<\sigma<0$. The proof of this lemma is finished. □

Lemma 2.3. *Let $p \geq 2$ and $H \in (\frac{1}{2}, 1)$. If $\phi : [0,T] \times \Omega \to L^2$ is a progressively measurable function satisfying $E\Big(\int_0^T \|\phi(s)\|_{L^2}^2 ds\Big)^{\frac{p}{2}} < \infty$, then for any $t \in [0,T]$,*

$$E\Big\|\int_0^t \phi(s)dB^H(s)\Big\|_{L^2}^p \leq C_p(M_t)^{\frac{p}{2}} E\Big(\int_0^t \|\phi(s)\|_{L^2}^2 ds\Big)^{\frac{p}{2}}, \tag{2.13}$$

where C_p is given in Lemma 2.1,

$$M_t = (C_H)^2(H - \frac{1}{2})\beta(2-2H, H-\frac{1}{2})t^{2H-1},$$

and C_H is given in Definition 2.2.

Proof. *Since the proof is similar to Lemma 2.2, we omit the details here.* □

Remark 2.1. For the case $\lambda = 0$, it follows from Lemma 2.2 that

$$\begin{aligned} &E\Big\|\int_0^t \phi(s)dB^{\sigma,0}(s)\Big\|_{L^2}^p \\ &\leq C_p\Big((H-\frac{1}{2})\beta(2-2H, H-\frac{1}{2})t^{2H-1}\Big)^{\frac{p}{2}} E\Big(\int_0^t \|\phi(s)\|_{L^2}^2 ds\Big)^{\frac{p}{2}}. \end{aligned} \tag{2.14}$$

Comparing (2.13) and (2.14), we find that the coefficient C_H in (2.13) appears because of the definition of the normalized FBM.

3. The global existence and uniqueness of mild solutions

To set our problem (1.1) in the abstract framework, we consider the following usual abstract space:

$$\mathcal{L}^2 = \big\{u \in L^2 : \nabla \cdot u = 0 \ \text{ in } \ \mathbb{R}^2\big\},$$

where L^2 denotes the vector-valued Lebesgue space with the norm $\|\cdot\|$, and

$$\|u\|^2 = \sum_{j=1}^2 \int_{\mathbb{R}^2} |u_j(x)|^2 dx.$$

For non-integer $\gamma > 0$, we define the Banach space $H^\gamma = D(A^\gamma)$, where A is the Stokes operator and $D(A^\gamma)$ denotes the domain of the fractional power operator $A^\gamma : \mathcal{L}^2 \to \mathcal{L}^2$. The norm is given by

$$\|f\|_\gamma := \big\|A^\gamma f\big\| \ \text{ for } \ f \in H^\gamma.$$

Moreover, we define the abstract phase space $\mathcal{C}^p(H^\gamma)$ by

$$\mathcal{C}^p(H^\gamma) = \Big\{\psi \in C\big(-\infty, 0; L^p(\Omega; H^\gamma)\big) : \lim_{\theta \to -\infty} \psi(\theta) \text{ exists in } L^p(\Omega; H^\gamma)\Big\},$$

for $p \geq 2$. If $\mathcal{C}^p(H^\gamma)$ is endowed with the norm

$$\|\psi\|_{\mathcal{C}^p(H^\gamma)} = \Big(\sup_{\theta \in (-\infty, 0]} E\|\psi(\theta)\|_\gamma^p\Big)^{\frac{1}{p}}, \quad \psi \in \mathcal{C}^p(H^\gamma),$$

then $\big(\mathcal{C}^p(H^\gamma), \|\cdot\|_{\mathcal{C}^p(H^\gamma)}\big)$ is a Banach space.

For the semigroup generated by the Stokes operator A, we have the following properties (see [20,26] for the similar results):

($\mathcal{P}_1$) There exist positive constants $C_0, C_{\gamma,0} \geq 1$ such that for any $u \in \mathcal{L}^2$,

$$
\begin{aligned}
&i) \qquad \big\|A^\gamma S(t)u\big\| \leq C_{\gamma,0} e^{-\delta t} t^{-\gamma} \|u\|, \quad t > 0,\\
&ii) \qquad \big\|S(t)u\big\| \leq C_0 e^{-\delta t} \|u\|, \quad t \geq 0.
\end{aligned}
$$

($\mathcal{P}_2$) There exists a positive constant $C_\gamma \geq 1$ such that for any $0 < \gamma < 1$ and $u \in H^\gamma$,

$$
\big\|S(t)u - u\big\| \leq C_\gamma t^\gamma \|A^\gamma u\|.
$$

In order to prove the global existence and uniqueness of mild solutions to problem (1.1), we impose the following assumptions:

(H_1) For any $\mu \in \mathcal{C}^p(H^\gamma)$, the mappings $[0,\infty) \ni t \mapsto f(t,\mu) \in \mathcal{L}^2$ and $[0,\infty) \ni t \mapsto g(t,\mu) \in \mathcal{L}^2$ are measurable.

(H_2) There exist $l_f, l_g > 0$ such that for any $\mu \in \mathcal{C}^p(H^\gamma)$ and $t \geq 0$,

$$
\begin{aligned}
E\|f(t,\mu)\|^p &\leq l_f\big(1 + \|\mu\|^p_{\mathcal{C}^p(H^\gamma)}\big),\\
E\|g(t,\mu)\|^p &\leq l_g\big(1 + \|\mu\|^p_{\mathcal{C}^p(H^\gamma)}\big).
\end{aligned}
$$

(H_3) There exist two positive constants L_f and L_g such that for any $\mu, \nu \in \mathcal{C}^p(H^\gamma)$ and $t \geq 0$,

$$
\begin{aligned}
E\big\|f(t,\mu) - f(t,\nu)\big\|^p &\leq L_f \|\mu - \nu\|^p_{\mathcal{C}^p(H^\gamma)},\\
E\big\|g(t,\mu) - g(t,\nu)\big\|^p &\leq L_g \|\mu - \nu\|^p_{\mathcal{C}^p(H^\gamma)}.
\end{aligned}
$$

For a real number $T > 0$, each $\tau \in [0,T]$ and $\nu \in C\big(-\infty, T; L^p(\Omega; H^\gamma)\big)$, we denote by $\nu_\tau \in C\big(-\infty, 0; L^p(\Omega; H^\gamma)\big)$ the function defined by $\nu_\tau(s) = \nu(\tau + s)$ ($s \leq 0$). We now introduce the following notation. Let $u \in C\big(0, T; L^p(\Omega; H^\gamma)\big)$ with $u(0) = \varphi(0)$ and $\varphi \in \mathcal{C}^p(H^\gamma)$. Then for $\tau \in [0,T]$, we denote by $u \vee_\tau \varphi$ the mapping from $\mathbb{R}^-$ to $L^p(\Omega; H^\gamma)$ defined by

$$
u \vee_\tau \varphi(s) = \begin{cases} u(\tau + s), & s \in (-\tau, 0],\\ \varphi(\tau + s), & s \leq -\tau. \end{cases} \tag{3.1}
$$

For our aims, let us state the definition of mild solution to Eq. (1.1).

Definition 3.1. Let $\varphi \in \mathcal{C}^p(H^\gamma)$ be an initial process with $\mathcal{F}_t = \mathcal{F}_0$ for all $t \leq 0$. An $\mathcal{F}_t$-adapted stochastic process $u(t)$ is called a mild solution of Eq. (1.1) if $u \in C\big(-\infty, T; L^p(\Omega; H^\gamma)\big)$, $u(t) = \varphi(t)(t \leq 0)$ and the following integral equation is fulfilled with probability one:

$$
u(t) = S(t)\varphi(0) + \int_0^t S(t-\tau) f(\tau, u_\tau) d\tau + \int_0^t S(t-\tau) g(\tau, u_\tau) dB^{\sigma,\lambda}(\tau), \tag{3.2}
$$

for $t \in [0,T]$.

Theorem 3.1. *Let $p \geq 2$ and $T > 0$ be given arbitrarily. Suppose that the assumptions (H_1)-(H_3) and $0 < \gamma < \frac{1}{p}$ hold. Then for each $\varphi \in \mathcal{C}^p(H^\gamma)$, problem (1.1) has a unique mild solution on $[0, T]$.*

Proof. Let us fix some $\varphi \in \mathcal{C}^p(H^\gamma)$, and let $R = 3^{p-1}C_0^p\big(E\|\varphi(0)\|_\gamma^p + 1\big)$. Note that for any $\rho > 0$, the norms $\big(\sup_{t\in[0,T]} E\|u(t)\|_\gamma^p\big)^{\frac{1}{p}}$ and $\big(\sup_{t\in[0,T]} e^{-\rho t} E\|u(t)\|_\gamma^p\big)^{\frac{1}{p}}$ are equivalent. Now we consider

$$B(R) = \big\{u \in C\big(0, T; L^p(\Omega; H^\gamma)\big) : u(0) = \varphi(0), \sup_{t\in[0,T]} e^{-\rho t} E\|u(t)\|_\gamma^p \leq R\big\},$$

and define the mapping $\mathcal{M}$ by

$$(\mathcal{M}u)(t) = S(t)\varphi(0) + \int_0^t S(t-\tau)f(\tau, u\vee_\tau\varphi)d\tau + \int_0^t S(t-\tau)g(\tau, u\vee_\tau\varphi)dB^{\sigma,\lambda}(\tau). \tag{3.3}$$

In order to show that $\mathcal{M}$ has a fixed point in $B(R)$, we split the proof into three steps.

Step 1. $\mathcal{M}$ maps $B(R)$ into $C\big(0, T; L^p(\Omega; H^\gamma)\big)$.

Let $0 < t < T$ and $u \in B(R)$ be given arbitrarily. Then, for $s > 0$ small enough, we have

$$\begin{aligned}
E\big\|(\mathcal{M}u)(t+s) - (\mathcal{M}u)(t)\big\|_\gamma^p &\leq 5^{p-1}E\big\|S(t+s)\varphi(0) - S(t)\varphi(0)\big\|_\gamma^p \\
&+ 5^{p-1}E\Big\|\int_0^t \big(S(t+s-\tau) - S(t-\tau)\big)f(\tau, u\vee_\tau\varphi)d\tau\Big\|_\gamma^p \\
&+ 5^{p-1}E\Big\|\int_t^{t+s} S(t+s-\tau)f(\tau, u\vee_\tau\varphi)d\tau\Big\|_\gamma^p \\
&+ 5^{p-1}E\Big\|\int_0^t \big(S(t+s-\tau) - S(t-\tau)\big)g(\tau, u\vee_\tau\varphi)dB^{\sigma,\lambda}(\tau)\Big\|_\gamma^p \\
&+ 5^{p-1}E\Big\|\int_t^{t+s} S(t+s-\tau)g(\tau, u\vee_\tau\varphi)dB^{\sigma,\lambda}(\tau)\Big\|_\gamma^p \\
&:= V_1 + V_2 + V_3 + V_4 + V_5,
\end{aligned} \tag{3.4}$$

where we have used $\big(\sum_{i=1}^m b_i\big)^l \leq m^{l-1}\sum_{i=1}^m b_i^l$ for $1 \leq l < \infty$. Using the properties $(\mathcal{P}_1)$-$(\mathcal{P}_2)$, we obtain

$$\begin{aligned}
V_1 &= 5^{p-1}E\big\|A^\gamma S(t)\big(S(s) - I\big)\varphi(0)\big\|^p \\
&\leq \mathbb{C}(\gamma, p)E\big(e^{-\delta t}t^{-\gamma}s^\gamma\big\|A^\gamma\varphi(0)\big\|\big)^p \\
&\leq \mathbb{C}(\gamma, p)e^{-p\delta t}t^{-p\gamma}s^{p\gamma}\|\varphi\|_{\mathcal{C}^p(H^\gamma)}^p \to 0 \quad \text{as} \quad s \to 0.
\end{aligned} \tag{3.5}$$

Given $\epsilon > 0$, in view of Hölder's inequality, the properties $(\mathcal{P}_1)$-$(\mathcal{P}_2)$ and the assumption (H_2), we find that for χ and s small enough,

$$\begin{aligned}
V_2 &\le 5^{p-1}E\Big(\int_0^t \big\|\big(S(t+s-\tau)-S(t-\tau)\big)f(\tau,u\vee_\tau\varphi)\big\|_\gamma d\tau\Big)^p \\
&\le 10^{p-1}E\Big(\int_0^{t-\chi}\big\|A^\gamma S(t-\chi-\tau)\big(S(s)-I\big)S(\chi)f(\tau,u\vee_\tau\varphi)\big\|d\tau\Big)^p \\
&\quad + 10^{p-1}E\Big(\int_{t-\chi}^t\big\|A^\gamma S(t-\tau)\big(S(s)-I\big)f(\tau,u\vee_\tau\varphi)\big\|d\tau\Big)^p \\
&\le 10^{p-1}E\Big(\int_0^{t-\chi}C_{\gamma,0}e^{-\delta(t-\chi-\tau)}(t-\chi-\tau)^{-\gamma}C_\gamma s^\gamma \\
&\quad\times\big\|A^\gamma S(\chi)f(\tau,u\vee_\tau\varphi)\big\|d\tau\Big)^p \\
&\quad + 10^{p-1}E\Big(\int_{t-\chi}^t C_{\gamma,0}e^{-\delta(t-\tau)}(t-\tau)^{-\gamma}\big\|\big(S(s)-I\big)f(\tau,u\vee_\tau\varphi)\big\|d\tau\Big)^p \\
&\le \mathbb{C}(\gamma,p)s^{p\gamma}\Big(\int_0^{t-\chi}(t-\chi-\tau)^{-\frac{p\gamma}{p-1}}\chi^{-\frac{p\gamma}{p-1}}d\tau\Big)^{p-1} \\
&\quad\times\int_0^{t-\chi}E\big\|f(\tau,u\vee_\tau\varphi)\big\|^p d\tau \\
&\quad + \mathbb{C}(\gamma,p)\Big(\int_{t-\chi}^t(t-\tau)^{-\frac{p\gamma}{p-1}}d\tau\Big)^{p-1}\int_{t-\chi}^t E\big\|f(\tau,u\vee_\tau\varphi)\big\|^p d\tau \\
&\le \mathbb{C}(\gamma,p,l_f)s^{p\gamma}\chi^{-p\gamma}\Big(\int_0^{t-\chi}(t-\chi-\tau)^{-\frac{p\gamma}{p-1}}d\tau\Big)^{p-1} \\
&\quad\times\int_0^{t-\chi}e^{\rho\tau}e^{-\rho\tau}\big(1+\|u\vee_\tau\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)d\tau \\
&\quad + \mathbb{C}(\gamma,p,l_f)\Big(\int_{t-\chi}^t(t-\tau)^{-\frac{p\gamma}{p-1}}d\tau\Big)^{p-1}\int_{t-\chi}^t e^{\rho\tau}e^{-\rho\tau}\big(1+\|u\vee_\tau\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)d\tau \\
&\le \mathbb{C}(\gamma,p,l_f)s^{p\gamma}\chi^{-p\gamma}e^{\rho T}\big(1+R+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)\frac{(t-\chi)^{p-p\gamma}}{(1-\frac{p\gamma}{p-1})^{p-1}} \\
&\quad + \mathbb{C}(\gamma,p,l_f)e^{\rho T}\big(1+R+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)\frac{\chi^{p-p\gamma}}{(1-\frac{p\gamma}{p-1})^{p-1}} < \epsilon.
\end{aligned} \tag{3.6}$$

By Hölder's inequality, the property $(\mathcal{P}_1)$ and the assumption (H_2), we deduce that

$$\begin{aligned}
V_3 &\le 5^{p-1}E\Big(\int_t^{t+s}\big\|A^\gamma S(t+s-\tau)f(\tau,u\vee_\tau\varphi)\big\|d\tau\Big)^p \\
&\le 5^{p-1}C^p_{\gamma,0}\int_t^{t+s}E\|f(\tau,u\vee_\tau\varphi)\|^p d\tau\Big(\int_t^{t+s}(t+s-\tau)^{-\frac{p\gamma}{p-1}}d\tau\Big)^{p-1} \\
&\le \mathbb{C}(\gamma,p,l_f)\int_t^{t+s}e^{\rho\tau}e^{-\rho\tau}\big(1+\|u\vee_\tau\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)d\tau\Big(\int_t^{t+s}(t+s-\tau)^{-\frac{p\gamma}{p-1}}d\tau\Big)^{p-1} \\
&\le \mathbb{C}(\gamma,p,l_f)e^{\rho T}\big(1+R+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)\frac{s^{p-p\gamma}}{(1-\frac{p\gamma}{p-1})^{p-1}} \to 0 \quad \text{as} \quad s\to 0.
\end{aligned} \tag{3.7}$$

Similar to the above arguments, by using Lemma 2.2, we conclude that for χ and s small enough,

$$\begin{aligned}
V_4 &\le 10^{p-1}E\Big\|\int_0^{t-\chi}A^\gamma S(t-\chi-\tau)\big(S(s)-I\big)S(\chi)g(\tau,u\vee_\tau\varphi)dB^{\sigma,\lambda}(\tau)\Big\|^p\\
&\quad+10^{p-1}E\Big\|\int_{t-\chi}^{t}A^\gamma S(t-\tau)\big(S(s)-I\big)g(\tau,u\vee_\tau\varphi)dB^{\sigma,\lambda}(\tau)\Big\|^p\\
&\le \mathbb{C}(\gamma,p)(N_{t-\chi})^{\frac{p}{2}}E\Big(\int_0^{t-\chi}(t-\chi-\tau)^{-2\gamma}e^{-2\delta(t-\chi-\tau)}\\
&\quad\times\big\|\big(S(s)-I\big)S(\chi)g(\tau,u\vee_\tau\varphi)\big\|^2d\tau\Big)^{\frac{p}{2}}\\
&\quad+\mathbb{C}(\gamma,p)(N_\chi)^{\frac{p}{2}}E\Big(\int_{t-\chi}^{t}e^{-2\delta(t-\tau)}(t-\tau)^{-2\gamma}\big\|\big(S(s)-I\big)g(\tau,u\vee_\tau\varphi)\big\|^2d\tau\Big)^{\frac{p}{2}}\\
&\le \mathbb{C}(\gamma,p)(N_{t-\chi})^{\frac{p}{2}}s^{p\gamma}E\Big(\int_0^{t-\chi}(t-\chi-\tau)^{-2\gamma}\big\|A^\gamma S(\chi)g(\tau,u\vee_\tau\varphi)\big\|^2d\tau\Big)^{\frac{p}{2}}\\
&\quad+\mathbb{C}(\gamma,p)(N_\chi)^{\frac{p}{2}}E\Big(\int_{t-\chi}^{t}e^{-2\delta(t-\tau)}(t-\tau)^{-2\gamma}\|g(\tau,u\vee_\tau\varphi)\|^2d\tau\Big)^{\frac{p}{2}}\\
&\le \mathbb{C}(\gamma,p,l_g)(N_{t-\chi})^{\frac{p}{2}}s^{p\gamma}(t-\chi)^{\frac{p-2}{2}}\int_0^{t-\chi}(t-\chi-\tau)^{-p\gamma}\chi^{-p\gamma}e^{\rho\tau}\\
&\quad\times e^{-\rho\tau}\big(1+\|u\vee_\tau\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)d\tau\\
&\quad+\mathbb{C}(\gamma,p,l_g)(N_\chi)^{\frac{p}{2}}\chi^{\frac{p-2}{2}}\int_{t-\chi}^{t}(t-\tau)^{-p\gamma}e^{\rho\tau}e^{-\rho\tau}\big(1+\|u\vee_\tau\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)d\tau\\
&\le \mathbb{C}(\gamma,p,l_g)(N_{t-\chi})^{\frac{p}{2}}s^{p\gamma}\chi^{-p\gamma}e^{\rho T}\big(1+R+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)\frac{(t-\chi)^{\frac{p}{2}-p\gamma}}{1-p\gamma}\\
&\quad+\mathbb{C}(\gamma,p,l_g)(N_\chi)^{\frac{p}{2}}e^{\rho T}\big(1+R+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)\frac{\chi^{\frac{p}{2}-p\gamma}}{1-p\gamma}<\epsilon,
\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}
V_5 &\le 5^{p-1}C_p(N_s)^{\frac{p}{2}}E\Big(\int_t^{t+s}\big\|A^\gamma S(t+s-\tau)g(\tau,u\vee_\tau\varphi)\big\|^2d\tau\Big)^{\frac{p}{2}}\\
&\le \mathbb{C}(\gamma,p,l_g)(N_s)^{\frac{p}{2}}s^{\frac{p-2}{2}}\int_t^{t+s}(t+s-\tau)^{-p\gamma}e^{\rho\tau}e^{-\rho\tau}\big(1+\|u\vee_\tau\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)d\tau\\
&\le \mathbb{C}(\gamma,p,l_g)(N_s)^{\frac{p}{2}}e^{\rho T}\big(1+R+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)\frac{s^{\frac{p}{2}-p\gamma}}{1-p\gamma}\to 0\quad\text{as}\quad s\to 0.
\end{aligned}\tag{3.9}$$

Substituting the estimates of terms V_1-V_5 into (3.4) yields that $E\big\|(\mathcal{M}u)(t+s)-(\mathcal{M}u)(t)\big\|^p_\gamma\to 0$ as $s\to 0$, which implies that $\mathcal{M}u\in C\big(0,T;L^p(\Omega;H^\gamma)\big)$.

Step 2. $\mathcal{M}$ maps $B(R)$ into itself.

Let $u \in B(R)$. Then for $t \in [0,T]$, the definition of $\mathcal{M}$ immediately implies

$$\begin{aligned} e^{-\rho t}E\big\|\big(\mathcal{M}u\big)(t)\big\|_\gamma^p &\le 3^{p-1}e^{-\rho t}E\big\|A^\gamma S(t)\varphi(0)\big\|^p \\ &\quad + 3^{p-1}e^{-\rho t}E\Big\|\int_0^t A^\gamma S(t-\tau)f(\tau, u\vee_\tau \varphi)d\tau\Big\|^p \\ &\quad + 3^{p-1}e^{-\rho t}E\Big\|\int_0^t A^\gamma S(t-\tau)g(\tau, u\vee_\tau \varphi)dB^{\sigma,\lambda}(\tau)\Big\|^p \\ &:= \widehat{V}_1+\widehat{V}_2+\widehat{V}_3. \end{aligned} \tag{3.10}$$

On account of the property $(\mathcal{P}_1)$, we obtain

$$\widehat{V}_1 \le 3^{p-1}C_0^p e^{-\rho t}E\|\varphi(0)\|_\gamma^p. \tag{3.11}$$

Using again the property $(\mathcal{P}_1)$ and the assumption (H_2), in view of Hölder's inequality, it follows that

$$\begin{aligned} \widehat{V}_2 &\le 3^{p-1}E\Big(\int_0^t e^{-\frac{\rho(t-\tau)}{p}}e^{-\frac{\rho\tau}{p}}\big\|A^\gamma S(t-\tau)f(\tau,u\vee_\tau\varphi)\big\|d\tau\Big)^p \\ &\le 3^{p-1}C_{\gamma,0}^p\Big(\int_0^t e^{-\frac{\rho(t-\tau)}{p-1}}(t-\tau)^{-\frac{p\gamma}{p-1}}d\tau\Big)^{p-1}\int_0^t e^{-\rho\tau}E\|f(\tau,u\vee_\tau\varphi)\|^p d\tau \\ &\le 6^{p-1}C_{\gamma,0}^p t l_f\big(1+R+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)\big(\frac{p-1}{\rho}\big)^{p-1-p\gamma}\big(\Gamma(1-\frac{p\gamma}{p-1})\big)^{p-1}. \end{aligned} \tag{3.12}$$

In a similar way as in (3.12), by Lemma 2.2 we have

$$\begin{aligned} \widehat{V}_3 &\le 3^{p-1}C_p(N_t)^{\frac{p}{2}}E\Big(\int_0^t e^{-\frac{2\rho(t-\tau)}{p}}e^{-\frac{2\rho\tau}{p}}\big\|A^\gamma S(t-\tau)g(\tau,u\vee_\tau\varphi)\big\|^2d\tau\Big)^{\frac{p}{2}} \\ &\le 3^{p-1}C_pC_{\gamma,0}^p(N_t)^{\frac{p}{2}}t^{\frac{p-2}{2}}\int_0^t e^{-\rho(t-\tau)}(t-\tau)^{-p\gamma}e^{-\rho\tau}E\big\|g(\tau,u\vee_\tau\varphi)\big\|^p d\tau \\ &\le 6^{p-1}C_p(N_t)^{\frac{p}{2}}C_{\gamma,0}^p l_g\big(1+R+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)t^{\frac{p-2}{2}}\int_0^t(t-\tau)^{-p\gamma}e^{-\rho(t-\tau)}d\tau \\ &\le 6^{p-1}C_{\gamma,0}^pC_p(N_t)^{\frac{p}{2}}l_g\big(1+R+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)t^{\frac{p-2}{2}}\big(\frac{1}{\rho}\big)^{1-p\gamma}\Gamma(1-p\gamma). \end{aligned} \tag{3.13}$$

Therefore, given $T>0$, we can choose $\rho>0$ sufficiently large such that

$$\begin{aligned} &6^{p-1}C_{\gamma,0}^pTl_f\big(1+R+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)\big(\frac{p-1}{\rho}\big)^{p-1-p\gamma}\big(\Gamma(1-\frac{p\gamma}{p-1})\big)^{p-1} \\ &+6^{p-1}C_{\gamma,0}^pC_p(N_T)^{\frac{p}{2}}l_g\big(1+R+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)T^{\frac{p-2}{2}}\big(\frac{1}{\rho}\big)^{1-p\gamma}\Gamma(1-p\gamma)<3^{p-1}C_0^p. \end{aligned} \tag{3.14}$$

Then it follows directly from (3.10)-(3.14) that $\mathcal{M}$ maps $B(R)$ into itself.
Step 3. The operator $\mathcal{M}: B(R)\to B(R)$ is a contraction mapping.

By applying Hölder's inequality, Lemma 2.2, the property $(\mathcal{P}_1)$ and the assumption (H_3), we have that for $u, v \in B(R)$,

$$\begin{aligned}
& e^{-\rho t} E\big\|\big(\mathcal{M}u\big)(t) - \big(\mathcal{M}v\big)(t)\big\|_\gamma^p \\
& \leq 2^{p-1} E\Big(\int_0^t e^{-\frac{\rho(t-\tau)}{p}} e^{-\frac{\rho\tau}{p}} \big\|A^\gamma S(t-\tau)\big(f(\tau, u\vee_\tau \varphi) - f(\tau, v \vee_\tau \varphi)\big)\big\| d\tau\Big)^p \\
& \quad + 2^{p-1} C_p (N_t)^{\frac{p}{2}} E\Big(\int_0^t e^{-\frac{2\rho(t-\tau)}{p}} e^{-\frac{2\rho\tau}{p}} \big\|A^\gamma S(t-\tau)\big(g(\tau, u \vee_\tau \varphi) \\
& \quad - g(\tau, v\vee_\tau \varphi)\big)\big\|^2 d\tau\Big)^{\frac{p}{2}} \\
& \leq 2^{p-1} C_{\gamma,0}^p E\Big(\int_0^t e^{-\frac{\rho(t-\tau)}{p}} e^{-\delta(t-\tau)} (t-\tau)^{-\gamma} e^{-\frac{\rho\tau}{p}} \big\|f(\tau, u\vee_\tau \varphi) \\
& \quad - f(\tau, v\vee_\tau\varphi)\big\| d\tau\Big)^p + 2^{p-1} C_p (N_t)^{\frac{p}{2}} C_{\gamma,0}^p t^{\frac{p-2}{2}} \int_0^t e^{-\rho(t-\tau)} (t-\tau)^{-p\gamma} e^{-\rho\tau} \\
& \quad \times E\big\|g(\tau, u\vee_\tau\varphi) - g(\tau, v\vee_\tau \varphi)\big\|^p d\tau \\
& \leq 2^{p-1} C_{\gamma,0}^p \big(\frac{p-1}{\rho}\big)^{p-1-p\gamma} \big(\Gamma(1 - \frac{p\gamma}{p-1})\big)^{p-1} t L_f \sup_{\tau\in[0,t]} e^{-\rho\tau} E\|u(\tau) - v(\tau)\|_\gamma^p \\
& \quad + 2^{p-1} C_p C_{\gamma,0}^p (N_t)^{\frac{p}{2}} t^{\frac{p-2}{2}} \big(\frac{1}{\rho}\big)^{1-p\gamma} \Gamma(1-p\gamma) L_g \sup_{\tau\in[0,t]} e^{-\rho\tau} E\|u(\tau) - v(\tau)\|_\gamma^p.
\end{aligned} \tag{3.15}$$

Notice that for sufficiently large $\rho > 0$,

$$\begin{aligned}
& 2^{p-1} C_{\gamma,0}^p \big(\frac{p-1}{\rho}\big)^{p-1-p\gamma} \big(\Gamma(1-\frac{p\gamma}{p-1})\big)^{p-1} T L_f \\
& + 2^{p-1} C_p C_{\gamma,0}^p (N_T)^{\frac{p}{2}} T^{\frac{p-2}{2}} \big(\frac{1}{\rho}\big)^{1-p\gamma} \Gamma(1-p\gamma) L_g < 1,
\end{aligned} \tag{3.16}$$

which means that the mapping $\mathcal{M} : B(R) \to B(R)$ is contractive. Thus, the assertion of this theorem follows immediately from the Banach fixed point theorem. □

Remark 3.1. Note that Theorem 3.1 ensures that for any given $T > 0$, problem (1.1) has a unique mild solution u on $[0, T]$ for each initial data φ. Thus the solution u can be globally defined.

In view of (2.13) and (2.14), the following result can be obtained by slightly modifying the proof of Theorem 3.1.

Corollary 3.1. *Let $p \geq 2$. Suppose that assumptions (H_1)-(H_3) and $0 < \gamma < \frac{1}{p}$ hold. Then for each $\varphi \in \mathcal{C}^p(H^\gamma)$, there exists a unique global mild solution for problem* (1.1) *with FBM or Brownian motion instead of TFBM.*

4. Continuity of solutions with respect to tempered parameter λ

In this section we shall show that mild solutions to Eq. (1.1) are continuous with respect to tempered parameter λ at 0. First, we state the following technical lemma.

Lemma 4.1. *Let $p \geq 2$, $-\frac{1}{2} < \sigma < 0$ and $\lambda > 0$. If $\phi_1, \phi_2 : [0,T] \times \Omega \to \mathcal{L}^2$ are progressively measurable functions satisfying $\int_0^T E\|\phi_1(s)\|^p ds < \infty$ and $\int_0^T E\|\phi_2(s)\|^p ds < \infty$, then for any $t \in [0,T]$,*

$$
\begin{aligned}
&E\Big\|\int_0^t \phi_1(s)dB^{\sigma,\lambda}(s) - \int_0^t \phi_2(s)dB^{\sigma,0}(s)\Big\|^p\\
&\leq 2^{p-2}\big(4H-2\big)^{\frac{p}{2}} C_p t^{pH-1}\big(\beta(2-2H, H-\frac{1}{2})\big)^{\frac{p}{2}} \int_0^t E\|\phi_1(s)-\phi_2(s)\|^p ds\\
&\quad + 2^{p-2}\big(4H-2\big)^{\frac{p}{2}} \lambda^p t^{p(H+1)-1} C_p(\beta(2-2H, H+\frac{1}{2}))^{\frac{p}{2}} \int_0^t E\|\phi_2(s)\|^p ds\\
&\quad + 2^{2p-1}\lambda^p C_p \frac{t^{p(H+1)-1}}{(4H-2)^{\frac{p}{2}}}(\beta(2-2H, H+\frac{1}{2}))^{\frac{p}{2}} \int_0^t E\|\phi_1(s)\|^p ds,
\end{aligned}
$$

where $H = \frac{1}{2} - \sigma$.

Proof. Following similar arguments as in the proof of Lemma 2.2, we obtain that $\big({}_s\mathbb{I}_t^{-\sigma,\lambda}\phi_1(s) - \lambda {}_s\mathbb{I}_t^{1-\sigma,\lambda}\phi_1(s)\big)$ and ${}_s\mathbb{I}_t^{-\sigma,0}\phi_2(s)$ are progressively measurable. Then by using Definitions 2.4-2.5 and Lemma 2.1, we find that

$$
\begin{aligned}
&E\Big\|\int_0^t \phi_1(s)dB^{\sigma,\lambda}(s) - \int_0^t \phi_2(s)dB^{\sigma,0}(s)\Big\|^p\\
&\leq 2^{p-1}\lambda^p(\Gamma(1-\sigma))^p E\Big\|\int_0^t {}_s\mathbb{I}_t^{1-\sigma,\lambda}\phi_1(s)dB(s)\Big\|^p\\
&\quad + 2^{p-1}(\Gamma(1-\sigma))^p E\Big\|\int_0^t {}_s\mathbb{I}_t^{-\sigma,\lambda}\phi_1(s) - {}_s\mathbb{I}_t^{-\sigma,0}\phi_2(s)dB(s)\Big\|^p\\
&= 2^{p-1}(-\sigma)^p E\Big\|\int_0^t\int_s^t \big(\phi_1(u)e^{-\lambda(u-s)} - \phi_2(u)\big)(u-s)^{-\sigma-1}dudB(s)\Big\|^p \qquad (4.1)\\
&\quad + 2^{p-1}\lambda^p E\Big\|\int_0^t\int_s^t \phi_1(u)(u-s)^{-\sigma}e^{-\lambda(u-s)}dudB(s)\Big\|^p\\
&\leq 2^{p-1}(-\sigma)^p C_p E\Big(\int_0^t\Big(\int_s^t \big\|\phi_1(u)e^{-\lambda(u-s)} - \phi_2(u)\big\|(u-s)^{-\sigma-1}du\Big)^2 ds\Big)^{\frac{p}{2}}\\
&\quad + 2^{p-1}\lambda^p C_p E\Big(\int_0^t\Big(\int_s^t \|\phi_1(u)\|(u-s)^{-\sigma}e^{-\lambda(u-s)}du\Big)^2 ds\Big)^{\frac{p}{2}} := \Upsilon_1 + \Upsilon_2.
\end{aligned}
$$

For the term Υ_1, we have

$$
\begin{aligned}
\Upsilon_1 &= 2^{p-1}(-\sigma)^p C_p E\Big(\int_0^t \Big(\int_s^t \big\|\phi_1(u)e^{-\lambda(u-s)} - \phi_2(u)e^{-\lambda(u-s)} \\
&\quad + \phi_2(u)e^{-\lambda(u-s)} - \phi_2(u)\big\|(u-s)^{-\sigma-1}du\Big)^2 ds\Big)^{\frac{p}{2}} \\
&\le 2^{\frac{3p}{2}-1}(-\sigma)^p C_p E\Big(\int_0^t \Big[\Big(\int_s^t \big\|\phi_1(u)-\phi_2(u)\big\|e^{-\lambda(u-s)}(u-s)^{-\sigma-1}du\Big)^2 \\
&\quad + \Big(\int_s^t \|\phi_2(u)\|\big(e^{-\lambda(u-s)}-1\big)(u-s)^{-\sigma-1}du\Big)^2\Big]ds\Big)^{\frac{p}{2}} \\
&\le 2^{2p-2}(-\sigma)^p C_p E\Big(\int_0^t \Big(\int_s^t \big\|\phi_1(u)-\phi_2(u)\big\|e^{-\lambda(u-s)}(u-s)^{-\sigma-1}du\Big)^2 ds\Big)^{\frac{p}{2}} \\
&\quad + 2^{2p-2}(-\sigma)^p C_p E\Big(\int_0^t \Big(\int_s^t \|\phi_2(u)\|\big(e^{-\lambda(u-s)}-1\big)(u-s)^{-\sigma-1}du\Big)^2 ds\Big)^{\frac{p}{2}} \\
&:= \Upsilon_1^1 + \Upsilon_1^2.
\end{aligned} \tag{4.2}
$$

Applying inequality (2.12), the mean value theorem and Hölder's inequality to the term Υ_1^2, we obtain

$$
\begin{aligned}
\Upsilon_1^2 &= 2^{2p-2}(-\sigma)^p \lambda^p C_p E\Big(\int_0^t \Big(\int_s^t \|\phi_2(u)\|e^{-\xi(u-s)}(u-s)^{-\sigma}du\Big)^2 ds\Big)^{\frac{p}{2}} \\
&\le \frac{C_p}{4}(4\lambda(-\sigma))^p E\Big(\int_0^t\int_s^t\int_s^t \|\phi_2(u)\|\|\phi_2(r)\|(u-s)^{-\sigma}(r-s)^{-\sigma}drduds\Big)^{\frac{p}{2}} \\
&= \frac{C_p}{4}(4\lambda(-\sigma))^p E\Big(\int_0^t\int_0^t\int_0^{u\wedge r} \|\phi_2(u)\|\|\phi_2(r)\|(u-s)^{-\sigma}(r-s)^{-\sigma}dsdudr\Big)^{\frac{p}{2}} \\
&\le \frac{C_p}{4}(4\lambda t(-\sigma))^p E\Big(\int_0^t\int_0^t \|\phi_2(r)\|^2|u-r|^{-2\sigma-1}\beta(1+2\sigma,1-\sigma)dudr\Big)^{\frac{p}{2}} \\
&\le 2^{2p-2}(-\sigma)^{\frac{p}{2}}\lambda^p t^{p(H+1)-1}C_p(\beta(2-2H,H+\frac{1}{2}))^{\frac{p}{2}}\int_0^t E\|\phi_2(r)\|^p dr,
\end{aligned} \tag{4.3}
$$

where $0 < \xi < \lambda$. Using inequality (2.11) and Hölder's inequality, we have

$$\begin{aligned}
\Upsilon_1^1 &\le 2^{2p-2}(-\sigma)^p C_p E\Big(\int_0^t\int_s^t\int_s^t \|\phi_1(u)-\phi_2(u)\|\|\phi_1(r)-\phi_2(r)\| \\
&\quad \times (u-s)^{-\sigma-1}(r-s)^{-\sigma-1}dudrds\Big)^{\frac{p}{2}} \\
&= 2^{2p-2}(-\sigma)^p C_p E\Big(\int_0^t\int_0^t\int_0^{u\wedge r} \|\phi_1(u)-\phi_2(u)\|\|\phi_1(r)-\phi_2(r)\| \\
&\quad \times (u-s)^{-\sigma-1}(r-s)^{-\sigma-1}dsdudr\Big)^{\frac{p}{2}} \\
&\le \frac{C_p}{4}(-4\sigma)^p E\Big(\int_0^t\int_0^t \|\phi_1(r)-\phi_2(r)\|^2|u-r|^{-2\sigma-1}\beta(1+2\sigma,-\sigma)dudr\Big)^{\frac{p}{2}} \\
&\le 2^{2p-2}(-\sigma)^{\frac{p}{2}} C_p t^{pH-1}\big(\beta(2-2H,H-\frac{1}{2})\big)^{\frac{p}{2}}\int_0^t E\|\phi_1(r)-\phi_2(r)\|^p dr.
\end{aligned} \tag{4.4}$$

Then for the term Υ_2, by inequality (2.12) and Hölder's inequality, we deduce that

$$\begin{aligned}
\Upsilon_2 &\le 2^{p-1}\lambda^p C_p E\Big(\int_0^t\int_s^t\int_s^t \|\phi_1(u)\|\|\phi_1(r)\|(u-s)^{-\sigma}(r-s)^{-\sigma}dudrds\Big)^{\frac{p}{2}} \\
&= 2^{p-1}\lambda^p C_p E\Big(\int_0^t\int_0^t\int_0^{u\wedge r} \|\phi_1(u)\|\|\phi_1(r)\|(u-s)^{-\sigma}(r-s)^{-\sigma}dsdudr\Big)^{\frac{p}{2}} \\
&\le 2^{p-1}\lambda^p t^p C_p E\Big(\int_0^t\int_0^t \|\phi_1(r)\|^2|u-r|^{-2\sigma-1}\beta(1+2\sigma,1-\sigma)dudr\Big)^{\frac{p}{2}} \\
&\le 2^{p-1}\lambda^p C_p \frac{t^{p(H+1)-1}}{(-\sigma)^{\frac{p}{2}}}(\beta(2-2H,H+\frac{1}{2}))^{\frac{p}{2}}\int_0^t E\|\phi_1(r)\|^p dr.
\end{aligned} \tag{4.5}$$

Inserting (4.2)-(4.5) into (4.1) gives the assertion of the lemma. □

Furthermore, we need the following uniform (w.r.t. $\lambda \in (0,1]$) estimates of solutions.

Theorem 4.1. *Let u be the mild solution to Eq.* (1.1) *and let assumptions in Theorem* 3.1 *hold. Then for each $\varphi \in \mathcal{C}^p(H^\gamma)$, any $T > 0$ and all $\lambda \in (0,1]$,*

$$\sup_{r\in[0,T]} E\|u(r)\|_\gamma^p \le \mathbb{C}\big(1+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big), \tag{4.6}$$

where $\mathbb{C}$ is independent of λ.

Proof. By Definition 3.1, we obtain that for $t \in [0,T]$,

$$\begin{aligned}
E\|u(t)\|_\gamma^p &\le 3^{p-1}E\big\|S(t)\varphi(0)\big\|_\gamma^p + 3^{p-1}E\Big\|\int_0^t S(t-\tau)f(\tau,u_\tau)d\tau\Big\|_\gamma^p \\
&\quad + 3^{p-1}E\Big\|\int_0^t S(t-\tau)g(\tau,u_\tau)dB^{\sigma,\lambda}(\tau)\Big\|_\gamma^p
\end{aligned} \tag{4.7}$$

$$:= \widetilde{V}_1 + \widetilde{V}_2 + \widetilde{V}_3.$$

In view of the assumption (H_1), we have

$$\widetilde{V}_1 \le \mathbb{C}(p) E\|\varphi(0)\|_\gamma^p. \tag{4.8}$$

Since $p\gamma$ takes values in $(0,1)$, we can choose $q' > 1$ such that $p\gamma q' < 1$. Using Hölder's inequality, the property $(\mathcal{P}_1)$ and the assumption (H_2), we find that

$$\begin{aligned}
\widetilde{V}_2 &\le \mathbb{C}(\gamma, p) t^{p-1} \int_0^t (t-\tau)^{-p\gamma} E\|f(\tau, u_\tau)\|^p d\tau \\
&\le \mathbb{C}(\gamma, p, l_f)\big(1 + \|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big) \frac{t^{p(1-\gamma)}}{1-p\gamma} \\
&\quad + \mathbb{C}(\gamma, p, l_f) t^{p-1} \int_0^t (t-\tau)^{-p\gamma} \sup_{r\in[0,\tau]} E\|u(r)\|_\gamma^p d\tau \\
&\le \mathbb{C}(\gamma, p, l_f)\big(1 + \|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big) \frac{t^{p(1-\gamma)}}{1-p\gamma} \\
&\quad + \mathbb{C}(\gamma, p, l_f) t^{p-1} \Big(\int_0^t (t-\tau)^{-pq'\gamma} d\tau\Big)^{\frac{1}{q'}} \Big(\int_0^t \big(\sup_{r\in[0,\tau]} E\|u(r)\|_\gamma^p\big)^{p'} d\tau\Big)^{\frac{1}{p'}} \\
&= \mathbb{C}(\gamma, p, l_f)\big(1 + \|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big) \frac{t^{p-p\gamma}}{1-p\gamma} \\
&\quad + \frac{\mathbb{C}(\gamma, p, l_f) t^{p(1-\gamma)-\frac{1}{p'}}}{(1-pq'\gamma)^{\frac{1}{q'}}} \Big(\int_0^t \big(\sup_{r\in[0,\tau]} E\|u(r)\|_\gamma^p\big)^{p'} d\tau\Big)^{\frac{1}{p'}},
\end{aligned} \tag{4.9}$$

where $p' > 1$ is a constant such that $1/p' + 1/q' = 1$. Thanks to Lemma 2.2, by a similar way as in (4.9), it follows that

$$\begin{aligned}
\widetilde{V}_3 &\le \mathbb{C}(p)(N_t)^{\frac{p}{2}} E\Big(\int_0^t \big\|A^\gamma S(t-\tau) g(\tau, u_\tau)\big\|^2 d\tau\Big)^{\frac{p}{2}} \\
&\le \mathbb{C}(\gamma, p)(N_t)^{\frac{p}{2}} t^{\frac{p}{2}-1} \int_0^t (t-\tau)^{-p\gamma} E\|g(\tau, u_\tau)\|^p d\tau \\
&\le \mathbb{C}(\gamma, p, l_g)(N_t)^{\frac{p}{2}} \frac{t^{\frac{p}{2}-p\gamma}}{1-p\gamma}\big(1 + \|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big) + \mathbb{C}(\gamma, p, l_g)(N_t)^{\frac{p}{2}} t^{\frac{p}{2}-1} \\
&\quad \times \Big(\int_0^t (t-\tau)^{-pq'\gamma} d\tau\Big)^{\frac{1}{q'}} \Big(\int_0^t \big(\sup_{r\in[0,\tau]} E\|u(r)\|_\gamma^p\big)^{p'} d\tau\Big)^{\frac{1}{p'}} \\
&\le \mathbb{C}(\gamma, p, l_g)\Big((2H-1) t^{2H-1} \beta(2-2H, H-\frac{1}{2}) \\
&\quad + 4t^{2H+1} \frac{\beta(2-2H, H+\frac{1}{2})}{2H-1}\Big)^{\frac{p}{2}} \Big(\frac{t^{\frac{p}{2}-p\gamma}}{1-p\gamma}\big(1 + \|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big) \\
&\quad + \frac{t^{p(\frac{1}{2}-\gamma)-\frac{1}{p'}}}{(1-pq'\gamma)^{\frac{1}{q'}}} \Big(\int_0^t \big(\sup_{r\in[0,\tau]} E\|u(r)\|_\gamma^p\big)^{p'} d\tau\Big)^{\frac{1}{p'}}\Big).
\end{aligned} \tag{4.10}$$

Inserting (4.8)-(4.10) into (4.7) yields

$$\begin{aligned}\big(\sup_{r\in[0,t]} E\|u(r)\|_\gamma^p\big)^{p'} \le{}& \mathbb{C}(\gamma,p,l_g,l_f,H,q',p',T)\\ &\times\bigg(\big(1+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)^{p'}+\int_0^t\big(\sup_{r\in[0,\tau]}E\|u(r)\|_\gamma^p\big)^{p'}d\tau\bigg).\end{aligned}\tag{4.11}$$

The assertion of this theorem follows immediately by applying Gronwall's lemma to (4.11). □

Arguing as in the proof of Theorem 4.1, we have

Corollary 4.1. *Let u be the mild solution to Eq.* (1.1) *with FBM $B^{\sigma,0}$ instead of TFBM $B^{\sigma,\lambda}$. Suppose that the assumptions in Corollary* 3.1 *hold. Then for each $\varphi\in\mathcal{C}^p(H^\gamma)$ and any $T>0$,*

$$\sup_{r\in[0,T]} E\|u(r)\|_\gamma^p \le \mathbb{C}\big(1+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big),\tag{4.12}$$

where $\mathbb{C}$ is a constant.

Now we are ready to prove that the mild solution $u^{\sigma,\lambda}$ of (1.1) converges to the mild solution $u^{\sigma,0}$ of (1.1) but with FBM $B^{\sigma,0}$ instead of TFBM $B^{\sigma,\lambda}$ as tempered parameter $\lambda\to 0$.

Theorem 4.2. *Suppose that the assumptions in Theorem* 3.1 *hold. Then for any $T>0$,*

$$\sup_{0\le\tau\le T} E\|u^{\sigma,\lambda}(\tau)-u^{\sigma,0}(\tau)\|_\gamma^p\to 0 \quad as \quad \lambda\to 0,$$

where $u^{\sigma,\lambda}$ and $u^{\sigma,0}$, respectively, are mild solutions to Eq. (1.1) *driven by TFBM $B^{\sigma,\lambda}$ and FBM $B^{\sigma,0}$ instead of $B^{\sigma,\lambda}$ with the same initial data $\varphi\in\mathcal{C}^p(H^\gamma)$.*

Proof. By Hölder's inequality and the property $(\mathcal{P}_1)$, we have

$$
\begin{aligned}
&E\big\|u^{\sigma,\lambda}(t)-u^{\sigma,0}(t)\big\|_\gamma^p\\
&\le 2^{p-1}\Big[E\Big(\int_0^t \big\|A^\gamma S(t-\tau)\big(f(\tau,u_\tau^{\sigma,\lambda})-f(\tau,u_\tau^{\sigma,0})\big)\big\|d\tau\Big)^p\\
&+E\Big\|\int_0^t A^\gamma S(t-\tau)g(\tau,u_\tau^{\sigma,\lambda})dB^{\sigma,\lambda}(\tau)-\int_0^t A^\gamma S(t-\tau)g(\tau,u_\tau^{\sigma,0})dB^{\sigma,0}(\tau)\Big\|^p\Big]\\
&\le 2^{p-1}C_{\gamma,0}^p t^{p-1}\int_0^t (t-\tau)^{-p\gamma}E\big\|f(\tau,u_\tau^{\sigma,\lambda})-f(\tau,u_\tau^{\sigma,0})\big\|^p d\tau\\
&+2^{2p-3}C_{\gamma,0}^p\big(4H-2\big)^{\frac{p}{2}}C_p t^{pH-1}\big(\beta(2-2H,H-\frac{1}{2})\big)^{\frac{p}{2}}\\
&\quad\times\int_0^t (t-\tau)^{-p\gamma}E\|g(\tau,u_\tau^{\sigma,\lambda})-g(\tau,u_\tau^{\sigma,0})\|^p d\tau\\
&+2^{2p-3}C_{\gamma,0}^p\big(4H-2\big)^{\frac{p}{2}}\lambda^p t^{p(H+1)-1}C_p(\beta(2-2H,H+\frac{1}{2}))^{\frac{p}{2}}\\
&\quad\times\int_0^t (t-\tau)^{-p\gamma}E\|g(\tau,u_\tau^{\sigma,0})\|^p d\tau\\
&+2^{3p-2}C_{\gamma,0}^p\lambda^p C_p\frac{t^{p(H+1)-1}}{(4H-2)^{\frac{p}{2}}}(\beta(2-2H,H+\frac{1}{2}))^{\frac{p}{2}}\\
&\quad\times\int_0^t (t-\tau)^{-p\gamma}E\|g(\tau,u_\tau^{\sigma,\lambda})\|^p d\tau,
\end{aligned}
\tag{4.13}
$$

thanks to Lemma 4.1. In view of $u^{\sigma,\lambda}(t)=u^{\sigma,0}(t)=\varphi(t)$ for each $t\in(-\infty,0]$, we obtain

$$
\begin{aligned}
\|u_\tau^{\sigma,\lambda}-u_\tau^{\sigma,0}\|_{\mathcal{C}^p(H^\gamma)}^p &= \sup_{r\in(-\infty,\tau]} E\|u^{\sigma,\lambda}(r)-u^{\sigma,0}(r)\|_\gamma^p\\
&= \sup_{r\in[0,\tau]} E\|u^{\sigma,\lambda}(r)-u^{\sigma,0}(r)\|_\gamma^p.
\end{aligned}
$$

It follows from the assumptions (H_2)-(H_3), (4.6) and (4.12) that

$$
\begin{aligned}
&E\|u^{\sigma,\lambda}(t)-u^{\sigma,0}(t)\|_\gamma^p \\
&\le 2^{p-1}\mathbb{C}(\gamma,L_f)t^{p-1}\int_0^t (t-\tau)^{-p\gamma}\sup_{r\in[0,\tau]} E\|u^{\sigma,\lambda}(r)-u^{\sigma,0}(r)\|_\gamma^p d\tau \\
&\quad + 2^{2p-3}\mathbb{C}(\gamma,L_g)C_p t^{pH-1}\big(\beta(2-2H,H-\frac{1}{2})\big)^{\frac{p}{2}}\big(4H-2\big)^{\frac{p}{2}}\int_0^t (t-\tau)^{-p\gamma} \\
&\quad\times \sup_{r\in[0,\tau]} E\|u^{\sigma,\lambda}(r)-u^{\sigma,0}(r)\|_\gamma^p d\tau \qquad\qquad (4.14)\\
&\quad + \lambda^p\mathbb{C}\big(1+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)\big(\beta(2-2H,H+\frac{1}{2})\big)^{\frac{p}{2}}C_p t^{p(H+1)-p\gamma} \\
&\quad\times\Big[\frac{2^{3p-4}(4H-2)^{\frac{p}{2}}}{1-p\gamma}+\frac{2^{4p-3}}{(1-p\gamma)(4H-2)^{\frac{p}{2}}}\Big] \\
&\le \lambda^p\widetilde{\Upsilon}_1(t)+\widetilde{\Upsilon}_2(t)\int_0^t (t-\tau)^{-p\gamma}\sup_{r\in[0,\tau]} E\|u^{\sigma,\lambda}(r)-u^{\sigma,0}(r)\|_\gamma^p d\tau,
\end{aligned}
$$

where we have used the notations

$$
\begin{aligned}
\widetilde{\Upsilon}_1(t) := \mathbb{C}(\beta(2-2H,H+\frac{1}{2}))^{\frac{p}{2}}\big(1+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)C_p t^{p(H+1)-p\gamma} \\
\times\Big[\frac{2^{3p-4}(4H-2)^{\frac{p}{2}}}{1-p\gamma}+\frac{2^{4p-3}}{(1-p\gamma)(4H-2)^{\frac{p}{2}}}\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{\Upsilon}_2(t) &:= 2^{p-1}\mathbb{C}(\gamma,L_f)t^{p-1} \\
&\quad + 2^{2p-3}\mathbb{C}(\gamma,L_g)C_p t^{pH-1}\big(\beta(2-2H,H-\frac{1}{2})\big)^{\frac{p}{2}}\big(4H-2\big)^{\frac{p}{2}}.
\end{aligned}
$$

Note that $p\gamma$ takes values in $(0,1)$, hence we can choose $q'>1$ such that $p\gamma q'<1$ and $1/p'+1/q'=1$. Then by applying Hölder's inequality to the last term on the right hand side of (4.14), we deduce that

$$
\sup_{r\in[0,t]} E\|u^{\sigma,\lambda}(r)-u^{\sigma,0}(r)\|_\gamma^p
$$

$$
\le \lambda^p\widetilde{\Upsilon}_1(T)+\widetilde{\Upsilon}_2(T)\frac{T^{\frac{1}{q'}-p\gamma}}{(1-pq'\gamma)^{\frac{1}{q'}}}\Big(\int_0^t\Big(\sup_{r\in[0,\tau]} E\|u^{\sigma,\lambda}(r)-u^{\sigma,0}(r)\|_\gamma^p\Big)^{p'}d\tau\Big)^{\frac{1}{p'}}. \quad (4.15)
$$

Consequently, the assertion of the theorem holds by using Gronwall's lemma. □

5. Continuity of solutions with respect to parameter σ

This section is devoted to showing continuity of solutions with respect to parameter σ. To this end, we first present the following lemma which is crucial for proving Theorem 5.1.

Lemma 5.1. *Let $p > 2$, $-\frac{1}{2} < \sigma_1, \sigma_2 < 0$ and $\lambda > 0$. If $\phi_1, \phi_2 : [0,T] \times \Omega \to \mathcal{L}^2$ are progressively measurable functions satisfying $\int_0^t E\|\phi_1(s)\|^p ds < \infty$ and $\int_0^t E\|\phi_2(s)\|^p ds < \infty$, then for any $t \in [0,T]$,*

$$
\begin{aligned}
&E\Big\| \int_0^t \phi_1(s) dB^{\sigma_1,\lambda}(s) - \int_0^t \phi_2(s) dB^{\sigma_2,\lambda}(s) \Big\|^p \\
&\leq 2^{2p-2} C_p (\sigma_2 - \sigma_1)^p (-\sigma_1)^{-\frac{p}{2}} \big(\beta(1+2\sigma_1, -\sigma_1)\big)^{\frac{p}{2}} t^{(\frac{1}{2}-\sigma_1)p-1} \int_0^t E\|\phi_1(s)\|^p ds \\
&+ 2^{3p-3} C_p (-\sigma_2)^{\frac{p}{2}} \big(\beta(1+2\sigma_2, -\sigma_2)\big)^{\frac{p}{2}} t^{(\frac{1}{2}-\sigma_2)p-1} \int_0^t E\|\phi_1(s) - \phi_2(s)\|^p ds \\
&+ 2^{2p-2} C_p \frac{\lambda^p}{(-\sigma_1)^{\frac{p}{2}}} \big(\beta(1+2\sigma_1, 1-\sigma_1)\big)^{\frac{p}{2}} t^{(\frac{3}{2}-\sigma_1)p-1} \int_0^t E\big\|\phi_1(s) - \phi_2(s)\big\|^p ds \\
&+ 2^{3p-3} C_p (-\sigma_2)^p \eth_1(t)^{\frac{p-2}{2}} \int_0^t E\|\phi_1(s)\|^p ds \\
&+ 2^{2p-2} \lambda^p C_p \eth_2(t)^{\frac{p-2}{2}} \int_0^t E\|\phi_2(s)\|^p ds,
\end{aligned}
$$

where

$$
\begin{aligned}
\eth_1(t) := \int_0^t \Big(\int_0^t \int_0^{u\wedge r} & \big((r-s)^{-\sigma_1-1} - (r-s)^{-\sigma_2-1}\big) \\
& \times \big((u-s)^{-\sigma_1-1} - (u-s)^{-\sigma_2-1}\big) ds du \Big)^{\frac{p}{p-2}} dr,
\end{aligned}
$$

$$
\begin{aligned}
\eth_2(t) := \int_0^t \Big(\int_0^t \int_0^{u\wedge r} & \big((r-s)^{-\sigma_1} - (r-s)^{-\sigma_2}\big) \\
& \times \big((u-s)^{-\sigma_1} - (u-s)^{-\sigma_2}\big) ds du \Big)^{\frac{p}{p-2}} dr,
\end{aligned}
$$

and C_p is given in Lemma 2.1.

Proof. Applying Definitions 2.4-2.5 and Lemma 2.1 results in

$$
\begin{aligned}
&E\Big\| \int_0^t \phi_1(s) dB^{\sigma_1,\lambda}(s) - \int_0^t \phi_2(s) dB^{\sigma_2,\lambda}(s) \Big\|^p \\
&= E\Big\| \Gamma(1-\sigma_1) \int_0^t {}_s\mathbb{I}_t^{-\sigma_1,\lambda} \phi_1(s) - \lambda {}_s\mathbb{I}_t^{1-\sigma_1,\lambda} \phi_1(s) dB(s)
\end{aligned}
$$

$$
\begin{aligned}
&- \Gamma(1-\sigma_2)\int_0^t {}_s\mathbb{I}_t^{-\sigma_2,\lambda}\phi_2(s) - \lambda_s\mathbb{I}_t^{1-\sigma_2,\lambda}\phi_2(s)dB(s)\Big\|^p \\
&= E\Big\| \int_0^t \Big((-\sigma_1)\int_s^t (r-s)^{-\sigma_1-1}e^{-\lambda(r-s)}\phi_1(r)dr \\
&- \lambda\int_s^t (u-s)^{-\sigma_1}e^{-\lambda(u-s)}\phi_1(u)du\Big)dB(s) \\
&- \int_0^t \Big((-\sigma_2)\int_s^t (r-s)^{-\sigma_2-1}e^{-\lambda(r-s)}\phi_2(r)dr \\
&- \lambda\int_s^t (u-s)^{-\sigma_2}e^{-\lambda(u-s)}\phi_2(u)du\Big)dB(s)\Big\|^p \\
&\le 2^{p-1}E\Big\| \int_0^t (-\sigma_1)\int_s^t (r-s)^{-\sigma_1-1}e^{-\lambda(r-s)}\phi_1(r)drdB(s) \\
&- \int_0^t (-\sigma_2)\int_s^t (r-s)^{-\sigma_2-1}e^{-\lambda(r-s)}\phi_2(r)drdB(s)\Big\|^p \\
&+ 2^{p-1}\lambda^p E\Big\| \int_0^t\int_s^t (u-s)^{-\sigma_1}e^{-\lambda(u-s)}\phi_1(u)dudB(s) \\
&- \int_0^t\int_s^t (u-s)^{-\sigma_2}e^{-\lambda(u-s)}\phi_2(u)dudB(s)\Big\|^p \\
&:= \Upsilon_3 + \Upsilon_4.
\end{aligned} \tag{5.1}
$$

It follows from similar arguments as in the proof of Lemma 2.2 that ${}_s\mathbb{I}_t^{-\sigma_1,\lambda}\phi_1(s) - {}_s\mathbb{I}_t^{1-\sigma_1,\lambda}\phi_1(s)$ and ${}_s\mathbb{I}_t^{-\sigma_2,\lambda}\phi_2(s) - {}_s\mathbb{I}_t^{1-\sigma_2,\lambda}\phi_2(s)$ are progressively measurable. Then, by using Lemma 2.1, we obtain that

$$
\begin{aligned}
\Upsilon_3 &\le 2^{2p-2}(\sigma_2-\sigma_1)^p E\Big\| \int_0^t\int_s^t (r-s)^{-\sigma_1-1}e^{-\lambda(r-s)}\phi_1(r)drdB(s)\Big\|^p \\
&+ 2^{2p-2}(-\sigma_2)^p E\Big\| \int_0^t\int_s^t e^{-\lambda(r-s)}\big((r-s)^{-\sigma_2-1}\phi_2(r) \\
&- (r-s)^{-\sigma_1-1}\phi_1(r)\big)drdB(s)\Big\|^p \\
&\le 2^{2p-2}C_p(\sigma_2-\sigma_1)^p E\Big(\int_0^t\int_s^t\int_s^t \|\phi_1(r)\|\|\phi_1(u)\| \\
&\times (r-s)^{-\sigma_1-1}(u-s)^{-\sigma_1-1}drduds\Big)^{\frac{p}{2}} \\
&+ 2^{3p-3}(-\sigma_2)^p E\Big\| \int_0^t\int_s^t (r-s)^{-\sigma_2-1}e^{-\lambda(r-s)}\big(\phi_2(r)-\phi_1(r)\big)drdB(s)\Big\|^p \\
&+ 2^{3p-3}(-\sigma_2)^p E\Big\| \int_0^t\int_s^t \big((r-s)^{-\sigma_1-1} - (r-s)^{-\sigma_2-1}\big) \\
&\times e^{-\lambda(r-s)}\phi_1(r)drdB(s)\Big\|^p \\
&:= \Upsilon_3^1 + \Upsilon_3^2 + \Upsilon_3^3.
\end{aligned} \tag{5.2}
$$

For Υ_3^1, we deduce from inequality (2.11) and Hölder's inequality that

$$\begin{aligned}
\Upsilon_3^1 &= 2^{2p-2}C_p(\sigma_2-\sigma_1)^p E\Big(\int_0^t\int_0^t\int_0^{u\wedge r}\|\phi_1(r)\|\|\phi_1(u)\|(u-s)^{-\sigma_1-1} \\
&\quad \times (r-s)^{-\sigma_1-1}dsdudr\Big)^{\frac{p}{2}} \\
&\le 2^{2p-2}C_p(\sigma_2-\sigma_1)^p E\Big(\int_0^t\int_0^t\|\phi_1(r)\|^2|r-u|^{-2\sigma_1-1} \\
&\quad \times \beta(1+2\sigma_1,-\sigma_1)dudr\Big)^{\frac{p}{2}} \\
&\le 2^{2p-2}C_p(\sigma_2-\sigma_1)^p(-\sigma_1)^{-\frac{p}{2}}\big(\beta(1+2\sigma_1,-\sigma_1)\big)^{\frac{p}{2}} \\
&\quad \times t^{(\frac{1}{2}-\sigma_1)p-1}\int_0^t E\|\phi_1(s)\|^p ds.
\end{aligned} \tag{5.3}$$

For Υ_3^2, using inequality (2.11) and Hölder's inequality again we have

$$\begin{aligned}
\Upsilon_3^2 &\le 2^{3p-3}C_p(-\sigma_2)^p E\Big(\int_0^t\int_s^t\int_s^t\|\phi_2(r)-\phi_1(r)\|\|\phi_2(u)-\phi_1(u)\| \\
&\quad \times (r-s)^{-\sigma_2-1}(u-s)^{-\sigma_2-1}drduds\Big)^{\frac{p}{2}} \\
&= 2^{3p-3}C_p(-\sigma_2)^p E\Big(\int_0^t\int_0^t\int_0^{u\wedge r}\|\phi_2(r)-\phi_1(r)\|\|\phi_2(u)-\phi_1(u)\| \\
&\quad \times (r-s)^{-\sigma_2-1}(u-s)^{-\sigma_2-1}dsdudr\Big)^{\frac{p}{2}} \\
&\le 2^{3p-3}C_p(-\sigma_2)^p E\Big(\int_0^t\int_0^t\|\phi_2(r)-\phi_1(r)\|^2|r-u|^{-2\sigma_2-1} \\
&\quad \times \beta(1+2\sigma_2,-\sigma_2)dudr\Big)^{\frac{p}{2}} \\
&\le 2^{3p-3}C_p(-\sigma_2)^{\frac{p}{2}}\big(\beta(1+2\sigma_2,-\sigma_2)\big)^{\frac{p}{2}}t^{(\frac{1}{2}-\sigma_2)p-1}\int_0^t E\|\phi_2(s)-\phi_1(s)\|^p ds.
\end{aligned} \tag{5.4}$$

Then for Υ_3^3, by repeatedly using Hölder's inequality we find that

$$\begin{aligned}
\Upsilon_3^3 &\le 2^{3p-3}C_p(-\sigma_2)^p E\bigg(\int_0^t\Big(\int_s^t\big((r-s)^{-\sigma_1-1}-(r-s)^{-\sigma_2-1}\big) \\
&\quad \times e^{-\lambda(r-s)}\|\phi_1(r)\|dr\Big)^2 ds\bigg)^{\frac{p}{2}} \\
&\le 2^{3p-3}C_p(-\sigma_2)^p E\Big(\int_0^t\int_s^t\int_s^t\big((r-s)^{-\sigma_1-1}-(r-s)^{-\sigma_2-1}\big) \\
&\quad \times \big((u-s)^{-\sigma_1-1}-(u-s)^{-\sigma_2-1}\big)\|\phi_1(r)\|\|\phi_1(u)\|drduds\Big)^{\frac{p}{2}} \\
&\le 2^{3p-3}C_p(-\sigma_2)^p E\Big(\int_0^t\|\phi_1(r)\|^2\int_0^t\int_0^{u\wedge r}\big((r-s)^{-\sigma_1-1}-(r-s)^{-\sigma_2-1}\big)
\end{aligned}$$

$$
\begin{aligned}
&\times \Big((u-s)^{-\sigma_1-1}-(u-s)^{-\sigma_2-1}\Big)dsdudr\Big)^{\frac{p}{2}} \\
&\le 2^{3p-3}C_p(-\sigma_2)^p\int_0^t E\|\phi_1(r)\|^p dr\bigg(\int_0^t\Big(\int_0^t\int_0^{u\wedge r}\Big((r-s)^{-\sigma_1-1} \\
&-(r-s)^{-\sigma_2-1}\Big)\Big((u-s)^{-\sigma_1-1}-(u-s)^{-\sigma_2-1}\Big)dsdu\Big)^{\frac{p}{p-2}}dr\bigg)^{\frac{p-2}{2}}.
\end{aligned}
\tag{5.5}
$$

We next estimate the term Υ_4,

$$
\begin{aligned}
\Upsilon_4 &\le 2^{2p-2}\lambda^p E\Big\|\int_0^t\int_s^t (u-s)^{-\sigma_1}e^{-\lambda(u-s)}\big(\phi_1(u)-\phi_2(u)\big)dudB(s)\Big\|^p \\
&+2^{2p-2}\lambda^p E\Big\|\int_0^t\int_s^t \big((u-s)^{-\sigma_1}-(u-s)^{-\sigma_2}\big)e^{-\lambda(u-s)}\phi_2(u)dudB(s)\Big\|^p \\
&:= \Upsilon_4^1+\Upsilon_4^2.
\end{aligned}
\tag{5.6}
$$

Applying Lemma 2.1, Hölder's inequality and the inequality (2.12) results in

$$
\begin{aligned}
\Upsilon_4^1 &\le 2^{2p-2}\lambda^p C_p E\bigg(\int_0^t\Big(\int_s^t (u-s)^{-\sigma_1}e^{-\lambda(u-s)}\big\|\phi_1(u)-\phi_2(u)\big\|du\Big)^2 ds\bigg)^{\frac{p}{2}} \\
&\le 2^{2p-2}\lambda^p C_p E\Big(\int_0^t\int_s^t\int_s^t (u-s)^{-\sigma_1}(r-s)^{-\sigma_1} \\
&\times \big\|\phi_1(r)-\phi_2(r)\big\|\big\|\phi_1(u)-\phi_2(u)\big\|drduds\Big)^{\frac{p}{2}} \\
&\le 2^{2p-2}\lambda^p C_p E\Big(\int_0^t\int_0^t\int_0^{u\wedge r} (u-s)^{-\sigma_1}(r-s)^{-\sigma_1} \\
&\times \big\|\phi_1(r)-\phi_2(r)\big\|\big\|\phi_1(u)-\phi_2(u)\big\|dsdudr\Big)^{\frac{p}{2}} \\
&\le 2^{2p-2}\lambda^p C_p t^p E\Big(\int_0^t\int_0^t \big\|\phi_1(r)-\phi_2(r)\big\|^2|u-r|^{-2\sigma_1-1} \\
&\times \beta(1+2\sigma_1,1-\sigma_1)dudr\Big)^{\frac{p}{2}} \\
&\le 2^{2p-2}\frac{C_p\lambda^p}{(-\sigma_1)^{\frac{p}{2}}}\big(\beta(1+2\sigma_1,1-\sigma_1)\big)^{\frac{p}{2}}t^{(\frac{3}{2}-\sigma_1)p-1}\int_0^t E\big\|\phi_1(s)-\phi_2(s)\big\|^p ds.
\end{aligned}
\tag{5.7}
$$

Arguing as in (5.5) we obtain

$$
\begin{aligned}
\Upsilon_4^2 &\le 2^{2p-2}\lambda^p C_p E\bigg(\int_0^t\Big(\int_s^t \big((u-s)^{-\sigma_1}-(u-s)^{-\sigma_2}\big) \\
&\times e^{-\lambda(u-s)}\|\phi_2(u)\|du\Big)^2 ds\bigg)^{\frac{p}{2}} \\
&\le 2^{2p-2}\lambda^p C_p E\Big(\int_0^t\int_s^t\int_s^t \|\phi_2(u)\|\|\phi_2(r)\|\big((r-s)^{-\sigma_1}-(r-s)^{-\sigma_2}\big) \\
&\times \big((u-s)^{-\sigma_1}-(u-s)^{-\sigma_2}\big)drduds\Big)^{\frac{p}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\le 2^{2p-2}\lambda^p C_p E\Big(\int_0^t \|\phi_2(r)\|^2 \int_0^t\int_0^{u\wedge r}\big((r-s)^{-\sigma_1}-(r-s)^{-\sigma_2}\big)\\
&\quad\times\big((u-s)^{-\sigma_1}-(u-s)^{-\sigma_2}\big)dsdudr\Big)^{\frac{p}{2}}\\
&\le 2^{2p-2}\lambda^p C_p\int_0^t E\|\phi_2(r)\|^p dr\bigg(\int_0^t\Big(\int_0^t\int_0^{u\wedge r}\big((r-s)^{-\sigma_1}-(r-s)^{-\sigma_2}\big)\\
&\quad\times\big((u-s)^{-\sigma_1}-(u-s)^{-\sigma_2}\big)dsdu\Big)^{\frac{p}{p-2}}dr\bigg)^{\frac{p-2}{2}}.
\end{aligned}
\tag{5.8}
$$

Collecting (5.2)-(5.5) and (5.6)-(5.8) together, the assertion of this lemma follows immediately from (5.1). □

Theorem 5.1. *Let $u^{\sigma,\lambda}$ denote the mild solution to Eq.* (1.1) *driven by $B^{\sigma,\lambda}(t)$ with $-1/2<\sigma<0$ and $\lambda>0$. Suppose that the assumptions in Thereom* 3.1 *hold. Then for any $T>0$ and $\lambda>0$,*

$$\sup_{0\le\tau\le T} E\|u^{\sigma_1,\lambda}(\tau)-u^{\sigma_2,\lambda}(\tau)\|_\gamma^p\to 0 \quad as \quad \sigma_1\to\sigma_2.$$

Proof. By Lemma 5.1, Hölder's inequality and the property $(\mathcal{P}_1)$, we deduce that

$$
\begin{aligned}
&E\big\|u^{\sigma_1,\lambda}(t)-u^{\sigma_2,\lambda}(t)\big\|_\gamma^p\\
&\le 2^{p-1}E\Big(\int_0^t\big\|A^\gamma S(t-\tau)\big(f(\tau,u_\tau^{\sigma_1,\lambda})-f(\tau,u_\tau^{\sigma_2,\lambda})\big)\big\|d\tau\Big)^p\\
&+2^{p-1}E\Big\|\int_0^t A^\gamma S(t-\tau)g(\tau,u_\tau^{\sigma_1,\lambda})dB^{\sigma_1,\lambda}(\tau)\\
&-\int_0^t A^\gamma S(t-\tau)g(\tau,u_\tau^{\sigma_2,\lambda})dB^{\sigma_2,\lambda}(\tau)\Big\|^p\\
&\le 2^{p-1}C_{\gamma,0}^p t^{p-1}\int_0^t(t-\tau)^{-p\gamma}E\big\|f(\tau,u_\tau^{\sigma_1,\lambda})-f(\tau,u_\tau^{\sigma_2,\lambda})\big\|^p d\tau\\
&+2^{3p-3}C_{\gamma,0}^p C_p(\sigma_2-\sigma_1)^p(-\sigma_1)^{-\frac{p}{2}}\big(\beta(1+2\sigma_1,-\sigma_1)\big)^{\frac{p}{2}}t^{(\frac{1}{2}-\sigma_1)p-1}\\
&\times\int_0^t(t-\tau)^{-p\gamma}E\|g(\tau,u_\tau^{\sigma_1,\lambda})\|^p d\tau\\
&+2^{4p-4}C_{\gamma,0}^p C_p(-\sigma_2)^{\frac{p}{2}}\big(\beta(1+2\sigma_2,-\sigma_2)\big)^{\frac{p}{2}}t^{(\frac{1}{2}-\sigma_2)p-1}\\
&\times\int_0^t(t-\tau)^{-p\gamma}E\big\|g(\tau,u_\tau^{\sigma_1,\lambda})-g(\tau,u_\tau^{\sigma_2,\lambda})\big\|^p d\tau\\
&+2^{3p-3}\lambda^p C_{\gamma,0}^p C_p\Big(\frac{\beta(1+2\sigma_1,1-\sigma_1)}{-\sigma_1}\Big)^{\frac{p}{2}}t^{(\frac{3}{2}-\sigma_1)p-1}\\
&\times\int_0^t(t-\tau)^{-p\gamma}E\big\|g(\tau,u_\tau^{\sigma_1,\lambda})-g(\tau,u_\tau^{\sigma_2,\lambda})\big\|^p d\tau\\
&+2^{4p-4}C_{\gamma,0}^p C_p(-\sigma_2)^p\eth_1(t)^{\frac{p-2}{2}}\int_0^t(t-\tau)^{-p\gamma}E\|g(\tau,u_\tau^{\sigma_1,\lambda})\|^p d\tau
\end{aligned}
\tag{5.9}
$$

$$+ 2^{3p-3}\lambda^p C_{\gamma,0}^p C_p \eth_2(t)^{\frac{p-2}{2}} \int_0^t (t-\tau)^{-p\gamma} E\|g(\tau, u_\tau^{\sigma_2,\lambda})\|^p d\tau.$$

Since $u^{\sigma_1,\lambda}(t) = u^{\sigma_2,\lambda}(t) = \varphi(t)$ for each $t \in (-\infty, 0]$, we obtain

$$\begin{aligned} \|u_\tau^{\sigma_1,\lambda} - u_\tau^{\sigma_2,\lambda}\|_{\mathcal{C}^p(H^\gamma)}^p &= \sup_{r\in(-\infty,\tau]} E\|u^{\sigma_1,\lambda}(r) - u^{\sigma_2,\lambda}(r)\|_\gamma^p \\ &= \sup_{r\in[0,\tau]} E\|u^{\sigma_1,\lambda}(r) - u^{\sigma_2,\lambda}(r)\|_\gamma^p. \end{aligned} \tag{5.10}$$

Then, by using Hölder's inequality, the assumptions (H_2)-(H_3) and (4.6), we obtain

$$\begin{aligned} &E\|u^{\sigma_1,\lambda}(t) - u^{\sigma_2,\lambda}(t)\|_\gamma^p \\ &\leq 2^{p-1} C_{\gamma,0}^p t^{p-1} L_f \int_0^t (t-\tau)^{-p\gamma} \sup_{r\in[0,\tau]} E\|u^{\sigma_1,\lambda}(r) - u^{\sigma_2,\lambda}(r)\|_\gamma^p d\tau \\ &\quad + (\sigma_2 - \sigma_1)^p t^{(\frac{1}{2}-\sigma_1-\gamma)p} \mathbb{C}(\sigma_1, p, \gamma, l_g)\big(1 + \|\varphi\|_{\mathcal{C}^p(H^\gamma)}^p\big) \\ &\quad + t^{(\frac{1}{2}-\sigma_2)p-1} \mathbb{C}(\sigma_2, p, \gamma) L_g \int_0^t (t-\tau)^{-p\gamma} \sup_{r\in[0,\tau]} E\|u^{\sigma_1,\lambda}(r) - u^{\sigma_2,\lambda}(r)\|_\gamma^p d\tau \\ &\quad + \lambda^p t^{(\frac{3}{2}-\sigma_1)p-1} \mathbb{C}(\sigma_1, p, \gamma) L_g \int_0^t (t-\tau)^{-p\gamma} \sup_{r\in[0,\tau]} E\|u^{\sigma_1,\lambda}(r) - u^{\sigma_2,\lambda}(r)\|_\gamma^p d\tau \\ &\quad + t^{1-p\gamma}\big(\eth_1(t)^{\frac{p-2}{2}} + \lambda^p \eth_2(t)^{\frac{p-2}{2}}\big) \mathbb{C}(\sigma_2, p, \gamma, l_g)\big(1 + \|\varphi\|_{\mathcal{C}^p(H^\gamma)}^p\big) \\ &\leq \widetilde{\Upsilon}_5(t)\Big(\int_0^t \big(\sup_{r\in[0,\tau]} E\|u^{\sigma_1,\lambda}(r) - u^{\sigma_2,\lambda}(r)\|_\gamma^p\big)^{p'} d\tau\Big)^{\frac{1}{p'}} \\ &\quad + (\sigma_1 - \sigma_2)^p \widetilde{\Upsilon}_6(t) + \widetilde{\Upsilon}_7(t), \end{aligned} \tag{5.11}$$

where we choose $q' > 1$ such that $pq'\gamma < 1$, $1/p' + 1/q' = 1$ and $pp'(\frac{1}{2} - \sigma_2 - \gamma) > 1$. Here we have used the notations

$$\begin{aligned} \widetilde{\Upsilon}_5(t) := {} & 2^{p-1} C_{\gamma,0}^p t^{p(1-\gamma)-\frac{1}{p'}} L_f + t^{(\frac{1}{2}-\sigma_2-\gamma)p-\frac{1}{p'}} \mathbb{C}(\sigma_2, p, \gamma) L_g \\ & + \lambda^p t^{(\frac{3}{2}-\sigma_1-\gamma)p-\frac{1}{p'}} \mathbb{C}(\sigma_1, p, \gamma) L_g, \end{aligned}$$

$$\widetilde{\Upsilon}_6(t) := t^{(\frac{1}{2}-\sigma_1-\gamma)p} \mathbb{C}(\sigma_1, p, \gamma, l_g)\big(1 + \|\varphi\|_{\mathcal{C}^p(H^\gamma)}^p\big),$$

and

$$\widetilde{\Upsilon}_7(t) := t^{1-p\gamma}\big(\eth_1(t)^{\frac{p-2}{2}} + \lambda^p \eth_2(t)^{\frac{p-2}{2}}\big) \mathbb{C}(\sigma_2, p, \gamma, l_g)\big(1 + \|\varphi\|_{\mathcal{C}^p(H^\gamma)}^p\big).$$

Therefore, by applying Gronwall's lemma to (5.11), the assertion of this theorem follows immediately from the dominated convergence theorem. □

6. Time regularity of mild solutions

The goal of this section is to show mean-p Hölder regularity of mild solutions.

Theorem 6.1. *Let $p \geq 2$, $\gamma \in (0, \frac{1}{p})$ and $\varphi \in \mathcal{C}^p(H^\gamma)$. Suppose that the assumptions in Theorem 3.1 hold. Then there exists $\mathbb{C} > 0$ depending on l_f, l_g, γ, p, T such that for all $t_1, t_2 \in [0, T]$,*

$$\big\|u(t_1) - u(t_2)\big\|_{L^p(\Omega;H^\gamma)} \leq \mathbb{C}|t_1 - t_2|^{\frac{1}{p}-\gamma}, \tag{6.1}$$

where u is the unique mild solution of problem (1.1) on $[0, T]$.

Proof. Let $0 < t < t + s < T$. Then we have

$$\begin{aligned}
\|u(t+s) - u(t)\|_{L^p(\Omega;H^\gamma)} &\leq \big\|S(t+s)\varphi(0) - S(t)\varphi(0)\big\|_{L^p(\Omega;H^\gamma)} \\
&+ \Big\|\int_0^t \big(S(t+s-\tau) - S(t-\tau)\big)f(\tau, u_\tau)d\tau\Big\|_{L^p(\Omega;H^\gamma)} \\
&+ \Big\|\int_0^t S(t+s-\tau)g(\tau, u_\tau) - S(t-\tau)g(\tau, u_\tau)dB^{\sigma,\lambda}(\tau)\Big\|_{L^p(\Omega;H^\gamma)} \\
&+ \Big\|\int_t^{t+s} S(t+s-\tau)f(\tau, u_\tau)d\tau\Big\|_{L^p(\Omega;H^\gamma)} \\
&+ \Big\|\int_t^{t+s} S(t+s-\tau)g(\tau, u_\tau)dB^{\sigma,\lambda}(\tau)\Big\|_{L^p(\Omega;H^\gamma)} \\
&:= V_6 + V_7 + V_8 + V_9 + V_{10}.
\end{aligned} \tag{6.2}$$

We now estimate each term V_i $(i = 6, \ldots, 10)$. By making use of the property $(\mathcal{P}_1)$ and Hölder's inequality, we can choose $a \in (0, \frac{1}{p})$ such that

$$\begin{aligned}
V_6 &= \Big\|\int_t^{t+s} \dot{S}(\tau)\varphi(0)d\tau\Big\|_{L^p(\Omega;H^\gamma)} \\
&\leq \int_t^{t+s} \big\|AS(\tau)\varphi(0)\big\|_{L^p(\Omega;H^\gamma)}d\tau \\
&\leq \int_t^{t+s} C_{1,0}e^{-\delta\tau}\tau^{-1}\big\|\varphi(0)\big\|_{L^p(\Omega;H^\gamma)}d\tau \\
&\leq C_{1,0}\|\varphi\|_{\mathcal{C}^p(H^\gamma)}\Big(\int_t^{t+s} \tau^{-pa}d\tau\Big)^{\frac{1}{p}}\Big(\int_t^{t+s} \tau^{-\frac{p(1-a)}{p-1}}d\tau\Big)^{\frac{p-1}{p}} \\
&\leq C_{1,0}\|\varphi\|_{\mathcal{C}^p(H^\gamma)}t^{a-1}\frac{s^{1-a}}{(1-pa)^{\frac{1}{p}}},
\end{aligned} \tag{6.3}$$

where we have used the inequality

$$a^\theta - b^\theta \leq (a-b)^\theta \quad \text{for} \quad a > b > 0 \quad \text{and} \quad \theta \in (0, 1). \tag{6.4}$$

Using the property $(\mathcal{P}_1)$, the assumption (H_2) and (4.6), the term V_7 can be bounded by

$$\begin{aligned} V_7 &\le \int_0^t \int_t^{t+s} \big\|A^{\gamma+1}S(r-\tau)f(\tau,u_\tau)\big\|_{L^p(\Omega;\mathcal{L}^2)} dr d\tau \\ &\le \int_0^t \int_t^{t+s} C_{1+\gamma,0}(r-\tau)^{-(1+\gamma)}\|f(\tau,u_\tau)\|_{L^p(\Omega;\mathcal{L}^2)} dr d\tau \\ &\le \mathbb{C}(\gamma,p,l_g,l_f,H,q',p',T)\big(1+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)^{\frac{1}{p}} \int_t^{t+s}\int_0^t (r-\tau)^{-1-\gamma} d\tau dr \\ &\le \mathbb{C}(\gamma,p,l_g,l_f,H,q',p',T)\big(1+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)^{\frac{1}{p}} \Big(\frac{(r-t)^{1-\gamma}}{\gamma(1-\gamma)}\Big|_t^{t+s}\Big) \\ &\le \mathbb{C}(\gamma,p,l_g,l_f,H,q',p',T)\big(1+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)^{\frac{1}{p}} s^{1-\gamma}. \end{aligned} \tag{6.5}$$

For V_8, by using Hölder's inequality, (4.6), the property $(\mathcal{P}_1)$ and the assumption (H_2) again, we deduce from Lemma 2.2 that

$$\begin{aligned} V_8 &= \Big\|\int_0^t \int_t^{t+s} \dot{S}(r-\tau)g(\tau,u_\tau) dr dB^{\sigma,\lambda}(\tau)\Big\|_{L^p(\Omega;H^\gamma)} \\ &\le \int_t^{t+s} \Big\|\int_0^t AS(r-\tau)g(\tau,u_\tau)dB^{\sigma,\lambda}(\tau)\Big\|_{L^p(\Omega;H^\gamma)} dr \\ &\le (C_p)^{\frac{1}{p}}\sqrt{N_t}\, t^{\frac{p-2}{2p}} \int_t^{t+s}\Big(\int_0^t \big\|AS(r-\tau)g(\tau,u_\tau)\big\|^p_{L^p(\Omega;H^\gamma)} d\tau\Big)^{\frac{1}{p}} dr \\ &\le (C_p)^{\frac{1}{p}}\sqrt{N_t}\, t^{\frac{p-2}{2p}} \int_t^{t+s}\Big(\int_0^t \big\|A^{1+\gamma}S(r-\tau)g(\tau,u_\tau)\big\|^p_{L^p(\Omega;\mathcal{L}^2)} d\tau\Big)^{\frac{1}{p}} dr \\ &\le (C_p)^{\frac{1}{p}}\sqrt{N_t}\, t^{\frac{p-2}{2p}} C_{1+\gamma,0}\int_t^{t+s}\Big(\int_0^t (r-\tau)^{-p-p\gamma}\|g(\tau,u_\tau)\|^p_{L^p(\Omega;\mathcal{L}^2)} d\tau\Big)^{\frac{1}{p}} dr \\ &\le \mathbb{C}(\gamma,p,l_g,l_f,H,q',p',T)\sqrt{N_t}\, t^{\frac{p-2}{2p}}\big(1+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)^{\frac{1}{p}} \\ &\quad \times \int_t^{t+s}\Big(\int_0^t (r-\tau)^{-p-p\gamma} d\tau\Big)^{\frac{1}{p}} dr \\ &\le \mathbb{C}(\gamma,p,l_g,l_f,H,q',p',T)\sqrt{N_t}\, t^{\frac{p-2}{2p}}\big(1+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)^{\frac{1}{p}} s^{\frac{1}{p}-\gamma}. \end{aligned} \tag{6.6}$$

Analogous to the arguments as in (6.5) and (6.6), we conclude that

$$\begin{aligned} V_9 &\le \int_t^{t+s} \big\|S(t+s-\tau)f(\tau,u_\tau)\big\|_{L^p(\Omega;H^\gamma)} d\tau \\ &\le \int_t^{t+s} C_{\gamma,0}(t+s-\tau)^{-\gamma}\|f(\tau,u_\tau)\|_{L^p(\Omega;\mathcal{L}^2)} d\tau \\ &\le \mathbb{C}(\gamma,p,l_g,l_f,H,q',p',T)\big(1+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)^{\frac{1}{p}} \int_t^{t+s}(t+s-\tau)^{-\gamma} d\tau \\ &\le \mathbb{C}(\gamma,p,l_g,l_f,H,q',p',T)\big(1+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)^{\frac{1}{p}} s^{1-\gamma}, \end{aligned} \tag{6.7}$$

and

$$
\begin{aligned}
V_{10} &\le (C_p)^{\frac{1}{p}}\sqrt{N_s}s^{\frac{p-2}{2p}}\Big(\int_t^{t+s}\big\|S(t+s-\tau)g(\tau,u_\tau)\big\|^p_{L^p(\Omega;H^\gamma)}d\tau\Big)^{\frac{1}{p}}\\
&= (C_p)^{\frac{1}{p}}\sqrt{N_s}s^{\frac{p-2}{2p}}\Big(\int_t^{t+s}\big\|A^\gamma S(t+s-\tau)g(\tau,u_\tau)\big\|^p_{L^p(\Omega;\mathcal{L}^2)}d\tau\Big)^{\frac{1}{p}}\\
&\le (C_p)^{\frac{1}{p}}\sqrt{N_s}s^{\frac{p-2}{2p}}C_{\gamma,0}\Big(\int_t^{t+s}(t+s-\tau)^{-p\gamma}\|g(\tau,u_\tau)\|^p_{L^p(\Omega;\mathcal{L}^2)}d\tau\Big)^{\frac{1}{p}}\\
&\le \mathbb{C}(\gamma,p,l_g,l_f,H,q',p',T)\sqrt{N_s}\big(1+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)^{\frac{1}{p}}s^{\frac{1}{2}-\gamma}.
\end{aligned}
\tag{6.8}
$$

Inserting (6.3)-(6.8) into (6.2) yields

$$
\begin{aligned}
\|u(t+s)-u(t)\|_{L^p(\Omega;H^\gamma)} &\le C\|\varphi(0)\|_{\mathcal{C}^p(H^\gamma)}t^{a-1}\frac{s^{1-a}}{(1-pa)^{\frac{1}{p}}}\\
&+\mathbb{C}(\gamma,p,l_g,l_f,H,q',p',T)\big(1+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)^{\frac{1}{p}}s^{1-\gamma}\\
&+\mathbb{C}(\gamma,p,l_g,l_f,H,q',p',T)\sqrt{N_t}t^{\frac{p-2}{2p}}\big(1+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)^{\frac{1}{p}}s^{\frac{1}{p}-\gamma}\\
&+\mathbb{C}(\gamma,p,l_g,l_f,H,q',p',T)\big(1+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)^{\frac{1}{p}}s^{1-\gamma}\\
&+\mathbb{C}(\gamma,p,l_g,l_f,H,q',p',T)\sqrt{N_s}\big(1+\|\varphi\|^p_{\mathcal{C}^p(H^\gamma)}\big)^{\frac{1}{p}}s^{\frac{1}{2}-\gamma},
\end{aligned}
\tag{6.9}
$$

and thus the proof is complete. □

Arguing as in the proof of Theorem 6.1, we have

Corollary 6.1. *Let $p \ge 2$, $\gamma \in (0, \frac{1}{p})$ and $\varphi \in \mathcal{C}^p(H^\gamma)$. Suppose that the assumptions in Corollary 3.1 hold. Then there exists $\mathbb{C} > 0$ depending on l_f, l_g, γ, p, T such that for all $t_1, t_2 \in [0, T]$,*

$$
\big\|u(t_1)-u(t_2)\big\|_{L^p(\Omega;H^\gamma)} \le \mathbb{C}|t_1-t_2|^{\frac{1}{p}-\gamma},
$$

where u is the unique mild solution of problem (1.1) with FBM or Brownian motion instead of TFBM.

It is worth mentioning that the results in Sections 3-6 can be obtained for problem (1.1) but with $\delta = 0$, i.e., $A = -P\Delta$.

7. Polynomial stability for a special case

In this section we will start our analysis of the asymptotic behavior of solutions and will provide some significant results. Due to the fact that the right hand side of inequalities (2.7) and (2.13) are dependent of t, it is difficult to show that mild solutions to problem (1.1) with tempered fractional Gaussian noise or fractional Gaussian noise are polynomially stable in the space $\mathcal{C}^p(H^\gamma)$. However, it is still possible to provide insightful results for the special case of proportional delay when the function g becomes independent of the state variable.

Therefore, we shall study polynomial stability of mild solutions for the following stochastic 2D-Stokes equation with proportional delay (also called of pantograph type) and additive tempered fractional Gaussian noise:

$$\begin{cases} du(t) = -Au(t)dt + f(t, u(\eta t))dt + \tilde{g}(t)dB^{\sigma,\lambda}(t), \quad t \geq 0, \quad \eta \in (0,1), \\ u(0) = u_0. \end{cases} \tag{7.1}$$

First, we need the following assumptions on functions f and $\tilde{g}$.

(H_6) There exists a nonnegative function $L_1 \in L^\infty(\mathbb{R}^+)$ such that for any $\mu, \nu \in L^p(\Omega; H^\gamma)$ and $t \geq 0$,

$$E\big\|f(t,\mu) - f(t,\nu)\big\|^p \leq L_1(t)E\|\mu - \nu\|_\gamma^p.$$

(H_7) There exist nonnegative functions $l_1, l_2 \in L^q(\mathbb{R}^+)$ such that for any $\mu \in L^p(\Omega; H^\gamma)$ and $t \geq 0$,

$$E\|f(t,\mu)\|^p \leq l_1(t) + l_2(t)E\|\mu\|_\gamma^p,$$

and

$$\Big(\int_0^\infty \tau^{q\xi} l_1^q(\tau)d\tau\Big)^{\frac{1}{q}} < \infty, \qquad \Big(\int_0^\infty \tau^{-q\xi} l_2^q(\tau)d\tau\Big)^{\frac{1}{q}} < \infty$$

for some $\xi \in (0,1)$, where $1/p + 1/q = 1$.

(H_8) There exists a constant $\tilde{q} > 1$ such that

$$\int_0^\infty \big(e^{\delta\tau} E\|\tilde{g}(\tau)\|^p\big)^{\tilde{q}} d\tau := \hbar < \infty,$$

where $1/\tilde{p} + 1/\tilde{q} = 1$ and $1 < \tilde{p} < \frac{1}{2\gamma}$.

Theorem 7.1. *Let $p \geq 2$, $\gamma \in (0, \frac{1}{p})$ and $u_0 \in L^p(\Omega; H^\gamma)$. Suppose that the assumptions (H_6)-(H_8) hold. Let $\|L_1\|_{L^\infty(\mathbb{R}^+)}$ be sufficiently small such that*

$$3^p C_{\gamma,0}^p \big(\delta^{\gamma-1}\Gamma(1-\gamma)\big)^p \eta^{-\xi} \|L_1\|_{L^\infty(\mathbb{R}^+)} < 1, \tag{7.2}$$

where δ and $C_{\gamma,0}$ are given in the property $(\mathcal{P}_1)$. Then problem (7.1) has a unique global mild solution u satisfying

$$\sup_{r\in[0,\infty)} r^\xi E\|u(r)\|_\gamma^p < \infty,$$

where ξ is given in the assumption (H_7).

Proof. Define $\|\mu\|_\vartheta = \sup_{t\in[0,\infty)} \vartheta(t)E\|\mu(t)\|_\gamma^p$ for any $\mu \in C\big(0,\infty; L^p(\Omega; H^\gamma)\big)$, where

$$\vartheta(t) = \begin{cases} T^\xi, & t \in [0,T], \\ t^\xi, & t \geq T, \end{cases}$$

with $T > 0$ given later. We consider the abstract phase space

$$C_\vartheta\big(0,\infty; L^p(\Omega; H^\gamma)\big) = \big\{\mu \in C\big(0,\infty; L^p(\Omega; H^\gamma)\big) : \|\mu\|_\vartheta < \infty\big\}.$$

Then $\big(C_\vartheta\big(0,\infty;L^p(\Omega;H^\gamma)\big),\|\cdot\|_\vartheta\big)$ is a Banach space. For our purpose, we define the mapping $\mathcal{T}$ by

$$(\mathcal{T}u)(t)=S(t)u_0+\int_0^t S(t-\tau)f(\tau,u(\eta\tau))d\tau+\int_0^t S(t-\tau)\tilde{g}(\tau)dB^{\sigma,\lambda}(\tau). \quad (7.3)$$

Step 1. We show that $\mathcal{T}$ is contractive.

In view of (7.3), Hölder's inequality, the property $(\mathcal{P}_1)$ and the assumption (H_6), we deduce that for $t\in[0,T]$ and any $u,v\in C_\vartheta\big(0,\infty;L^p(\Omega;H^\gamma)\big)$,

$$\begin{aligned}
&\vartheta(t)E\big\|(\mathcal{T}u)(t)-(\mathcal{T}v)(t)\big\|_\gamma^p\\
&\le T^\xi E\Big(\int_0^t\big\|S(t-\tau)\big(f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big)\big\|_\gamma d\tau\Big)^p\\
&\le T^\xi C_{\gamma,0}^p E\Big(\int_0^t e^{-\delta(t-\tau)}(t-\tau)^{-\gamma}\big\|f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big\|d\tau\Big)^p\\
&\le T^\xi C_{\gamma,0}^p\Big(\int_0^t e^{-\delta(t-\tau)}(t-\tau)^{-\gamma}d\tau\Big)^{p-1}\int_0^t e^{-\delta(t-\tau)}(t-\tau)^{-\gamma}\\
&\quad\times E\big\|f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big\|^p d\tau\\
&\le C_{\gamma,0}^p\big(\delta^{\gamma-1}\Gamma(1-\gamma)\big)^p\|L_1\|_{L^\infty(\mathbb{R}^+)}\|u-v\|_\vartheta.
\end{aligned}\quad (7.4)$$

Next we consider the case of $t\ge T$. Let M be a positive constant which will be fixed later. Then for any $u,v\in C_\vartheta\big(0,\infty;L^p(\Omega;H^\gamma)\big)$,

$$\begin{aligned}
&\vartheta(t)E\big\|(\mathcal{T}u)(t)-(\mathcal{T}v)(t)\big\|_\gamma^p\\
&\le t^\xi E\Big\|\int_0^t S(t-\tau)\big(f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big)d\tau\Big\|_\gamma^p\\
&\le 3^{p-1}t^\xi E\Big(\int_0^{\frac{t}{2}}\big\|S(t-\tau)\big(f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big)\big\|_\gamma d\tau\Big)^p\\
&\quad+3^{p-1}t^\xi E\Big(\int_{\frac{t}{2}}^{t-M}\big\|S(t-\tau)\big(f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big)\big\|_\gamma d\tau\Big)^p\\
&\quad+3^{p-1}t^\xi E\Big(\int_{t-M}^{t}\big\|S(t-\tau)\big(f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big)\big\|_\gamma d\tau\Big)^p\\
&:=V_{11}^1+V_{11}^2+V_{11}^3.
\end{aligned}\quad (7.5)$$

Applying Hölder's inequality, the property $(\mathcal{P}_1)$ and the assumption (H_6), we have

$$
\begin{aligned}
V_{11}^1 &\le 3^{p-1}t^{\xi}C^p_{\gamma,0}E\Big(\int_0^{\frac{t}{2}} e^{-\delta(t-\tau)}(t-\tau)^{-\gamma}\big\|f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big\|d\tau\Big)^p\\
&\le \frac{3^{p-1}C^p_{\gamma,0}t^{\xi}}{(t/2)^{p\gamma}}E\Big(\int_0^{\frac{t}{2}} e^{-\frac{p-1}{p}\delta(t-\tau)}e^{-\frac{1}{p}\delta(t-\tau)}\big\|f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big\|d\tau\Big)^p\\
&\le 3^{p-1}t^{\xi}C^p_{\gamma,0}\big(\frac{t}{2}\big)^{-p\gamma}\Big(\int_0^{\frac{t}{2}} e^{-\delta(t-\tau)}d\tau\Big)^{p-1}\\
&\quad\times\int_0^{\frac{t}{2}} e^{-\delta(t-\tau)}E\big\|f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big\|^p d\tau\\
&\le t^{\xi}C^p_{\gamma,0}\big(\frac{t}{2}\big)^{-p\gamma}\big(\frac{3e^{-\frac{\delta t}{2}}}{\delta}\big)^{p-1}\|u-v\|_{\vartheta}\|L_1\|_{L^\infty(\mathbb{R}^+)}\eta^{-\xi}\int_0^{\frac{t}{2}} e^{-\delta(t-\tau)}\tau^{-\xi}d\tau\\
&\le C^p_{\gamma,0}\big(\frac{t}{2}\big)^{-p\gamma}\big(\frac{3e^{-\frac{\delta t}{2}}}{\delta}\big)^{p-1}\|u-v\|_{\vartheta}\|L_1\|_{L^\infty(\mathbb{R}^+)}\frac{t^{\xi}}{\eta^{\xi}}\Big(\int_0^{\frac{t}{2}} e^{-\delta p'(t-\tau)}d\tau\Big)^{\frac{1}{p'}}\\
&\quad\times\Big(\int_0^{\frac{t}{2}}\tau^{-q'\xi}d\tau\Big)^{\frac{1}{q'}}\\
&\le 3^{p-1}C^p_{\gamma,0}\big(\frac{t}{2}\big)^{-p\gamma}\|L_1\|_{L^\infty(\mathbb{R}^+)}\|u-v\|_{\vartheta}\big(\frac{e^{-\frac{\delta t}{2}}}{\delta}\big)^{p-1}\frac{e^{-\frac{\delta t}{2}}\big(\frac{t}{2}\big)^{\frac{1}{q'}-\xi}t^{\xi}}{\eta^{\xi}(\delta p')^{\frac{1}{p'}}(1-q'\xi)^{\frac{1}{q'}}},
\end{aligned}
\tag{7.6}
$$

where we take ξ in $(0,1)$ and choose $q'>1$ such that $\xi q'<1$ and $1/p'+1/q'=1$. Using Hölder's inequality, the property $(\mathcal{P}_1)$ and the assumption (H_6) again, we obtain that

$$
\begin{aligned}
V_{11}^2 &\le 3^{p-1}C^p_{\gamma,0}t^{\xi}E\Big(\int_{\frac{t}{2}}^{t-M} e^{-\frac{p-1}{p}\delta(t-\tau)}(t-\tau)^{-\gamma}e^{-\frac{1}{p}\delta(t-\tau)}\\
&\quad\times\big\|f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big\|d\tau\Big)^p\\
&\le 3^{p-1}C^p_{\gamma,0}t^{\xi}M^{-p\gamma}\Big(\int_{\frac{t}{2}}^{t-M} e^{-\delta(t-\tau)}d\tau\Big)^{p-1}\\
&\quad\times\int_{\frac{t}{2}}^{t-M} e^{-\delta(t-\tau)}E\big\|f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big\|^p d\tau\\
&\le 3^{p-1}C^p_{\gamma,0}M^{-p\gamma}\big(\frac{e^{-\delta M}}{\delta}\big)^{p-1}\int_{\frac{t}{2}}^{t-M} e^{-\delta(t-\tau)}\big((t-\tau)^{\xi}+\tau^{\xi}\big)\\
&\quad\times L_1(\tau)E\|u(\eta\tau)-v(\eta\tau)\|^p_{\gamma}d\tau\\
&\le 3^{p-1}C^p_{\gamma,0}M^{-p\gamma}\|L_1\|_{L^\infty(\mathbb{R}^+)}\|u-v\|_{\vartheta}\big(\frac{e^{-\delta M}}{\delta}\big)^{p-1}\eta^{-\xi}\\
&\quad\times\int_{\frac{t}{2}}^{t-M} e^{-\delta(t-\tau)}\big((t-\tau)^{\xi}\tau^{-\xi}+1\big)d\tau
\end{aligned}
\tag{7.7}
$$

$$\leq 3^{p-1}C^p_{\gamma,0}M^{-p\gamma}\|L_1\|_{L^\infty(\mathbb{R}^+)}\|u-v\|_\vartheta\big(\frac{e^{-\delta M}}{\delta}\big)^{p-1}\eta^{-\xi}$$
$$\times\Big(\frac{e^{-\delta M}}{\delta}+\big(\frac{t}{2}\big)^{-\xi}\frac{\Gamma(1+\xi)}{\delta^{1+\xi}}\Big),$$

thanks to

$$(a+b)^\theta \leq a^\theta + b^\theta \quad \text{for} \quad a,b>0 \quad \text{and} \quad \theta\in(0,1). \tag{7.8}$$

For the term V^3_{11}, we find that

$$\begin{aligned}
V^3_{11} &\leq 3^{p-1}C^p_{\gamma,0}t^\xi E\Big(\int_{t-M}^t e^{-\frac{p-1}{p}\delta(t-\tau)}(t-\tau)^{-\frac{p-1}{p}\gamma}e^{-\frac{1}{p}\delta(t-\tau)}(t-\tau)^{-\frac{1}{p}\gamma}\\
&\quad\times\big\|f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big\|d\tau\Big)^p\\
&\leq 3^{p-1}C^p_{\gamma,0}t^\xi\Big(\int_{t-M}^t\frac{e^{-\delta(t-\tau)}}{(t-\tau)^\gamma}d\tau\Big)^{p-1}\int_{t-M}^t e^{-\delta(t-\tau)}(t-\tau)^{-\gamma}\\
&\quad\times E\big\|f(\tau,u(\eta\tau))-f(\tau,v(\eta\tau))\big\|^p d\tau\\
&\leq 3^{p-1}C^p_{\gamma,0}t^\xi\big(\delta^{\gamma-1}\Gamma(1-\gamma)\big)^{p-1}\|u-v\|_\vartheta\|L_1\|_{L^\infty(\mathbb{R}^+)}\eta^{-\xi}\\
&\quad\times\int_{t-M}^t e^{-\delta(t-\tau)}(t-\tau)^{-\gamma}\tau^{-\xi}d\tau\\
&\leq 3^{p-1}C^p_{\gamma,0}\big(\delta^{\gamma-1}\Gamma(1-\gamma)\big)^p\frac{\eta^{-\xi}t^\xi}{(t-M)^\xi}\|L_1\|_{L^\infty(\mathbb{R}^+)}\|u-v\|_\vartheta.
\end{aligned} \tag{7.9}$$

Inserting (7.6)-(7.9) into (7.5) gives

$$\begin{aligned}
&\vartheta(t)E\big\|(\mathcal{T}u)(t)-(\mathcal{T}v)(t)\big\|^p_\gamma\\
&\quad\leq 3^{p-1}C^p_{\gamma,0}\Bigg(\big(\frac{t}{2}\big)^{-p\gamma}\big(\frac{e^{-\frac{\delta t}{2}}}{\delta}\big)^{p-1}\frac{e^{-\frac{\delta t}{2}}\big(\frac{t}{2}\big)^{\frac{1}{q'}-\xi}t^\xi}{\eta^\xi(\delta p')^{\frac{1}{p'}}(1-q'\xi)^{\frac{1}{q'}}}\\
&\quad\quad+\frac{\eta^{-\xi}(e^{-\delta M}/\delta)^{p-1}}{M^{p\gamma}}\Big(\frac{e^{-\delta M}}{\delta}+\big(\frac{t}{2}\big)^{-\xi}\frac{\Gamma(1+\xi)}{\delta^{1+\xi}}\Big)\\
&\quad\quad+\big(\delta^{\gamma-1}\Gamma(1-\gamma)\big)^p\frac{\eta^{-\xi}t^\xi}{(t-M)^\xi}\Bigg)\|L_1\|_{L^\infty(\mathbb{R}^+)}\|u-v\|_\vartheta.
\end{aligned} \tag{7.10}$$

By using (7.2), we can choose $M>0$ and $T>2M$ sufficiently large such that for any $t>T$,

$$\vartheta(t)E\big\|(\mathcal{T}u)(t)-(\mathcal{T}v)(t)\big\|^p_\gamma \quad < \|u-v\|_\vartheta,$$

which together with (7.4) implies that $\mathcal{T}$ is contractive on the space $C_\vartheta\big(0,\infty; L^p(\Omega;H^\gamma)\big)$.

Step 2. We prove that $\mathcal{T}$ is bounded in $C_\vartheta\big(0,\infty;L^p(\Omega;H^\gamma)\big)$.

Due to (7.3) and the property $(\mathcal{P}_1)$, we deduce that for $u \in C_\vartheta\big(0,\infty; L^p(\Omega; H^\gamma)\big)$ and any $t \geq 0$,

$$
\begin{aligned}
\vartheta(t)E\big\|(\mathcal{T}u)(t)\big\|_\gamma^p \leq\ & 3^{p-1}\vartheta(t)E\big\|S(t)u_0\big\|_\gamma^p \\
& + 3^{p-1}\vartheta(t)E\Big\|\int_0^t S(t-\tau)f(\tau,u(\eta\tau))d\tau\Big\|_\gamma^p \\
& + 3^{p-1}\vartheta(t)E\Big\|\int_0^t S(t-\tau)\tilde{g}(\tau)dB^{\sigma,\lambda}(\tau)\Big\|_\gamma^p \\
\leq\ & 3^{p-1}\vartheta(t)C_0^p e^{-\delta pt}E\|u_0\|_\gamma^p + V_{12} + V_{13}.
\end{aligned}
\tag{7.11}
$$

Following similar calculations as in (7.4) and applying the assumption (H_7), we obtain that

$$
\begin{aligned}
V_{12} \leq\ & \mathbb{C}(\gamma,p)\vartheta(t)\Big(\int_0^t e^{-\delta(t-\tau)}(t-\tau)^{-\gamma}d\tau\Big)^{p-1} \\
& \times \int_0^t e^{-\delta(t-\tau)}(t-\tau)^{-\gamma}E\big\|f(\tau,u(\eta\tau))\big\|^p d\tau \\
\leq\ & \mathbb{C}(\gamma,p)\vartheta(t)\big(\delta^{\gamma-1}\Gamma(1-\gamma)\big)^{p-1} \\
& \times \int_0^t e^{-\delta(t-\tau)}(t-\tau)^{-\gamma}\big(l_1(\tau)+l_2(\tau)E\|u(\eta\tau)\|_\gamma^p\big)d\tau.
\end{aligned}
$$

Now we consider the case of $t \geq T$. Using inequality (7.8) and Hölder's inequality results in

$$
\begin{aligned}
V_{12} \leq\ & \mathbb{C}(p,\gamma,\delta)\bigg(\int_0^t e^{-\delta(t-\tau)}\big((t-\tau)^{-\gamma}\tau^{\xi}+(t-\tau)^{\xi-\gamma}\big) \\
& \times \big(l_1(\tau)+l_2(\tau)E\|u(\eta\tau)\|_\gamma^p\big)d\tau \\
\leq\ & \mathbb{C}(p,\gamma,\delta)\bigg(\Big((\delta p)^{(\gamma-\xi)p-1}\Gamma(1-(\gamma-\xi)p)\Big)^{\frac{1}{p}} \\
& \times \Big[\|l_1\|_q + \eta^{-\xi}\|u\|_\vartheta\Big(\int_0^\infty l_2^q(\tau)\tau^{-q\xi}d\tau\Big)^{\frac{1}{q}}\Big] \\
& + \big((p\delta)^{p\gamma-1}\Gamma(1-p\gamma)\big)^{\frac{1}{p}}\Big[\Big(\int_0^\infty \tau^{q\xi}l_1^q(\tau)d\tau\Big)^{\frac{1}{q}} + \eta^{-\xi}\|u\|_\vartheta\|l_2\|_q\Big]\bigg).
\end{aligned}
\tag{7.12}
$$

For the term V_{13}, by making use of Lemma 2.2, Hölder's inequality, the property $(\mathcal{P}_1)$ and the assumption (H_8), we have

$$
\begin{aligned}
V_{13} \leq\ & \mathbb{C}(p)\vartheta(t)(N_t)^{\frac{p}{2}}E\Big(\int_0^t \big\|S(t-\tau)\tilde{g}(\tau)\big\|_\gamma^2 d\tau\Big)^{\frac{p}{2}} \\
\leq\ & \mathbb{C}(\gamma,p)\vartheta(t)(N_t)^{\frac{p}{2}}E\Big(\int_0^t e^{-2\delta(t-\tau)}(t-\tau)^{-2\gamma}\|\tilde{g}(\tau)\|^2 d\tau\Big)^{\frac{p}{2}} \\
\leq\ & \mathbb{C}(\gamma,p)\vartheta(t)(N_t)^{\frac{p}{2}}\Big(\int_0^t e^{-2\delta(t-\tau)}(t-\tau)^{-2\gamma}d\tau\Big)^{\frac{p-2}{2}} \\
& \times \int_0^t e^{-2\delta(t-\tau)}(t-\tau)^{-2\gamma}E\|\tilde{g}(\tau)\|^p d\tau
\end{aligned}
\tag{7.13}
$$

$$\leq \mathbb{C}(p,\delta,\gamma)\vartheta(t)(N_t)^{\frac{p}{2}}e^{-\delta t}\Big(\int_0^t e^{-\tilde{p}\delta(t-\tau)}(t-\tau)^{-2\tilde{p}\gamma}d\tau\Big)^{\frac{1}{\tilde{p}}}$$
$$\times\Big(\int_0^t \big(e^{\delta\tau}E\|\tilde{g}(\tau)\|^p\big)^{\tilde{q}}d\tau\Big)^{\frac{1}{\tilde{q}}}$$
$$\leq \mathbb{C}(p,\delta,\gamma)\vartheta(t)(N_t)^{\frac{p}{2}}e^{-\delta t}\hbar^{\frac{1}{\tilde{q}}}.$$

By similar arguments as above, we can compute V_{12} and V_{13} in the case of $t \in [0,T]$. Hence $\mathcal{T}$ is bounded on the space $C_\vartheta\big(0,\infty;L^p(\Omega;H^\gamma)\big)$. The assertion of this theorem follows immediately by applying the Banach fixed point theorem. □

Remark 7.1. Indeed, there exist nonnegative functions l_1, l_2 satisfying the assumption (H_7). For example, we can take $l_1(t) = e^{-c_1 t}$, $l_2(t) = e^{-c_2 t}$, then it is easy to see that

$$\int_0^\infty \tau^{q\xi}l_1^q(\tau)d\tau < C\Gamma(1+q\xi), \quad \int_0^\infty \tau^{-q\xi}l_2^q(\tau)d\tau < C\Gamma(1-q\xi)$$

for some constant C.

The following result follows directly from Theorem 7.1.

Corollary 7.1. *Let $p \geq 2$, $\gamma \in (0,\frac{1}{p})$ and $u_0 \in L^p(\Omega;H^\gamma)$. Suppose that (7.2) and the assumptions (H_6)-(H_8) hold. Then problem (7.1) but with FBM or Brownian motion instead of TFBM has a unique global mild solution u satisfying*

$$\sup_{r\in[0,\infty)} r^\xi E\|u(r)\|_\gamma^p < \infty$$

where ξ is given in the assumption (H_7).

8. Polynomial and exponential stability of mild solutions in a more regular phase space

In this section we will analyze not only polynomial stability of our Eq. (1.1) but also will provide some exponential stability results. However, we need to consider a different phase space, $\mathcal{C}^{p,\zeta}(H^\gamma)$, defined below, in which the norm has an exponential weight which prevents, in general, that the case of variable delay can be included in this formulation (in particular the case of proportional delay considered in Section 7), since the Lipschitz assumption (H_6) cannot be proved with the new norm (see [30] for more details). Let us define the phase space $\mathcal{C}^{p,\zeta}(H^\gamma)$ by

$$\mathcal{C}^{p,\zeta}(H^\gamma) = \Big\{\psi \in C\big(-\infty,0;L^p(\Omega;H^\gamma)\big) : \lim_{\theta\to-\infty} e^{\zeta\theta}\psi(\theta) \text{ exists in } L^p(\Omega;H^\gamma)\Big\},$$

for $p \geq 2$, $\zeta > 0$. If $\mathcal{C}^{p,\zeta}(H^\gamma)$ is endowed with the norm

$$\|\psi\|_{\mathcal{C}^{p,\zeta}(H^\gamma)} = \big(\sup_{\theta\in(-\infty,0]} e^{\zeta\theta}E\|\psi(\theta)\|_\gamma^p\big)^{\frac{1}{p}}, \quad \psi \in \mathcal{C}^{p,\zeta}(H^\gamma),$$

then $\big(\mathcal{C}^{p,\zeta}(H^\gamma), \|\cdot\|_{\mathcal{C}^{p,\zeta}(H^\gamma)}\big)$ is a Banach space.

We now enumerate now the new assumptions on the delay terms f and g.

$(\widetilde{H}_1)$ For any $\mu \in \mathcal{C}^{p,\varsigma}(H^\gamma)$, the mappings $[0,\infty) \ni t \mapsto f(t,\mu) \in \mathcal{L}^2$ and $[0,\infty) \ni t \mapsto g(t,\mu) \in \mathcal{L}^2$ are measurable.

$(\widetilde{H}_2)$ There exist nonnegative functions $k_1, k_2 \in L^\infty(\mathbb{R}^+)$ such that for any $\mu, \nu \in \mathcal{C}^{p,\varsigma}(H^\gamma)$ and $t \geq 0$,

$$E\|f(t,\mu) - f(t,\nu)\|^p \leq k_1(t)\|\mu - \nu\|^p_{\mathcal{C}^{p,\varsigma}(H^\gamma)},$$
$$E\|g(t,\mu) - g(t,\nu)\|^p \leq k_2(t)\|\mu - \nu\|^p_{\mathcal{C}^{p,\varsigma}(H^\gamma)},$$

and

$$\|k_1\|_{L^\infty(\mathbb{R}^+)} := K_1 < \infty, \qquad \|k_2\|_{L^\infty(\mathbb{R}^+)} := K_2 < \infty.$$

$(\widetilde{H}_3)$ There exist nonnegative functions k_3, k_4 and $q' > 1$ such that for any $\mu \in \mathcal{C}^{p,\varsigma}(H^\gamma)$ and $t \geq 0$,

$$E\|f(t,\mu)\|^p \leq k_3(t) + k_4(t)\|\mu\|^p_{\mathcal{C}^{p,\varsigma}(H^\gamma)},$$

and

$$\int_0^\infty e^{\delta\tau} k_3(\tau) d\tau := K_3 < \infty, \qquad \int_0^\infty \left(e^{\delta\tau} k_4(\tau)\tau^{-\xi}\right)^{q'} d\tau := K_4 < \infty,$$

for some $\xi \in (0,1)$, and where δ is given in (1.1).

$(\widetilde{H}_4)$ There exist nonnegative functions k_5 and k_6 such that for any $\mu \in \mathcal{C}^{p,\varsigma}(H^\gamma)$ and $t \geq 0$,

$$E\|g(t,\mu)\|^p \leq k_5(t) + k_6(t)\|\mu\|^p_{\mathcal{C}^{p,\varsigma}(H^\gamma)},$$

and

$$\int_0^\infty e^{\delta p'\tau} k_5^{p'}(\tau) d\tau := K_5 < \infty, \qquad \int_0^\infty \left(e^{\delta\tau} k_6(\tau)\tau^{-\xi}\right)^{q_2'} d\tau := K_6 < \infty,$$

for some $\xi \in (0,1)$. Here $1/q_1' + 1/q_2' + 1/p' = 1$, $1/q' + 1/p' = 1$ and $1 < q', q_1' < \frac{1}{2\gamma}$.

Remark 8.1. Similar to the proof of Theorem 3.1, we can deduce from the assumptions $(\widetilde{H}_1)$-$(\widetilde{H}_4)$ that for each $\varphi \in \mathcal{C}^{p,\varsigma}(H^\gamma)$, there exists a unique global mild solution to Eq. (1.1).

Theorem 8.1. *Let $p \geq 2$, $\gamma \in (0, \frac{1}{p})$ and $\varphi \in \mathcal{C}^{p,\varsigma}(H^\gamma)$. Suppose that the assumptions $(\widetilde{H}_1)$-$(\widetilde{H}_4)$ and*

$$\varsigma > p\delta \tag{8.1}$$

hold. Then, mild solutions to Eq. (1.1) *are polynomially stable, that is, for any mild solution u of Eq.* (1.1) *with the initial condition $\varphi \in \mathcal{C}^{p,\varsigma}(H^\gamma)$,*

$$\sup_{t\in[0,\infty)} t^\xi \|u_t\|^p_{\mathcal{C}^{p,\varsigma}(H^\gamma)} < \infty, \tag{8.2}$$

where ξ is given in the assumptions $(\widetilde{H}_3)$ and $(\widetilde{H}_4)$.

Proof. It follows immediately from (3.2) that

$$E\|u(t)\|_\gamma^p \le 3^{p-1}E\big\|S(t)\varphi(0)\big\|_\gamma^p + 3^{p-1}E\Big(\int_0^t \big\|S(t-\tau)f(\tau,u_\tau)\big\|_\gamma d\tau\Big)^p \\ + 3^{p-1}E\Big\|\int_0^t S(t-\tau)g(\tau,u_\tau)dB^{\sigma,\lambda}(\tau)\Big\|_\gamma^p := G_1+G_2+G_3. \tag{8.3}$$

By the property $(\mathcal{P}_1)$, we have

$$G_1 \le 3^{p-1}C_0^p e^{-p\delta t}E\|\varphi(0)\|_\gamma^p. \tag{8.4}$$

In view of Hölder's inequality, the property $(\mathcal{P}_1)$ and the assumption $(\widetilde{H}_3)$, we deduce that

$$\begin{aligned} G_2 &\le 3^{p-1}C_{\gamma,0}^p E\Big(\int_0^t e^{-\frac{(p-1)\delta}{p}(t-\tau)}(t-\tau)^{-\gamma}e^{-\frac{\delta}{p}(t-\tau)}\|f(\tau,u_\tau)\|d\tau\Big)^p \\ &\le 3^{p-1}C_{\gamma,0}^p\Big(\int_0^t e^{-\delta(t-\tau)}(t-\tau)^{-\frac{p\gamma}{p-1}}d\tau\Big)^{p-1}\int_0^t e^{-\delta(t-\tau)}E\|f(\tau,u_\tau)\|^p d\tau \\ &\le 3^{p-1}C_{\gamma,0}^p(\delta^{\frac{p\gamma}{p-1}-1}\Gamma(1-\frac{p\gamma}{p-1}))^{p-1}e^{-\delta t} \\ &\quad\times\Big(\int_0^t e^{\delta\tau}k_3(\tau)d\tau + \int_0^t e^{\delta\tau}k_4(\tau)\tau^{-\xi}\tau^{\xi}\|u_\tau\|_{\mathcal{C}^{p,\varsigma}(H^\gamma)}^p d\tau\Big) \\ &\le 3^{p-1}C_{\gamma,0}^p(\delta^{\frac{p\gamma}{p-1}-1}\Gamma(1-\frac{p\gamma}{p-1}))^{p-1}e^{-\delta t}K_3 + 3^{p-1}C_{\gamma,0}^p \\ &\quad\times(\delta^{\frac{p\gamma}{p-1}-1}\Gamma(1-\frac{p\gamma}{p-1}))^{p-1}e^{-\delta t}K_4^{\frac{1}{q'}}\Big(\int_0^t\big(\tau^{\xi}\|u_\tau\|_{\mathcal{C}^{p,\varsigma}(H^\gamma)}^p\big)^{p'}d\tau\Big)^{\frac{1}{p'}}, \end{aligned} \tag{8.5}$$

where q' is given in the assumption $(\widetilde{H}_3)$ and $1/p'+1/q'=1$. For the stochastic term G_3, thanks to Lemma 2.2, by a similar way as in (8.5) we obtain that

$$\begin{aligned} G_3 &\le 3^{p-1}(N_t)^{\frac{p}{2}}E\Big(\int_0^t e^{-2\delta(t-\tau)}(t-\tau)^{-2\gamma}\|g(\tau,u_\tau)\|^2 d\tau\Big)^{\frac{p}{2}} \\ &\le 3^{p-1}(N_t)^{\frac{p}{2}}\Big(\int_0^t e^{-2\delta(t-\tau)}(t-\tau)^{-2\gamma}d\tau\Big)^{\frac{p-2}{2}} \\ &\quad\times\int_0^t e^{-2\delta(t-\tau)}(t-\tau)^{-2\gamma}E\|g(\tau,u_\tau)\|^p d\tau \\ &\le 3^{p-1}\frac{(N_t)^{\frac{p}{2}}}{e^{\delta t}}\big((2\delta)^{2\gamma-1}\Gamma(1-2\gamma)\big)^{\frac{p-2}{2}}\Big(\int_0^t e^{\delta\tau}k_5(\tau)e^{-\delta(t-\tau)}(t-\tau)^{-2\gamma}d\tau \\ &\quad+\int_0^t e^{-\delta(t-\tau)}(t-\tau)^{-2\gamma}e^{\delta\tau}k_6(\tau)\tau^{-\xi}\tau^{\xi}\|u_\tau\|_{\mathcal{C}^{p,\varsigma}(H^\gamma)}^p d\tau\Big) \\ &\le \frac{3^{p-1}(N_t)^{\frac{p}{2}}}{e^{\delta t}}\big((2\delta)^{2\gamma-1}\Gamma(1-2\gamma)\big)^{\frac{p-2}{2}}\Big(K_5^{\frac{1}{p'}}\Big(\int_0^t e^{-\delta q'(t-\tau)}(t-\tau)^{-2q'\gamma}d\tau\Big)^{\frac{1}{q'}} \\ &\quad+K_6^{\frac{1}{q_2'}}\Big(\int_0^t e^{-\delta q_1'(t-\tau)}(t-\tau)^{-2q_1'\gamma}d\tau\Big)^{\frac{1}{q_1'}}\Big(\int_0^t\big(\tau^{\xi}\|u_\tau\|_{\mathcal{C}^{p,\varsigma}(H^\gamma)}^p\big)^{p'}d\tau\Big)^{\frac{1}{p'}}\Big) \end{aligned} \tag{8.6}$$

$$\leq 3^{p-1}\big((2\delta)^{2\gamma-1}\Gamma(1-2\gamma)\big)^{\frac{p-2}{2}}\big((\delta q')^{2q'\gamma-1}\Gamma(1-2q'\gamma)\big)^{\frac{1}{q'}}K_5^{\frac{1}{p'}}(N_t)^{\frac{p}{2}}e^{-\delta t}$$
$$+3^{p-1}\big((2\delta)^{2\gamma-1}\Gamma(1-2\gamma)\big)^{\frac{p-2}{2}}\Big(\frac{\Gamma(1-2q_1'\gamma)}{(\delta q_1')^{1-2q_1'\gamma}}\Big)^{\frac{1}{q_1'}}K_6^{\frac{1}{q_2'}}(N_t)^{\frac{p}{2}}e^{-\delta t}$$
$$\times\Big(\int_0^t\big(\tau^{\xi}\|u_\tau\|^p_{\mathcal{C}^{p,\zeta}(H^\gamma)}\big)^{p'}d\tau\Big)^{\frac{1}{p'}}.$$

Inserting (8.4)-(8.6) into (8.3) yields

$$E\|u(t)\|^p_\gamma\leq 3^{p-1}C_0^pe^{-p\delta t}E\|\varphi(0)\|^p_\gamma+\mathbb{C}(p,p',q',\delta,\gamma,K_3,K_5)e^{-\delta t}\big(1+(N_t)^{\frac{p}{2}}\big)$$
$$+\mathbb{C}(p,q',q_1',q_2',\delta,\gamma,K_4,K_6)e^{-\delta t}\big(1+(N_t)^{\frac{p}{2}}\big)\Big(\int_0^t\big(\tau^{\xi}\|u_\tau\|^p_{\mathcal{C}^{p,\zeta}(H^\gamma)}\big)^{p'}d\tau\Big)^{\frac{1}{p'}}.\quad(8.7)$$

By the assumption (8.1), we have $e^{(\zeta-p\delta)\theta}<1$ for $\theta\leq 0$. Then multiplying (8.7) by $e^{\zeta\theta}e^{-\zeta\theta}$ and replacing t by $t+\theta$, in view of the monotonicity for N_t with respect to t, we conclude that for $\theta\in[-t,0]$,

$$\begin{aligned}&e^{\zeta\theta}E\|u(t+\theta)\|^p_\gamma\\&\leq 3^{p-1}C_0^pe^{-p\delta t}E\|\varphi(0)\|^p_\gamma+\mathbb{C}(p,p',q',\delta,\gamma,K_3,K_5)e^{-\delta t}\big(1+(N_t)^{\frac{p}{2}}\big)\\&+\mathbb{C}(p,q',q_1',q_2',\delta,\gamma,K_4,K_6)e^{-\delta t}\big(1+(N_t)^{\frac{p}{2}}\big)\Big(\int_0^t\big(\tau^{\xi}\|u_\tau\|^p_{\mathcal{C}^{p,\zeta}(H^\gamma)}\big)^{p'}d\tau\Big)^{\frac{1}{p'}}.\end{aligned}\quad(8.8)$$

On the other hand, we have for all $\theta\in(-\infty,-t]$,

$$\begin{aligned}e^{\zeta\theta}E\|u(t+\theta)\|^p_\gamma&\leq e^{-\zeta t}e^{\zeta(t+\theta)}E\|u(t+\theta)\|^p_\gamma\\&\leq e^{-\zeta t}\|\varphi\|^p_{\mathcal{C}^{p,\zeta}(H^\gamma)}\leq e^{-p\delta t}\|\varphi\|^p_{\mathcal{C}^{p,\zeta}(H^\gamma)}.\end{aligned}\quad(8.9)$$

Therefore,

$$\begin{aligned}&t^{\xi}\|u_t\|^p_{\mathcal{C}^{p,\zeta}(H^\gamma)}\\&\leq(3^{p-1}C_0^p+1)\frac{t^\xi}{e^{p\delta t}}\|\varphi\|^p_{\mathcal{C}^{p,\zeta}(H^\gamma)}+\mathbb{C}(p,p',q',\delta,\gamma,K_3,K_5)e^{-\delta t}t^{\xi}\big(1+(N_t)^{\frac{p}{2}}\big)\\&+\mathbb{C}(p,q',q_1',q_2',\delta,\gamma,K_4,K_6)e^{-\delta t}t^{\xi}\big(1+(N_t)^{\frac{p}{2}}\big)\Big(\int_0^t\big(\tau^{\xi}\|u_\tau\|^p_{\mathcal{C}^{p,\zeta}(H^\gamma)}\big)^{p'}d\tau\Big)^{\frac{1}{p'}}.\end{aligned}\quad(8.10)$$

By using Gronwall's lemma, the assertion of this theorem follows immediately. □

As a simple consequence of Theorem 8.1, we have

Corollary 8.1. *Let $p\geq 2$, $\gamma\in(0,\frac{1}{p})$ and $\varphi\in\mathcal{C}^{p,\zeta}(H^\gamma)$. Suppose that the assumptions $(\widetilde{H}_1)$-$(\widetilde{H}_4)$ and*

$$\zeta>p\delta\quad(8.11)$$

hold. Then mild solutions to Eq. (1.1) with FBM or Brownian motion instead of TFBM are polynomially stable.

Remark 8.2. In fact, by slightly modifying the conditions of Theorem 8.1, the exponential stability of mild solutions to Eq. (1.1) is established in the sense of $p-$th moment. More precisely, let $p \geq 2$, $\gamma \in (0, \frac{1}{p})$ and $\varphi \in \mathcal{C}^{p,\zeta}(H^\gamma)$, and assume that

($\widehat{H}_3$) There exist nonnegative functions k_3, $\hat{k}_4$ and $q' > 1$ such that for any $\mu \in \mathcal{C}^{p,\zeta}(H^\gamma)$ and $t \geq 0$,

$$E\|f(t,\mu)\|^p \leq k_3(t) + \hat{k}_4(t)\|\mu\|^p_{\mathcal{C}^{p,\zeta}(H^\gamma)},$$

and

$$\int_0^\infty e^{\delta\tau} k_3(\tau) d\tau := K_3 < \infty, \qquad \int_0^\infty \left(e^{(\delta-\xi)\tau}\hat{k}_4(\tau)\right)^{q'} d\tau := \widehat{K}_4 < \infty,$$

for some $\xi \in (0,1)$, where δ is given in problem (1.1).

($\widehat{H}_4$) There exist nonnegative functions k_5 and $\hat{k}_6$ such that for any $\mu \in \mathcal{C}^{p,\zeta}(H^\gamma)$ and $t \geq 0$,

$$E\|g(t,\mu)\|^p \leq k_5(t) + \hat{k}_6(t)\|\mu\|^p_{\mathcal{C}^{p,\zeta}(H^\gamma)},$$

and

$$\int_0^\infty e^{\delta p'\tau} k_5^{p'}(\tau) d\tau := K_5 < \infty, \qquad \int_0^\infty \left(e^{(\delta-\xi)\tau}\hat{k}_6(\tau)\right)^{q_2'} d\tau := \widehat{K}_6 < \infty,$$

for some $\xi \in (0,1)$. Here $1/q_1' + 1/q_2' + 1/p' = 1$, $1/q' + 1/p' = 1$ and $1 < q', q_1' < \frac{1}{2\gamma}$.

Furthermore, suppose that the assumptions ($\widetilde{H}_1$)-($\widetilde{H}_2$), and

$$\zeta > p\delta > p\xi \tag{8.12}$$

hold. Then mild solutions to Eq. (1.1) are exponentially stable, that is, for any mild solution u of Eq. (1.1) with the initial condition $\varphi \in \mathcal{C}^{p,\zeta}(H^\gamma)$,

$$\sup_{t\in[0,\infty)} e^{t\xi}\|u_t\|^p_{\mathcal{C}^{p,\zeta}(H^\gamma)} < \infty, \tag{8.13}$$

where ξ is given in the assumptions (8.12) and ($\widehat{H}_3$)-($\widehat{H}_4$).

9. Summary

There have been very few work in the literature on stochastic partial differential equations with unbounded delay driven by tempered fractional Gaussian noise. In this paper we have considered stochastic Stokes models with unbounded delay and multiplicative TFBM $B^{\sigma,\lambda}(t)$ in fractional power spaces and moments of order $p \geq 2$. The continuity of mild solutions is first studied in the case of $\lambda \to 0$, $\sigma \in (-1/2, 0)$ or $\lambda > 0$, $\sigma \to \sigma_0 \in (-1/2, 0)$ where λ is tempered parameter and $H := 1/2 - \sigma$ is Hurst index. It is worth mentioning that the global existence, continuity and p-th moment Hölder regularity in time can be obtained for stochastic delay Stokes models without damping term. One technical challenge is that we consider the stability of models in the sense of $p-$th moment. The presence of fractional power spaces and unbounded delay also makes the analysis more complicated. Another highlight of the work is that p-th polynomial stability of mild solutions can be obtained in two types of infinite delay phase spaces. By considering the phase space $\varphi \in \mathcal{C}^{p,\zeta}(H^\gamma)$ we prove, not only polynomial stability of mild solutions, but also exponential stability in the $p-$th moment. However, the assumptions imposed do not allow the case of variable delay be handled. At least the case of proportional delay can be analyzed considering the phase space more general infinite delay phase $\varphi \in \mathcal{C}^p(H^\gamma)$ and polynomial stability is successfully proved in this case.

Eventually, it is important to emphasize that our results hold not only for the Stokes problem, but for any other semilinear problem in which the operator A satisfies properties $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$.

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