Asymptotic behavior of a semilinear problem in heat conduction with long time memory and non-local diffusion

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Abstract

In this paper, the asymptotic behavior of a semilinear heat equation with long time memory and non-local diffusion is analyzed in the usual set-up for dynamical systems generated by differential equations with delay terms. This approach is different from the previous published literature on the long time behavior of heat equations with memory which is carried out by the Dafermos transformation. As a consequence, the obtained results provide complete information about the attracting sets for the original problem, instead of the transformed one. In particular, the proved results also generalize and complete previous literature in the local case.

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1. Introduction

The main objective of this paper is to analyze the asymptotic behavior of a semilinear heat equation with long time memory and non-local diffusion, which is an interesting situation with important applications in the real world.

On the one hand, the effects that memory terms (or the past history of a phenomenon) produce on the evolution of a dynamical system is obvious, since it is sensible to think that the evolution of any system depends not only on the current state but on its whole history (see, for instance, [1, 8, 12, 2, 6, 10, 15] and the references therein). On the other hand, many problems are better described by considering non-local terms, which created a great interest in the modeling of various real applications (see [3, 4, 5, 12] and the references therein).

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Motivated by some physical problems from thermal memory or materials with memory, one can find a significant literature devoted to the analysis of partial differential equations with long time memory. For example, the authors introduced in [12] a semilinear partial differential equation to model the heat flow in a rigid, isotropic, homogeneous heat conductor with linear memory, which is given by

$$\begin{cases} c_0\partial_t u - k_0\Delta u - \displaystyle\int_{-\infty}^{t} k(t-s)\Delta u(s)ds + f(u) = h, & \text{in } \ \Omega\times(\tau,+\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega\times(\tau,+\infty), \\ u(x,\tau+t) = u_0(x,t), & \text{in } \ \Omega\times(-\infty,0], \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with regular boundary, $u : \Omega\times\mathbb{R} \to \mathbb{R}$ is the temperature field, $k : \mathbb{R}^+ \to \mathbb{R}$ is the heat flux memory kernel, $\mathbb{R}^+$ denotes the interval $(0,+\infty)$, c_0 and k_0 denote the specific heat and the instantaneous conductivity, respectively. To solve (1.1) successfully, the authors considered this problem as a non-delay one by making the past history of u from $-\infty$ to 0^- be part of the forcing term given by the causal function g, which is defined by

$$g(x,t) = h(x,t) + \int_{-\infty}^{\tau} k(t-s)\Delta u_0(x,s)ds, \qquad x\in\Omega, \quad t\geq\tau.$$

In this way, (1.1) becomes an initial value problem without delay or memory,

$$\begin{cases} c_0\partial_t u - k_0\Delta u - \displaystyle\int_{\tau}^{t} k(t-s)\Delta u(s)ds + f(u) = g, & \text{in } \ \Omega\times(\tau,+\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega\times(\tau,+\infty), \\ u(x,\tau) = u_0(x,0), & \text{in } \ \Omega. \end{cases} \tag{1.2}$$

However, this problem does not generate a dynamical system in an appropriate phase space, since the equation in (1.2) depends on the past history and we are just fixing an initial value at time τ.

Therefore, two alternatives are possible. The first one is based on the idea introduced by Dafermos [7], for linear viscoelasticity, in the 70's. Let us define the new variables,

$$u^t(x,s) = u(x,t-s), \qquad s\geq 0, \quad t\geq\tau,$$

$$\eta^t(x,s) = \int_0^s u^t(x,r)dr = \int_{t-s}^{t} u(x,r)dr, \qquad s\geq 0, \quad t\geq\tau. \tag{1.3}$$

Besides, assuming $k(\infty)=0$, a change of variable and a formal integration by parts imply

$$\int_{-\infty}^{t} k(t-s)\Delta u(s)ds = -\int_0^{\infty} k'(s)\Delta\eta^t(s)ds.$$

Setting

$$\mu(s) = -k'(s),$$

the original equation (1.2) turns into the following autonomous system without delay,

$$\begin{cases} c_0 \frac{\partial u}{\partial t} - k_0 \Delta u - \int_0^\infty \mu(s) \Delta \eta^t(s) ds + f(u) = g, & \text{in } \Omega \times (\tau, \infty), \\ \eta_t^t(s) = -\eta_s^t(s) + u(t), & \text{in } \Omega \times (\tau, \infty) \times \mathbb{R}^+, \\ u(x,t) = \eta^t(x,s) = 0, & \text{on } \partial\Omega \times \mathbb{R} \times \mathbb{R}^+, \\ u(x,\tau) = u_0(0), & \text{in } \Omega, \\ \eta^\tau(x,s) = \eta_0(s), & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \tag{1.4}$$

where, η_s^t denotes the distributional derivative of $\eta^t(s)$ with respect to the internal variable s. It follows from the definition of $\eta^t(x,s)$ (see (1.3)) that

$$\eta_0(s) = \int_{\tau-s}^{\tau} u(r)dr = \int_{\tau-s}^{\tau} u_0(r-\tau)dr = \int_{-s}^{0} u_0(r)dr, \tag{1.5}$$

which is the initial integrated past history of u with vanishing boundary. Consequently, any solution to (1.2) is a solution to (1.4) for the corresponding initial values $(u_0(0), \eta_0)$ given by (1.5). It is worth emphasizing that problem (1.4) can be solved for arbitrary initial values (u_0, η_0) in a proper phase space $L^2(\Omega) \times L^2_\mu(\mathbb{R}^+; H^1_0(\Omega))$ (see Section 2 for more details), i.e., the second component η_0 does not necessarily depend on $u_0(\cdot)$. This permits us to construct a dynamical system in this phase space and prove the existence of global attractors. However, the transformed equation (1.4) is a generalization of problem (1.2), and therefore, not every solution to equation (1.4) possesses a corresponding one to (1.2). Notice that both problems are equivalent if and only if the initial value η_0 belongs to a proper subspace of $L^2_\mu(\mathbb{R}^+; H^1_0(\Omega))$, which coincides with the domain of the distributional derivative with respecto to s, denoted by $D(\mathbf{T})$ (for more details, see [10]). Hence, it is natural to construct a dynamical system generated by (1.4) in the phase space $L^2(\Omega) \times D(\mathbf{T})$ to prove the existence of attractors to the original problem, via the above relationship (see [12, 6, 10]). Nevertheless, as far as we know, it is not possible to prove the existence of attractors in this space unless solutions are proved to have more regularity. Thus, in principle, we cannot transfer the existence of attractors for system (1.4) to the original problem (1.2).

The idea of the second alternative comes from a simple case, which was successfully applied in [1] when the kernel is $k(t) = e^{-d_0 t}$, $d_0 > 0$ (non-singular kernel). Using this method, it is proved that the problem in [1] generates a dynamical system in the phase space $L^2_{H^1_0}$ given by the measurable functions $\varphi : (-\infty, 0] \to H^1_0(\Omega)$, such that $\int_{-\infty}^0 e^{\gamma s} \|\varphi(s)\|^2_{H^1_0} ds < +\infty$, for certain $\gamma > 0$. Under the construction of this phase space, there exists a global attractor to this problem (in fact, the problem in [1] is non-autonomous and the attractor is of pullback type). Notice that, for this kind of delay problems, in which the initial value at zero may not be related to the values for negative times, the standard and more appropriate phase space to construct a dynamical system is the cartesian product $L^2(\Omega) \times L^2_{H_0^1}$ (see [2] for more details). In such a way, for any initial values $u_0 \in L^2(\Omega)$ and $\varphi \in L^2_{H_0^1}$, there exists a unique solution to the following problem (we set $\tau = 0$ since the problem is autonomous),

$$\begin{cases} c_0 \frac{\partial u}{\partial t} - k_0 \Delta u - \displaystyle\int_{-\infty}^{t} k(t-s)\Delta u(s)ds + f(u) = g, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}, \\ u(x,0) = u_0(x), & \text{in } \Omega, \\ u(x,t) = \varphi(x,t), & \text{in } \Omega \times (-\infty, 0). \end{cases} \tag{1.6}$$

According to the regularity of solutions to the above equation, one can define a dynamical system $S(t) : L^2(\Omega) \times L^2_{H_0^1} \to L^2(\Omega) \times L^2_{H_0^1}$ by the relation

$$S(t)(u_0, \varphi) := (u(t; 0, u_0, \varphi), u_t(\cdot; 0, u_0, \varphi)),$$

where $u(\cdot; 0, u_0, \varphi)$ denotes the solution of problem (1.6) (see [2] for more details on this set-up). We emphasize that the two components of the dynamical system are the current state of the solution and the past history up to present, respectively, what is more sensible in a problem with delays or memory. By using this framework, the method in [1] can be successfully applied to prove the existence of attractors to problem (1.6) when k is of exponential type. However, this exponential behavior may be a big restriction on the kernel k, consequently, on the function μ, since in many real situations the latter often has singularities, for instance $k(t) = e^{-d_0 t} t^{-\alpha}, \alpha \in (0,1)$. Therefore, it is interesting to design a technique which allows us to handle the cases with this kind of singular kernels within the context of the phase space $L^2(\Omega) \times L^2_{H_0^1}$. We will obtain this result as a consequence of the analysis performed in this paper even for the more general case of non-local problems as described below.

Let us recall now that amongst many interesting results concerning non-local differential equations, we mention the pioneering work [9], in which a model of single-species dynamics incorporating non-local effects was analyzed, comparing with the standard approach to model a single-species domain Ω of "Kolmogorov" type,

$$\partial_t u = \Delta u + \lambda u g(u), \qquad \text{in} \quad \Omega, \quad t > 0.$$

Taking into account the following two natural assumptions: (i) a population in which individuals compete for a shared rapidly equilibrate resource; (ii) a population in which individuals communicate either visually or by chemical means, then the most straightforward way of introducing non-local effects is to consider, instead of $g(u)$, a "crowding" effect of the form $g(u, \bar{u})$, where

$$\bar{u}(x,t) = \int_{\Omega} G(x,y) u(y,t) dy,$$

and $G(x,y)$ is some reasonable kernel. Reasoning in a heuristic way, Chipot et al. [5] studied the behavior of a population of bacteria with non-local term $a(\int_\Omega u)$ in a container. Later, Chipot et al. (cf. [3, 4]) extended this term to a general non-local operator $a(l(u))$, where $l \in \mathcal{L}(L^2(\Omega);\mathbb{R})$, for instance, if $g \in L^2(\Omega)$,

$$l(u) = l_g(u) = \int_\Omega g(x)u(x)dx.$$

Motivated by these works, the dynamics of the following non-autonomous non-local partial differential equations with delay and memory was investigated in [20] by using the Galerkin method and energy estimations,

$$\begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + h(t,u_t), & \text{in } \Omega \times (\tau,\infty), \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}, \\ u_\tau(x,t) = \varphi(x,t), & \text{in } \Omega \times (-\rho, 0], \end{cases} \tag{1.7}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $\tau \in \mathbb{R}$, the function $a \in C(\mathbb{R};\mathbb{R}^+)$ is locally Lipschitz, $f \in C(\mathbb{R})$, h contains hereditary characteristics involving delays, and $u_t : (-\infty, 0] \to \mathbb{R}$ is a segment of the solution given by $u_t(x,s) = u(x,t+s)$, $s \leq 0$, which essentially represents the history of the solution up to time t. Moreover, $0 < \rho \leq \infty$, which implies, the authors considered both cases, bounded and unbounded delays, for this model. However, the technique applied in [20] is the same used in [1] and, therefore, it is valid only for non-singular memory terms of exponential kind (e.g., $k(t) = k_1 e^{-d_0 t}$, $k_1 \in \mathbb{R}, d_0 > 0$), for more details, see [1]. Whereas, this technique fails to deal with various important models with memory, whose kernels have singularities.

Consequently, very recently, a new model has been considered related to long time memory differential equations containing non-local diffusion,

$$\begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u - \displaystyle\int_{-\infty}^{t} k(t-s)\Delta u(s)ds + f(u) = g, & \text{in } \Omega \times (\tau,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}, \\ u(t+\tau) = \varphi(t), & \text{in } \Omega \times (-\infty, 0], \end{cases} \tag{1.8}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with regular boundary, the function $a \in C(\mathbb{R};\mathbb{R}^+)$ satisfies

$$0 < m \leq a(r), \qquad \forall r \in \mathbb{R}. \tag{1.9}$$

$k : \mathbb{R}^+ \to \mathbb{R}$ is the memory kernel, with or without singularities, whose properties will be specified later, $g \in L^2(\Omega)$ which is independent of time. Notice that, thanks to a change of variable, the long time memory term in problem (1.8) can be interpreted as an infinite delay term,

$$h(u_t) := \int_{-\infty}^{0} k(-s)\Delta u_t(x,s)ds = \int_{-\infty}^{0} k(-s)\Delta u(x,t+s)ds = \int_{-\infty}^{t} k(t-s)\Delta u(x,s)ds. \tag{1.10}$$

Obviously, our model is an autonomous non-local partial differential equation. The authors first proved in [21] the existence and uniqueness of solutions to (1.8) by using the Dafermos transformation. Next, they constructed an autonomous dynamical system in the phase space $L^2(\Omega) \times L^2_\mu(\mathbb{R}^+; H^1_0(\Omega))$ and proved the existence of a global attractor in this space. As in the local heat equation case mentioned above, the same lack of enough regularity does not allow us to obtain an appropriate attractor for the original problem (1.8) in the phase space $L^2(\Omega) \times L^2_{H^1_0}$. Therefore, our objective is to overcome this difficulty and we succeeded by proceeding in the following way: Consider problem (1.8) with initial values $u(\tau) = u_0$ and $u(t + \tau) = \varphi(t)$ for $t < 0$, where $(u_0, \varphi) \in L^2(\Omega) \times L^2_{H^1_0}$. Thus, for those kernels $\mu(\cdot)$ which guarantee that, when $\varphi \in L^2_{H^1_0}$ the corresponding η_φ, defined by $\eta_\varphi(s) = \int_{-s}^0 \varphi(r)\, dr$, $(s > 0)$ belongs to the space $L^2_\mu(\mathbb{R}^+; H^1_0(\Omega))$, we can perform the Dafermos transformation and obtain the initial value problem which was already analyzed in [21], and consequently we have the existence, uniqueness and regularity of solutions in a straightforward way. Thanks to this result, we are able to construct the dynamical system in the phase space $L^2(\Omega) \times L^2_{H^1_0}$ with the help of some additional technical results. The existence of global attractor is then proved by first showing the existence of a bounded absorbing set and the proof of the asymptotic compactness property which requires an appropriate adaptation of the technique used in [1]. These results proved in the non-local problem (1.8) improve and complete the ones in [1] by simply assuming that $a(\cdot)$ is a constant, and also improve the previous literature on the local case (see, e.g., [10, 11, 12]), where it is only provided the existence of attractors for the transformed equation (1.4) but not for the original one (1.1).

The content of this paper is as follows: In Section 2, we recall some preliminaries, notations and the framework in which we will carry out our analysis. Section 3 is devoted to proving the main results of our paper. First, we state the existence and uniqueness of solutions of our problem by rewriting it as an equivalent one thanks to the Dafermos transformation. The transformed problem has already been analyzed in [21], whence our result follows immediately. However, as some estimations we need for the subsequent results are based on the ones in the proof of this existence theorem, we have included the complete proof in the Appendix (at the end of the paper). Next, we prove that our model generates an autonomous dynamical system in the phase space $L^2(\Omega) \times L^2_{H^1_0}$. Eventually, the existence of a global attractor for the dynamical system is proved by working directly on our model with memory, instead of using any result already proved in [21] for the transformed problem.

2. Well-posedness to a non-local differential equation with memory

The following non-local differential equation associated with singular memory will be investigated,

$$\begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u - \int_{-\infty}^{t} k(t-s)\Delta u(x,s)ds + f(u) = g(x,t), & \text{in } \Omega \times (\tau, \infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}, \\ u(x,0) = u_0(x), & \text{in } \Omega \\ u(x,t+\tau) = \phi(x,t), & \text{in } \Omega \times (-\infty, 0], \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ is a fixed bounded domain with regular boundary. The function $a \in C(\mathbb{R};\mathbb{R}^+)$ satisfies

$$0 < m \leq a(r), \qquad \forall r \in \mathbb{R}, \tag{2.2}$$

$k : \mathbb{R}^+ = (0, +\infty) \to \mathbb{R}$ is the memory kernel, whose properties will be specified later. The initial values are $u_0 \in L^2(\Omega)$ and $\phi \in L^2_V$ (see Section 2.2 below).

Let us define the new variables

$$u^t(x,s) = u(x,t-s), \qquad s \geq 0,$$

and

$$\eta^t(x,s) = \int_0^s u^t(x,r)dr = \int_{t-s}^t u(x,r)dr, \qquad s \geq 0. \tag{2.3}$$

Assuming $k(\infty) = 0$, a change of variable and a formal integration by parts yield

$$\int_{-\infty}^t k(t-s)\Delta u(s)ds = -\int_0^\infty k'(s)\Delta\eta^t(s)ds,$$

here and in the sequel, the prime denotes derivation with respect to variable s. Setting

$$\mu(s) = -k'(s), \tag{2.4}$$

the above choice of variable leads to the following non-delay system,

$$\begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u - \int_0^\infty \mu(s)\Delta\eta^t(s)ds + f(u) = g(x,t), & \text{in } \Omega \times (\tau,\infty), \\ \frac{\partial}{\partial t}\eta^t(s) = u - \frac{\partial}{\partial s}\eta^t(s), & \text{in } \Omega \times (\tau,\infty) \times \mathbb{R}^+, \\ u(x,t) = \eta^t(x,s) = 0, & \text{on } \partial\Omega \times \mathbb{R} \times \mathbb{R}^+, \\ u(x,\tau) = u_0(x), & \text{in } \Omega, \\ \eta^\tau(x,s) = \eta_0(x,s), & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \tag{2.5}$$

where, by the definition of $\eta^t(x,s)$ (see (2.3)), it obviously follows

$$\eta^\tau(x,s) = \int_{\tau-s}^\tau u(x,r)dr = \int_{-s}^0 \phi(x,r)dr := \eta_0(x,s), \tag{2.6}$$

which is the initial integrated past history of u with vanishing boundary.

It is worth emphasizing that we will consider solutions of our problems in the weak (variational) sense.

2.1. Assumptions

In our analysis, we shall suppose the nonlinear term $f : \mathbb{R} \to \mathbb{R}$ is a polynomial of odd degree with positive leading coefficient,

$$f(u) = \sum_{k=1}^{2p} f_{2p-k} u^{k-1}, \qquad p \in \mathbb{N}. \tag{2.7}$$

This situation can be extended, without any additional difficulties, to a more general function satisfying suitable assumptions (see, for instance, [12]).

In view of the evolution problem (2.5), the variable μ is required to verify the following hypotheses:

(h_1) $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $\quad \mu(s) \geq 0$, $\quad \mu'(s) \leq 0$, $\quad \forall s \in \mathbb{R}^+$;

(h_2) $\mu'(s) + \delta\mu(s) \leq 0$, $\quad \forall s \in \mathbb{R}^+$, $\quad$ for some $\delta > 0$.

Remark 2.1. 1. *It is straightforward to check that conditions (h_1)-(h_2) are fulfilled by singular kernels given by*

$$\mu(t) = e^{-\delta t} t^{-\alpha}, \ t > 0,$$

for $\delta > 0$ and $\alpha \in (0, 1)$.

2. *Restriction (h_1) is equivalent to assuming $k(\cdot)$ is a non-negative, non-increasing, bounded, convex function of class C^2 vanishing at infinity. Moreover, from (h_1) it easily follows that*

$$k(0) = \int_0^\infty \mu(s)ds \qquad \text{is finite and non-negative.}$$

3. *Assumption (h_2) implies that $\mu(s)$ decays exponentially. Also, this condition allows the memory kernel $k(\cdot)$ to have a singularity at $t = 0$, which coincides with the intention to study problem (2.5).*

2.2. Notations

Let Ω be a fixed bounded domain in $\mathbb{R}^N$. On this set, we introduce the Lebesgue space $L^p(\Omega)$, where $1 \leq p \leq \infty$. Besides, $W^{1,p}(\Omega)$ is the subspace of $L^p(\Omega)$ consisting of functions such that the first order weak derivative belongs to $L^p(\Omega)$. In this paper, $L^2(\Omega)$ is denoted by H, $H_0^1(\Omega)$ is denoted by V and $H^{-1}(\Omega)$ is denoted by V^*. The norms in H, V and V^* will be denoted by $|\cdot|$, $\|\cdot\|$ and $\|\cdot\|_*$, respectively.

In view of system (2.5) and (h_1), we need to introduce some additional notations before proving our main theorems. Let $L^2_\mu(\mathbb{R}^+;H)$ be a Hilbert space of functions $w:\mathbb{R}^+\to H$ endowed with the inner product,

$$(w_1,w_2)_\mu=\int_0^\infty \mu(s)(w_1(s),w_2(s))ds,$$

and let $|\cdot|_\mu$ denote the corresponding norm. In a similar way, we introduce the inner products $((\cdot,\cdot))_\mu$, $(((\cdot,\cdot)))_\mu$ and relative norms $\|\cdot\|_\mu$, $|||\cdot|||_\mu$ on $L^2_\mu(\mathbb{R}^+;V)$, $L^2_\mu(\mathbb{R}^+;V\cap H^2(\Omega))$ respectively. It follows then that

$$((\cdot,\cdot))_\mu=(\nabla\cdot,\nabla\cdot)_\mu,\quad \text{and}\quad (((\cdot,\cdot)))_\mu=(\Delta\cdot,\Delta\cdot)_\mu.$$

We also define the Hilbert spaces

$$\mathcal{H}=H\times L^2_\mu(\mathbb{R}^+;V),$$

and

$$\mathcal{V}=V\times L^2_\mu(\mathbb{R}^+;V\cap H^2(\Omega)),$$

which are respectively endowed with inner products

$$(w_1,w_2)_{\mathcal{H}}=(w_1,w_2)+((w_1,w_2))_\mu,$$

and

$$(w_1,w_2)_{\mathcal{V}}=((w_1,w_2))+(((w_1,w_2)))_\mu,$$

where $w_i\in\mathcal{H}$ or $\mathcal{V}$ $(i=1,2)$ and usual norms.

At last, with standard notations, $\mathcal{D}(I;X)$ is the space of infinitely differentiable X-valued functions with compact support in $I\subset\mathbb{R}$, whose dual space is the distribution space on I with values in X^* (dual of X), denoted by $\mathcal{D}'(I;X^*)$. For convenience, we define L^2_V the space of functions $u(\cdot)$ satisfying

$$\int_{-\infty}^0 e^{\gamma s}\left\|u(s)\right\|^2 ds<\infty,$$

where $0<\gamma<\min\{m\lambda_1,\delta\}$ and δ comes from (h_2).

3. Main results

Let us start by proving a technical result which will be crucial to our analysis. To this end, we define the linear operator $\mathcal{J}:L^2_V\to L^2_\mu(\mathbb{R}^+;V)$ by

$$(\mathcal{J}\phi)(s)=\int_{-s}^0\phi(r)\,dr,\quad s\in\mathbb{R}^+.\tag{3.1}$$

Then we have the following result.

Lemma 3.1. *Assume* (h_1)-(h_2) *hold. Then, the operator* $\mathcal{J}$ *defined by* (3.1) *is a linear and continuous mapping. In particular, there exists a positive constant* K_μ *such that, for any* $\phi \in L^2_V$*, it holds*

$$\|\mathcal{J}\phi\|^2_{L^2_\mu(\mathbb{R}^+;V)} \leq K_\mu \|\phi\|^2_{L^2_V}. \tag{3.2}$$

Proof. The linearity of $\mathcal{J}$ is obvious, we only need to prove it is well defined and bounded. Indeed, taking into account the fact that $\phi \in L^2_V$, (h_1)-(h_2) and (3.1), we have

$$\begin{aligned}
\|\mathcal{J}\phi\|^2_{L^2_\mu(\mathbb{R}^+;V)} &= \int_0^\infty \mu(s) \left\| \int_{-s}^0 \phi(r)dr \right\|^2 ds \\
&= \int_0^1 \mu(s) \left\| \int_{-s}^0 \phi(r)dr \right\|^2 ds + \int_1^\infty \mu(s) \left\| \int_{-s}^0 \phi(r)dr \right\|^2 ds \\
&\leq \int_0^1 s\mu(s) \int_{-s}^0 \|\phi(r)\|^2 drds + \mu(1) \int_1^\infty e^{-\delta(s-1)} \left\| \int_{-s}^0 \phi(r)dr \right\|^2 ds \\
&\leq \int_{-1}^0 \|\phi(r)\|^2 \int_{-r}^1 s\mu(s)dsdr + \mu(1)e^\delta \int_0^\infty e^{-\delta s} s \int_{-s}^0 \|\phi(r)\|^2 drds \\
&\leq \int_0^1 s\mu(s)ds \int_{-1}^0 \|\phi(r)\|^2 dr + \mu(1)e^\delta \int_{-\infty}^0 e^{\gamma r} \|\phi(r)\|^2 \int_{-r}^\infty se^{-\gamma r} e^{-\delta s} dsdr \\
&\leq \int_0^1 \mu(s)ds \int_{-1}^0 e^{-\gamma r} e^{\gamma r} \|\phi(r)\|^2 dr \\
&\quad + \mu(1)e^\delta \int_{-\infty}^0 e^{\gamma r} \|\phi(r)\|^2 \int_{-r}^\infty se^{\gamma s} e^{-\delta s} dsdr \\
&\leq \left(e^\gamma \int_0^1 \mu(s)ds + \mu(1)e^\delta (\gamma - \delta)^{-2} \right) \|\phi\|^2_{L^2_V}.
\end{aligned}$$

Denoting $K_\mu = e^\gamma \int_0^1 \mu(s)ds + \mu(1)e^\delta(\gamma - \delta)^{-2}$, the proof is finished. □

Remark 3.2. *Notice that when we fix an initial value* $\phi \in L^2_V$ *for problem* (2.1)*, then the corresponding initial value for the second component of problem* (2.5) *becomes* $\eta_0 := \mathcal{J}\phi$*, which belongs to* $L^2_\mu(\mathbb{R}^+;V)$ *thanks to Lemma 3.1.*

Before stating the existence, uniqueness and regularity of solution to our problem (2.1), we first recall a general result proved in [21] for problem (2.5) with general initial data in $H \times L^2_\mu(\mathbb{R}^+;V)$. Let us denote

$$z(t) = (u(t), \eta^t) \qquad \text{and} \qquad z_0 = (u_0, \eta_0).$$

Set

$$\mathcal{L}z = \left(a(l(u))\Delta u + \int_0^\infty \mu(s)\Delta\eta(s)ds, \quad u - \eta_s \right),$$

and

$$\mathcal{G}(z) = (-f(u) + g, \quad 0).$$

Then problem (2.5) can be written in the following compact form,

$$\begin{cases} z_t = \mathcal{L}z + \mathcal{G}(z), & \text{in } \ \Omega \times (\tau, \infty), \\ z(x,t) = 0, & \text{on } \ \partial\Omega \times (\tau, \infty), \\ z(x,\tau) = z_0, & \text{in } \ \Omega. \end{cases} \tag{3.3}$$

Now we have the following result.

Theorem 3.3 ([21]). *Suppose* (2.2), (2.7) *and* (h_1)-(h_2) *hold true, also let* $g \in H$. *In addition, assume that* $a(\cdot)$ *is locally Lipschitz, and there exists a positive constant* $\tilde{m}$ *such that,*

$$a(s) \le \tilde{m}, \qquad \forall s \in \mathbb{R}. \tag{3.4}$$

Then:

(*i*) *For any* $z_0 \in \mathcal{H}$, *there exists a unique solution* $z(\cdot) = (u(\cdot), \eta^{\cdot})$ *to problem* (3.3) *which satisfies*

$$u(\cdot) \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^{2p}(\tau, T; L^{2p}(\Omega)), \qquad \forall T > \tau,$$
$$\eta^{\cdot} \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; V)), \qquad \forall T > \tau.$$

Furthermore, $z(\cdot) \in C(\tau, T; \mathcal{H})$ *for every* $T > \tau$, *and the mapping* $F : z_0 \in \mathcal{H} \to z(t) \in \mathcal{H}$ *is continuous for every* $t \in [\tau, T]$.

(*ii*) *For any* $z_0 \in \mathcal{V}$, *the unique solution* $z(\cdot) = (u(\cdot), \eta^{\cdot})$ *to problem* (3.3) *satisfies*

$$u(\cdot) \in L^\infty(\tau, T; V) \cap L^2(\tau, T; V \cap H^2(\Omega)), \qquad \forall T > \tau,$$
$$\eta^{\cdot} \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; V \cap H^2(\Omega))), \qquad \forall T > \tau.$$

In addtion, $z(\cdot) \in C(\tau, T; \mathcal{V})$ *for every* $T > \tau$.

Based on the previous theorem, we can state now the corresponding result for our original problem (2.1).

Theorem 3.4. *Assume (2.2), (2.7), and* (h_1)-(h_2) *hold. Let* $a(\cdot)$ *be locally Lipschitz satisfying* (3.4),

$$g \in H, \quad u_0 \in H \quad \text{and} \quad \phi \in L^2_V.$$

Then, there exists a unique function $z(\cdot) = (u(\cdot), \eta^{\cdot})$ *satisfying*

$$u(\cdot) \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^{2p}(\tau, T; L^{2p}(\Omega)), \qquad \forall T > \tau,$$
$$\eta^{\cdot} \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; V)), \qquad \forall T > \tau,$$

such that

$$\partial_t z = \mathcal{L}z + \mathcal{G}(z)$$

in the weak sense, and

$$z|_{t=\tau} = (u_0, \mathcal{J}\phi).$$

Furthermore, for every $t \in [\tau, T]$,

$$z(t) : \mathcal{H} \to \mathcal{H} \text{ is a continuous mapping.}$$

If we also assume that $u_0 \in V$, $\phi \in L^2_{V\cap H^2(\Omega)}$, then

$$\begin{aligned} &u \in L^\infty(\tau, T; V) \cap L^2(\tau, T; V \cap H^2(\Omega)), \qquad \forall T > \tau, \\ &\eta^{\cdot} \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; V \cap H^2(\Omega))), \qquad \forall T > \tau, \end{aligned}$$

and for each $t \in [\tau, T]$,

$$z(t) : \mathcal{V} \to \mathcal{V} \text{ is a continuous mapping.}$$

Proof. Thanks to Lemma 3.1, we obtain $\mathcal{J}\phi \in L^2_\mu(\mathbb{R}^+; V)$ since $\phi \in L^2_V$. Therefore, the first statement of Theorem 3.4 holds by applying (i) in Theorem 3.3 with initial value $z_0 = (u_0, \mathcal{J}\phi)$. If, in addition, we assume that initial values $u_0 \in V$ and $\phi \in L^2_{V\cap H^2(\Omega)}$, then it is straightforward to prove that $z_0 = (u_0, \mathcal{J}\phi) \in \mathcal{V}$ and the regularity result follows from statement (ii) in Theorem 3.3. □

Remark 3.5. *Although the proof of Theorem 3.4 follows directly from Theorem 3.3, some computations, that we need in the sequel, are based on some estimations carried out in the proof. For this reason, we have included the complete proof of Theorem 3.4 as an Appendix, so that the paper is self-contained and easier to read.*

In what follows, we construct the dynamical system generated by (2.1) assuming that g does not depend on t, which makes our problem be autonomous. Thus, the theory of autonomous dynamical systems is appropriate to carry out the analysis of the global asymptotic behavior. We emphasize that the non-autonomous case can also be studied by exploiting the theory of non-autonomous dynamical systems (either the theory of pullback attractors or the uniform attractors one). The autonomous framework is concerned with the phase space

$$X = H \times L^2_V,$$

endowed with the norm

$$\|(w_1, w_2)\|^2_X = |w_1|^2 + \|w_2\|^2_{L^2_V}.$$

Then, thanks to Theorem 3.4, we can define a semigroup $S : \mathbb{R}^+ \times X \to X$ by

$$S(t)\,(u_0, \phi) = (u(t; 0, (u_0, \mathcal{J}\phi)), u_t(\cdot; 0, (u_0, \mathcal{J}\phi))),$$

where $(u(\cdot; 0, (u_0, \mathcal{J}\phi)), \eta^{\cdot})$ is the unique solution to problem (2.5) with $u\,(0) = u_0$, $\eta_0 = \mathcal{J}\phi$.

Let us first prove that the dynamical system S is well defined. In what follows, we will take $\tau = 0$ since we are working on autonomous dynamical system.

Lemma 3.6. *Under assumptions of Theorem 3.4, if* $(u_0, \phi) \in X$*, then* $S(t)\,(u_0, \phi) \in X$.

Proof. Let $(u_0, \phi) \in X$ and, for simplicity, denote by $(u(\cdot), \eta^{\cdot})$ the solution to problem (2.5) corresponding to the initial value $(u_0, \mathcal{J}\phi)$. It follows from Theorem 3.4 that the first component $u(t)$ belongs to H, thus it only remains to show that the segment of solution $u_t(\cdot)$ belongs to L^2_V. Indeed,

$$\begin{aligned}
\int_{-\infty}^{0} e^{\gamma s}\|u_t(s)\|^2\,ds &= \int_{-\infty}^{0} e^{\gamma s}\|u(t+s)\|^2\,ds \\
&= \int_{-\infty}^{t} e^{\gamma(\sigma-t)}\|u(\sigma)\|^2\,d\sigma \\
&= e^{-\gamma t}\int_{-\infty}^{t} e^{\gamma\sigma}\|u(\sigma)\|^2\,d\sigma \\
&= e^{-\gamma t}\int_{-\infty}^{0} e^{\gamma\sigma}\|\phi(\sigma)\|^2\,d\sigma + \int_{0}^{t} e^{\gamma(\sigma-t)}\|u(\sigma)\|^2\,d\sigma \\
&< +\infty,
\end{aligned}$$

where the above estimation holds true since $\phi \in L^2_V$ and $u \in L^2(0,T;V)$ for all $T > 0$. The proof of this lemma is complete. □

Lemma 3.7. *Under assumptions of Theorem 3.4, there exist two positive constants* K_1 *and* K_2*, such that*

$$\|S(t)(u_0,\phi)\|_X^2 \le K_1\,\|(u_0,\phi)\|_X^2\,e^{-\gamma t} + K_2, \qquad \forall t \ge 0,\ (u_0,\phi) \in X. \tag{3.5}$$

Proof. Let $(u_0, \phi) \in X$ and denote by $z(\cdot) = (u(\cdot), \eta^{\cdot})$ the solution to (2.5) corresponding to the initial value $(u_0, \mathcal{J}\phi)$. Now, we multiply the first equation in (2.5) by $u\,(t)$ in H and the second equation in (2.5) by η^t in $L^2_\mu(\mathbb{R}^+; V)$. Then, by the same energy estimations as in the proof of Theorem 3.4 (see Appendix (3.29)), we obtain

$$\begin{aligned}
&\frac{d}{dt}\,\|z\|_{\mathcal{H}}^2 + m\lambda_1\,|u|^2 + m\,\|u\|^2 + f_0\,|u|_{2p}^{2p} + 2(((\eta^t)', \eta^t))_\mu \\
&\le 2a_0\,|\Omega| + \frac{2}{\sqrt{\lambda_1}}\,|g|\,\|u\| \\
&\le 2a_0\,|\Omega| + \frac{2}{m\lambda_1}\,|g|^2 + \frac{m}{2}\,\|u\|^2.
\end{aligned}$$

Since

$$2(((\eta^t)', \eta^t))_\mu = -\int_0^\infty \mu'(s)|\nabla\eta^t(s)|^2 ds \ge \delta\int_0^\infty \mu(s)|\nabla\eta^t(s)|^2 ds, \tag{3.6}$$

it follows that

$$\frac{d}{dt}\,\|z\|_{\mathcal{H}}^2 + \gamma\,\|z\|_{\mathcal{H}}^2 + \frac{m}{2}\,\|u\|^2 + f_0\,|u|_{2p}^{2p} \le K_0, \tag{3.7}$$

where $K_0 = 2a_0|\Omega| + \frac{2}{m\lambda_1}|g|^2$ and we recall that $\gamma < \min\{m\lambda_1, \delta\}$. Notice that inequality (3.6) has been deduced formally but can be fully justified by using mollifiers (see [12, p. 348]). Now multiplying the above inequality by $e^{\gamma t}$ and integrating over $(0,t)$, neglecting the last term of the left hand side of (3.7), we obtain

$$\begin{aligned} &\|z(t)\|_{\mathcal{H}}^2 + \frac{m}{2}\int_0^t e^{-\gamma(t-s)}\|u(s)\|^2 ds \\ &\le \|z(t)\|_{\mathcal{H}}^2 + \frac{m}{2}\int_{-t}^0 e^{\gamma s}\|u_t(s)\|^2 ds \\ &\le \|z_0\|_{\mathcal{H}}^2 e^{-\gamma t} + \frac{K_0}{\gamma}. \end{aligned} \tag{3.8}$$

Then

$$\begin{aligned} \frac{m}{2}\|u_t\|_{L_V^2}^2 &= \frac{m}{2}\int_{-\infty}^0 e^{-\gamma(t-s)}\|\phi(s)\|^2 ds + \frac{m}{2}\int_0^t e^{-\gamma(t-s)}\|u(s)\|^2 ds \\ &\le \frac{m}{2}e^{-\gamma t}\|\phi\|_{L_V^2}^2 + \|(u_0, \mathcal{J}\phi)\|_{\mathcal{H}}^2 e^{-\gamma t} + \frac{K_0}{\gamma}. \end{aligned}$$

In view of Lemma 3.1, we have that

$$\|z_0\|_{\mathcal{H}}^2 \le |u_0|^2 + \|\mathcal{J}\phi\|_{L_\mu^2(\mathbb{R}^+;V)}^2 \le |u_0|^2 + K_\mu\|\phi\|_{L_V^2}^2. \tag{3.9}$$

Hence, (3.8)-(3.9) imply the existence of positive constants K_1 and K_2, such that

$$\|S(t)(u_0,\phi)\|_X^2 := |u(t)|^2 + \|u_t\|_{L_V^2}^2 \le K_1\left(|u_0|^2 + \|\phi\|_{L_V^2}^2\right)e^{-\gamma t} + K_2.$$

The proof of this lemma is complete. □

From Lemma 3.7, we immediately have the following result.

Corollary 3.8. *The ball* $B_0 = \{v \in X : \|v\|_X^2 \le 2K_2\}$ *is absorbing for the semigroup* S.

Now we shall prove the asymptotic compactness of the semigroup S. To this end, we first state the next result.

Lemma 3.9. *Assume the hypotheses in Theorem 3.4. Let* $\{(u_0^n, \phi^n)\}$ *be a sequence, such that* $(u_0^n, \phi^n) \to (u_0, \phi)$ *weakly in* X *as* $n \to \infty$. *Then,* $S(t)(u_0^n, \phi^n) = (u^n(t), u_t^n)$ *fulfills:*

$$u^n(\cdot) \to u(\cdot) \quad \textit{in} \quad C([r,T], H) \quad \textit{for all} \quad 0 < r < T; \tag{3.10}$$

$$u^n(\cdot) \to u(\cdot) \quad \textit{weakly in} \quad L^2(0,T;V) \quad \textit{for all} \quad T > 0; \tag{3.11}$$

$$u^n \to u \quad \textit{in} \quad L^2(0,T;H) \quad \textit{for all} \quad T > 0; \tag{3.12}$$

$$\limsup_{n\to\infty}\ \|u_t^n - u_t\|^2_{L^2_V} \le K_5\, e^{-\gamma t} \limsup_{n\to\infty} \left(|u_0^n - u_0|^2 + \|\phi^n - \phi\|^2_{L^2_V} \right) \quad \textit{for all} \quad t \ge 0, \tag{3.13}$$

where $K_5 = \frac{1}{m}((\gamma+\delta)^2+1)$. *Moreover, if* $(u_0^n, \phi^n) \to (u_0, \phi)$ *strongly in* X *as* $n \to \infty$*, then*

$$u^n(\cdot) \to u(\cdot) \quad \textit{in} \quad L^2(0,T;V) \quad \textit{for all} \quad T > 0; \tag{3.14}$$

$$u_t^n(\cdot) \to u_t(\cdot) \quad \textit{in} \quad L^2_V \quad \textit{for all} \quad t \ge 0. \tag{3.15}$$

Proof. Let $T > 0$ be arbitrary. In view of (3.5) and integrating in (3.7) over $(0,T)$, we deduce that u^n is bounded in $L^\infty(0,T;H)$, $L^2(0,T;V)$ and $L^{2p}(0,T;L^{2p}(\Omega))$, η_n^t is bounded in $L^\infty(0,T;L^2_\mu(\mathbb{R}^+;V))$. Hence, passing to a subsequence, we have

$$\begin{aligned} &u^n \to u \quad \text{weak-star in} \quad L^\infty(0,T;H); \qquad (3.16)\\ &u^n \to u \quad \text{weakly in} \quad L^2(0,T;V);\\ &u^n \to u \quad \text{weakly in} \quad L^{2p}(0,T;L^{2p}(\Omega));\\ &\eta_n^t \to \eta^t \quad \text{weak-star in} \quad L^\infty(0,T;L^2_\mu(\mathbb{R}^+;V)); \end{aligned}$$

thus (3.11) holds. Also, by the same arguments in the proof of Theorem 3.4 (see Appendix), we deduce

$$\begin{aligned} &\frac{du^n}{dt} \to \frac{du}{dt} \quad \text{weakly in} \quad L^2(0,T;V^*) + L^q(0,T;L^q(\Omega)), \qquad (3.17)\\ &f(u^n) \to \chi \quad \text{weakly in} \quad L^q(0,T;L^q(\Omega)). \end{aligned}$$

In view of (3.11) and (3.17), making use of the Compactness Theorem [18] we infer that (3.12) holds true. Thus, $u^n(t,x) \to u(t,x)$, $f(u^n(t,x)) \to f(u(t,x))$ for a.a. $(t,x) \in (0,T)\times\Omega$, so Lemma 1.3 in [16] implies that $\chi = f(u)$.

By proceeding as in the proof of Theorem 3.4, we obtain that $z(\cdot) = (u(\cdot), \eta^{\cdot})$ is a solution to problem (2.5) with initial value $z(0) = (u_0, \mathcal{J}\phi)$. Thanks to the uniqueness of solution, a standard argument implies that the above convergences are true for the whole sequence.

Further, we will prove (3.10). Formally, we multiply the first equation of (2.5) by $-\Delta u(t)$ in H, and the second equation of (2.5) by $-\eta^t$ in $L^2_\mu(\mathbb{R}^+; V\cap H^2(\Omega))$ (these calculations can be rigorously justified via Galerkin approximations). Then, arguing as in the proof of Theorem 3.4, we obtain

$$\begin{aligned} &\frac{d}{dt}\|z\|^2_{\mathcal{V}} + 2a(l(u))|\Delta u|^2 + 2(((\eta^t,(\eta^t)')))_\mu\\ &= 2(-f(u)+g(t), -\Delta u)\\ &\le m|\Delta u|^2 + \frac{2}{m}|g|^2 + \frac{2}{m}f^2_{2p-1}|\Omega| + d_0\,\|u\|^2, \end{aligned}$$

where $d_0 > 0$. Under the suitable spatial regularity of η^t, integration by parts in time and condition (h_1) imply that

$$(((\eta^t,(\eta^t)')))_\mu = -\int_0^\infty \mu'(s)|\Delta\eta^t(s)|^2 ds \geq 0.$$

Hence, by (2.2), we have

$$\frac{d}{dt}\|z\|_{\mathcal{V}}^2 \leq \frac{2}{m}|g|^2 + \frac{2}{m}f_{2p-1}^2|\Omega| + d_0\|u\|^2 \leq K_3\left(1+\|u\|^2\right), \tag{3.18}$$

where we have used the notation,

$$K_3 = \max\left\{\frac{2}{m}|g|^2 + \frac{2}{m}f_{2p-1}^2|\Omega|, \quad d_0\right\}.$$

Integrating in (3.7) over $(t,t+r)$ for $t \geq 0$, $0 < r < T-t$ and using (3.8), we deduce that

$$\int_t^{t+r} \|u\|^2\, ds \leq \frac{2K_0}{m}r + \frac{2}{m}\|z(t)\|_{\mathcal{H}}^2 \leq K_4\,(1+r)\,, \quad \forall t \geq 0, \tag{3.19}$$

where we have used the notation

$$K_4 = \max\left\{\frac{2K_0}{m}, \frac{2}{m}\|z_0\|_{\mathcal{H}}^2 + \frac{2K_0}{m\gamma}\right\}.$$

We integrate in (3.18) over $(s,t+r)$, where $s \in (t,t+r)$. Thus, by (3.19),

$$\|z(t+r)\|_{\mathcal{V}}^2 \leq \|z\,(s)\,\|_{\mathcal{V}}^2 + K_3 r + K_3K_4\,(1+r)\,.$$

Integrating the above inequality now again over $(t,t+r)$ in s, with the help of (3.19), we have

$$r\|z(t+r)\|_{\mathcal{V}}^2 \leq \|z_0\|_{\mathcal{V}}^2 r + 2K_3r^2 + (K_3+1)K_4r(1+r), \quad \forall t \geq 0,$$

thus, $\|z(t)\|_{\mathcal{V}}$ is uniformly bounded in $[r,T]$. We observe that by a standard argument (see [1, p.195]), for any sequence $t_n \to t_0$ as $n \to \infty$, $t_n, t_0 \in [0,T]$, $u^n\,(t_n) \to u\,(t_0)$ weakly in V. Then the compact embedding $V \subset H$ ensures $u^n\,(t_n) \to u\,(t_0)$ strongly in H, for all $t_n, t_0 \in [r,T]$ and $t_n \to t_0$ as $n \to \infty$, therefore (3.10) holds true.

Define the functions $w^n = z^n - z$, $\beta_n^t = \eta_n^t - \eta^t$, similarly to the uniqueness step in the proof of Theorem 3.4, Step 5 in Appendix, we have

$$\begin{aligned}&\frac{d}{dt}\|w^n\|_{\mathcal{H}}^2 + 2(((\beta_n^t)', \beta_n^t))_\mu \\ &\leq -2\int_\Omega (f\,(u^n) - f\,(u))\,(u^n-u)\,dx - \int_\Omega (a\,(l\,(u^n))\,\nabla u^n - a(l\,(u))\nabla u)\cdot\nabla\,(u^n-u)\,dx.\end{aligned} \tag{3.20}$$

Since a is locally Lipschitz, by (2.2) and the Young inequality, we have

$$\begin{aligned}
&-2\int_{\Omega}\left(a\left(l\left(u^{n}\right)\right)\nabla u^{n}-a(l\left(u\right))\nabla u\right)\cdot\nabla\left(u^{n}-u\right)dx\\
&=-2\int_{\Omega}a\left(l\left(u^{n}\right)\right)\left|\nabla\left(u^{n}-u\right)\right|^{2}dx-2\left(a\left(l\left(u^{n}\right)\right)-a\left(l\left(u\right)\right)\right)\int_{\Omega}\nabla u\cdot\nabla\left(u^{n}-u\right)dx\\
&\leq-2m\left\|u^{n}-u\right\|^{2}+2L_{a}\left(R\right)|l|\left|u^{n}-u\right|\left\|u\right\|\left\|u^{n}-u\right\|\\
&\leq\left(\alpha-2m\right)\left\|u^{n}-u\right\|^{2}+\frac{L_{a}^{2}\left(R\right)|l|^{2}}{\alpha}\left|u^{n}-u\right|^{2}\left\|u\right\|^{2},
\end{aligned}\tag{3.21}$$

where $\alpha\leq\left(m\lambda_{1}-\gamma\right)/\lambda_{1}$, and for all $n\geq1$, $t\geq0$, we choose $R>0$ such that $\left|u^{n}\left(t\right)\right|,\ \left|u\left(t\right)\right|\leq R$ (cf. (3.10)). By (3.6), (3.20) and (3.21), we have

$$\begin{aligned}
&\frac{d}{dt}\left\|w^{n}\right\|_{\mathcal{H}}^{2}+\gamma\left\|w^{n}\right\|_{\mathcal{H}}^{2}+m\left\|u^{n}-u\right\|^{2}\\
&\leq\frac{d}{dt}\left\|w^{n}\right\|_{\mathcal{H}}^{2}+\left(2m-\alpha\right)\left\|u^{n}-u\right\|^{2}+\delta\int_{0}^{\infty}\mu(s)|\nabla\beta_{n}^{t}(s)|^{2}ds\\
&\leq\frac{L_{a}^{2}\left(R\right)|l|^{2}}{\alpha}\left|u^{n}-u\right|^{2}\left\|u\right\|^{2}-2\int_{\Omega}\left(f\left(u^{n}\right)-f\left(u\right)\right)\left(u^{n}-u\right)dx,
\end{aligned}$$

where we have used that $\gamma\leq\min\{\left(m-\alpha\right)\lambda_{1},\delta\}$ by the choice of α. Multiplying by $e^{\gamma t}$ on both sides of the above inequality and integrating over $(0,t)$, we obtain

$$\begin{aligned}
&\left\|w^{n}\left(t\right)\right\|_{\mathcal{H}}^{2}+m\int_{0}^{t}e^{-\gamma(t-s)}\left\|u^{n}-u\right\|^{2}ds\\
&\leq e^{-\gamma t}\left\|w^{n}\left(0\right)\right\|_{\mathcal{H}}^{2}+\frac{L_{a}^{2}\left(R\right)|l|^{2}}{\alpha}\int_{0}^{t}e^{-\gamma(t-s)}\left|u^{n}-u\right|^{2}\left\|u\right\|^{2}ds\\
&\quad-2\int_{0}^{t}e^{-\gamma(t-s)}\int_{\Omega}\left(f\left(u^{n}\right)-f\left(u\right)\right)\left(u^{n}-u\right)dxds.
\end{aligned}$$

On the one hand, by (3.10), we know that $\left|u^{n}\left(s\right)-u\left(s\right)\right|^{2}\left\|u\left(s\right)\right\|^{2}\to0$ for a.e. $s\in(0,t)$. On the other hand, $e^{-\gamma(t-s)}\left|u^{n}\left(s\right)-u\left(s\right)\right|^{2}\left\|u\left(s\right)\right\|^{2}$ is bounded by the integrable function $4R^{2}e^{-\gamma(t-s)}\left\|u\left(s\right)\right\|^{2}$. Hence, Lebesgue's theorem implies that

$$\int_{0}^{t}e^{-\gamma(t-s)}\left|u^{n}-u\right|^{2}\left\|u\right\|^{2}ds\to0\quad\text{as}\quad n\to\infty.$$

Since $f\left(u^{n}\right)\to f\left(u\right)$ weakly in $L^{q}\left(0,T;L^{q}\left(\Omega\right)\right)$, it follows that

$$\int_{0}^{t}e^{-\gamma(t-s)}\int_{\Omega}\left(f\left(u^{n}\right)-f\left(u\right)\right)udxds\to0\ \text{ as }\ n\to\infty.$$

Furthermore, as $f\left(u^{n}\left(t,x\right)\right)u^{n}\left(t,x\right)\geq-\kappa_{1}+\kappa_{2}|u^{n}(t,x)|^{2p}$ (see (3.28)) and $u^{n}\left(t,x\right)\to u\left(t,x\right)$, $f\left(u^{n}\left(t,x\right)\right)\to f\left(u\left(t,x\right)\right)$ for a.a. $(t,x)\in(0,T]\times\Omega$, Lebesgue-Fatous's theorem implies

$$\begin{aligned}
&\limsup_{n\to\infty}\left(-2\int_0^t e^{-\gamma(t-s)}\int_\Omega f(u^n)\,u^n dxds\right)\\
&\le -2\liminf_{n\to\infty}\int_0^t e^{-\gamma(t-s)}\int_\Omega f(u^n)\,u^n dxds\\
&\le -2\int_0^t e^{-\gamma(t-s)}\int_\Omega \liminf_{n\to\infty} f(u^n)\,u^n dxds\\
&= 2\int_0^t e^{-\gamma(t-s)}\int_\Omega f(u)\,udxds.
\end{aligned}$$

This inequality, together with

$$\int_0^t e^{-\gamma(t-s)}\int_\Omega f(u)\,(u^n-u)\,dxds\to 0 \quad \text{as} \quad n\to\infty, \tag{3.22}$$

shows that

$$\limsup_{n\to\infty}\left(-2\int_0^t e^{-\gamma(t-s)}\int_\Omega (f(u^n)-f(u))u^n dxds\right)\le 0 \quad \text{as} \quad n\to\infty.$$

Notice that (3.22) follows from the facts $f(u(\cdot))\in L^q(0,T;L^q(\Omega))$ and $u^n\to u$ weakly in $L^{2p}\left(0,T;L^{2p}(\Omega)\right)$.

Collecting all inequalities and using (3.2),

$$\begin{aligned}
&\limsup_{n\to\infty}\int_0^t e^{-\gamma(t-s)}\left\|u^n(s)-u(s)\right\|^2 ds\\
&\le \frac{1}{m}e^{-\gamma t}\limsup_{n\to\infty}\left\|w^n(0)\right\|^2_{\mathcal{H}}\\
&= \frac{1}{m}e^{-\gamma t}\limsup_{n\to\infty}\left(\left|u^n(0)-u_0\right|^2+\int_0^\infty \mu(s)\,\|\beta_n^0(s)\,\|^2 ds\right)\\
&\le \frac{1}{m}e^{-\gamma t}\limsup_{n\to\infty}\left(\left|u^n(0)-u_0\right|^2+K_\mu\int_{-\infty}^0 e^{\gamma s}\left\|\phi^n(s)-\phi(s)\right\|^2 ds\right).
\end{aligned}$$

Finally, (3.13) follows from

$$\begin{aligned}
\|u_t^n-u_t\|^2_{L_V^2} &= \int_{-t}^0 e^{\gamma s}\left\|u^n(t+s)-u(t+s)\right\|^2 ds+\int_{-\infty}^{-t} e^{\gamma s}\left\|u^n(t+s)-u(t+s)\right\|^2 ds\\
&= \int_0^t e^{-\gamma(t-s)}\left\|u^n(s)-u(s)\right\|^2 ds+e^{-\gamma t}\int_{-\infty}^0 e^{\gamma s}\left\|\phi^n(s)-\phi(s)\right\|^2 ds.
\end{aligned}$$

If $(u_0^n,\phi^n)\to(u_0,\phi)$ in X, then (3.13) implies (3.14) and (3.15). □

As a consequence, we obtain the continuous dependence with respect to the initial data.

Corollary 3.10. *Assume conditions of Theorem 3.4 are true. Then, for any $t \geq 0$, the mapping $(u_0, \phi) \mapsto S(t)(u_0, \phi)$ is continuous.*

Finally, we are ready to prove the asymptotic compactness of the semigroup.

Lemma 3.11. *Under assumptions of Theorem 3.4, the semigroup S is asymptotically compact.*

Proof. Let $B \subset X$ be a bounded set, we need to prove that for any sequences $\{(y_n, \phi_n)\}_{n\in\mathbb{N}} \subset B$ and $t_n \to +\infty$ as $n \to +\infty$, the sequence $\{S(t_n)(y_n, \phi_n)\}_{n\in\mathbb{N}}$ is relatively compact. Recall that

$$S(t_n)(y_n, \phi_n) = (u(t_n; 0, (y_n, \mathcal{J}\phi_n)), u_{t_n}(\cdot; 0, (y_n, \mathcal{J}\phi_n))) := (u^n(t_n), u^n_{t_n}(\cdot))$$

Pick now $T > 0$, and assume that $t_n > T$ for all $n \in \mathbb{N}$ (there is no loss of generality in assuming this since $t_n \to +\infty$). Now we can define $v^n(t) = u^n(t + t_n - T)$, observe that $v^n(T) = u^n(t_n)$ and $v^n_T(t) = v^n(T + t) = u^n(t + t_n) = u^n_{t_n}(t)$. Therefore

$$S(t_n)(y_n, \phi_n) = (u^n(t_n), u^n_{t_n}(\cdot)) = (v^n(T), v^n_T(\cdot)).$$

Let us denote now

$$\mathcal{Y}_n = (v^n(T), v^n_T) = (u^n(t_n), u^n_{t_n}(\cdot)),\ \xi^T_n = (v^n(0), v^n_0(\cdot)) = (u^n(t_n - T), u^n_{t_n-T}(\cdot)).$$

By Lemma 3.7, the sequences $\{\mathcal{Y}_n\}$, $\{\xi^T_n\}$ are bounded in X, so up to a subsequence $\mathcal{Y}_n \to \mathcal{Y} := (y, \phi)$, $\xi^T_n \to \xi^T$ weakly in X. In addition, by Lemma 3.9, $\mathcal{V}(t) := S(t)\xi^T = (v(t), v_t(\cdot))$ satisfies (3.10)-(3.13). It follows from the above convergences that, $\phi = v_T$ in L^2_V and $y = v_T(0)$, $\phi(s) = v_T(s)$ for almost all $s \in (-\infty, 0)$. Also, in view of (3.10) we infer that

$$u^n(t_n) = v^n(T) \to v(T) = y.$$

Hence, in order to prove that $\mathcal{Y}_n \to \mathcal{Y}$ in X, it remains to check that $u^n_{t_n}(\cdot) \to \phi$ in L^2_V (up to a subsequence). Notice that $u^n_{t_n}(\cdot) = v^n_T$ for all $t_n > T$ and $v_T = \phi$. Thanks to (3.13) we have, for each $T > 0$,

$$\begin{aligned}
\limsup_{n\to\infty} \left\|u^n_{t_n}(\cdot) - \phi\right\|^2_{L^2_V} &= \limsup_{n\to\infty} \|v^n_T - v_T\|^2_{L^2_V} \\
&\leq K_5\, e^{-\gamma(T-\tau)} \limsup_{n\to\infty} \left(\left\|\xi^T_n - \xi^T\right\|^2_X\right) \\
&\leq \widetilde{K} e^{-\gamma T},
\end{aligned}$$

where the last inequality follows from Lemma 3.7. For every $k > 0$, there exists $T := T(k)$ such that for all $T \geq T(k)$,

$$\limsup_{n\to\infty} \left\|u^n_{t_n}(\cdot) - \phi\right\|^2_{L^2_V} = \limsup_{n\to\infty} \|v^n_T - v_T\|^2_{L^2_V} \leq \frac{1}{k}.$$

Taking $k \to \infty$ and using a diagonal argument, we obtain that there exists a subsequence $\{u^{n_k}_{t_{n_k}}(\cdot)\}$ such that $u^{n_k}_{t_{n_k}}(\cdot) \to \phi$ in L^2_V. $\square$

By Corollaries 3.8, 3.10 and Lemma 3.11 the general theory of attractors (see [14, Theorem 3.1]) implies the following result.

Theorem 3.12. *Under the assumptions of Theorem 3.4, the semigroup S possesses a global connected attractor $\mathcal{A} \subset X$.*

As a straightforward consequence of the previous results, we can provide information for the local problem analyzed, amongst others, in the papers [10, 11, 12] by simply assuming that $a(\cdot)$ is a constant function.

Corollary 3.13. *Under the hypotheses of Theorem 3.4, assume also that $a(t) = k_0 > 0$ for all $t \geq 0$. Then the local problem* (2.1) *poseesses a global connected attractor $\mathcal{A} \subset X$.*

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Appendix

Proof of Theorem 3.4. We follow a standard Faedo-Galerkin method. Recall that there exists a smooth orthonormal basis $\{w_j\}_{j=1}^{\infty}$ in H which also belongs to $V \cap L^{2p}(\Omega)$ ([12, Proposition 2.3]). Let us take a complete set of normalized eigenfunctions for $-\Delta$ in V, such that $-\Delta w_j = \lambda_j w_j$ (λ_j the eigenvalue corresponding to w_j). Next we will select an orthonormal basis $\{\zeta_j\}_{j=1}^{\infty}$ of $L^2_\mu(\mathbb{R}^+; V)$ which also belongs to $\mathcal{D}(\mathbb{R}^+; V)$.

The proof is divided into 6 steps.

Step 1. (Faedo-Galerkin scheme) Fix $T > \tau$, for a given integer n, denote by P_n and Q_n the projections on the subspaces

$$\text{span}\{w_1, \cdots, w_n\} \subset V \qquad \text{and} \qquad \text{span}\{\zeta_1, \cdots, \zeta_n\} \subset L^2_\mu(\mathbb{R}^+; V),$$

respectively. We look for a function $z_n = (u_n, \eta^t_n)$ of the form

$$u_n(t) = \sum_{j=1}^{n} b_j(t) w_j \qquad \text{and} \qquad \eta^t_n(s) = \sum_{j=1}^{n} c_j(t)\zeta_j(s),$$

satisfying

$$\begin{cases}(\partial_t z_n, (w_k, \zeta_j))_{\mathcal{H}} = (\mathcal{L}z_n, (w_k, \zeta_j)) + (\mathcal{G}(z), (w_k, \zeta_j)), \quad k, j = 0, \cdots, n, \\ z_n|_{t=\tau} = (P_n u_0, Q_n \eta_0),\end{cases} \tag{3.23}$$

for a.e. $\tau \leq t \leq T$, where w_0 and ζ_0 are the zero vectors in the respective spaces. Taking (w_k, ζ_0) and (w_0, ζ_k) in (3.23), applying the divergence theorem, we derive a system of ODE in the variables

$$\begin{cases}\frac{d}{dt}b_k(t) = -\lambda_k a(l(\sum_{j=1}^n b_j(t)w_j))b_k - \sum_{j=1}^n c_j((\zeta_j, w_k))_\mu - (f(\sum_{j=1}^n b_j(t)w_j), w_k) + (g, w_k), \\ \frac{d}{dt}c_k(t) = \sum_{j=1}^n b_j((w_j, \zeta_k))_\mu - \sum_{j=1}^n c_j((\zeta_j', \zeta_k))_\mu,\end{cases} \tag{3.24}$$

subject to the initial conditions,

$$b_k(\tau) = (u_0, w_k), \qquad c_k(\tau) = ((\eta_0, \zeta_k))_\mu. \tag{3.25}$$

According to the standard existence theory for ODE, there exists a continuous solution of (3.24)-(3.25) on some interval (τ, t_n). Then a priori estimates imply $t_n = \infty$.

Step 2. (Energy estimate) Multiplying the first equation of (3.24) by b_k and the second one by c_k, summing over k ($k = 1, 2, \cdots, n$) and adding the results, we have

$$\frac{1}{2}\frac{d}{dt}\|z_n\|_{\mathcal{H}}^2 = (\mathcal{L}z_n, z_n)_{\mathcal{H}} + (\mathcal{G}(z_n), z_n)_{\mathcal{H}}. \tag{3.26}$$

On the one hand, by the divergence theorem,

$$\left(\int_0^\infty \mu(s)\Delta\eta_n^t(s)ds, u_n\right) = -\int_0^\infty \mu(s)\int_\Omega \nabla\eta_n^t(s)\cdot\nabla u_n(s)dxds = -((u_n, \eta_n^t))_\mu,$$

therefore,

$$(\mathcal{L}z_n, z_n)_{\mathcal{H}} = -a(l(u_n))|\nabla u_n|^2 - (((\eta_n^t)', \eta_n^t))_\mu. \tag{3.27}$$

On the other hand, (2.7) yields that there exists a constant a_0, such that

$$f(u)\cdot u \geq \frac{1}{2}f_0 u^{2p} - a_0,$$

hence,

$$(\mathcal{G}(z_n), z_n)_{\mathcal{H}} = (-f(u_n) + g, u_n) \leq -\frac{1}{2}f_0|u_n|_{2p}^{2p} + a_0|\Omega| + (g, u_n). \tag{3.28}$$

It follows from (2.2), (3.26)-(3.28) and the Young inequality that

$$\frac{d}{dt}\|z_n\|_{\mathcal{H}}^2 + 2m|\nabla u_n|^2 + 2(((\eta_n^t)', \eta_n^t))_\mu + f_0|u_n|_{2p}^{2p} \leq 2a_0|\Omega| + \frac{1}{m\lambda_1}|g|^2 + m|\nabla u_n|^2. \tag{3.29}$$

Integration by parts and (h_1) yield that,

$$2(((\eta_n^t)', \eta_n^t))_\mu = -\int_0^\infty \mu'(s)|\nabla \eta_n^t(s)|^2 ds \geq 0,$$

thus the third term of the right hand side of (3.29) can be neglected, we obtain

$$\frac{d}{dt}\|z_n\|_{\mathcal{H}}^2 + m|\nabla u_n|^2 + f_0|u_n|_{2p}^{2p} \leq 2a_0|\Omega| + \frac{1}{m\lambda_1}|g|^2.$$

Integrating the above inequality between τ and t, $t \in (\tau, T]$, we have

$$\|z_n(t)\|_{\mathcal{H}}^2 + \int_\tau^t \left[m\|u_n\|^2 + f_0|u_n|_{2p}^{2p}\right] dr \leq \|z_0\|_{\mathcal{H}}^2 + \Lambda(T-\tau), \tag{3.30}$$

where we have used the notation $\Lambda := 2a_0|\Omega| + \frac{1}{m\lambda_1}|g|^2$. Therefore, it arrives that

$$\begin{aligned} &u_n \quad \text{is bounded in} \quad L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^{2p}(\tau, T; L^{2p}(\Omega)), \\ &\eta_n \quad \text{is bounded in} \quad L^\infty(\tau, T; L_\mu^2(\mathbb{R}^+; V)). \end{aligned}$$

Passing to a subsequence, there exists a function $z = (u, \eta)$ such that

$$\begin{cases} u_n \to u & \text{weak-star in} & L^\infty(\tau, T; H); \\ u_n \to u & \text{weakly in} & L^2(\tau, T; V); \\ u_n \to u & \text{weakly in} & L^{2p}(\tau, T; L^{2p}(\Omega)); \\ \eta_n^t \to \eta^t & \text{weak-star in} & L^\infty(\tau, T; L_\mu^2(\mathbb{R}^+; V)). \end{cases} \tag{3.31}$$

Step 3. (Passage to limit) For a fixed integer m, choose a function

$$v = (\sigma, \pi) \in \mathcal{D}((\tau, T); V \cap L^{2p}(\Omega)) \times \mathcal{D}((\tau, T); \mathcal{D}(\mathbb{R}^+; V))$$

of the form

$$\sigma(t) = \sum_{j=1}^m \tilde{b}_j(t) w_j \qquad \text{and} \qquad \pi^t(s) = \sum_{j=1}^m \tilde{c}_j(t)\zeta_j(s),$$

where $\{\tilde{b}_j\}_{j=1}^m$ and $\{\tilde{c}_j\}_{j=1}^m$ are given functions in $\mathcal{D}(\tau, T)$, then (3.23) holds with (σ, π) in place of (ω_k, ζ_j).

Our main target is to prove problem (2.5) has a solution in the weak sense, i.e., for arbitrary $v \in \mathcal{D}((\tau, T); V \cap L^{2p}(\Omega)) \times \mathcal{D}((\tau, T); \mathcal{D}(\mathbb{R}^+; V))$ (here, specially, we pick up $v = (\sigma, \pi) \in \mathcal{D}(\tau, T)$ as a test function), the following equality

$$\begin{aligned} \int_\tau^t (\partial_r z_n, v)_{\mathcal{H}} dr = \int_\tau^t \bigg[&- a(l(u_n))(\nabla u_n, \nabla \sigma) - ((\eta_n^t, \sigma))_\mu - (f(u_n), \sigma) \\ &+ (g, \sigma) + ((u_n, \pi^t))_\mu - \ll (\eta_n^t)', \pi^t \gg \bigg] dr \end{aligned} \tag{3.32}$$

holds in the sense of $\mathcal{D}'(\tau,T)$. Here, we denote by $\ll \cdot,\cdot \gg$ the duality map between $H^1_\mu(\mathbb{R}^+;V)$ and its dual space.

Firstly, using the same argument as in [20, Theorem 2.7] and $(3.31)_2$, we know

$$\int_\tau^t a(l(u_n))(\nabla u_n,\nabla\sigma)dr \to \int_\tau^t a(l(u))(\nabla u,\nabla\sigma)dr \quad \text{as} \quad n\to\infty.$$

Similarly, by means of $(3.31)_4$ and $(3.31)_2$, we have

$$\int_\tau^t ((\eta_n^t,\sigma))_\mu dr \to \int_\tau^t ((\eta^t,\sigma))_\mu dr \qquad \text{as} \quad n\to\infty,$$

and

$$\int_\tau^t ((u_n,\pi^t))_\mu dr \to \int_\tau^t ((u,\pi^t))_\mu dr \qquad \text{as} \quad n\to\infty,$$

respectively.

Secondly, we now show that

$$\lim_{n\to\infty} \ll (\eta_n^t)',\pi^t \gg = \ll (\eta^t)',\pi^t \gg .$$

Notice that, for every $\upsilon\in L^2_\mu(\mathbb{R}^+;V)$, making use of integration by parts, we derive

$$\ll \upsilon',\pi^t \gg = -\int_0^\infty \mu'(s)(\nabla\upsilon(s),\nabla\pi^t(s))ds - \int_0^\infty \mu(s)(\nabla\upsilon(s),\nabla(\pi^t)'(s))ds. \tag{3.33}$$

Replacing υ by η_n^t in (3.33), together with $(3.31)_4$, it is clear the right hand side of (3.33) converges to $\ll (\eta^t)',\pi^t \gg$ as $n\to\infty$.

Thirdly, we are going to prove that

$$\lim_{n\to\infty}\int_\tau^T\int_\Omega |f(u_n)\sigma|dxdt = \int_\tau^T\int_\Omega |f(u)\sigma|dxdt.$$

Based on the dominated convergence theorem, it is sufficient to show

$$f(u_n(t,x))\to f(u(t,x)) \qquad \text{for a.e. } (t,x)\in(\tau,T)\times\Omega,$$

and

$$|f(u_n)|_{L^q((\tau,T)\times\Omega)} \le C,$$

where $q=\frac{2p}{2p-1}\in(1,2)$, which is the conjugate exponent of $2p$ and the constant C is independent of n. Observe that

$$\begin{aligned}\|\partial_t u_n\|_{L^2(\tau,T;V^*)+L^q(\tau,T;L^q(\Omega))} \le \|a(l(u_n))\Delta u_n\|_{L^2(\tau,T;V^*)} + \left\|\int_0^\infty \mu(s)\Delta\eta_n^t(s)ds\right\|_{L^2(\tau,T;V^*)} \\ + \|g\|_{V^*} + \|f(u)\|_{L^q(\tau,T;L^q(\Omega))}.\end{aligned} \tag{3.34}$$

It follows from (2.7), there exists a constant $K > 0$ such that

$$|f(u_n)|^q \le K(1 + |u_n|^{2p}). \tag{3.35}$$

Together with (3.4), (3.31) and the assumption $g \in H$, we know that $\{\partial_t u_n\}$ is bounded in $L^2(\tau, T; V^*) + L^q(\tau, T; L^q(\Omega))$. Thus, up to a subsequence, we infer

$$\partial_t u_n \to \tilde{u} \qquad \text{weakly in} \quad L^2(\tau, T; V^*) + L^q(\tau, T; L^q(\Omega)). \tag{3.36}$$

By a standard argument we infer that $\tilde{u} = u_t$. Since

$$L^2(\tau, T; V^*) + L^q(\tau, T; L^q(\Omega)) \subset L^q(\tau, T; V^* + L^q(\Omega))$$

and

$$L^2(\tau, T; V) \subset L^q(\tau, T; V),$$

by (3.31) and (3.36), we deduce

$$u_n \to u \qquad \text{weakly in} \quad W^{1,q}(\tau, T; V^* + L^q(\Omega)) \cap L^q(\tau, T; V). \tag{3.37}$$

Applying a compactness argument [16], we derive that the injection

$$W^{1,q}(\tau, T; V^* + L^q(\Omega)) \cap L^q(\tau, T; V) \hookrightarrow L^q(\tau, T; L^q(\Omega))$$

is compact. Therefore, (3.37) implies that

$$u_n \to u \qquad \text{strongly in} \quad L^q(\tau, T; L^q(\Omega)).$$

By the continuity of f we obtain that (up to a subsequence)

$$f(u_n(t,x)) \to f(u(t,x)) \qquad \text{for a.e. } (t,x) \in (\tau, T) \times \Omega.$$

In virtue of (3.35), we obtain

$$|f(u_n)|^q_{L^q((\tau,T)\times\Omega)} = \int_\tau^T \int_\Omega |f(u_n)|^q dxdt \le K|\Omega|(T-\tau) + K\int_\tau^T |u_n|^{2p}_{2p} dt,$$

which is bounded uniformly with respect to n.

Eventually, by a standard argument, we derive

$$\partial_t z_n \to z_t \quad \text{in} \quad \mathcal{D}'(\tau, T; V \cap L^{2p}(\Omega)) \times \mathcal{D}'(\tau, T; \mathcal{D}(\mathbb{R}^+; V)).$$

Therefore, using a density argument, (3.32) is proved by the previous statements.

Step 4. (Continuity of solution) By (3.33)-(3.34), it is immediate to see that $z_t = (u_t, \eta_t)$ fulfills

$$u_t \in L^2(\tau, T; V^*) + L^q(\tau, T; L^q(\Omega));$$
$$\eta_t \in L^2(\tau, T; H^{-1}_\mu(\mathbb{R}^+; V)),$$

where $L^2(\tau,T;V^*) + L^q(\tau,T;L^q(\Omega))$ is the dual space of $L^2(\tau,T,V)\cap L^{2p}(\tau,T;L^{2p}(\Omega))$. Using a slightly modified version of [19, Lemma III.1.2], together with (3.31), we infer that $u \in C([\tau,T];H)$.

As for the second component, by means of the same argument as [12, Theorem, Section 2], we obtain that $\eta^t \in C([\tau,T];L^2_\mu(\mathbb{R}^+;V))$. Thus, $z(\tau)$ makes sense, and the equality $z(\tau)=z_0$ follows from the fact that $(P_n u_0, Q_n \eta_0)$ converges to z_0 strongly.

Step 5. (Continuity with respect to the initial value and uniqueness) Let $z_1=(u_1,\eta_1)$ and $z_2=(u_2,\eta_2)$ be the two solutions of (3.3) with initial data z_{10} and z_{20}, respectively. Due to the a priori estimates on the first component of solutions u, see (3.30), together with the fact that $u\in C(\tau,T;H)$, we can ensure that there exists a bounded set $S\subset H$, such that $u_i(t)\in S$ for all $t\in[\tau,T]$ and $i=1,2$. In addition, taking into account that $l\in\mathcal{L}(H;\mathbb{R})$, we have $\{l(u_i(t))\}_{t\in[\tau,T]}\subset[-R,R]$ for $i=1,2$, for some $R>0$. Therefore, let $\bar z = z_1 - z_2 = (\bar u,\bar\eta) = (u_1-u_2,\eta_1-\eta_2)$ and $\bar z_0 = z_{10}-z_{20}$. Thanks to (2.2), the locally Lipschitz continuity of function a with Lipschitz constant $L_a(R)$ and the Poincaré inequality, we have

$$
\begin{aligned}
\frac{d}{dt}\|\bar z\|^2_{\mathcal{H}} &\le 2a(l(u_1))|\nabla\bar u|^2 + 2L_a(R)|l||\bar u||\nabla u_2||\nabla \bar u|\\
&\quad -2<f(u_1)-f(u_2),\bar u>_{L^{p,q}} -2(((\bar\eta)',\bar\eta))_\mu\\
&\le -2m|\nabla\bar u|^2 + 2L_a(R)|l||\bar u||\nabla u_2||\nabla\bar u|\\
&\quad -2<f(u_1)-f(u_2),\bar u>_{L^{p,q}} -2(((\bar\eta)',\bar\eta))_\mu\\
&\le -2m|\nabla\bar u|^2 + 2m|\nabla\bar u|^2 + \frac{1}{2m}L_a^2(R)|l|^2|\bar u|^2\|u_2\|^2\\
&\quad -2<f(u_1)-f(u_2),\bar u>_{L^{p,q}} -2(((\bar\eta)',\bar\eta))_\mu\\
&\le \frac{1}{2m}L_a^2(R)|l|^2\|\bar z\|^2_{\mathcal{H}}\|u_2\|^2 - 2<f(u_1)-f(u_2),\bar u>_{L^{p,q}} -2(((\bar\eta)',\bar\eta))_\mu,
\end{aligned}
\tag{3.38}
$$

where $<\cdot,\cdot>_{L^{p,q}}$ is the duality between L^{2p} and L^q. The previous calculation is obtained formally taking the product in $\mathcal{H}$ between $\bar z$ and the difference of (3.3) with z_1 and z_2 in place of z, and it can be made rigorous with the use of mollifiers, see [12, Theorem, Section 2]. In fact, integrating by parts and by the fact that $\mu'<0$ (see again [12, Section 2]), we have

$$2(((\bar\eta)',\bar\eta))_\mu = -\lim_{s\to 0}\mu(s)|\nabla\bar\eta^t(s)|^2 - \int_0^\infty \mu'(s)|\nabla\bar\eta^t(s)|^2 ds \ge 0.$$

Hence, the last term of the right hand side of (3.38) can be neglected.

At last, from (2.7) we know that $f(u)$ is increasing for $|u|\ge M$, for some $M>0$. Fix $t\in(\tau,T]$, and let

$$\Omega_1 := \{x\in\Omega : |u_1(t,x)|\le M \;\text{ and }\; |u_2(t,x)|\le M\},$$

and

$$N = 2 \sup_{|s|\le M} |f'(s)|.$$

Let $x \in \Omega_1$, then we have

$$2|f(u_1(x)) - f(u_2(x))| \le N|\bar{u}(x)|.$$

Then, by the monotonicity of $f(u)$ for $|u| \ge M$ and the Poincaré inequality, it follows that

$$\begin{aligned} -2 < f(u_1) - f(u_2), \bar{u} >_{L^{p,q}} &\le -2\int_{\Omega_1} (f(u_1(x)) - f(u_2(x)))\bar{u}(x)dx \\ &\le \int_{\Omega_1} N|\bar{u}(x)|^2 dx \\ &\le N\|\bar{z}\|^2_{\mathcal{H}}. \end{aligned} \tag{3.39}$$

(3.38)-(3.39) imply that

$$\frac{d}{dt}\|\bar{z}\|^2_{\mathcal{H}} \le \left(\frac{1}{2m}L_a^2|l|^2\|u_2\|^2 + N\right)\|\bar{z}\|^2_{\mathcal{H}}.$$

The uniqueness and continuous dependence on initial data of solution to problem (3.3) follow from the Gronwall inequality. Till now, we finish the proof of the first assertion.

Step 6. (Further regularity) We are going to study further regularity of (u, η). To this end, let us first consider the linear operator $\mathcal{I} : L^2_{V\cap H^2(\Omega)} \to L^2_\mu(\mathbb{R}^+; D(V))$ defined by

$$(\mathcal{I}\phi)(s) = \int_{-s}^0 \phi(r)\, dr, \quad s \in \mathbb{R}^+.$$

Then, the operator $\mathcal{I}$ defined above is a linear and continuous mapping. In particular, there exists a positive constant K_μ, which is the same as in Lemma 3.1, such that, for any $\phi \in L^2_{V\cap H^2(\Omega)}$, it holds

$$\|\mathcal{I}\phi\|^2_{L^2_\mu(\mathbb{R}^+;D(A))} \le K_\mu \|\phi\|^2_{L^2_{V\cap H^2(\Omega)}}.$$

Next, multiplying $(2.5)_1$ by $-\Delta u$ with respect to the inner product of H, the Laplacian of $(2.5)_2$ by η with respect to the inner product of $L^2_\mu(\mathbb{R}^+; D(A))$, and adding the two terms, we obtain

$$\frac{d}{dt}\|z\|^2_{\mathcal{V}} + 2a(l(u))|\Delta u|^2 + 2(((\eta^t, (\eta^t)')))_\mu = 2(-f(u) + g, \Delta u). \tag{3.40}$$

Since f is a polynomial of odd degree, there exists a constant $d_0 > 0$, such that

$$f'(u) \ge -\frac{d_0}{2}, \qquad \forall u \in \mathbb{R}. \tag{3.41}$$

Then, it follows from the above inequality, (2.7), the Green formula and the Young inequality that

$$\begin{aligned} 2(f(u), \Delta u) &= 2\int_\Omega f_{2p-1}\Delta u dx - 2\int_\Omega f'(u)\nabla u\cdot\nabla u dx \\ &\le \frac{2}{m}f_{2p-1}^2|\Omega| + \frac{m}{2}|\Delta u|^2 + d_0|\nabla u|^2. \end{aligned}$$

Again by the Young inequality, we have

$$2(g, \Delta u) \le \frac{m}{2}|\Delta u|^2 + \frac{2}{m}|g|^2.$$

Together with (2.2), (3.40) becomes

$$\frac{d}{dt}\|z\|_{\mathcal{V}}^2 + m|\Delta u|^2 + 2(((\eta^t, (\eta^t)')))_\mu \le \Theta, \tag{3.42}$$

where we have used the notation $\Theta = \frac{2}{m}f_{2p-1}^2|\Omega| + d_0|\nabla u|^2 + \frac{2}{m}|g|^2$, which belongs to $L^1(\tau, T)$. Under the suitable spatial regularity assumptions on η, integration by parts in time and using (h_1), we obtain

$$(((\eta^t, (\eta^t)')))_\mu = -\int_0^\infty \mu'(s)|\Delta\eta^t(s)|^2 ds \ge 0.$$

Therefore, the term $2(((\eta^t, (\eta^t)')))_\mu$ in (3.42) can be neglected, we integrate (3.42) between τ and t, where $t \in (\tau, T)$, which leads to

$$\|z(t)\|_{\mathcal{V}}^2 + m\int_\tau^t |\Delta u(s)|^2 ds \le \|z(\tau)\|_{\mathcal{V}}^2 + \int_\tau^t \Theta(s)ds. \tag{3.43}$$

From the above estimation, we conclude that

$$\begin{aligned} &u \in L^\infty(\tau, T, V) \cap L^2(\tau, T; D(A)); \\ &\eta \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; D(A))). \end{aligned}$$

Concerning the assertion (ii) of this theorem, the continuity of u follows again using a slightly modified version of [19, Lemma III.1.2]. The continuity of η can be proved mimicking the idea of the proof of Step 4 of (i), with $D(A)$ in place of V. The proof of this theorem is complete. □

References

[1] T. Caraballo, M. J. Garrido-Atienza, B. Schmalfuß, J. Valero, Global attractor for a non-autonomous integro-differential equation in materials with memory, *Nonlinear Analysis* 73 (2010), 183-201.

[2] T. Caraballo, J. Real, Attractors for 2D-Navier-Stokes models with delays, *J. Differential Equations* 205 (2004), 271-297.

[3] M. Chipot, T. Savistka, Nonlocal p-Laplace equations depending on the L^p norm of the gradient, *Adv. Differ. Equ.* 19 (2014), 997-1020.

[4] M. Chipot, V. Valente, G. V. Caffarelli, Remarks on a nonlocal problem involving the Dirichlet energy, *Rend. Semin. Mat. Univ. Padova* 110 (2003), 199-220.

[5] M. Chipot, J. F. Rodrigues, On a class of nonlocal nonlinear elliptic problems, *RAIRO Modél. Math. Anal. Numér.* 26 (1992), 447-467.

[6] M. Conti, V. Pata, M. Squassina, Singular limit of differential systems with memory, *Indiana U. Math. J.* 1 (2006), 169-215.

[7] C. M. Dafermos, Asymptotic stability in viscoelasticity, *Arch. Rational Mech. Anal.* 37 (1970), 297-308.

[8] M. Fabrizio, C. Giorgi, V. Pata, A new approach to equations with memory, *Arch. Rational Mech. Anal.* 198(2010), 189-232.

[9] J. Furter, M. Grinfeld, Local vs. nonlocal interactions in population dynamics, *J. Math. Biol.* 27 (1989), 65-80.

[10] M. Grasselli, V. Pata, Uniform attractors of nonautonomous dynamical systems with memory, Evolution Equations, Semigroups and Functional Analysis, *Progr. Nonlinear Differential Equations Appl.* 50 (2002), 155-178.

[11] S. Gatti, M. Grasselli, V. Pata, Exponential attractors for a phased-field model with memory and quadratic nonlinearities, *Indiana Univ. Math. J.* 53 (2004), 719-753.

[12] C. Giorgi, Vittorino Pata, A. Marzochi, Asymptotic behavior of a semilinear problem in heat conduction with memory, *Nonlinear Differ. Equ. Appl.* 5 (1998), 333-354.

[13] J. K. Hale, S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Applied Mathematical Sciences, 99, Springer-Verlag, New York, 1993.

[14] O. A. Ladyzhenskaya, *Attractors for Semigroups and Evolution Equations*, Cambridge University Press, Cambridge, New York, 1991.

[15] Y. J. Li, Y. J. Wang, The existence and asymptotic behavior of solutions to fractional stochastic evolution equations with infinite delay, *J. Differential Equations* 266 (2019), 3514-3558.

[16] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Gauthier-Villar, Paris, 1969.

[17] V. Pata, A. Zucchi, Attractors for a damped hyperbolic equation with linear memory, *Adv. Math. Sci. Appl.* 11 (2001), 505-529.

[18] J. Robinson, *Infinite Dimensional Dynamical Systems*, Cambridge University Press, Cambridge, 2001.

[19] R. Temam, *Navier-Stokes Equations*, Theory and Numerical Analysis, North-Holland, Amsterdam, 1979.

[20] J. H. Xu, Z. C. Zhang, T. Caraballo, Non-autonomous nonlocal partial differential equations with delay and memory, *J. Differential Equations* 270 (2021), 505-546.

[21] J. H. Xu, T. Caraballo, Asymptotic behavior of non-local partial differential equations with long time memory, submitted (2021).