η-stability of hybrid neutral stochastic differential equations with infinite delay

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Abstract

In this paper, we study the η-stability in q-th moment (η.s.q.m) of hybrid neutral stochastic differential equations with infinite delay (HNSDEID) using the Lyapunov techniques and the method of M-matrix. Finally, we apply the main result to some examples.

1 Introduction

A stochastic delay differential equation (SDDE) is a specific model of stochastic functional differential equations (SFDE). SDDE and SFDE have attracted a great interest in the recent decades. Many researchers from all disciplines have been investigating SDDE and SFDE arising in problems from physics, biology, economics, logistics, computer science and others (see [3]-[16]). The stability theory was initiated at the end of the 19-th century by Lyapunov. This theory has found wide application in various fields of physics and mathematical sciences (see [8]-[9]).

From a mathematical point of view, the theory of stability presents a particular case of the qualitative theory of stochastic differential equations (SDE). Liapunov's method was the ultimate object to study the stability for SDE and partial SDE.

As a specific case of SDE is the neutral SFDE (NSFDE) which has been applied to model an equation whose future depends on the present state, the past state and the derivatives in the terms containing variables delays (see [8], [9] and [13]).

For hybrid neutral SDDE with different structures, [17] studied p-th moment asymptotic boundedness, p-th moment and almost sure exponential stability of solutions.

To the best of our knowledge, there is no existing result about the η–stability of HNSDEID. We point out that in the works [1], [2], [12], [13] and [16] the authors have investigated the stability with general decay rate of nonlinear stochastic evolution equations, SFDE with finite and infinite delay and also of NSFDE using a Razumikhin approach.

Based on the work of [11], we will extend these results by studying the η–stability of HNSDEID using the method of M-matrix and the Lyapunov function.

In this sense, our results generalize those in the paper [11] in the η–stability concept and by using some appropriate conditions on the neutral term of the system.

Comparing with the existing articles on the literature, it is worth emphasizing that the η-stability is closely related to the analysis of stability with general decay rate (see e.g. [6],[1] and the references therein). However, the conditions impose on the general decay rate functions are usually stronger than the ones imposed on the η–function, this is why the η–stability can be more significant and more general. Now, we will describe the main contributions of our paper:

(1) Although there is an equivalence between η-stability and stability with general decay rate (see [1], [2], [12], [13] and [16]) in the sense of Definition 2.2, the main differences can be seen in Theorem 3.2, where it is only necessary to impose conditions (i) and (ii) on the function η.

(2) Namely, under assumptions Theorem 3.2, the theory of stability with general decay rate cannot be applied as done in other papers in the literature due to conditions (i) and (ii). In other words, in [1] we imposed more restrictive conditions on the general decay rate function $\lambda(t)$, for example, see conditions (d) and (e) in theorems 3.3 and 3.5, but in this paper we will just impose conditions (i) and (ii) in theorem 3.2 which are much weaker than the ones imposed in [1], [2], [12], [13] and [16].

(3) In [6], the class of ψ-function is more restrictive than the ones in this paper due to the condition $\psi(0) = 1$, condition (iii) in Definition 3.1 and condition (3.1) in assumption 3.3 (in [6]).

(4) Different from the previous results in [17], taking into account the influence of the neutral term, the infinite delay, the new class of η-function and the M-matrix technique in the stability theory make our results more general.

The content of our paper is as follows. After introducing some necessary preliminaries and basic notions in Section 2, we establish in Section 3 the η.s.q.m for HNSDEID. Finally, in Section 4, two examples are analyzed to show the interest of our results.

2 Preliminaries and basic notions

Let $\mathcal{C}([\rho,0],\mathbb{R}^i)$ be the set of all continuous functions from $[\rho,0]$ into $\mathbb{R}^i$.
Let $w(\tau)=(w_1(\tau),\ldots,w_p(\tau))^T$ be an p-dimensional Brownian motion defined on a complete probability space $(\Omega,\mathbb{F},\mathbb{P})$ with the usual filtration $\{\mathbb{F}_\tau\}_{\tau\geq 0}$ generated by $\{w(\varrho):0\leq\varrho\leq\tau\}$. We denote by $|y|$ the Euclidean norm of $y\in\mathbb{R}^i$. For a matrix B, its trace norm is $|B|=\sqrt{trace(B^TB)}$.
Let $\mathbb{BC}((-\infty,0],\mathbb{R}^i)$ be the set of all bounded continuous functions $\mu:(-\infty,0]\to\mathbb{R}^i$ equipped with the norm $||\mu||=\sup_{\zeta\leq 0}|\mu(\zeta)|$. For $q>0$, denote by $\mathbb{L}^q((-\infty,0],\mathbb{R}^i)$ the set of all functions $\nu:(-\infty,0]\to\mathbb{R}^i$ such that $\int_{-\infty}^0|\nu(\tau)|^qd\tau<\infty$. Let $\mathcal{L}((-\infty,0),\mathbb{R}_+)$ be the set of continuous bounded non-negative functions $\sigma(\cdot)$ with $-\infty<\sigma\leq 0$ such that $\int_{-\infty}^0\sigma(\tau)d\tau=1$.

Let $\{l(\tau),\tau\in[0,+\infty)\}$ be a right-continuous Markov chain on the probability space $\{\Omega,\mathbb{F},(\mathbb{F}_\tau)_{\tau\geq 0},\mathbb{P}\}$ taking values in a finite state space $S=\{1,2,\ldots,N\}$ with a generator $\Lambda=(\lambda_{mn})_{\mathbb{N}\times\mathbb{N}}$ given by

$$\mathbb{P}\left(l(\tau+\Delta)=n|l(\tau)=m\right)=\begin{cases}\lambda_{mn}\Delta+o(\Delta), & \text{if}\, m\neq n\\ \\ 1+\lambda_{mm}\Delta+o(\Delta), & \text{if}\, m=n\end{cases}$$

where $\Delta>0$. Here $\lambda_{mn}\geq 0$ is the transition rate from m to n, if $m\neq n$, while

$$\lambda_{mm}=-\sum_{n\neq m}\lambda_{mn}.$$

For a vector $z>0$ or a matrix $D>0$, we mean that all elements in this vector or matrix are positive. A Z-matrix is a square matrix $D=(d_{mn})_{\mathbb{N}\times\mathbb{N}}$ that has non-positive off-diagonal entries (namely $d_{mn}\leq 0$ for all $m\neq n$) and all positive diagonal entries.

Definition 2.1. A square matrix $D=(d_{mn})_{\mathbb{N}\times\mathbb{N}}$ is said to be a nonsingular M-matrix if D can be expressed in the form $D=tI-A$ with $t>\rho(A)$ while all elements of A are nonnegative, where I is the identity matrix and $\rho(A)$ the spectral radius of A.

Lemma 2.1. *If D is a Z-matrix, the following statements are equivalent:*

(i) *D is a nonsingular M-matrix.*

(ii) *D is semi-positive; that is, there exists $z>0$ in $\mathbb{R}^N$ such that $Dz>0$.*

(iii) *D^{-1} exists and its elements are all nonnegative.*

We assume that the Markov chain $l(\cdot)$ is independent of the Brownian motion $w(\cdot)$.

Consider the following HNSDEID:

$$\begin{cases} d(\alpha(\tau) - G(\tau, \alpha_\tau)) = f_1\left(\alpha(\tau), \alpha_\tau, \tau, l(\tau)\right) d\tau + f_2\left(\alpha(\tau), \alpha_\tau, \tau, l(\tau)\right) dw(\tau), \qquad \tau \geq 0 \\ \alpha_0 = \chi, \end{cases} \tag{2.1}$$

where the initial condition $\chi \in \mathbb{L}^q((-\infty, 0], \mathbb{R}^i)$, $l(0) = m_0 \in S$, $\alpha(\tau) = (\alpha_1(\tau), \alpha_2(\tau), \ldots, \alpha_i(\tau))^T$ and $\alpha_\tau = \{\alpha(\tau + \theta) : \theta \in (-\infty, 0]\}$ is a $\mathbb{BC}((-\infty, 0], \mathbb{R}^i)$-valued stochastic process, with $\theta \in (-\infty, 0]$.
In particular when $\theta = -\infty$, the interval $[\tau + \theta, \tau]$ be replaced by $(-\infty, \tau]$.
We suppose that

$$f_1 : \mathbb{R}^i \times \mathbb{BC}((-\infty, 0], \mathbb{R}^i) \times \mathbb{R}_+ \times S \to \mathbb{R}^i,$$

$$f_2 : \mathbb{R}^i \times \mathbb{BC}((-\infty, 0], \mathbb{R}^i) \times \mathbb{R}_+ \times S \to \mathbb{R}^{i \times p}, \quad G : \mathbb{R}_+ \times \mathbb{BC}((-\infty, 0], \mathbb{R}^i) \to \mathbb{R}^i.$$

Assume that $G(\tau, 0) = f_1\left(0, 0, \tau, r\right) = f_2\left(0, 0, \tau, r\right) = 0$ for all $(\tau, r) \in \mathbb{R}_+ \times S$. Then, system (2.1) possesses the trivial solution $\alpha(\tau) = 0$ with the initial condition $\chi = 0$.

Denote by $C^{1,2}\left(\mathbb{BC}((-\infty, 0], \mathbb{R}^i) \times \mathbb{R}_+ \times S, \mathbb{R}_+\right)$ the family of all non-negative functions $V(\alpha, \tau, j)$ on $\mathbb{BC}((-\infty, 0], \mathbb{R}^i) \times \mathbb{R}_+ \times S$, which are twice continuously differentiable with respect to α and once continuously differentiable with respect to τ.

Define the operator $\mathcal{L}V : \mathbb{R}^i \times \mathbb{BC}((-\infty, 0], \mathbb{R}^i) \times \mathbb{R}_+ \times S \to \mathbb{R}$ by (see [10])

$$\begin{aligned} \mathcal{L}V(\alpha, \psi, \tau, j) &= V_\tau(\alpha - G(\tau, \psi), \tau, j) + V_\alpha(\alpha - G(\tau, \psi), \tau, j) f_1\left(\psi, \tau, j\right) \\ &\quad + \frac{1}{2} trace\left(f_2^T(\psi, \tau, j) V_{\alpha\alpha}(\alpha - G(\tau, \psi), \tau, j) f_2(\psi, \tau, j)\right) \\ &\quad + \sum_{k=1}^{N} \gamma_{jk} V(\alpha - G(\tau, \psi), \tau, k), \end{aligned}$$

where

$$V_\tau(\alpha, \tau, j) = \frac{\partial V(\alpha, \tau, j)}{\partial \tau}, V_\alpha(\alpha, \tau, j) = \left(\frac{\partial V(\alpha, \tau, j)}{\partial \alpha_1}, \ldots, \frac{\partial V(\alpha, \tau, j)}{\partial \alpha_i}\right),$$

$$V_{\alpha\alpha}(\alpha, \tau, j) = \left(\frac{\partial^2 V(\alpha, \tau, j)}{\partial \alpha_m \partial \alpha_n}\right)_{i \times i}.$$

Definition 2.2. System (2.1) is said to be η-stable in q-th moment (η**-s.q.m**), if there exists an increasing differentiable function $\eta : \mathbb{R} \to \mathbb{R}^*{}_+$, such that $\eta(\tau) \uparrow \infty$ as $\tau \to \infty$, and a positive constant δ such that, for any $\chi \in \mathbb{L}^q((-\infty, 0], \mathbb{R}^i)$,

$$\limsup_{\tau \to \infty} \frac{\ln \mathbb{E}\left(|\alpha(\tau, \chi)|^q\right)}{\ln \eta(\tau)} \leq -\delta, \tag{2.2}$$

when $q = 2$, it is said to be η-stable in mean square.

Definition 2.3. System (2.1) is said to be almost surely η-stable (**a.s.**η**-s**), if there exists an increasing differentiable function $\eta : \mathbb{R} \to \mathbb{R}_+^*$, such that $\eta(\tau) \uparrow \infty$ as $\tau \to \infty$, and a positive constant δ such that, for any $\chi \in \mathbb{L}^q((-\infty, 0], \mathbb{R}^i)$,

$$\limsup_{\tau\to\infty} \frac{\ln|\alpha(\tau,\chi)|}{\ln\eta(\tau)} \leq -\delta. \tag{2.3}$$

Lemma 2.2. *Let $q > 1$, $\varepsilon > 0$ and $(a_1, a_2) \in \mathbb{R}^2$. Then,*

$$|a_1 + a_2|^q \leq \left[1 + \varepsilon^{\frac{1}{q-1}}\right]^{q-1}\left(|a_1|^q + \frac{|a_2|^q}{\varepsilon}\right).$$

Proof. See ([8]). □

Remark 2.3. Let $q > 1$ and $(a_1, a_2) \in \mathbb{R}^2$. By taking $\varepsilon = 1$, in Lemma 2.2, we obtain

$$|a_1 + a_2|^q \leq 2^{q-1}\left(|a_1|^q + |a_2|^q\right).$$

Assumption 2.4. *For each $a > 0$, there exists $\overline{K}_a > 0$ such that*

$$|f_1(\psi_1(0), \psi_1, \tau, j) - f_1(\psi_2(0), \psi_2, \tau, j)| \vee |f_2(\psi_1(0), \psi_1, \tau, j) - f_2(\psi_2(0), \psi_2, \tau, j)| \leq \overline{K}_a||\psi_1 - \psi_2|| \tag{2.4}$$

$\forall \psi_1, \psi_2 \in \mathbb{BC}((-\infty, 0], \mathbb{R}^i)$ *with* $||\psi_1|| \vee ||\psi_2|| \leq a$ *and* $\forall (\tau, j) \in \mathbb{R}_+ \times S$.

Assumption 2.5. *Assume that there exists a constant $\kappa \in (0, 1)$ such that for all $\psi_1, \psi_2 \in \mathbb{BC}((-\infty, 0], \mathbb{R}^i)$, we have*

$$|G(\psi_1) - G(\psi_2)| \leq \kappa|\psi_1(0) - \psi_2(0)|. \tag{2.5}$$

Assumption 2.6. *Let $q \geq 2$ and $\sigma \in \mathcal{L}((-\infty, 0), \mathbb{R}_+)$ be the same as in Assumption 2.5. Assume that for each $m \in S$ there exist a constants $\xi_{m3} \in \mathbb{R}$, ξ_{m1}, ξ_{m2} and ξ_{m4} such that $\xi_{m2} \geq \xi_{m1}$ and*

$$|\psi(0) - G(\tau, \psi)|^{q-2} \left[(\psi(0) - G(\tau, \psi))^T f_1(\psi(0), \psi, \tau, m) + \frac{q-1}{2}|f_2(\psi(0), \psi, \tau, m)|^2\right]$$

$$\leq \xi_{m4}|\psi(0) - G(\tau, \psi)|^{q-2} + \xi_{m3}|\psi(0) - G(\tau, \psi)|^q + \xi_{m1}\int_{-\infty}^{0} |\psi(\theta)|^q\sigma(\theta)d\theta - \xi_{m2}|\psi(0)|^q. \tag{2.6}$$

Assume that

$$\mathcal{B} = -diag(q\xi_{13}, q\xi_{23}, ..., q\xi_{N3}) - \Lambda, \tag{2.7}$$

and define

$$(\phi_1, \phi_2, ..., \phi_N)^T = \mathcal{B}^{-1}(\beta, \beta, ..., \beta)^T, \tag{2.8}$$

where $\{\phi_j\}_{1\leq j\leq N}$ are positive constants and β is a given positive number.

3 Main results

Under the assumption that $\mathcal{B}$ is a non singular M-matrix, for all ϕ_j, $j \in S$, which are positive, we can establish now our first main result about the existence and uniqueness of solution.

Theorem 3.1. *Assume that Assumptions 2.4-2.6 hold. Assume also that $\kappa \in (0, 2^{\frac{1}{q}-1})$. Then, for any initial condition $\chi \in \mathbb{BC}((-\infty,0],\mathbb{R}^i) \cap \mathbb{L}^q((-\infty,0],\mathbb{R}^i)$ and $l(0) = m_0 \in S$, there is a unique global solution $\alpha(\tau)$ to the HNSDEID (2.1).*

Proof. We will split the proof into two steps.

Step 1: Let $V : \mathbb{BC}((-\infty,0],\mathbb{R}^i) \times S \to \mathbb{R}_+$ be the Lyapunov function:

$$V(\alpha, j) = \phi_j |\alpha|^q, \tag{3.1}$$

where ϕ_j is defined in (2.8). It is easy to see that

$$\varphi_1 |\alpha|^q \leq V(\alpha, j) \leq \varphi_2 |\alpha|^q, \tag{3.2}$$

where $\varphi_1 = \min_{j\in S} \phi_j$ and $\varphi_2 = \max_{j\in S} \phi_j$.

Let $\widetilde{\psi} = \psi(0) - G(\tau,\psi)$. Applying the generalized Itô formula to $V(\widetilde{\psi}, l(\tau))$, we have

$$V(\widetilde{\psi}, l(\tau)) = V(\widetilde{\alpha}_0, m_0) + \int_0^\tau LV\left(\psi(0),\psi,t,l(t)\right) dt + \mathcal{M}(\tau), \tag{3.3}$$

where $\mathcal{M}(\tau)$ is a continuous local martingale with $\mathcal{M}(0) = 0$.

Then,

$$\begin{aligned} LV\left(\psi(0),\psi,\tau,j\right) &= q\phi_j |\psi(0) - G(\tau,\psi)|^{q-2} \left(\psi(0) - G(\tau,\psi)\right)^T f_1\left(\psi(0),\psi,\tau,j\right) \\ &+ \frac{1}{2} q\phi_j |\psi(0) - G(\tau,\psi)|^{q-2} |f_2\left(\psi(0),\psi,\tau,j\right)|^2 \\ &+ \frac{1}{2} q(q-2)\phi_j |\psi(0) - G(\tau,\psi)|^{q-4} |\left(\psi(0) - G(\tau,\psi)\right)^T f_2\left(\psi(0),\psi,\tau,j\right)|^2 \\ &+ \sum_{n=1}^N \lambda_{jn}\phi_n |\psi(0) - G(\tau,\psi)|^q. \end{aligned} \tag{3.4}$$

Using the inequality

$$|\left(\psi(0) - G(\tau,\psi)\right)^T f_2\left(\psi(0),\psi,\tau,j\right)|^2 \leq |\psi(0) - G(\tau,\psi)|^2 |f_2\left(\psi(0),\psi,\tau,j\right)|^2, \tag{3.5}$$

we deduce

$$LV\left(\psi(0),\psi,\tau,j\right)$$

$$
\begin{aligned}
&= q\phi_j|\psi(0)-G(\tau,\psi)|^{q-2}\left[(\psi(0)-G(\tau,\psi))^T f_1(\psi(0),\psi,\tau,j)+\frac{q-1}{2}|f_2(\psi(0),\psi,\tau,j)|^2\right] \\
&+ \sum_{n=1}^{N}\lambda_{jn}\phi_n|\psi(0)-G(\tau,\psi)|^q \\
&\le q\phi_j\xi_{j4}|\psi(0)-G(\tau,\psi)|^{q-2}+q\phi_j\xi_{j3}|\psi(0)-G(\tau,\psi)|^q+\sum_{n=1}^{N}\lambda_{jn}\phi_n|\psi(0)-G(\tau,\psi)|^q \\
&+ q\phi_j\xi_{j1}\int_{-\infty}^{0}|\psi(\theta)|^q\sigma(\theta)d\theta-q\phi_j\xi_{j2}|\psi(0)|^q. \qquad (3.6)
\end{aligned}
$$

By the definition of ϕ_j, we can derive that

$$
q\phi_j\xi_{j3}+\sum_{n=1}^{N}\lambda_{jn}\phi_n=-\beta, \quad j\in S.
$$

Hence,

$$
LV(\psi(0),\psi,\tau,j)\le-\beta|\psi(0)-G(\tau,\psi)|^q+\varphi_3|\psi(0)-G(\tau,\psi)|^{q-2}+q\phi_j\xi_{j1}\left(\int_{-\infty}^{0}|\psi(\theta)|^q\sigma(\theta)d\theta-|\psi(0)|^q\right), \qquad (3.7)
$$

where $\varphi_3=\max_{j\in S} q\phi_j\xi_{j4}$.

Note that $\vartheta(t)=\varphi_3 t^{q-2}-\beta t^q$ will have a finite supremum value over $\mathbb{R}_+$ denoted by $\varphi_4=\sup_{t\in[0,\infty)}\vartheta(t)$.

Therefore,

$$
LV(\psi(0),\psi,\tau,j)\le\varphi_4+q\phi_j\xi_{j1}\left(\int_{-\infty}^{0}|\psi(\theta)|^q\sigma(\theta)d\theta-|\psi(0)|^q\right). \qquad (3.8)
$$

Step 2: Since coefficients of HNSDEID (2.1) are locally Lipschitz continuous, for any given initial condition $\chi\in\mathbb{BC}((-\infty,0],\mathbb{R}^i)\cap\mathbb{L}^q((-\infty,0],\mathbb{R}^i)$, by the standard truncation method, there is a unique maximal local strong solution of equation (2.1) on $\tau\in(-\infty,\sigma_e)$, where σ_e is the explosion time (see, e.g., [9, Theorem 3.2.2, p.95] and [15, Theorem 3.3]).

Let $i_0>0$ be sufficiently large such that $||\chi||<i_0$. For each $i\in\mathbb{N}^*$ with $i\ge i_0$, we define the stopping time:

$$
\sigma_i=\inf\{\tau\in(-\infty,\sigma_e);|\alpha(\tau)|\ge i\}.
$$

It is obvious that σ_i is increasing. Define $\sigma_\infty=\lim_{i\to\infty}\sigma_i$. It is clear that $\sigma_\infty\le\sigma_e$ a.s. We will show that $\sigma_\infty=\infty$ a.s, which implies that $\sigma_e=\infty$ a.s.

Let $\widetilde{\alpha}_\tau = \alpha(\tau) - G(\tau, \alpha_\tau)$. By the Itô formula and (3.8),

$$\begin{aligned}
\mathbb{E}V(\widetilde{\alpha}_{\tau\wedge\sigma_i}, l(\tau\wedge\sigma_i)) &= \mathbb{E}V(\widetilde{\alpha}_0, m_0) + \mathbb{E}\Big(\int_0^{\tau\wedge\sigma_i} LV\left(\alpha(t), \alpha_t, t, l(t)\right) dt\Big), \\
&\leq \mathbb{E}V(\widetilde{\alpha}_0, m_0) + \varphi_4\tau, \\
&+ q\phi_j\xi_{j1}\mathbb{E}\Big(\int_0^{\tau\wedge\sigma_i}\big(\int_{-\infty}^0 |\alpha(t+\theta)|^q\sigma(\theta)d\theta - |\alpha(t)|^q\big)dt\Big). \qquad (3.9)
\end{aligned}$$

Moreover, by the Fubini theorem and the fact that $\theta \leq 0$, we have

$$\begin{aligned}
&\mathbb{E}\Big(\int_0^{\tau\wedge\sigma_i}\big(\int_{-\infty}^0 |\alpha(t+\theta)|^q\sigma(\theta)d\theta - |\alpha(t)|^q\big)dt\Big) \\
&= \mathbb{E}\Big(\int_{-\infty}^0\big(\int_0^{\tau\wedge\sigma_i} |\alpha(t+\theta)|^q dt\big)\sigma(\theta)d\theta\Big) - \mathbb{E}\Big(\int_0^{\tau\wedge\sigma_i} |\alpha(t)|^q dt\Big), \\
&\leq \mathbb{E}\Big(\int_{-\infty}^0\big(\int_{-\infty}^{\tau\wedge\sigma_i} |\alpha(t)|^q dt\big)\sigma(\theta)d\theta\Big) - \mathbb{E}\Big(\int_0^{\tau\wedge\sigma_i} |\alpha(t)|^q dt\Big), \\
&= \mathbb{E}\int_{-\infty}^0 |\chi(t)|^q dt. \qquad (3.10)
\end{aligned}$$

Therefore, by (2.5), we obtain

$$\begin{aligned}
\mathbb{E}V(\widetilde{\alpha}_{\tau\wedge\sigma_i}, l(\tau\wedge\sigma_i)) &\leq \mathbb{E}V(\widetilde{\alpha}_0, m_0) + \varphi_4\tau + q\phi_j\xi_{j1}\mathbb{E}\int_{-\infty}^0 |\chi(t)|^q dt, \\
&\leq \varphi_2\mathbb{E}|\widetilde{\alpha}_0|^q + \varphi_4\tau + q\phi_j\xi_{j1}\mathbb{E}\int_{-\infty}^0 |\chi(t)|^q dt, \\
&\leq 2^{q-1}\varphi_2\mathbb{E}|\alpha_0|^q + 2^{q-1}\varphi_2\mathbb{E}|G(0,\alpha_0)|^q + \varphi_4\tau + q\phi_j\xi_{j1}\mathbb{E}\int_{-\infty}^0 |\chi(t)|^q dt, \\
&\leq 2^{q-1}\varphi_2\mathbb{E}||\chi||^q\left(1+\kappa^q\right) + \varphi_4\tau + q\phi_j\xi_{j1}\mathbb{E}\int_{-\infty}^0 |\chi(t)|^q dt. \qquad (3.11)
\end{aligned}$$

Thus,

$$\mathbb{E}|\widetilde{\alpha}_{\tau\wedge\sigma_i}|^q \leq L + \frac{\varphi_4}{\varphi_1}\tau, \qquad (3.12)$$

where

$$L = 2^{q-1}\frac{\varphi_2}{\varphi_1}\mathbb{E}||\chi||^q\left(1+\kappa^q\right) + \frac{1}{\varphi_1}q\phi_j\xi_{j1}\mathbb{E}\int_{-\infty}^0 |\chi(t)|^q dt.$$

On the other hand, for $-\infty < \theta \leq 0$, we derive

$$\begin{aligned}
\mathbb{E}|\alpha(\tau\wedge\sigma_i)|^q &\leq 2^{q-1}\mathbb{E}|\widetilde{\alpha}_{\tau\wedge\sigma_i}|^q + 2^{q-1}\mathbb{E}|G(\tau\wedge\sigma_i, \alpha_{\tau\wedge\sigma_i})|^q, \\
&\leq 2^{q-1}\mathbb{E}|\widetilde{\alpha}_{\tau\wedge\sigma_i}|^q + 2^{q-1}\kappa^q\mathbb{E}|\alpha(\tau\wedge\sigma_i)|^q. \qquad (3.13)
\end{aligned}$$

Then, we have

$$\mathbb{E}|\alpha(\tau\wedge\sigma_i)|^q \le 2^{q-1}\left(L+\frac{\varphi_4}{\varphi_1}\tau\right)+2^{q-1}\kappa^q\mathbb{E}|\alpha(\tau\wedge\sigma_i)|^q. \tag{3.14}$$

Hence,

$$\sup_{0\le t\le\tau}\mathbb{E}|\alpha(t\wedge\sigma_i)|^q \le \frac{1}{1-2^{q-1}\kappa^q}2^{q-1}\left(L+\frac{\varphi_4}{\varphi_1}\tau\right). \tag{3.15}$$

Therefore,

$$\mathbb{E}|\alpha(\tau\wedge\sigma_i)|^q \le \frac{1}{1-2^{q-1}\kappa^q}2^{q-1}\left(L+\frac{\varphi_4}{\varphi_1}\tau\right). \tag{3.16}$$

Since, $\mathbb{E}|\alpha(\tau\wedge\sigma_i)|^q \ge i^q\mathbb{P}\left(\sigma_i\le\tau\right)$, then, $\mathbb{P}\left(\sigma_i\le\tau\right)\le\dfrac{Q(\tau)}{i^q}$,
where $Q(\tau)=\dfrac{1}{1-2^{q-1}\kappa^q}2^{q-1}\left(L+\dfrac{\varphi_4}{\varphi_1}\tau\right)$. Letting $i\to\infty$, we get $\mathbb{P}\left(\sigma_\infty\le\tau\right)=0$. Thus, $\mathbb{P}\left(\sigma_\infty>\tau\right)=1$, $\forall\tau\ge0$, which implies that $\mathbb{P}\left(\sigma_\infty=\infty\right)=1$, as desired. □

Now, we will discuss the η.s.q.m and the a.s.η-s of solutions for system (2.1).

Theorem 3.2. *Let Assumption (2.5) holds. Assume that there exists an increasing and twice differentiable continuous function $\eta:\mathbb{R}_+\to\mathbb{R}_+^*$ satisfying $\eta(\tau)\uparrow\infty$ as $\tau\to\infty$ and a constant $c_\eta>0$ such that*

(i) $0<\sup_{\tau>0}\frac{\eta'(\tau)}{\eta(\tau)}\le c_\eta$.

(ii) $\eta''(\tau)\eta(\tau)<\eta'(\tau)^2$, *for all* $\tau>0$.

Assume that there exists $\sigma\in\mathcal{L}\left((-\infty,0),\mathbb{R}_+\right)$ and a constant $0\le p_2c_\eta\le p_1$ such that $\forall\tau\ge0$ and $\forall\psi\in\mathbb{BC}((-\infty,0],\mathbb{R}^i)$

$$|\psi(0)-G(\tau,\psi)|^{q-2}\left[\left(\psi(0)-G(\tau,\psi)\right)^T f_1\left(\psi(0),\psi,\tau,m\right)+\frac{q-1}{2}|f_2\left(\psi(0),\psi,\tau,m\right)|^2\right]$$

$$\le -p_1|\psi(0)|^q+\xi_{m3}|\psi(0)-G(\tau,\psi)|^q+p_2\frac{\eta'(\tau)}{\eta(\tau)}\int_{-\infty}^0|\psi(\theta)|^q\sigma(\theta)d\theta. \tag{3.17}$$

Then, system (2.1) is η-s.q.m and a.s.η-s.

Proof. Consider $\chi\in\mathbb{BC}((-\infty,0],\mathbb{R}^i)\cap\mathbb{L}^q((-\infty,0],\mathbb{R}^i)$ and $l(0)=m_0\in S$. Set $\nu(\tau)=\ln(\eta(\tau))$, then $\nu'(\tau)\le c_\eta$. Let $\overline{\delta}\in(0,1)$ be arbitrary such that $\overline{\delta}c_\eta\max_{j\in S}\phi_j\le\beta$ and $\eta''(\tau)\eta(\tau)<(1-\overline{\delta})\eta'(\tau)^2$. Applying the Itô formula, we have

$$\mathbb{E}\left(\phi_j e^{\overline{\delta}\nu(\tau)}|\alpha(\tau)-G(\tau,\alpha_\tau)|^q\right)$$

$$
\begin{aligned}
&= \phi_j\mathbb{E}\left(e^{\overline{\delta}\nu(0)}|\alpha(0)-G(0,\alpha_0)|^q\right)+\overline{\delta}\phi_j\mathbb{E}\left(\int_0^\tau \nu^{'}(t)e^{\overline{\delta}\nu(t)}|\alpha(t)-G(t,\alpha_t)|^q dt\right),\\
&+ q\phi_m\mathbb{E}\left(\int_0^\tau e^{\overline{\delta}\nu(t)}|\alpha(t)-G(t,\alpha_t)|^{q-2}\left(\alpha(t)-G(t,\alpha_t)\right)^T f_1\left(\alpha(t),\alpha_t,t,l(t)\right)\right)dt,\\
&+ \frac{1}{2}q\phi_j\mathbb{E}\left(\int_0^\tau e^{\overline{\delta}\nu(t)}|\alpha(t)-G(t,\alpha_t)|^{q-2}|f_2\left(\alpha(t),\alpha_t,t,l(t)\right)|^2dt\right),\\
&+ \frac{1}{2}q(q-2)\phi_j\mathbb{E}\left(\int_0^\tau e^{\overline{\delta}\nu(t)}|\alpha(t)-G(t,\alpha_t)|^{q-4}|\left(\alpha(t)-G(t,\alpha_t)\right)^T f_2\left(\alpha(t),\alpha_t,t,l(t)\right)|^2dt\right),\\
&\qquad +\sum_{n=1}^{N}\lambda_{jn}\phi_n\mathbb{E}\left(\int_0^\tau e^{\overline{\delta}\nu(t)}|\alpha(t)-G(t,\alpha_t)|^q dt\right). \qquad (3.18)
\end{aligned}
$$

Using inequality (3.5) and condition (3.17), we can derive that

$$
\begin{aligned}
&\mathbb{E}\left(\phi_j e^{\overline{\delta}\nu(\tau)}|\alpha(\tau)-G(\tau,\alpha_\tau)|^q\right)\\
&\quad\leq \phi_j\mathbb{E}\left(e^{\overline{\delta}\nu(0)}|\alpha(0)-G(0,\alpha_0)|^q\right)+\overline{\delta}\phi_m\mathbb{E}\left(\int_0^\tau \nu^{'}(t)e^{\overline{\delta}\nu(t)}|\alpha(t)-G(t,\alpha_t)|^q dt\right),\\
&\quad+ q\phi_j\mathbb{E}\left(\int_0^\tau e^{\overline{\delta}\nu(t)}|\alpha(t)-G(t,\alpha_t)|^{q-2}\left(\alpha(t)-G(t,\alpha_t)\right)^T f_1\left(\alpha(t),\alpha_t,t,l(t)\right)\right)dt,\\
&\quad+ \frac{1}{2}q(q-1)\phi_j\mathbb{E}\left(\int_0^\tau e^{\overline{\delta}\nu(t)}|\alpha(t)-G(t,\alpha_t)|^{q-2}|f_2\left(\alpha(t),\alpha_t,t,l(t)\right)|^2dt\right),\\
&\quad+ \sum_{n=1}^{N}\lambda_{jn}\phi_n\mathbb{E}\left(\int_0^\tau e^{\overline{\delta}\nu(t)}|\alpha(t)-G(t,\alpha_t)|^q dt\right),\\
&\quad\leq \phi_j\mathbb{E}\left(e^{\overline{\delta}\nu(0)}|\alpha(0)-G(0,\alpha_0)|^q\right)+\overline{\delta}c_\eta\phi_m\mathbb{E}\left(\int_0^\tau e^{\overline{\delta}\nu(t)}|\alpha(t)-G(t,\alpha_t)|^q dt\right),\\
&\quad+ q\phi_j\xi_{m3}\mathbb{E}\left(\int_0^\tau e^{\overline{\delta}\nu(t)}|\alpha(t)-G(t,\alpha_t)|^q dt\right)-p_1q\phi_m\mathbb{E}\left(\int_0^\tau e^{\overline{\delta}\nu(t)}|\alpha(t)|^q\right)dt,\\
&\quad+ p_2q\phi_j\mathbb{E}\left(\int_0^\tau \nu^{'}(t)e^{\overline{\delta}\nu(t)}\left(\int_{-\infty}^0 \sigma(\theta)|\alpha(t+\theta)|^q d\theta\right)dt\right),\\
&\quad+ \sum_{n=1}^{N}\lambda_{jn}\phi_n\mathbb{E}\left(\int_0^\tau e^{\overline{\delta}\nu(t)}|\alpha(t)-G(t,\alpha_t)|^q dt\right).
\end{aligned}
$$

Hence, by equation (2.8), we have

$$
\begin{aligned}
&\mathbb{E}\left(\phi_j e^{\overline{\delta}\nu(\tau)}|\alpha(\tau)-G(\tau,\alpha_\tau)|^q\right)\\
&\quad\leq\ \phi_j\mathbb{E}\left(e^{\overline{\delta}\nu(0)}|\alpha(0)-G(0,\alpha_0)|^q\right)+\overline{\delta}c_\eta\phi_m\mathbb{E}\left(\int_0^\tau e^{\overline{\delta}\nu(t)}|\alpha(t)-G(t,\alpha_t)|^q dt\right),\\
&\quad-\ \beta\mathbb{E}\left(\int_0^\tau e^{\overline{\delta}\nu(t)}|\alpha(t)-G(t,\alpha_t)|^q dt\right)-p_1q\phi_m\mathbb{E}\left(\int_0^\tau e^{\overline{\delta}\nu(t)}|\alpha(t)|^q\right)dt,\\
&\quad+\ p_2q\phi_j\mathbb{E}\left(\int_0^\tau \nu^{'}(t)e^{\overline{\delta}\nu(t)}\left(\int_{-\infty}^0\sigma(\theta)|\alpha(t+\theta)|^q d\theta\right)dt\right),\\
&\quad\leq\ \phi_j\mathbb{E}\left(e^{\overline{\delta}\nu(0)}|\alpha(0)-G(0,\alpha_0)|^q\right)-p_1q\phi_m\mathbb{E}\left(\int_0^\tau e^{\overline{\delta}\nu(t)}|\alpha(t)|^q\right)dt,\\
&\quad+\ p_2q\phi_j\mathbb{E}\left(\int_0^\tau \nu^{'}(t)e^{\overline{\delta}\nu(t)}\left(\int_{-\infty}^0\sigma(\theta)|\alpha(t+\theta)|^q d\theta\right)dt\right). \qquad (3.19)
\end{aligned}
$$

Let $\vartheta(\tau)=\nu^{'}(\tau)e^{\overline{\delta}\nu(\tau)}$, thus $\vartheta'(\tau)=\dfrac{\eta''(\tau)\eta(\tau)-(1-\overline{\delta})\eta'(\tau)^2}{\eta^2(\tau)}e^{\overline{\delta}\nu(\tau)}<0$. Hence, ϑ is a decreasing function. Using the Fubini theorem,

$$
\begin{aligned}
&\int_0^\tau e^{\overline{\delta}\nu(t)}\nu'(t)\left(\int_{-\infty}^0|\alpha(t+\theta)|^q\sigma(\theta)d\theta\right)dt\\
&=\int_0^\tau e^{\overline{\delta}\nu(t)}\nu'(t)\left(\int_{-\infty}^{-t}|\alpha(t+\theta)|^q\sigma(\theta)d\theta+\int_{-t}^0|\alpha(t+\theta)|^q\sigma(\theta)d\theta\right)dt\\
&=\int_0^\tau e^{\overline{\delta}\nu(t)}\nu'(t)\left(\int_{-\infty}^{-t}|\alpha(t+\theta)|^q\sigma(\theta)d\theta\right)dt+\int_{-\tau}^0\left(\int_{-\theta}^\tau e^{\overline{\delta}\nu(t)}\nu'(t)|\alpha(t+\theta)|^q dt\right)\sigma(\theta)d\theta\\
&\leq\int_0^\tau e^{\overline{\delta}\nu(t)}\nu'(t)\left(\int_{-\infty}^0|\alpha(m)|^q\sigma(m-t)dm\right)dt+\int_{-\infty}^0\left(\int_0^\tau e^{\overline{\delta}\nu(r-\theta)}\nu'(r-\theta)|\alpha(r)|^q dr\right)\sigma(\theta)d\theta\\
&\leq\nu'(0)e^{\overline{\delta}\nu(0)}\int_{-\infty}^0|\alpha(m)|^q\left(\int_0^\tau\sigma(m-t)dt\right)dm+\int_{-\infty}^0\left(\int_0^\tau e^{\overline{\delta}\nu(r)}\nu'(r)|\alpha(r)|^q dr\right)\sigma(\theta)d\theta\\
&\leq\nu'(0)e^{\overline{\delta}\nu(0)}\int_{-\infty}^0|\alpha(m)|^q\left(\int_{m-\tau}^m\sigma(r)dr\right)dm+\int_0^\tau e^{\overline{\delta}\nu(r)}\nu'(r)|\alpha(r)|^q dr\\
&\leq\nu'(0)e^{\overline{\delta}\nu(0)}\int_{-\infty}^0|\alpha(m)|^q\left(\int_{-\infty}^0\sigma(r)dr\right)dm+\int_0^\tau e^{\overline{\delta}\nu(r)}\nu'(r)|\alpha(r)|^q dr\\
&\leq c_\eta e^{\overline{\delta}\nu(0)}\int_{-\infty}^0|\chi(m)|^q dm+c_\eta\int_0^\tau e^{\overline{\delta}\nu(r)}|\alpha(r)|^q dr \qquad (3.20)
\end{aligned}
$$

By taking the expectation on both sides of (3.20), we deduce

$$\begin{aligned}&\mathbb{E}\left(\int_0^\tau e^{\overline{\delta}\nu(t)}\nu'(t)\left(\int_{-\infty}^0|\alpha(t+\theta)|^q\sigma(\theta)d\theta\right)dt\right),\\&\le c_\eta e^{\overline{\delta}\nu(0)}\mathbb{E}\int_{-\infty}^0|\chi(m)|^q dm+c_\eta\mathbb{E}\left(\int_0^\tau e^{\overline{\delta}\nu(t)}|\alpha(t)|^q dt\right).\end{aligned}\tag{3.21}$$

Substituting (3.21) into (3.19) and using (2.5), we obtain

$$\begin{aligned}\mathbb{E}\left(\phi_j e^{\overline{\delta}\nu(\tau)}|\alpha(\tau)-G(\tau,\alpha_\tau)|^q\right) &\le \phi_j\mathbb{E}\left(e^{\overline{\delta}\nu(0)}|\alpha(0)-G(0,\alpha_0)|^q\right)+p_2qc_\eta\phi_j e^{\overline{\delta}\nu(0)}\mathbb{E}\int_{-\infty}^0|\chi(s)|^q ds,\\&+ q\phi_j\left(p_2c_\eta-p_1\right)\mathbb{E}\left(\int_0^\tau e^{\overline{\delta}\nu(t)}|\alpha(t)|^q dt\right),\\&\le p_2qc_\eta\phi_j e^{\overline{\delta}\nu(0)}\mathbb{E}\int_{-\infty}^0|\chi(s)|^q ds+\phi_j\mathbb{E}\left(e^{\overline{\delta}\nu(0)}|\alpha(0)-G(0,\alpha_0)|^q\right),\\&\le p_2qc_\eta\phi_j e^{\overline{\delta}\nu(0)}\mathbb{E}\int_{-\infty}^0|\chi(s)|^q ds+2^{q-1}\phi_j e^{\overline{\delta}\nu(0)}\left(1+\kappa^q\right)\mathbb{E}\left\|\chi\right\|^q,\\&\le \phi_j\left(p_2qc_\eta\mathbb{E}\int_{-\infty}^0|\chi(s)|^q ds+2^{q-1}\left(1+\kappa^q\right)\mathbb{E}\left\|\chi\right\|^q\right)e^{\overline{\delta}\nu(0)}.\end{aligned}\tag{3.22}$$

Therefore,

$$\mathbb{E}\left(e^{\overline{\delta}\nu(\tau)}|\alpha(\tau)-G(\tau,\alpha_\tau)|^q\right)\le\left(p_2qc_\eta\mathbb{E}\int_{-\infty}^0|\chi(s)|^q ds+2^{q-1}\left(1+\kappa^q\right)\mathbb{E}\left\|\chi\right\|^q\right)e^{\overline{\delta}\nu(0)}\le Me^{\overline{\delta}\nu(0)},\tag{3.23}$$

where $M=\left(p_2qc_\eta\mathbb{E}\int_{-\infty}^0|\chi(s)|^q ds+2^{q-1}\left(1+\kappa^q\right)\mathbb{E}\left\|\chi\right\|^q\right)$. Moreover, using (iv), for any $T>0$, we have

$$\begin{aligned}\sup_{0\le\tau\le T}\left(e^{\overline{\delta}\nu(\tau)}\mathbb{E}\left|\alpha(\tau)\right|^q\right) &\le 2^{q-1}\sup_{0\le\tau\le T}\left(e^{\overline{\delta}\nu(\tau)}\mathbb{E}\left|\alpha(\tau)-G(\tau,\alpha_\tau)\right|^q\right)+2^{q-1}\sup_{0\le\tau\le T}\left(e^{\overline{\delta}\nu(\tau)}\mathbb{E}\left|G(\tau,\alpha_\tau)\right|^q\right)\\&\le 2^{q-1}Me^{\overline{\delta}\nu(0)}+2^{q-1}\kappa^q\sup_{0\le\tau\le T}\left(e^{\overline{\delta}\nu(\tau)}\mathbb{E}\left|\alpha(\tau)\right|^q\right).\end{aligned}\tag{3.24}$$

Thus, by choosing κ such that $\kappa<2^{\frac{1}{q}-1}$, we have

$$\sup_{0\le\tau\le T}\left(e^{\overline{\delta}\nu(\tau)}\mathbb{E}\left|\alpha(\tau)\right|^q\right)\le\frac{1}{1-2^{q-1}\kappa^q}\times 2^{q-1}Me^{\overline{\delta}\nu(0)}.$$

Letting $T\to+\infty$, we have

$$\sup_{0\le\tau\le+\infty}\left(e^{\overline{\delta}\nu(\tau)}\mathbb{E}\left|\alpha(\tau)\right|^q\right)\le M_1$$

where

$$M_1 = \frac{1}{1-2^{q-1}\kappa^q} \times 2^{q-1} M e^{\overline{\delta}\nu(0)}.$$

This implies that for all $\tau \geq 0$

$$\mathbb{E}\,|\alpha(\tau)|^q \leq M_1 e^{-\overline{\delta}\nu(\tau)}.$$

Therefore,

$$\limsup_{\tau\to\infty} \frac{\ln \mathbb{E}\left(|\alpha(\tau)|^q\right)}{\ln \eta(\tau)} \leq -\overline{\delta}.$$

Applying the Itô formula and proceeding as (3.23), we have

$$\begin{aligned} e^{\overline{\delta}\nu(\tau)}|\alpha(\tau) - G(\tau,\alpha_\tau)|^q &\leq \left(p_2 q c_\eta \int_{-\infty}^{0} |\chi(s)|^q ds + 2^{q-1}\left(1+\kappa^q\right)\|\chi\|^q\right) e^{\overline{\delta}\nu(0)} + L(\tau) \\ &\leq M_2 e^{\overline{\delta}\nu(0)} + L(\tau), \end{aligned} \tag{3.25}$$

where $L(\tau)$ is a local continuous martingale with initial value $L(0) = 0$ and

$$M_2 = p_2 q c_\eta \int_{-\infty}^{0} |\chi(s)|^q ds + 2^{q-1}\left(1+\kappa^q\right)\|\chi\|^q.$$

Using the non-negative semi-martingale convergence theorem, we obtain

$$\limsup_{\tau\to\infty} e^{\overline{\delta}\nu(\tau)}|\alpha(\tau) - G(\tau,\alpha_\tau)|^q < \infty, \quad a.s. \tag{3.26}$$

Thus, there exists a finite positive random variable ϖ such that

$$\sup_{0\leq\tau<\infty} e^{\overline{\delta}\nu(\tau)}|\alpha(\tau) - G(\tau,\alpha_\tau)|^q \leq \varpi, \quad a.s. \tag{3.27}$$

Now proceeding as (3.24), we can derive that

$$\sup_{0\leq\tau\leq T}\left(e^{\overline{\delta}\nu(\tau)}\mathbb{E}\,|\alpha(\tau)|^q\right) \leq \frac{1}{1-2^{q-1}\kappa^q} \times 2^{q-1} \sup_{0\leq\tau\leq T} e^{\overline{\delta}\nu(\tau)}|\alpha(\tau) - G(\tau,\alpha_\tau)|^q. \tag{3.28}$$

Letting $T \to +\infty$, we have

$$\sup_{0\leq\tau\leq+\infty}\left(e^{\overline{\delta}\nu(\tau)}\,|\alpha(\tau)|^q\right) \leq M_3$$

where

$$M_3 = \frac{1}{1-2^{q-1}\kappa^q} \times 2^{q-1}\varpi,$$

which implies that for all $\tau \geq 0$

$$|\alpha(\tau)|^q \leq M_3 e^{-\overline{\delta}\nu(\tau)}.$$

Hence,

$$\limsup_{\tau\to\infty} \frac{\ln|\alpha(\tau)|}{\ln \eta(\tau)} \leq -\frac{\overline{\delta}}{q},$$

as desired. □

4 Examples

In this section we illustrate our abstract results with two examples.

Example 4.1. *We consider the following HNSDEID:*

$$d\left[\alpha(\tau)+\frac{1}{2+\tau}\alpha(\tau)\right]=f_1\left(\alpha(\tau),\alpha_\tau,\tau,l(\tau)\right)d\tau+f_2\left(\alpha(\tau),\alpha_\tau,\tau,l(\tau)\right)dw(\tau), \tag{4.1}$$

where the initial condition is

$$\chi=\begin{cases}4e^{0.02\tau}, & if \quad \tau\in(-150,0]\\ 0, & if \quad \tau\in(-\infty,-150]\end{cases}$$

and $\omega(t)$ is a one dimensional Brownian motions. Let

$$f_1\left(\alpha(\tau),\alpha_\tau,\tau,1\right)=-4\frac{2+\tau}{3+\tau}\alpha(\tau),$$

$$f_2\left(\alpha(\tau),\alpha_\tau,\tau,1\right)=\frac{1}{\sqrt{1+\tau}}\int_{-\infty}^{0}|\alpha(\tau+\theta)|e^{\theta}d\theta,$$

$$f_1\left(\alpha(\tau),\alpha_\tau,\tau,2\right)=-5\alpha(\tau),$$

$$f_2\left(\alpha(\tau),\alpha_\tau,\tau,2\right)=\frac{1}{\sqrt{2(1+\tau)}}\int_{-\infty}^{0}|\alpha(\tau+\theta)|e^{\theta}d\theta.$$

Let $S=\{1,2\}$ and the matrix $\Lambda=(\lambda_{mn})_{1\le m,n\le 2}$ define by

$$\begin{pmatrix}-1 & 1\\ 7 & -7\end{pmatrix}$$

Then, for $m=1$

$$\begin{aligned}&\left(\psi(0)-G(\tau,\psi)\right)^T f_1\left(\psi(0),\psi,\tau,1\right)+\frac{1}{2}|f_2\left(\psi(0),\psi,\tau,1\right)|^2\\ &\le -4|\psi(0)|^2+\frac{1}{2}\frac{1}{1+\tau}\Big(\int_{-\infty}^{0}|\psi(\theta)|e^{\theta}d\theta\Big)^2.\end{aligned} \tag{4.2}$$

For $m=2$

$$\begin{aligned}&\left(\psi(0)-G(\tau,\psi)\right)^T f_1\left(\psi(0),\psi,\tau,2\right)+\frac{1}{2}|f_2\left(\psi(0),\psi,\tau,2\right)|^2\\ &\le -5\frac{3+\tau}{2+\tau}|\psi(0)|^2+\frac{1}{4}\frac{1}{1+\tau}\Big(\int_{-\infty}^{0}|\psi(\theta)|e^{\theta}d\theta\Big)^2\end{aligned}$$

$$\leq -4|\psi(0)|^2 + \frac{1}{2}\frac{1}{1+\tau}\Big(\int_{-\infty}^{0}|\psi(\theta)|e^{\theta}d\theta\Big)^2. \tag{4.3}$$

Using the Hölder inequality, we obtain

$$\Big(\int_{-\infty}^{0}|\psi(\theta)|e^{\theta}d\theta\Big)^2 \leq \int_{-\infty}^{0}|\psi(\theta)|^2e^{\theta}d\theta. \tag{4.4}$$

Furthermore,

$$|\psi(0) - G(\tau,\psi)| = (1+\frac{1}{2+\tau})|\psi(0)| \geq |\psi(0)|. \tag{4.5}$$

Substituting (4.4) and (4.5) into (4.2) and (4.3), it yields that for all $m \in \{1,2\}$

$$(\psi(0) - G(\tau,\psi))^T f_1(\psi(0),\psi,\tau,m) + \frac{1}{2}|f_2(\psi(0),\psi,\tau,m)|^2$$

$$\leq -5|\psi(0)|^2 + |\psi(0)|^2 + \frac{1}{2}\frac{\eta'(\tau)}{\eta(\tau)}\int_{-\infty}^{0}|\psi(\theta)|^2e^{\theta}d\theta$$

$$\leq -5|\psi(0)|^2 + |\psi(0) - G(\tau,\psi)|^2 + \frac{1}{2}\frac{\eta'(\tau)}{\eta(\tau)}\int_{-\infty}^{0}|\psi(\theta)|^2e^{\theta}d\theta. \tag{4.6}$$

Thus assumption is satisfied with $p_1 = 5$*,* $p_2 = \frac{1}{2}$*,* $\xi_{m3} = 1$ $(\forall m \in \{1,2\})$*,* $\eta(\tau) = 1+\tau$*,* $q = 2$ *and* $c_\eta = 1$*. Using the fact that* $|\psi(0) - G(\tau,\psi)|^{2-2} = 1$*, it is easy to see that* $\forall m \in \{1,2\}$

$$(\psi(0) - G(\tau,\psi))^T f_1(\psi(0),\psi,\tau,m) + \frac{1}{2}|f_2(\psi(0),\psi,\tau,m)|^2$$

$$\leq 1 + |\psi(0) - G(\tau,\psi)|^2 + \int_{-\infty}^{0}|\psi(\theta)|^2e^{\theta}d\theta - 5|\psi(0)|^2. \tag{4.7}$$

Which proves that Assumption 2.6 is satisfied with $\xi_{m1} = \xi_{m4} = 1$ *and* $\xi_{m2} = 5$*. Moreover, the parameter* κ *in Theorem 3.1 is* $\frac{1}{2}$*.*
We have

$$\mathcal{B} = -diag(2\xi_{13}, 2\xi_{23}) - \Lambda = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 7 & -7 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -7 & 9 \end{pmatrix}.$$

Then,

$$\mathcal{B}^{-1} = \begin{pmatrix} \frac{9}{20} & \frac{1}{20} \\ \\ \frac{7}{20} & \frac{3}{20} \end{pmatrix}.$$

Then, $\mathcal{B}$ *is a nonsingular* M*-matrix. It is easy to check that all conditions of theorems 3.1 and 3.2 are satisfied. Therefore system (4.8) is* η*-stable in mean square and almost surely* η*-stable.*

Example 4.2. *We consider the following HNSDEID:*

$$d\left[\alpha(\tau) - \frac{1}{2+\tau}|\sin\alpha(\tau)|\right] = f_1\left(\alpha(\tau), \alpha_\tau, \tau, l(\tau)\right) d\tau + f_2\left(\alpha(\tau), \alpha_\tau, \tau, l(\tau)\right) dw(\tau), \tag{4.8}$$

where the initial condition is

$$\chi = \begin{cases} 3e^{0.03\tau}, & \text{if} \quad \tau \in (-120, 0] \\ 0, & \text{if} \quad \tau \in (-\infty, -120] \end{cases}$$

and $\omega(t)$ is a one dimensional Brownian motions. Let

$$f_1\left(\alpha(\tau), \alpha_\tau, \tau, 1\right) = -3\frac{2+\tau}{3+\tau}\alpha(\tau),$$

$$f_2\left(\alpha(\tau), \alpha_\tau, \tau, 1\right) = \sqrt{\frac{2+\tau}{1+\tau}} \int_{-\infty}^{0} |\alpha(\tau+\theta)| e^{\theta} d\theta,$$

$$f_1\left(\alpha(\tau), \alpha_\tau, \tau, 2\right) = -4\frac{2+\tau}{3+\tau}\alpha(\tau),$$

$$f_2\left(\alpha(\tau), \alpha_\tau, \tau, 2\right) = \sqrt{\frac{2+\tau}{3(1+\tau)}} \int_{-\infty}^{0} |\alpha(\tau+\theta)| e^{\theta} d\theta.$$

Let $S = \{1, 2\}$ and the matrix $\Lambda = (\lambda_{mn})_{1 \le m,n \le 2}$ define by

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Then, for $m = 1$

$$\left(\psi(0) - G(\tau, \psi)\right)^T f_1\left(\psi(0), \psi, \tau, 1\right) + \frac{1}{2}|f_2\left(\psi(0), \psi, \tau, 1\right)|^2$$

$$\begin{aligned} &= -3\frac{2+\tau}{3+\tau}|\psi(0)|^2 + \frac{3}{3+\tau}\psi(0)|\sin\psi(0)| + \frac{1}{2}\frac{2+\tau}{1+\tau}\left(\int_{-\infty}^{0} |\psi(\theta)| e^{\theta} d\theta\right)^2 \\ &\le -3\frac{2+\tau}{3+\tau}|\psi(0)|^2 + \frac{3}{3+\tau}|\psi(0)|^2 + \frac{1}{2}\frac{2+\tau}{1+\tau}\left(\int_{-\infty}^{0} |\psi(\theta)| e^{\theta} d\theta\right)^2 \\ &\le -|\psi(0)|^2 + \left(\int_{-\infty}^{0} |\psi(\theta)| e^{\theta} d\theta\right)^2. \end{aligned} \tag{4.9}$$

For $m = 2$

$$\left(\psi(0) - G(\tau, \psi)\right)^T f_1\left(\psi(0), \psi, \tau, 2\right) + \frac{q-1}{2}|f_2\left(\psi(0), \psi, \tau, 2\right)|^2$$

$$
\begin{aligned}
&= -4\frac{2+\tau}{3+\tau}|\psi(0)|^2 + \frac{4}{3+\tau}\psi(0)|\sin\psi(0)| + \frac{1}{2}\frac{2+\tau}{3(1+\tau)}\left(\int_{-\infty}^{0}|\psi(\theta)|e^{\theta}d\theta\right)^2 \\
&\leq -4\frac{2+\tau}{3+\tau}|\psi(0)|^2 + \frac{4}{3+\tau}|\psi(0)|^2 + \frac{1}{2}\frac{2+\tau}{3(1+\tau)}\left(\int_{-\infty}^{0}|\psi(\theta)|e^{\theta}d\theta\right)^2 \\
&\leq -|\psi(0)|^2 + \Big(\int_{-\infty}^{0}|\psi(\theta)|e^{\theta}d\theta\Big)^2. \qquad (4.10)
\end{aligned}
$$

Substituting (4.4) into (4.9) and (4.10), we can derive that for all $m \in \{1,2\}$

$$
\begin{aligned}
&(\psi(0) - G(\tau,\psi))^T f_1(\psi(0),\psi,\tau,m) + \frac{1}{2}|f_2(\psi(0),\psi,\tau,m)|^2 \\
&\leq -|\psi(0)|^2 + |\psi(0) - G(\tau,\psi)|^2 + \frac{1}{2}\frac{\eta'(\tau)}{\eta(\tau)}\int_{-\infty}^{0}|\psi(\theta)|^2 e^{\theta}d\theta. \qquad (4.11)
\end{aligned}
$$

Thus assumption is satisfied with $p_1 = 1$, $p_2 = \frac{1}{2}$, $\xi_{m3} = 1$ $(\forall m \in \{1,2\})$, $\eta(\tau) = (1+\tau)e^{\tau}$, $q = 2$ *and* $c_\eta = 2$. *Using the fact that* $|\psi(0) - G(\tau,\psi)|^{2-2} = 1$, *it is easy to check that* $\forall m \in \{1,2\}$

$$
\begin{aligned}
&(\psi(0) - G(\tau,\psi))^T f_1(\psi(0),\psi,\tau,m) + \frac{1}{2}|f_2(\psi(0),\psi,\tau,m)|^2 \\
&\leq 1 + |\psi(0) - G(\tau,\psi)|^2 + \int_{-\infty}^{0}|\psi(\theta)|^2 e^{\theta}d\theta - |\psi(0)|^2. \qquad (4.12)
\end{aligned}
$$

Which proves that assumptions 2.6 is satisfied with $\xi_{m1} = \xi_{m2} = \xi_{m4} = 1$. *Moreover, the parameter* κ *in theorem 3.1 equal* $\frac{1}{2}$.
We have

$$
\mathcal{B} = -diag(2\xi_{13}, 2\xi_{23}) - \Lambda = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}.
$$

Then,

$$
\mathcal{B}^{-1} = \begin{pmatrix} \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} \end{pmatrix}.
$$

Then, $\mathcal{B}$ *is a nonsingular M-matrix. It is easily to check that all conditions of theorems 3.1 and 3.2 are satisfied. Therefore system (4.8) is* η*-stable in mean square and a.s.*η*-stable.*

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