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New criteria for global asymptotic stability of linear neutral differential equations by a fixed point approach

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Abstract New criteria ensuring global asymptotic stability of the zero solution for a class of linear neutral differential equations in C^1 are proved, by using two auxiliary functions on a contraction condition. Necessary and sufficient conditions for the stability of our equation which also improves recent results on this field are shown. Finally, an example is provided to illustrate the feasibility and advantage of our results.

Keywords Contraction mapping principle · Asymptotic stability · Neutral differential equations

Mathematics Subject Classification (2010) 34K20 · 34K13 · 92B20

1 Introduction

Theory and applications of functional differential equations with delay have been studied by many authors (see, for example, [19, 20] and the references therein). More recently, researchers have paid special attention to the study of equations in which the delay argument occurs in the derivative of the state variable as well as in the independent variable, the so-called neutral differential equations, see for instance [18, 19]. Practical examples of neutral delay differential systems include biological models of single species growth [24], distributed networks containing lossless transmission lines [11], population ecology [20], and other engineering systems [21]. In particular, qualitative analysis such as

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stability of solutions of neutral differential equations has received much attention over the last few decades. We refer to [1–4, 13, 14, 17, 22, 23, 25] for some recent work on the subject of stability of neutral equations.

Lyapunov's direct method has been successfully used to investigate stability properties of a wide variety of differential equations. Nevertheless, the application of this method to problems of stability in differential and integro-differential equations with delays has encountered serious obstacles when the delay is unbounded or when the equation has unbounded terms [5]-[7]. In recent years, several investigators have analyzed stability by using a new technique. Particularly, Burton and other researchers studied stability using various fixed point theorems which overcame the difficulties encountered in the study of stability by means of Lyapunov's direct method. We can refer to [5, 15, 25, 26] and [28]-[32] for more details. It turns out that the fixed point method is becoming a powerful technique in dealing with stability problems for stochastic differential equations with delays [12, 16, 22, 32].

Previously, almost all scholars, who used fixed point theory to study the asymptotic stability of zero solutions of nonlinear neutral differential equations with variable delays, imposed that c must be differentiable and τ twice differentiable and $\tau'(t) \neq 1$ for $t \geq 0$. As distinguished from this line, in [17], Jin and Luo stated a sufficient and necessary condition for the asymptotic stability in the space C^0 of the following equation (1.1), by using a fixed point method of continuous functions:

$$x'(t) = -a(t)x(t) + c(t)x'(t-\tau(t)) - b(t)\,x(t-\tau(t)), t \geq 0. \tag{1.1}$$

In [26], Raffoul obtained sufficient conditions for the asymptotic stability of the zero solution, under appropriate conditions, to the following equation

$$x'(t) = -a(t)x(t) + c(t)x'(t-\tau(t)) + Q\left(x(t), x(t-\tau(t))\right), t \geq 0. \tag{1.2}$$

by using the contraction mapping theorem.

On the other hand, Liu and Yang [23] were the first to establish a necessary and sufficient condition for the global asymptotic stability in C^1 of the zero solutions of the following nonlinear neutral functional differential equation

$$x'(t) = -a(t)x(t) + c(t)x'(t-\tau_1(t)) + Q\left(t, x(t), x(t-\tau_2(t))\right), \tag{1.3}$$

by the fixed point theory, where Q is a Lipschitz continuous function with the respect to x.

Very recently, by the same method of Jin and Luo [23], Ardjouni and Djoudi [1] improved the results of Liu *et al.* [23] to the generalized nonlinear neutral differential equation with variable delays of the form

$$\begin{aligned} x'(t) = &-a(t)x(t) + f\left(t, x(t-\tau_1(t)), ..., x(t-\tau_N(t))\right) \\ &+h\left(t, x'(t-\tau_1(t)), ..., x'(t-\tau_N(t))\right), \end{aligned} \tag{1.4}$$

where $f(t,0,...,0) = h(t,0,...,0) = 0$ and there exist bounded functions $b_i, c_i \in C([0,\infty),(0,\infty))$, $a \in C(\mathbb{R}^+,\mathbb{R})$, such that

$$|f(t,x_1,...,x_N) - f(t,y_1,...,y_N)| \leq \sum_{i=1}^{N} b_i(t)|x_i - y_i|, \tag{1.5}$$

$$|h(t,x_1',...,x_N') - h(t,y_1',...,y_N')| \leq \sum_{i=1}^{N} c_i(t)|x_i' - y_i'|, \tag{1.6}$$

for all $x_i, y_i \in \mathbb{R}, i = 1,...,N$. More precisely, the following result was established.

Theorem A. (Ardjouni and Djoudi [1]) *Suppose that assumptions (1.5),(1.6) hold, and there exists a constant* $\eta \in (0,1)$ *such that for* $t \geq t_0$,

$$\liminf_{t\to\infty} \int_{t_0}^{t} a(s)\,ds > -\infty, \tag{1.4}$$

and

$$\int_{t_0}^{t} e^{-\int_s^t a(u)du} \sum_{i=1}^{N} (|b_i(s)| + |c_i(s)|)\,ds \leq \eta, \tag{1.5}$$

$$|a(t)| \int_{t_0}^{t} e^{-\int_s^t a(u)du} \sum_{i=1}^{N} (|b_i(s)| + |c_i(s)|)\,ds + \sum_{i=1}^{N} (|b_i(t)| + |c_i(t)|) \leq \eta. \tag{1.6}$$

Then the zero solution of equation (1.4) *is globally asymptotically stable in* C^1 *if and only if*

$$\int_0^t a(s)ds \to \infty \text{ as } t \to \infty. \tag{1.7}$$

By using the contraction mapping principle, the authors established some new conditions to ensure that the zero solution of equation (1.4) is globally asymptotically stable in C^1. Unlike most research methods, these conditions do not require a quadratic differentiability of delay τ and $\tau'(t) \neq 1$ for $t \geq 0$. In addition, in [1,2,28], they all studied the global asymptotic stability in C^1.

Inspired by the application of the fixed point method mentioned above, in this paper, we will state some new conditions, which make stability conditions more feasible and the results in [1,17,23,26] are improved. By using two auxiliary functions g and p to construct a contraction mapping on a complete metric space S, defined below, which may depend on the initial condition φ, we obtain Theorem 1.3 which will be proved in Section 2. Namely, a necessary and sufficient condition ensuring the global asymptotic stability in C^1 is proved. In addition, an example is eventually analyzed to illustrate the effectiveness of the proved results.

Notice that the condition (1.6) in Theorem A is mainly dependent of the constraint

$$\sum_{i=1}^{N} |b_i(t)| + \sum_{i=1}^{N} |c_i(t)| < 1.$$

However, there are some interesting examples where the constraint is not satisfied. It is our aim in this paper to remove this constraint condition and consider the global stability in C^1 of the special case of (1.4) when

$$f\left(t, x(t-\tau_1(t)), ..., x(t-\tau_N(t))\right) = \sum_{i=1}^{N} b_i(t) x(t-\tau_i(t)),$$

and

$$h\left(t, x'(t-\tau_1(t)), ..., x'(t-\tau_N(t))\right) = \sum_{i=1}^{N} c_i(t) x'(t-\tau_i(t)).$$

In particular, we introduce two auxiliary continuous functions g and p to define an appropriate mapping, and present new criteria for the global asymptotic stability of equation (1.2) which can be applied to the case

$$\sum_{i=1}^{N} |b_i(t)| + \sum_{i=1}^{N} |c_i(t)| \geq 1,$$

as well.

2 Preliminaries

Let us consider the following class of neutral differential equations with variable delays,

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{N} b_i(t) x'(t-\tau_i(t)) + \sum_{i=1}^{N} c_i(t) x(t-\tau_i(t)), t \geq t_0, \tag{2.1}$$

denote $x(t) \in \mathbb{R}$ the solution to (2.1) with the initial condition

$$x(t) = \varphi(t) \text{ for } t \in [m(t_0), t_0],$$

where $\varphi \in C([m(t_0), t_0], \mathbb{R})$. We assume that $a, b_i, c_i \in C(\mathbb{R}^+, \mathbb{R})$, $\tau_i \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfy

$$t - \tau_i(t) \to \infty \text{ as } t \to \infty, i = 1, 2, ..., N, \tag{2.2}$$

and for each $t_0 \geq 0$, $m_i(t_0) = \inf\{t - \tau_i(t), t \geq t_0\}$, $m(t_0) = \min\{m_i(t_0), i = 1, 2, ..., N\}$.

For each $t_0 \in [0, \infty)$, denote $C^1_{t_0} = C^1([m(t_0), t_0], \mathbb{R})$ with the norm defined by

$$|x|_{t_0} := \max_{t \in [m(t_0), t_0]} \{|x(t)|, |x'(t)|\},$$

for $x \in C^1_{t_0} = C^1([m(t_0), t_0], \mathbb{R})$. In addition, denote Φ_{t_0}, where

$$\Phi_{t_0} := \left\{ \varphi \in C^1_{t_0} : \varphi'_-(t_0) = -a(t)\varphi(t_0) + \sum_{i=1}^{N} b_i(t_0)\varphi'(t_0 - \tau_i(t_0)) \right.$$
$$\left. + \sum_{i=1}^{N} c_i(t_0)\varphi(t_0 - \tau_i(t_0)) \right\}.$$

For each $t_0 \in [0, \infty)$, we choose initial functions for equation (2.1) of the type $\varphi \in \Phi_{t_0}$.

Let us recall the definitions of stability that will be used in the next section.

Definition 1.1. *For each initial value $(t_0, \varphi) \in [0, \infty) \times \Phi_{t_0}$, x is said to be a solution of equation* (2.1) *through (t_0, φ) if $x \in C^1([m(t_0), \infty), \mathbb{R})$ satisfies equation (2.1) on $[t_0, \infty)$ and $x(t) = \varphi(t)$ for $t \in [m(t_0), t_0]$. Such a solution will be denoted by $x(t) = x(t, t_0, \varphi)$.*

Definition 1.2. *i) The Zero solution of equation* (2.1) *is said to be stable in C^1 if, for any $t_0 \in [0, \infty)$, $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon, t_0)$ such that $\varphi \in \Phi_{t_0}$ and $|\varphi|_{t_0} < \delta$ implies*

$$\max_{s \in [m(t_0), t_0]} \{|x(s, t_0, \varphi)|, |x'(s, t_0, \varphi)|\} < \varepsilon \text{ for } t \geq t_0.$$

ii) The zero solution of equation (2.1) *is said to be globally asymptotically stable in C^1 if it is stable in C^1, and for any $t_0 \in [0, \infty), \varphi \in \Phi_{t_0}$ implies*

$$\lim_{t \to \infty} x(t, t_0, \varphi) = \lim_{t \to \infty} x'(t, t_0, \varphi) = 0.$$

At light of the previous definition of solution of equation (2.1), it is clear that the conditions imposed on the initial functions are sensible.

3 Stability by contraction mapping

As we mentioned previously, the results of this work extend and improve previously known results. More exactly, we will consider a linear scalar neutral delay differential equation with variable delays and give new conditions to ensure that the zero solution is global asymptotically stable in C^1 by means of fixed point theory. By weakening the assumptions on the neutral coefficient c_i and delays τ_i, $\tau_i'(t) \neq 1, \forall t \geq 0$, and by obtaining some criteria, easier to check in applications, which does not satisfy the constraint

$$\sum_{i=1}^{N} |b_i(t)| + \sum_{i=1}^{N} |c_i(t)| < 1.$$

However, the mathematical analysis used in this research to construct the mapping to employ fixed point theorem is different from that of [1]. The results of this article are new and they extend and improve previously known results.

To the best of our knowledge, there are few authors who have used the fixed point theorem to prove the existence and uniqueness of solution and the stability of trivial equilibrium of several special cases of (*2.1*) all at once [3, 29,31]. In our study, as we are mainly concerned with the stability analysis of our model, we will assume that there exists a unique solution of (*2.1*) globally defined in time.

Theorem 3.1. *Consider the neutral delay differential equation (2.1) and suppose the following conditions are satisfied:*
H1) Suppose there exists a bounded function $p : [m(t_0), \infty[\to (0, \infty)$ *with* $p(t) = 1$ *for* $t \in [m(t_0), t_0]$ *such that* $p'(t)$ *exists for all* $t \in [m(t_0), \infty[$,
H2) there exists an arbitrary bounded continuous function $g \in C([m(t_0), \infty[, \mathbb{R}^+)$ *and*

$$\liminf_{t\to\infty} \int_{t_0}^{t} g(s)\, ds > -\infty, \tag{3.1}$$

H3) there exists a constant $\eta \in (0, \frac{1}{2})$ *such that for* $t \geq t_0$,

$$\begin{aligned} r_1(t) &:= \int_{t_0}^{t} e^{-\int_s^t g(s)ds} \left[\left| g(s) - \left(a(s) + \frac{p'(s)}{p(s)} \right) \right| \right. \\ &+ \sum_{i=1}^{N} \left| \frac{b_i(s)\, p(s - \tau_i(s))}{p(s)} \right| \\ &+ \left. \sum_{i=1}^{N} \left| \frac{b_i(s)\, p'(s - \tau_i(s))}{p(s)} \right| + \sum_{i=1}^{N} \left| \frac{c_i(s)\, p(s - \tau_i(s))}{p(s)} \right| \right] ds \leq \eta, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} r_2(t) &:= |g(t)| \int_{t_0}^{t} e^{-\int_s^t g(u)du} \left\{ \left| g(s) - \left(a(s) + \frac{p'(s)}{p(s)} \right) \right| \right. \\ &+ \sum_{i=1}^{N} \left| \frac{b_i(s)\, p(s - \tau_i(s))}{p(s)} \right| + \sum_{i=1}^{N} \left| \frac{b_i(s)\, p'(s - \tau_i(s))}{p(s)} \right| \\ &+ \left. \sum_{i=1}^{N} \left| \frac{c_i(s)\, p(s - \tau_i(s))}{p(s)} \right| \right\} ds \\ &+ \left| g(t) - \left(a(t) + \frac{p'(t)}{p(t)} \right) \right| + \sum_{i=1}^{N} \left| \frac{b_i(t)\, p(t - \tau_i(t))}{p(t)} \right| \\ &+ \sum_{i=1}^{N} \left| \frac{b_i(t)\, p'(t - \tau_i(t))}{p(t)} \right| + \sum_{i=1}^{N} \left| \frac{c_i(t)\, p(t - \tau_i(t))}{p(t)} \right| \leq \eta. \end{aligned} \tag{3.3}$$

Then the zero solution of equation (2.1) *is globally asymptotically stable in* C^1 *if and only if*

$$\int_0^t g(s)\, ds \to \infty \text{ as } t \to \infty. \tag{3.4}$$

Proof. ($\Leftarrow$:) First, suppose that $\int_0^t g(s)ds \to \infty$ as $t \to \infty$. For each $t_0 \in [0, \infty)$, we define S as the following space

$$S = \left\{ z \in C^1\left([m(t_0), \infty), \mathbb{R}\right) : \lim_{t\to\infty} z(t) = \lim_{t\to\infty} z'(t) = 0 \right\},$$

with the metric defined by

$$\|z\| := \max_{t\in[m(t_0),\infty)} \left\{ |z(t)|, |z'(t)| \right\}.$$

Then S is a complete metric space.
For any initial function $\varphi \in \Phi_{t_0}$, let

$$D_\varphi^l = \left\{ z \in S : z(t) = \varphi(t) \text{ for } t \in [m(t_0), t_0] \text{ and } \max_{t\geq t_0} \left\{ |z(t)|, |z'(t)| \right\} \leq l \right\},$$

which is a nonempty, closed convex subset of S.

The technique for constructing a contraction mapping comes from an idea in [31]. Indeed, let $z(t) = \varphi(t)$ on $t \in [m(t_0), t_0]$ and for $t \geq t_0$

$$x(t) = p(t)z(t). \tag{3.5}$$

Replacing (3.5) into (2.1), we have

$$\begin{aligned} z'(t) = &- \left(a(t) + \frac{p'(t)}{p(t)} \right) z(t) \\ &+ \sum_{i=1}^N \frac{b_i(t)\, p(t - \tau_i(t))}{p(t)} z'(t - \tau_i(t)) \\ &+ \sum_{i=1}^N \frac{c_i(t)\, p(t - \tau_i(t)) + b_i(t)\, p'(t - \tau_i(t))}{p(t)} z(t - \tau_i(t)). \end{aligned} \tag{3.6}$$

If z satisfies (3.6) then it can be verified that x satisfies (2.1).
Since p is a positive bounded function, to obtain global asymptotic stability of the zero solution of (2.1), it remains to prove that the zero solution of (3.6) is globally asymptotically stable in C^1.

Multiplying both sides of (3.6) by $e^{\int_0^t g(u)du}$ and integrating from t_0 to t,

$$\begin{aligned} &\int_{t_0}^t \left[e^{\int_0^s g(u)du} z(s) \right]' ds \\ &= \int_{t_0}^t e^{\int_0^s g(u)du} \left(g(s) - \left(a(s) + \frac{p'(s)}{p(s)} \right) \right) z(s) ds \\ &\quad + \int_{t_0}^t e^{\int_0^s g(u)du} \sum_{i=1}^N \frac{b_i(s)\, p(s - \tau_i(s))}{p(s)} z'(s - \tau_i(s)) ds \\ &\quad + \int_{t_0}^t e^{\int_0^s g(u)du} \sum_{i=1}^N \frac{c_i(s)\, p(s - \tau_i(s)) + b_i(s)\, p'(s - \tau_i(s))}{p(s)} z(s - \tau_i(s)) ds. \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}
&z(t)e^{\int_0^t g(u)du} \\
&\quad= \varphi(t_0) + \int_{t_0}^{t} e^{\int_0^s g(u)du}\left(g\left(s\right) - \left(a\left(s\right) + \frac{p'(s)}{p(s)}\right)\right) z(s)ds \\
&\qquad+ \int_{t_0}^{t} e^{\int_0^s g(u)du}\sum_{i=1}^{N}\frac{b_i\left(s\right)p\left(s-\tau_i\left(s\right)\right)}{p(s)}z'\left(s-\tau_i\left(s\right)\right)ds \\
&\qquad+ \int_{t_0}^{t} e^{\int_0^s g(u)du}\sum_{i=1}^{N}\frac{c_i\left(s\right)p\left(s-\tau_i\left(s\right)\right) + b_i\left(s\right)p'\left(s-\tau_i\left(s\right)\right)}{p(s)}z\left(s-\tau_i\left(s\right)\right)ds.
\end{aligned}$$

Dividing both sides of the above equation by $e^{\int_0^t g(s)ds}$, we obtain

$$\begin{aligned}
z(t) &= e^{-\int_{t_0}^t g(s)ds}\varphi(t_0) + \int_{t_0}^{t} e^{-\int_s^t g(u)du}\left(g\left(s\right) - \left(a\left(s\right) + \frac{p'(s)}{p(s)}\right)\right) z(s)ds \\
&\quad+ \int_{t_0}^{t} e^{-\int_s^t g(u)du}\sum_{i=1}^{N}\frac{b_i\left(s\right)p\left(s-\tau_i\left(s\right)\right)}{p(s)}z'\left(s-\tau_i\left(s\right)\right)ds \\
&\quad+ \int_{t_0}^{t} e^{-\int_s^t g(u)du}\sum_{i=1}^{N}\frac{c_i\left(s\right)p\left(s-\tau_i\left(s\right)\right) + b_i\left(s\right)p'\left(s-\tau_i\left(s\right)\right)}{p(s)} \\
&\quad\times z\left(s-\tau_i\left(s\right)\right)ds.
\end{aligned}$$

Clearly, $\Psi\left(z\right) : \mathbb{R} \to \mathbb{R}$ is continuous with $\left(\Psi z\right)\left(t\right) = \varphi\left(t\right)$ for $t \in \left[m\left(t_0\right), t_0\right]$, and for $t \geq t_0$,

$$\begin{aligned}
\left(\Psi z\right)\left(t\right) &= e^{-\int_{t_0}^t g(s)ds}\varphi(t_0) + \int_{t_0}^{t} e^{-\int_s^t g(u)du}\left(g\left(s\right) - \left(a\left(s\right) + \frac{p'(s)}{p(s)}\right)\right) z(s)ds \\
&\quad+ \int_{t_0}^{t} e^{-\int_s^t g(u)du}\sum_{i=1}^{N}\frac{b_i\left(s\right)p\left(s-\tau_i\left(s\right)\right)}{p(s)}z'\left(s-\tau_i\left(s\right)\right)ds \\
&\quad+ \int_{t_0}^{t} e^{-\int_s^t g(u)du}\sum_{i=1}^{N}\frac{c_i\left(s\right)p\left(s-\tau_i\left(s\right)\right) + b_i\left(s\right)p'\left(s-\tau_i\left(s\right)\right)}{p(s)} \\
&\quad\times z\left(s-\tau_i\left(s\right)\right)ds. \qquad (3.7)
\end{aligned}$$

Initially, we show that, $\Psi : D^l_\varphi \to D^l_\varphi$. In view of (3.7), we can derive, for $t \geq t_0$,

$$\begin{aligned}
(\Psi z)'(t) &= -\varphi(t_0) g(t) e^{-\int_{t_0}^t g(s)ds} \\
&\quad + \left(g(t) - \left(a(t) + \frac{p'(t)}{p(t)} \right) \right) z(t) \\
&\quad + \sum_{i=1}^N \frac{b_i(t) p(t - \tau_i(t))}{p(t)} z'(t - \tau_i(t)) \\
&\quad + \sum_{i=1}^N \frac{c_i(t) p(t - \tau_i(t)) + b_i(t) p'(t - \tau_i(t))}{p(t)} z(t - \tau_i(t)) \\
&\quad - g(t) \int_{t_0}^t e^{-\int_s^t g(u)du} \left(g(s) - \left(a(s) + \frac{p'(s)}{p(s)} \right) \right) z(s) ds \\
&\quad - g(t) \int_{t_0}^t e^{-\int_s^t g(u)du} \sum_{i=1}^N \frac{b_i(s) p(s - \tau_i(s))}{p(s)} z'(s - \tau_i(s)) ds \\
&\quad - g(t) \int_{t_0}^t e^{-\int_s^t g(u)du} \sum_{i=1}^N \frac{c_i(s) p(s - \tau_i(s)) + b_i(s) p'(s - \tau_i(s))}{p(s)} \\
&\quad \times z(s - \tau_i(s)) ds.
\end{aligned}$$

Thus

$$\begin{aligned}
(\Psi z)'(t) &= -g(t)(\Psi z)(t) + \left(g(t) - \left(a(t) + \frac{p'(t)}{p(t)} \right) \right) z(t) \\
&\quad + \sum_{i=1}^N \frac{b_i(t) p(t - \tau_i(t))}{p(t)} z'(t - \tau_i(t)) \\
&\quad + \sum_{i=1}^N \frac{c_i(t) p(t - \tau_i(t)) + b_i(t) p'(t - \tau_i(t))}{p(t)} z(t - \tau_i(t)).
\end{aligned} \tag{3.8}$$

By the definition of Φ_{t_0}, (3.8) yields

$$\begin{aligned}
(\Psi z)'_+(t_0) &= -g(t_0) \varphi(t_0) \\
&\quad + \left(g(t_0) - \left(a(t_0) + \frac{p'(t_0)}{p(t_0)} \right) \right) z(t_0) \\
&\quad + \sum_{i=1}^N \frac{b_i(t_0) p(t_0 - \tau_i(t_0))}{p(t_0)} z'(t_0 - \tau_i(t_0)) \\
&\quad + \sum_{i=1}^N \frac{c_i(t_0) p(t_0 - \tau_i(t_0)) + b_i(t_0) p'(t_0 - \tau_i(t_0))}{p(t_0)} \\
&\quad \times z(t_0 - \tau_i(t_0)) \\
&= \varphi'_-(t_0).
\end{aligned}$$

Hence, $\Psi z \in C^1([m(t_0), \infty))$ for $z \in D^l_\varphi$.

Next, we verify that $\max_{t \geq t_0} \left\{ \left|(\Psi z)'(t)\right|, \left|(\Psi z)(t)\right| \right\} < l$. Let

$$K = \sup_{t \geq t_0} e^{-\int_{t_0}^t g(s)ds} \text{ and } A = \sup_{t \geq t_0} \{|g(t)|\}.$$

From (3.4) and (3.1), $K, A \in [0, \infty)$. Let φ be a small bounded initial function with $|\varphi|_{t_0} < \delta_0$, where $\delta_0 > 0$ satisfies

$$\delta_0 < l \min\left\{1, \frac{1-\eta}{K}, \frac{1-2\eta}{KA}\right\}. \tag{3.9}$$

Let $z \in D^l_\varphi$, then $\max_{t \geq t_0} \{|z'(t)|, |z(t)|\} \leq l$. It follows from (3.7) and condition (3.2), (3.9) that

$$\begin{aligned}
|(\Psi z)(t)| \leq{}& e^{-\int_{t_0}^t g(s)ds} |\varphi(t_0)| + \int_{t_0}^t e^{-\int_s^t g(u)du} \left| g(s) - \left(a(s) + \frac{p'(s)}{p(s)}\right)\right| |z(s)|\, ds \\
&+ \int_{t_0}^t e^{-\int_s^t g(u)du} \sum_{i=1}^N \left| \frac{b_i(s)\, p(s-\tau_i(s))}{p(s)} \right| |z'(s-\tau_i(s))|\, ds \\
&+ \int_{t_0}^t e^{-\int_s^t g(u)du} \sum_{i=1}^N \left[\left| \frac{b_i(s)\, p'(s-\tau_i(s))}{p(s)} \right| + \left| \frac{c_i(s)\, p(s-\tau_i(s))}{p(s)} \right| \right] \\
&\times |z(s-\tau_i(s))|\, ds \\
\leq{}& K\delta_0 + \eta l < l.
\end{aligned}$$

Now, (3.8), (3.2) , (3.3) and (3.9) imply that

$$
\begin{aligned}
&\left|(\Psi z)'(t)\right| \\
&\leq |g(t)|\, e^{-\int_{t_0}^{t} g(s)ds} |\varphi(t_0)| \\
&\quad + |g(t)| \int_{t_0}^{t} e^{-\int_s^t g(u)du} \left[\left| g(s) - \left(a(s) + \frac{p'(s)}{p(s)} \right) \right| |z(s)|\, ds \right. \\
&\quad + \sum_{i=1}^{N} \left| \frac{b_i(s)\, p(s-\tau_i(s))}{p(s)} \right| |z'(s-\tau_i(s))|\, ds \\
&\quad \left. + \left[\sum_{i=1}^{N} \left| \frac{c_i(s)\, p(s-\tau_i(s))}{p(s)} \right| + \sum_{i=1}^{N} \left| \frac{b_i(s)\, p'(s-\tau_i(s))}{p(s)} \right| \right] |z(s-\tau_i(s))| \right] ds \\
&\quad + \left| g(t) - \left(a(t) + \frac{p'(t)}{p(t)} \right) \right| |z(t)| \\
&\quad + \sum_{i=1}^{N} \left| \frac{b_i(t)\, p(t-\tau_i(t))}{p(t)} \right| |z'(t-\tau_i(t))| \\
&\quad + \left[\sum_{i=1}^{N} \left| \frac{c_i(t)\, p(t-\tau_i(t))}{p(t)} \right| + \sum_{i=1}^{N} \left| \frac{b_i(t)\, p'(t-\tau_i(t))}{p(t)} \right| \right] |z(t-\tau_i(t))| \\
&\leq KA\delta_0 \\
&\quad + l\, |g(t)| \int_{t_0}^{t} e^{-\int_s^t g(u)du} \left(\left| g(s) - \left(a(s) + \frac{p'(s)}{p(s)} \right) \right| \right. \\
&\quad + \sum_{i=1}^{N} \left| \frac{b_i(s)\, p(s-\tau_i(s))}{p(s)} \right| \\
&\quad \left. + \left[\sum_{i=1}^{N} \left| \frac{c_i(s)\, p(s-\tau_i(s))}{p(s)} \right| + \sum_{i=1}^{N} \left| \frac{b_i(s)\, p'(s-\tau_i(s))}{p(s)} \right| \right] \right) ds \\
&\quad + l \left| g(t) - \left(a(t) + \frac{p'(t)}{p(t)} \right) \right| + l \sum_{i=1}^{N} \left| \frac{b_i(t)\, p(t-\tau_i(t))}{p(t)} \right| \\
&\quad + l \left[\sum_{i=1}^{N} \left| \frac{c_i(t)\, p(t-\tau_i(t))}{p(t)} \right| + \sum_{i=1}^{N} \left| \frac{b_i(t)\, p'(t-\tau_i(t))}{p(t)} \right| \right] \Bigg) \\
&\leq KA\delta_0 + l \left| g(t) - \left(a(t) + \frac{p'(t)}{p(t)} \right) \right| + l \sum_{i=1}^{N} \left| \frac{b_i(t)\, p(t-\tau_i(t))}{p(t)} \right| \\
&\quad + l \left[\sum_{i=1}^{N} \left| \frac{c_i(t)\, p(t-\tau_i(t))}{p(t)} \right| + \sum_{i=1}^{N} \left| \frac{b_i(t)\, p'(t-\tau_i(t))}{p(t)} \right| \right] + \eta l \\
&\leq KA\delta_0 + 2\eta l < l,
\end{aligned}
$$

by the choice of δ_0. This implies, $\max_{t \geq t_0} \left\{ |(\Psi z)(t)|, \left|(\Psi z)'(t)\right| \right\} < l$. Now we show that $(\Psi z)(t) \to 0$ as $t \to \infty$.

For $z \in D_{\varphi}^{l}$,

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} z'(t) = 0.$$

Note that $\lim_{t\to\infty} (t - \tau_i(t)) = \infty$, $i = 1, 2..., N$. Therefore, for any $\varepsilon > 0$, there exists $T_1 > 0$ such that for $t \geq T_1$,

$$\max\{|z(t)|, |z'(t-\tau_i(t))|, |z(t-\tau_i(t))|\} \leq \varepsilon,\ i = 1, 2..., N, \tag{3.10}$$

and the fact $z \in D_{\varphi}^{l}$ implies that $\max\{|z(t)|, |z'(t)|\} < l$ for all $t \geq t_0$. It follows from (3.2), (3.3), (3.7) and (3.10) that for $t > T_1$,

$$\begin{aligned}
&|(\Psi z)(t)| \\
&\quad \leq e^{-\int_{t_0}^{t} g(s)ds} |\varphi(t_0)| + \int_{t_0}^{T_1} e^{-\int_s^t g(u)du} \left[\left| g(s) - \left(a(s) + \frac{p'(s)}{p(s)} \right) \right| |z(s)| \right. \\
&\qquad + \sum_{i=1}^{N} \left| \frac{b_i(s)\, p(s-\tau_i(s))}{p(s)} \right| |z'(s-\tau_i(s))| \\
&\qquad + \left[\sum_{i=1}^{N} \left| \frac{c_i(s)\, p(s-\tau_i(s))}{p(s)} \right| + \sum_{i=1}^{N} \left| \frac{b_i(s)\, p'(s-\tau_i(s))}{p(s)} \right| \right] \\
&\qquad \left. \times |z(s-\tau_i(s))| \right] ds \\
&\qquad + \int_{T_1}^{t} e^{-\int_s^t g(u)du} \left[\left| g(s) - \left(a(s) + \frac{p'(s)}{p(s)} \right) \right| |z(s)| \right. \\
&\qquad + \sum_{i=1}^{N} \left| \frac{b_i(s)\, p(s-\tau_i(s))}{p(s)} \right| |z'(s-\tau_i(s))| \\
&\qquad + \left[\sum_{i=1}^{N} \left| \frac{c_i(s)\, p(s-\tau_i(s))}{p(s)} \right| + \sum_{i=1}^{N} \left| \frac{b_i(s)\, p'(s-\tau_i(s))}{p(s)} \right| \right] \\
&\qquad \left. \times |z(s-\tau_i(s))| \right] ds \\
&\quad \leq e^{-\int_{t_0}^{t} g(u)du} \left\{ |\varphi(t_0)| + \int_{t_0}^{T_1} e^{-\int_{t_0}^{s} g(u)du} \left[\left| g(s) - \left(a(s) + \frac{p'(s)}{p(s)} \right) \right| |z(s)| \right. \right. \\
&\qquad + \sum_{i=1}^{N} \left| \frac{b_i(s)\, p(s-\tau_i(s))}{p(s)} \right| |z'(s-\tau_i(s))| \\
&\qquad \left. \left. + \sum_{i=1}^{N} \left[\left| \frac{c_i(s)\, p(s-\tau_i(s))}{p(s)} \right| + \left| \frac{b_i(s)\, p'(s-\tau_i(s))}{p(s)} \right| \right] |z(s-\tau_i(s))| \right\} ds \\
&\qquad + \eta\varepsilon.
\end{aligned}$$

On the other hand, by using condition (3.4), there exists $T \geq T_1$ such that, for $t \geq T$, we have

$$
\begin{aligned}
&le^{-\int_{t_0}^{t} g(s)ds} \left\{ |\varphi(t_0)| + \int_{t_0}^{T} e^{-\int_{t_0}^{s} g(u)du} \left[\left| g(s) - \left(a(s) + \frac{p'(s)}{p(s)} \right) \right| \right. \right. \\
&+ \sum_{i=1}^{N} \left| \frac{b_i(s)\, p(s-\tau_i(s))}{p(s)} \right| \\
&\left. \left. + \sum_{i=1}^{N} \left| \frac{c_i(s)\, p(s-\tau_i(s))}{p(s)} \right| + \sum_{i=1}^{N} \left| \frac{b_i(s)\, p'(s-\tau_i(s))}{p(s)} \right| \right\} ds \leq \varepsilon.
\end{aligned}
$$

This yields $\lim_{t\to\infty} (\Psi z)(t) = 0$ for $z \in D_\varphi^l$.

In addition, we have from (3.8) ,

$$
\begin{aligned}
\left| (\Psi z)'(t) \right| \leq |g(t)|\, |(\Psi z)(t)| &+ \left| g(t) - \left(a(t) + \frac{p'(t)}{p(t)} \right) \right| |z(t)| \\
&+ \sum_{i=1}^{N} \left| \frac{b_i(t)\, p(t-\tau_i(t))}{p(t)} \right| |z'(t-\tau_i(t))| \\
&+ \sum_{i=1}^{N} \left[\left| \frac{c_i(t)\, p(t-\tau_i(t))}{p(t)} \right| + \left| \frac{b_i(t)\, p'(t-\tau_i(t))}{p(t)} \right| \right] |z(t-\tau_i(t))| .
\end{aligned}
$$

This, together with (3.1) − (3.3), leads to $\lim_{t\to\infty} (\Psi z)'(t) = 0$ for $z \in D_\varphi^l$. Therefore, $\Psi z \in D_\varphi^l$ for $z \in D_\varphi^l$, *i.e.* $\Psi : D_\varphi^l \to D_\varphi^l$.

Now, we will show that $\Psi : D_\varphi^l \to D_\varphi^l$ is a contraction mapping. For any $z, y \in D_\varphi^l$, it follows from (3.2) , (3.3) , (3.7) that, for $t \in [t_0, \infty)$,

$$
\begin{aligned}
&|(\Psi z)(t) - (\Psi y)(t)| \\
&\quad \leq \int_{t_0}^{t} e^{-\int_{s}^{t} g(u)du} \left[\left| g(s) - \left(a(s) + \frac{p'(s)}{p(s)} \right) \right| \right. \\
&\qquad + \sum_{i=1}^{N} \left| \frac{b_i(s)\, p(s-\tau_i(s))}{p(s)} \right| + \sum_{i=1}^{N} \left| \frac{c_i(s)\, p(s-\tau_i(s))}{p(s)} \right| \\
&\qquad \left. + \sum_{i=1}^{N} \left| \frac{b_i(s)\, p'(s-\tau_i(s))}{p(s)} \right| \right] ds\, \|z - y\| \\
&\quad \leq \eta\, \|z - y\| .
\end{aligned} \tag{3.11}
$$

In addition, we can derive

$$
\begin{aligned}
&\left|(\Psi z)'(t)-(\Psi y)'(t)\right| \\
&\quad\leq\left\{|g(t)|\int_{t_0}^{t}e^{-\int_s^t g(u)du}\left\{\left|g(s)-\left(a(s)+\frac{p'(s)}{p(s)}\right)\right|\right.\right. \\
&\qquad+\sum_{i=1}^{N}\left|\frac{b_i(s)\,p(s-\tau_i(s))}{p(s)}\right| \\
&\qquad\left.+\sum_{i=1}^{N}\left|\frac{c_i(s)\,p(s-\tau_i(s))}{p(s)}\right|+\sum_{i=1}^{N}\left|\frac{b_i(s)\,p'(s-\tau_i(s))}{p(s)}\right|\right\}ds \\
&\qquad+\left|g(t)-\left(a(t)+\frac{p'(t)}{p(t)}\right)\right|+\sum_{i=1}^{N}\left|\frac{b_i(t)\,p(t-\tau_i(t))}{p(t)}\right| \\
&\qquad+\sum_{i=1}^{N}\left|\frac{c_i(t)\,p(t-\tau_i(t))}{p(t)}\right| \\
&\qquad\left.+\sum_{i=1}^{N}\left|\frac{b_i(t)\,p'(t-\tau_i(t))}{p(t)}\right|\right\}\times\|z-y\| \\
&\quad\leq\eta\|z-y\|.
\end{aligned}
\tag{3.12}
$$

From (3.11) and (3.12), $\Psi : D^l_\varphi \to D^l_\varphi$ is a contraction mapping with constant η. Thanks to the contraction mapping principle (Smart [27, p. 2]), we deduce that Ψ possesses a unique fixed point z in D^l_φ which solves (3.6) through (t_0,φ), is bounded and tends to zero as t goes to infinity.

Referring to [5,14,26], except for the fixed point method, we know of another way to prove that solutions of (3.6) are stable. Let $\varepsilon>0$, by proceeding now in the opposite way as before, that is, choosing a fixed $l=\varepsilon>0$, we obtain that there is $\delta>0$ such that for $|\varphi|_{t_0}<\delta$ implies that the unique solution z of (3.6) with $z_{t_0}=\varphi$ on $[m(t_0),t_0]$ satisfies $\max\limits_{t\geq t_0}\{|z(t)|,|z'(t)|\}<\varepsilon$. Moreover $\lim\limits_{t\to\infty}z(t)=\lim\limits_{t\to\infty}z'(t)=0$.

Finally, we show that the zero solution of equation (3.6) is stable in C^1.

For any $\varepsilon>0$, let $\delta>0$ such that

$$
\delta<\varepsilon\min\left\{1,\frac{1-\eta}{K},\frac{1-\eta}{KA}\right\}.
$$

If $z(t)=z(t,t_0,\varphi)$ is a solution of equation (3.6) with $|\varphi|_{t_0}<\delta$, then $z(t)=(\Psi z)(t)$ on $[t_0,\infty)$. We claim that $\|z\|<\varepsilon$. Otherwise, there would exist $t^*>t_0$ such that

$$
\max\{|z(t^*,t_0,\varphi)|,|z'(t^*,t_0,\varphi)|\}=\varepsilon,
$$

and

$$
\max\{|z(t,t_0,\varphi)|,|z'(t,t_0,\varphi)|\}<\varepsilon,
$$

for $t \in [m(t_0), t^*]$, if $|z(t^*, t_0, \varphi)| = \varepsilon$, then it follows from (3.7) and (3.2) that

$$\begin{aligned}
&|z(t^*, t_0, \varphi)| \\
&\quad \leq e^{-\int_{t_0}^{t^*} g(s)ds} |\varphi(t_0)| + \int_{t_0}^{t^*} e^{-\int_s^{t^*} g(u)du} \left| g(s) - \left(a(s) + \frac{p'(s)}{p(s)} \right) \right| |z(s)| \, ds \\
&\qquad + \int_{t_0}^{*} e^{-\int_s^{*} g(u)du} \sum_{i=1}^{N} \left| \frac{b_i(s) \, p(s - \tau_i(s))}{p(s)} \right| |z'(s - \tau_i(s))| \, ds \\
&\qquad + \int_{t_0}^{*} e^{-\int_s^{*} g(u)du} \left[\sum_{i=1}^{N} \left| \frac{c_i(s) \, p(s - \tau_i(s))}{p(s)} \right| + \sum_{i=1}^{N} \left| \frac{b_i(s) \, p'(s - \tau_i(s))}{p(s)} \right| \right] \\
&\qquad\quad \times |z(s - \tau_i(s))| \, ds \\
&\quad \leq K\delta + \eta\varepsilon < \varepsilon,
\end{aligned}$$

and this is a contradiction.

If $|z'(t^*, t_0, \varphi)| = \varepsilon$, then it follows from (3.8) and (3.3) that

$$\begin{aligned}
&|z'(t^*, t_0, \varphi)| \\
&\quad \leq e^{-\int_{t_0}^{t^*} g(s)ds} |\varphi(t_0)| \, |g(t)| \\
&\quad + |g(t)| \int_{t_0}^{t^*} e^{-\int_s^{t^*} g(u)du} \left| g(s) - \left(a(s) + \frac{p'(s)}{p(s)} \right) \right| |z(s)| \, ds \\
&\quad + |g(t)| \int_{t_0}^{*} e^{-\int_s^{*} g(u)du} \sum_{i=1}^{N} \left| \frac{b_i(s) \, p(s - \tau_i(s))}{p(s)} \right| |z'(s - \tau_i(s))| \, ds \\
&\quad + |g(t)| \int_{t_0}^{*} e^{-\int_s^{*} g(u)du} \left[\sum_{i=1}^{N} \left| \frac{c_i(s) \, p(s - \tau_i(s))}{p(s)} \right| + \sum_{i=1}^{N} \left| \frac{b_i(s) \, p'(s - \tau_i(s))}{p(s)} \right| \right] \\
&\qquad \times |z(s - \tau_i(s))| \, ds \\
&\quad + \left| g(t) - \left(a(t) + \frac{p'(t)}{p(t)} \right) \right| |z(t)| \\
&\quad + \sum_{i=1}^{N} \left| \frac{b_i(t) \, p(t - \tau_i(t))}{p(t)} \right| |z'(t - \tau_i(t))| \\
&\quad + \left[\sum_{i=1}^{N} \left| \frac{b_i(t) \, p'(t - \tau_i(t))}{p(t)} \right| + \sum_{i=1}^{N} \left| \frac{c_i(t) \, p(t - \tau_i(t))}{p(t)} \right| \right] |z(t - \tau_i(t))| \\
&\quad \leq KA\delta + \eta\varepsilon < \varepsilon,
\end{aligned}$$

and this is a contradiction too. Hence, the zero solution of equation (3.6) is stable in C^1. This, together with

$$\lim_{t \to \infty} z(t) = \lim_{t \to \infty} z'(t) = 0,$$

implies that the zero solution of equation (3.6) is globally asymptotically stable in C^1. This shows that the zero solution of (2.1) is asymptotically stable if (3.4) holds.

($:\Rightarrow$) Assume that the zero solution of equation (2.1) is globally asymptotically stable in C^1. Now, we prove that (3.4) holds. If not, let us assume that (3.1) does not hold. Otherwise, set

$$J = \liminf_{t\to\infty} \int_0^t g(s)\,ds, \overset{0}{K} = \sup_{t\geq t_0} e^{-\int_{t_0}^t g(s)ds} \text{ and } \overset{0}{A} = \sup_{t\geq t_0}\{|g(t)|\}.$$

Thus, it follows from (3.1) that $J \in (-\infty,\infty)$, $\overset{0}{K}, \overset{0}{A} \in [0,\infty)$.

Therefore, there exists an increasing sequence $\{t_n\} \subset [0,\infty)$ such that $\lim\limits_{n\to\infty} t_n = \infty$ and

$$\lim_{n\to\infty} \int_0^{t_n} g(s)\,ds = J, n = 1, 2, ... \tag{3.13}$$

Denote

$$\begin{aligned} I_n = & \int_0^{t_n} e^{\int_0^s g(u)du} \left(\left| g(s) - \left(a(s) + \frac{p'(s)}{p(s)} \right) \right| \right. \\ & + \sum_{i=1}^N \left| \frac{b_i(s)\,p(s-\tau_i(s))}{p(s)} \right| \\ & \left. + \sum_{i=1}^N \left| \frac{c_i(s)\,p(s-\tau_i(s))}{p(s)} \right| + \sum_{i=1}^N \left| \frac{b_i(s)\,p'(s-\tau_i(s))}{p(s)} \right| \right) ds, \\ n = & 1, 2, ... \end{aligned}$$

From (3.2), it follows that

$$\begin{aligned} I_n = & e^{\int_0^{t_n} g(u)du} \int_0^{t_n} e^{\int_0^s g(u)du} \left(\left| g(s) - \left(a(s) + \frac{p'(s)}{p(s)} \right) \right| \right. \\ & + \sum_{i=1}^N \left| \frac{b_i(s)\,p(s-\tau_i(s))}{p(s)} \right| \\ & \left. + \sum_{i=1}^N \left| \frac{c_i(s)\,p(s-\tau_i(s))}{p(s)} \right| + \sum_{i=1}^N \left| \frac{b_i(s)\,p'(s-\tau_i(s))}{p(s)} \right| \right) ds \\ \leq & \eta e^{\int_0^{t_n} g(u)du} < e^J. \end{aligned}$$

This, together with (3.13), implies that the sequence $\{I_n\}$ is bounded. Furthermore, there exists a convergent subsequence. For brevity of notation, we still assume that $\{I_n\}$ is convergent. Therefore, there exists a positive integer m such that for any integer $n > m$,

$$\begin{aligned}
&\int_{t_m}^{t_n} e^{\int_0^s g(u)du} \left(\left| g\left(s\right) - \left(a\left(s\right) + \frac{p'(s)}{p(s)} \right) \right| \right. \\
&+ \sum_{i=1}^{N} \left| \frac{b_i\left(s\right) p\left(s - \tau_i\left(s\right)\right)}{p(s)} \right| + \sum_{i=1}^{N} \left| \frac{c_i\left(s\right) p\left(s - \tau_i\left(s\right)\right)}{p(s)} \right| \\
&\left. + \sum_{i=1}^{N} \left| \frac{b_i\left(s\right) p'\left(s - \tau_i\left(s\right)\right)}{p(s)} \right| \right) ds \\
&< \frac{1-\eta}{8B\left(e^{-J}+1\right)},
\end{aligned} \tag{3.14}$$

and

$$e^{-\int_{t_m}^{t_n} g(u)du} > \frac{1}{2},\ e^{-\int_0^{t_n} g(u)du} < e^{-J}+1,\ e^{\int_0^{t_m} g(u)du} < e^{J}+1, \tag{3.15}$$

where

$$B = \max\left\{ \overset{0}{K}\left(e^J+1\right), \overset{0}{K}\overset{0}{A}\left(e^J+1\right), 1 \right\}.$$

For any $\delta_0 > 0$, consider the solution $z(t) = z(t, t_m, \varphi)$ of equation (3.6) with $|\varphi|_{t_m} < \delta_0$ and $|\varphi\left(t_m\right)| > \dfrac{\delta_0}{2}$. It follows from $(3.7), (3.8), (3.15)$ and $(3.1) - (3.3)$, that for $t \in [t_m, \infty)$,

$$\begin{aligned}
|z\left(t\right)| &\le \delta_0 e^{-\int_{t_m}^{t} g(s)ds} + \int_{t_m}^{t} e^{-\int_s^t g(u)du} \left(\left| g\left(s\right) - \left(a\left(s\right) + \frac{p'(s)}{p(s)} \right) \right| |z(s)| \right. \\
&+ \sum_{i=1}^{N} \left| \frac{b_i\left(s\right) p\left(s - \tau_i\left(s\right)\right)}{p(s)} \right| |z'\left(s - \tau_i\left(s\right)\right)| \\
&\left. + \left[\sum_{i=1}^{N} \left| \frac{c_i\left(s\right) p\left(s - \tau_i\left(s\right)\right)}{p(s)} \right| + \sum_{i=1}^{N} \left| \frac{b_i\left(s\right) p'\left(s - \tau_i\left(s\right)\right)}{p(s)} \right| \right] |z\left(s - \tau_i\left(s\right)\right)| \right) ds \\
&\le \overset{0}{K}\left(e^J+1\right)\delta_0 + \|z\|_{t_m} \int_{t_m}^{t} e^{-\int_s^t g(u)du} \left(\left| g\left(s\right) - \left(a\left(s\right) + \frac{p'(s)}{p(s)} \right) \right| \right. \\
&+ \sum_{i=1}^{N} \left| \frac{b_i\left(s\right) p\left(s - \tau_i\left(s\right)\right)}{p(s)} \right| \\
&\left. + \left[\sum_{i=1}^{N} \left| \frac{c_i\left(s\right) p\left(s - \tau_i\left(s\right)\right)}{p(s)} \right| + \sum_{i=1}^{N} \left| \frac{b_i\left(s\right) p'\left(s - \tau_i\left(s\right)\right)}{p(s)} \right| \right] \right) ds \\
&\le B\delta_0 + \eta \|z\|_{t_m},
\end{aligned}$$

and

$$
\begin{aligned}
|z'(t)| \leq{} & |z(t_m)|\,|g(t)|\,e^{-\int_{t_m}^{t} g(s)ds} + \left|g(t) - \left(a(t) + \frac{p'(t)}{p(t)}\right)\right| |z(t)| \\
& + \sum_{i=1}^{N} \left|\frac{b_i(t)\,p(t-\tau_i(t))}{p(t)}\right| |z'(t-\tau_i(t))| \\
& + \left[\sum_{i=1}^{N} \left|\frac{c_i(t)\,p(t-\tau_i(t))}{p(t)}\right| + \sum_{i=1}^{N} \left|\frac{b_i(t)\,p'(t-\tau_i(t))}{p(t)}\right|\right] |z(t-\tau_i(t))| \\
& + |g(t)| \int_{t_m}^{t} e^{-\int_s^t g(u)du} \left(\left|g(s) - \left(a(s) + \frac{p'(s)}{p(s)}\right)\right| |z(s)| \right. \\
& + \sum_{i=1}^{N} \left|\frac{b_i(s)\,p(s-\tau_i(s))}{p(s)}\right| |z'(s-\tau_i(s))| \\
& \left. + \sum_{i=1}^{N} \left[\left|\frac{c_i(s)\,p(s-\tau_i(s))}{p(s)}\right| + \left|\frac{b_i(s)\,p'(s-\tau_i(s))}{p(s)}\right|\right] |z(s-\tau_i(s))| \right) ds
\end{aligned}
$$

$$
\begin{aligned}
& \leq \overset{0}{K}\overset{0}{A}\left(e^{J}+1\right)\delta_0 \\
& \quad + \|z\|_{t_m} |g(t)| \int_{t_m}^{t} e^{-\int_s^t g(u)du} \left[\left|g(s) - \left(a(s) + \frac{p'(s)}{p(s)}\right)\right| \right. \\
& \quad + \sum_{i=1}^{N} \left|\frac{b_i(s)\,p(s-\tau_i(s))}{p(s)}\right| \\
& \quad \left. + \sum_{i=1}^{N} \left|\frac{c_i(s)\,p(s-\tau_i(s))}{p(s)}\right| + \sum_{i=1}^{N} \left|\frac{b_i(s)\,p'(s-\tau_i(s))}{p(s)}\right| \right] \\
& \quad + \left|g(t) - \left(a(t) + \frac{p'(t)}{p(t)}\right)\right| + \sum_{i=1}^{N} \left|\frac{c_i(t)\,p(t-\tau_i(t))}{p(t)}\right| \\
& \quad + \sum_{i=1}^{N} \left|\frac{b_i(t)\,p'(t-\tau_i(t))}{p(t)}\right| \\
& \leq B\delta_0 + \eta \|z\|_{t_m}.
\end{aligned}
$$

Hence, $\|z\|_{t_m} \leq B\delta_0 + \eta \|z\|_{t_m}$, thus we have

$$
\|z\|_{t_m} \leq \frac{B}{1-\eta}\delta_0, \text{ for all } t \geq t_m. \tag{3.16}
$$

It follows from $(3.7), (3.14)-(3.16)$ that ,for any $n>m$,

$$
\begin{aligned}
&|z(t_n)| \\
&\geq |\varphi(t_m)|\, e^{-\int_{t_m}^{t_n} g(s)ds} - \int_{t_m}^{t_n} e^{-\int_s^{t_n} g(u)du} \Bigg| \Bigg[\left(g(s) - \left(a(s) + \frac{p'(s)}{p(s)} \right) \right) z(s) \\
&\quad + \sum_{i=1}^{N} \frac{b_i(s)\, p(s-\tau_i(s))}{p(s)} z'(s-\tau_i(s)) \\
&\quad + \sum_{i=1}^{N} \frac{c_i(s)\, p(s-\tau_i(s))}{p(s)} z(s-\tau_i(s)) \Bigg] ds \Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&\geq \delta_0 e^{-\int_{t_m}^{t_n} g(u)du} - \int_{t_m}^{t_n} e^{-\int_s^{t_n} g(u)du} \Bigg| \Bigg[\left(g(s) - \left(a(s) + \frac{p'(s)}{p(s)} \right) \right) z(s) \\
&\quad + \sum_{i=1}^{N} \frac{b_i(s)\, p(s-\tau_i(s))}{p(s)} z'(s-\tau_i(s)) \\
&\quad + \sum_{i=1}^{N} \frac{c_i(s)\, p(s-\tau_i(s))}{p(s)} z(s-\tau_i(s)) \Bigg] ds \Bigg| \\
&\qquad \geq \delta_0 e^{-\int_{t_m}^{t_n} g(u)du} \\
&\qquad\quad - \|z\|_{t_m} e^{-\int_0^{t_n} g(u)du} \int_{t_m}^{t_n} e^{\int_0^s g(u)du} \Bigg[\left| g(s) - \left(a(s) + \frac{p'(s)}{p(s)} \right) \right| \\
&\qquad\quad + \sum_{i=1}^{N} \left| \frac{b_i(s)\, p(s-\tau_i(s))}{p(s)} \right| \\
&\qquad\quad + \sum_{i=1}^{N} \left| \frac{c_i(s)\, p(s-\tau_i(s))}{p(s)} \right| \Bigg] ds \\
&\qquad \geq \frac{1}{4}\delta_0 - \frac{\delta_0 B}{1-\eta} \left(e^{-J}+1 \right) \frac{1-\eta}{8B\left(e^{-J}+1\right)} = \frac{1}{4}\delta_0.
\end{aligned}
$$

The facts that $\lim_{n\to\infty} t_n = \infty$ and the zero solution of equation (3.6) is globally asymptotically stable in C^1 imply $\lim_{n\to\infty} z(t, t_n, \varphi) = \lim_{n\to\infty} z'(t, t_n, \varphi) = 0$, which is in contradiction with (3.17). Hence condition (3.4) is necessary in order that (3.6) has a solution asymptotically stable in C^1. Thus, the zero solution of (3.6) is asymptotically stable, and hence the zero solution of (2.1) is asymptotically stable in C^1. The proof is complete.

Remark 3.1. When

$$f(t, x(t-\tau_1(t))), ..., x(t-\tau_N(t)))) = \sum_{i=1}^{N} b_i(t) x(t-\tau_i(t))$$

and

$$h(t, x(t-\tau_1(t))), ..., x(t-\tau_N(t)))) = \sum_{i=1}^{N} c_i(t) x'(t-\tau_i(t)),$$

with $g(t) \equiv a(t)$ and $p(t) = 1$, Theorem 3.1 reduces to Theorem A.

Remark 3.2. It follows from the first part of the proof of Theorem 3.1 that the zero solution of (2.1) is globally asymptotically stable in C^1 under (3.1), (3.2), and (3.3). Moreover, Theorem 3.1 still holds true if (3.2), (3.3) are satisfied for $t \geq t_\sigma$ for some $t_\sigma \in \mathbb{R}^+$.

4 An Example

In this section, we analyse an example to illustrate two facts. On the one hand, we will show how to apply our main result in this paper, Theorem 3.1. On the other hand and most importantly, we will highlight the real interest and importance of our result because the previous theory developed by Ardjouni and Djoudi [1] cannot be applied to this example.

Example 4.1. Consider the following linear neutral delay differential equation

$$x'(t) = -a(t) x(t) + b_1(t) x'(t-\tau_1(t)) + c_1(t) x(t-\tau_1(t)), \tag{4.1}$$

for $t \geq 0$, corresponding to equation (2.1) when $N = 1$, $\tau_1(t) = \dfrac{\pi}{2}$ and $a(t) = 1 - \dfrac{6\sin t\cos t}{1+\sin^2 t}$, $c_1(t) = \sin^6 t$, $b_1(t) = \dfrac{1}{10}\sin^6 t$. By choosing $g(t) = 1$ and $p(t) = \left(1+\sin^2 t\right)^3$ in Theorem 3.1, we obtain that

$$\left| g(t) - \left(a(t) + \frac{p'(t)}{p(t)} \right) \right| = 0.$$

By straightforward computations, we have

$$\left| \frac{p(t-\tau_1(t))}{p(t)} c_1(t) \right| = \left| \frac{\left(2-\sin^2 t\right)^3}{\left(1+\sin^2 t\right)^3} \sin^6 t \right| \leq 0.1539,$$

and

$$\begin{aligned} \left| \frac{p(t-\tau_1(t))}{p(t)} b_1(t) \right| &= \left| \frac{\left(2-\sin^2 t\right)^3}{\left(1+\sin^2 t\right)^3} \frac{\sin^6 t}{10} \right| \leq \frac{0.1539}{10} \\ &\leq 0.0153. \end{aligned}$$

Since $|\sin t \cos t| < 1$ for $t \in \mathbb{R}$, then we deduce

$$\begin{aligned} \left| \frac{p'(t-\tau_1(t))}{p(t)} b_1(t) \right| &= \frac{6}{10} \left| \frac{\left(2-\sin^2 t\right)^2}{\left(1+\sin^2 t\right)^3} \sin^6 t \right| |\sin t \cos t| \\ &\leq \frac{6}{10} \times 0.1286 \times |\sin t \cos t| \\ &\leq 0.0771. \end{aligned}$$

According to the definition of r_1, r_2 under conditions (3.2) and (3.3) of Theorem 3.1, for $t \in [0, \infty)$, we can calculate and obtain

$$r_1(t) := 0.1539 + 0.0153 + 0.0771 = 0.2463,$$

and

$$r_2(t) := 2(0.1539 + 0.0153 + 0.0771) = 0.4926,$$

then, we have $r_1(t) < 1/2$ and $r_2(t) < 1/2$. Hence, since $t - \dfrac{\pi}{2} \to \infty$ as $t \to \infty$, and it is easy to verify that $\int_0^t g(s)\,ds \to \infty$ as $t \to \infty$, $|p(t)| \leq 2$, then assumptions of Theorem 3.1 are fulfilled. Therefore, the zero solution of (4.1) is globally asymptotically stable in C^1 thanks to Theorem 3.1.

Note that

$$|b_1(t)| + |c_1(t)| = \frac{1}{10}\left|\sin^6 t\right| + \left|\sin^6 t\right| = 1.1 \text{ when } t = k\pi + \frac{\pi}{2} \text{ for } k = 0, 1, 2, ...$$

Thus, Theorem A cannot be applied to equation (4.1), when $f(t, x(t-\tau_1(t)))) = b_1(t)x(t - \tau_1(t))$ and $h(t, x(t - \tau_1(t))) = c_1(t)x'(t - \tau_1(t))$ in (1.4).

Remark 3.3: The method in this paper can be applied to more general neutral differential systems than Eq. (2.1).

Conflict of interest

The authors declare that they have no conflict of interest.

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