On the existence of positive periodic solutions for $n-$species Lotka-Volterra competitive systems with distributed delays and impulses

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Abstract

In this paper, we investigate the existence of positive periodic solutions for an n-species Lotka-Volterra system with distributed delays and impulsive effect. In the process we use integrating factors and convert the given Lotka-Volterra differential equation into an equivalent integral equation. Then we construct appropriate mappings and use Krasnoselskii's fixed point theorem to show the existence of a positive periodic solution of this system. In particular, the results improve some previous ones in the literature. Finally, as an application, we exhibit an example to illustrate the effectiveness of our abstract results.

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1 Introduction

It is well known that the theory of impulsive differential equations has become an important area of scientific activity. Many evolution processes are characterized by the fact that at certain moments of time they experience an abrupt change of state. These short term perturbations act instantaneously, that is in the form of impulses. Equations of this kind are found in almost every domain of applied sciences, numerous examples can be found in, e.g., [2, 3, 18, 19, 27]. For example, many biological phenomena involving fields such as economics, mechanics, electronics, telecommunications, medicine and biology, etc. (see [19]). Thus, impulsive differential equations appear as a natural description of observed evolution phenomena of several real world problems. However, besides impulsive effects, time delay is present in many fields in our society. In recent years, non-autonomous delay differential equations have been used in the study of population ecology and infectious diseases, population dynamics. Indeed, a famous model for population dynamics is the Lotka-Volterra competition system. Due to its theoretical and practical significance, Lotka-Volterra systems have been extensively and intensively studied for the past few years (see, [5, 6, 14, 23, 24, 25, 26, 29, 30]). On the other hand, a very basic and important qualitative problem is the study of periodic solutions of delay differential equations with or without impulsive effects which has attracted the interest of many mathematicians (we refer the reader to [1, 7, 10, 12, 15, 16, 17, 20, 21, 22, 28]). For instance, in 2006, by using the method of Krasnoselskii's fixed point theorem, Tang and Zhou [25] investigated the existence of positive periodic solutions of the following system with deviating arguments:

$$x_i'(t) = x_i\left(t\right)\left[r_i\left(t\right) - \sum_{j=1}^{n} a_{ij}(t)x_j(t-\tau_{ij}\left(t\right))\right],\ i = 1, 2, ..., n. \tag{1.1}$$

By the same method as the one in [25], Zhang *et al.* investigated in [30] the existence and global attractivity of positive periodic solutions of 3-species Lotka-Volterra predator-prey systems with infinite delays as follows:

$$\begin{cases} x_1'(t) = x_1\left(t\right)\Big(r_1\left(t\right) - c_{11}(t)x_1(t) - c_{12}(t)\int_{-\infty}^{t} K_{12}\left(s-t\right)x_2(s)ds \\ \qquad\qquad + c_{13}(t)\int_{-\infty}^{t} K_{13}\left(s-t\right)x_3(s)ds\Big), \\ x_2'(t) = x_2\left(t\right)\Big(r_2\left(t\right) - c_{21}(t)\int_{-\infty}^{t} K_{21}\left(s-t\right)x_1(s)ds - c_{22}(t)x_2(t) \\ \qquad\qquad + c_{23}(t)\int_{-\infty}^{t} K_{23}\left(s-t\right)x_3(s)ds\Big), \\ x_3'(t) = x_3\left(t\right)\Big(r_3\left(t\right) + c_{31}(t)\int_{-\infty}^{t} K_{31}\left(s-t\right)x_1(s)ds \\ \qquad\qquad + c_{32}(t)\int_{-\infty}^{t} K_{32}\left(s-t\right)x_2(s)ds - c_{33}(t)x_3(t)\Big). \end{cases} \tag{1.2}$$

Very recently, Benhadri *et al.* improved in [1] the results of Zhang *et al.* [25] to the generalized nonimpulsive nonlinear Lotka-Volterra competition with deviating arguments of the form:

$$\begin{aligned} x_i'(t) &= x_i\left(t\right)\left\{r_i\left(t\right) - \sum_{j=1}^{n} a_{ij}(t)x_j(t) - \sum_{j=1}^{n} b_{ij}(t)f_j\left(x_j(t)\right)\right. \\ &\quad \left. - \sum_{j=1}^{n} c_{ij}(t)g_j\left(x_j(t-\tau_{ij}\left(t\right))\right)\right\}, \qquad (1.3) \\ &\quad i = 1, 2, ..., n. \end{aligned}$$

The authors derived some sufficient conditions for the existence of positive periodic solutions of (1.3).

In this paper, motivated by the content in [1] and [25], we generalize system (1.1) to a model with variable and distributed delays and impulses,

$$
\begin{aligned}
x_i'(t) = x_i(t) \Bigg\{ & r_i(t) - \sum_{j=1}^{n} a_{ij}(t) h_j(x_j(t)) - \sum_{j=1}^{n} b_{ij}(t) f_j(x_j(t-\tau_{ij}(t))) \\
& - \sum_{j=1}^{n} c_{ij}(t) \int_{-\infty}^{t} D_{ij}(t,s) g_j(x_j(s)\, ds \Bigg\}, \qquad (1.4) \\
& i = 1,2,...,n, \ t \neq t_k, t > 0, \\
x_i(t_k^+) - x_i(t_k^-) & = I_{ik}(t_k, x_i(t_k)), \ t = t_k, \ k \in \mathbb{Z}^+,
\end{aligned}
$$

where $x(t) = [x_1(t), x_2(t), .., x_n(t)]^T \in \mathbb{R}^n$. The expression $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = I_{ik}(t_{ik}, x_i(t_k))$ denotes the impulse at moment t_k, and $t_1 < t_2 < ...$, is a strictly increasing sequence such that t_k goes to infinity, $x_i(t_k^+)$ and $x_i(t_k^-)$ stand for the right-hand and the left-hand limits of $x_i(t)$ at the impulsive moment t_k respectively. Consider that $I_{ik}(\cdot,\cdot) \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $i = 1,2,...,n$, $k = 1,2,...$, shows the impulsive perturbation at the moment t_k. Since we are searching for the existence of periodic solutions for equation (1.4), it is natural to assume that $r_i, a_{ij}, b_{ij}, c_{ij} \in C(\mathbb{R}^+, \mathbb{R}^+)$ are all $\omega-$periodic functions $(\omega > 0)$ with respect to t,

$$
\begin{aligned}
a_{ij}(t+\omega) &= a_{ij}(t), \tau_{ij}(t+\omega) = \tau_{ij}(t), \ D_{ij}(t+\omega, s+\omega) = D_{ij}(t,s), \\
b_{ij}(t+\omega) &= b_{ij}(t) \ \text{ and } c_{ij}(t+\omega) = c_{ij}(t) \qquad (1.5)
\end{aligned}
$$

for, $i,j = 1,2,...,n$, with τ_{ij} being scalar functions, continuous, and $\tau_{ij}(t) \geq \tau_{ij}^* > 0$ with

$$
\begin{aligned}
\widehat{r}_i &= \frac{1}{\omega}\int_0^{\omega} r_i(s) ds > 0, \\
\widehat{a}_{ij} &= \frac{T_j}{\omega}\int_0^{\omega} a_{ij}(s) ds \geq 0, \\
\widehat{b}_{ij} &= \frac{F_j}{\omega}\int_0^{\omega} b_{ij}(s) ds \geq 0, \ \widehat{c}_{ij} = \frac{R_j E_{ij}}{\omega}\int_0^{\omega} c_{ij}(s) ds \geq 0, \\
\widehat{\beta}_{ik} &= \frac{1}{\omega}\sum_{0 \leq t_k < \omega} \lambda_{ik}(t_k) \geq 0, k \in \mathbb{Z}^+, \qquad (1.6)
\end{aligned}
$$

for $i,j = 1,2,...,n$, where T_j, F_j and $\lambda_{ik}, R_j, E_{ij}$ are given in $(H_1) - (H_5)$. We also assume that the functions $D_{ij} \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ and $f_i, g_i, h_i \in$

$C\left(\mathbb{R}^{+},\mathbb{R}^{+}\right),\ i,j=1,2,...,n,$

$$\begin{aligned} f(x\left(t\right)) &= \left[f_1\left(x_1(t)\right),f_2\left(x_2(t)\right),...,f_n\left(x_n(t)\right)\right]^T\in\mathbb{R}_{+}^{n}, \\ g(x\left(t\right)) &= \left[g_1\left(x_1(t)\right),g_2\left(x_2(t)\right),...,g_n\left(x_n(t)\right)\right]^T\in\mathbb{R}_{+}^{n}, \\ h(x\left(t\right)) &= \left[h_1\left(x_1(t)\right),h_2\left(x_2(t)\right),...,h_n\left(x_n(t)\right)\right]^T\in\mathbb{R}_{+}^{n}, \end{aligned}$$

are positive and continuous in their respective arguments.

Throughout this paper, we will assume the following hypotheses:

(H_1) There exist nonnegative constants $\overline{T}_j,T_j$, such that for all $x\in\mathbb{R}^{+}$,

$$\overline{T}_j x\leq h_j\left(x\right)\leq T_j x,\ \ j=1,2,...,n. \tag{1.7}$$

(H_2) There exist nonnegative constants $\overline{F}_j,F_j$ such that for all $x\in\mathbb{R}^{+}$,

$$\overline{F}_j x\leq f_j\left(x\right)\leq F_j x,\ \ j=1,2,...,n. \tag{1.8}$$

(H_3) There exist nonnegative constants $\overline{R}_j,R_j$ such that for all $x\in\mathbb{R}^{+}$,

$$\overline{R}_j x\leq g_j\left(x\right)\leq R_j x,\ \ j=1,2,...,n. \tag{1.9}$$

(H_4) There exist nonnegative constants $\overline{E}_{ij},E_{ij}$ such that for all $t\in\mathbb{R}^{+}$,

$$\overline{E}_{ij}\leq\int_{-\infty}^{t}D_{ij}(t,s)ds\leq E_{ij},\ \ i,j=1,2,...,n. \tag{1.10}$$

(H_5) There exists an integer $q>0$ such that $t_{k+q}=t_k+\omega,\ I_{i(k+q)}=I_{ik}$, $k\in\mathbb{Z}^{+}$, where

$$[0,\omega]\cap\left\{t_k,k=1,2,...\right\}=\left\{t_1,t_2,...,t_q\right\}. \tag{1.11}$$

For convenience, we introduce the notion

$$f^M=\max_{t\in[0,\omega]}\left\{\left|f\left(t\right)\right|\right\},\ \ \ \delta_i=e^{-\int_0^{\omega}r_i(t)dt},\ \ i=1,2...,$$

where f is a continuous and $\omega-$periodic function.

The paper is organized as follows. In Section 2, we recall some results which are necessary for our analysis. The existence of positive periodic solutions of system (1.4) by using the Krasonoselskii fixed point theorem is proved in Section 3. Finally, in Section 4, we exhibit an example to show the validity of our result.

2 Preliminaries

Throughout this paper, a vector $x=(x_1,x_2,...,x_n)^T\in\mathbb{R}^n$ is said to be positive if $x_i\geq 0,1\leq i\leq n$.

First, we recall the following definitions. Let $I \subset \mathbb{R}$ be an interval, and denote by $PC(I, \mathbb{R}^n)$ the set of operators $x : I \to \mathbb{R}^n$ which are continuous for $t \in I$, $t \neq t_k$ and have discontinuities of the first kind at the points $t_k \in I, (k \in \mathbb{Z}^+)$ but are continuous from the left at these points.

The proofs of the main results in this paper are based on an application of Krasnoselskii's fixed point theorem in cones. Firstly, we need to introduce some definitions and lemmas.

Definition 2.1 (See [10, 18]) *A function $x_i : \mathbb{R} \to (0, +\infty)$ is said to be a positive solution of* (1.4)*, if the following conditions are satisfied*

1) $x_i(t)$ is absolutely continuous on each (t_k, t_{k+1});

2) for each $k \in \mathbb{Z}^+$, $x_i(t_k^+)$ and $x_i(t_k^-)$ exist and $x_i(t_k^-) = x_i(t_k)$;

3) $x_i(t)$ satisfies the first equation of (1.4) *for almost everywhere in $\mathbb{R}$ and $x_i(t_k)$ satisfies the second equation of* (1.4) *at impulsive point t_k, $k \in \mathbb{Z}^+$.*

Definition 2.2 *Let X be a Banach space and let K be a closed, nonempty subset of X. K is a cone if*

i) $\alpha x + \beta y \in K$ for all $x, y \in K$ and all $\alpha, \beta \geq 0$;

ii) $x, -x \in K$ imply $x = 0$.

Theorem 2.1. (Krasnoselskii, [13]). *Let X be a Banach space, and let $K \subset X$ be a cone in X. Assume that Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let*

$$\Phi : K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to K,$$

be a completely continuous operator such that either

a) $\|\Phi x\| \leq \|x\|$ for $x \in K \cap \partial\Omega_1$ and $\|\Phi x\| \geq \|x\|$ for $x \in K \cap \partial\Omega_2$; or

b) $\|\Phi x\| \geq \|x\|$ for $x \in K \cap \partial\Omega_1$ and $\|\Phi x\| \leq \|x\|$ for $x \in K \cap \partial\Omega_2$.

Then Φ has a fixed point in $K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

3 Existence of positive periodic solutions

As we mentioned previously, one of our main objectives in this paper is to improve the work carried out in [25], and to extend it to investigate a wider class of differential equations with impulsive effects presented in (1.4). In particular, by using Kranoselskii's fixed point theorem on cones, we will establish a sufficient condition ensuring the existence of positive $\omega-$periodic solutions of equation (1.4). This section will be splitted into two parts: in the first one, we will focuse on the existence of positive periodic solutions when we use subquadratic impulse functions, while in the second part, we will consider the case in which the impulse effects are sublinear (most frequently used in the published literature).

Let us start by obtaining an equivalent formulation for our problem (1.4).

Lemma 3.1. *The function $x(\cdot)$ is an $\omega-$periodic solution of equation* (1.4) *if* *and only if $x(\cdot)$ is an $\omega-$periodic solution of the following equation:*

$$\begin{aligned} x_i(t) &= \int_t^{t+\omega} G_i(t,s)\, x_i(s) \times \left[\sum_{j=1}^{n} a_{ij}(s) h_j(x_j(s)) \right. \\ &\quad + \sum_{j=1}^{n} b_{ij}(s) f_j(x_j(s-\tau_{ij}(s))) \\ &\quad \left. + \sum_{j=1}^{n} c_{ij}(s) \int_{-\infty}^{s} D_{ij}(s,z) g_j(x_j(z)\, dz \right\} ds \qquad (3.1) \\ &\quad + \sum_{t \le t_k < t+\omega} G_i(t,t_k)\, I_{ik}(t_k, x_i(t_k)), \end{aligned}$$

where

$$G_i(t,s) = \frac{e^{-\int_t^s r_i(\xi)d\xi}}{1 - e^{-\int_0^\omega r_i(\xi)d\xi}}, \; s \in [t, t+\omega], i = 1,2,...,n, \qquad (3.2)$$

and we assume

$$e^{-\int_0^\omega r_i(\xi)d\xi} \neq 1.$$

Proof. Unlike the procedure carried out in [25], where the authors used the variation of constants formula to rewrite the original equation as an integral equation, we have to proceed in a very different way which is motivated and justified by the appearance of the impulsive terms in our problem. Assume that $x = (x_1, x_2, ..., x_n)^T \in X$, is a periodic solution of equation (1.4). Then, we have

$$\begin{aligned} &\frac{d}{dt}\left[x_i(t) \exp\left(-\int_0^t r_i(s)\, ds \right) \right] \\ &= \exp\left(-\int_0^t r_i(s)\, ds \right) x_i(t) \\ &\quad \times \left\{ -\sum_{j=1}^{n} a_{ij}(t) h_j(x_j(t)) - \sum_{j=1}^{n} b_{ij}(t) f_j(x_j(t-\tau_{ij}(t))) \right. \\ &\quad \left. - \sum_{j=1}^{n} c_{ij}(t) \int_{-\infty}^{t} D_{ij}(t,s) g_j(x_j(s))\, ds \right\}, \qquad (3.3) \\ &t \neq t_k, \; i = 1,2,...,n. \end{aligned}$$

Integrating the above equation over $[t, t+\omega]$, we have

$$\begin{aligned}
& x_i(s)e^{-\int_0^s r_i(\xi)d\xi}\Big|_t^{t_{m_1}+n\omega} \\
& +x_i(s)e^{-\int_0^s r_i(\xi)d\xi}\Big|_{t_{m_1}+n\omega}^{t_{m_2}+n\omega} + ... \\
& +x_i(s)e^{-\int_0^s r_i(\xi)d\xi}\Big|_{t_{m_q}+n\omega}^{t+\omega} \\
= & \int_t^{t+\omega} x_i(s)\exp\left(-\int_0^s r_i(\xi)\,d\xi\right)\left\{-\sum_{j=1}^n a_{ij}(s)h_j(x_j(s))\right. \\
& \left.-\sum_{j=1}^n b_{ij}(s)f_j(x_j(s-\tau_{ij}(s)) - \sum_{j=1}^n c_{ij}(s)\int_{-\infty}^s D_{ij}(s,z)g_j(x_j(z))\,dz\right\} ds,
\end{aligned}$$

where $t_{m_k}+n\omega \in (t, t+\omega)$, $m_k \in \{1,2,...,q\}$, $k = 1,2,...,q$, $n \in \mathbb{Z}^+$. Therefore,

$$\begin{aligned}
& x_i(t)e^{-\int_0^t r_i(\xi)d\xi}\left[1 - e^{-\int_t^{t+\omega} r_i(\xi)d\xi}\right] \\
& + \sum_{t\le t_k<t+\omega} \Delta x_i(t_{m_k})\, e^{-\int_0^{t_{m_k}+n\omega} r_i(\xi)d\xi} \\
= & \int_t^{t+\omega} x_i(s)\exp\left(-\int_0^s r_i(\xi)\,d\xi\right) \\
& \times\left\{\sum_{j=1}^n a_{ij}(s)h_j(x_j(s)) + \sum_{j=1}^n b_{ij}(s)f_j(x_j(s-\tau_{ij}(s))\right. \\
& \left.+\sum_{j=1}^n c_{ij}(s)\int_{-\infty}^s D_{ij}(s,z)g_j(x_j(z))\,dz\right\} ds,
\end{aligned}$$

which can be transformed into

$$\begin{aligned}
x_i(t) = & \int_t^{t+\omega} G_i(t,s)\,x_i(s) \\
& \times\left\{\sum_{j=1}^n a_{ij}(s)h_j(x_j(s)) + \sum_{j=1}^n b_{ij}(s)f_j(x_j(s-\tau_{ij}(s))\right. \\
& \left.+\sum_{j=1}^n c_{ij}(s)\int_{-\infty}^s D_{ij}(s,z)g_j(x_j(z))\,dz\right\} ds \\
& + \sum_{t\le t_k<t+\omega} G_i(t,t_k)\,I_{ik}(t_k, x_i(t_k)),\ \ i = 1,2,,...,n. \qquad (3.4)
\end{aligned}$$

Thus, x_i is a periodic solution of (3.1). If $x = (x_1(t), x_2(t), ..., x_n(t))^T \in K$, is a periodic solution of (3.1), for any $t = t_k$, from (3.1) we obtain

$$\begin{aligned}
x_i'(t) &= G_i(t,t+\omega)\,x_i(t+\omega) \\
&\quad\times\left(\sum_{j=1}^{n} a_{ij}(t+\omega)h_j\left(x_j(t+\omega)\right)+\sum_{j=1}^{n} b_{ij}(t+\omega)f_j\left(x_j(t+\omega-\tau_j\left(t+\omega\right)\right)\right. \\
&\qquad\left.+\sum_{j=1}^{n} c_{ij}(t+\omega)\int_{-\infty}^{t+\omega} D_{ij}(t+\omega,s)g_j\left(x_j(s)\right)ds\right) \\
&\quad -G_i(t,t)\,x_i(t)\left(\sum_{j=1}^{n} a_{ij}(t)h_j\left(x_j(t)\right)+\sum_{j=1}^{n} b_{ij}(t)f_j\left(x_j(t-\tau_{ij}\left(t\right)\right)\right. \\
&\qquad\left.+\sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{t} D_{ij}(t,s)g_j\left(x_j(s)\right)ds\right)+r_i(t)x_i(t) \\
&= x_i(t)\left(r_i(t)-\sum_{j=1}^{n} a_{ij}(t)h_j\left(x_j(t)\right)-\sum_{j=1}^{n} b_{ij}(t)f_j\left(x_j(t-\tau_{ij}\left(t\right)\right)\right. \\
&\quad\left.-\sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{t} D_{ij}(t,s)g_j\left(x_j(s)\right)ds\right).
\end{aligned}$$

For any $t = t_j, j \in \mathbb{Z}^+$, we have from (3.1) that

$$\begin{aligned}
x_i(t_j^+)-x_i(t_j) &= \int_{t_j}^{t_j+\omega}\left[G_i\left(t_j^+,s\right)-G_i\left(t_j,s\right)\right]x_i(s) \\
&\qquad\times\left\{\sum_{j=1}^{n} a_{ij}(s)h_j\left(x_j(s)\right)\right. \\
&\qquad+\sum_{j=1}^{n} b_{ij}(s)f_j\left(x_j(s-\tau_{ij}\left(s\right))\right) \\
&\qquad\left.+\sum_{j=1}^{n} c_{ij}(s)\int_{-\infty}^{s} D_{ij}(s,z)g_j\left(x_j(z\right)dz\right\}ds \\
&\qquad+\sum_{t_j^+\le t_k<t_j+\omega} G_i\left(t_j^+,t_k\right)I_{ik}\left(t_k,x_i\left(t_k\right)\right) \\
&\qquad-\sum_{t_j\le t_k<t_j+\omega} G_i\left(t_j,t_k\right)I_{ik}\left(t_k,x_i\left(t_k\right)\right) \\
&= I_{ik}\left(t_k,x_i\left(t_k\right)\right).
\end{aligned}$$

Hence x_i is a positive $\omega-$periodic solution of (1.4). Thus, the proof of Lemma 3.1 is completed. ■

Define now

$$PC(\mathbb{R},\mathbb{R}^n) = \Big\{x=(x_1,x_2,...,x_n)^T : \mathbb{R}\to\mathbb{R}^n \mid x\in C((t_k,\ t_{k+1}),\mathbb{R}^n),\ \text{such that } x(t_k^-),\ x(t_k^+) \text{ exist and } x(t_k^-)=x(t_k), k\in\mathbb{Z}_+\Big\}. \tag{3.5}$$

To apply Theorem 2.1, we need to define a Banach space C_ω, a closed subset S of C_ω and construct one mapping. Thus, we let $(C_\omega, \|.\|) = (X, \|.\|)$ where

$$C_\omega = \Big\{x=(x_1,x_2,...,x_n)^T : x\in PC(\mathbb{R},\mathbb{R}^n), x(t+\omega)=x(t), t\in\mathbb{R}\Big\}, \tag{3.6}$$

with the norm

$$\|x\| = \sum_{i=1}^{n}|x_i|_0,\ |x_i|_0 = \max_{t\in[0,\omega]}|x_i(t)|, i=1,2,...,n,\ \forall x\in C_\omega. \tag{3.7}$$

Then, C_ω with the norm $\|.\|$ is a Banach space.
We denote $\theta = \min(1,\theta_1,\theta_2,\theta_3)$, where

$$\theta_1 = \min_{j=\overline{1,n}}\left(\frac{\overline{T}_j}{T_j}\right),\ \theta_2 = \min_{j=\overline{1,n}}\left(\frac{\overline{F}_j}{F_j}\right),\ \theta_3 = \min_{j=\overline{1,n}}\left\{\min_{i=\overline{1,n}}\left(\frac{\overline{E}_{ij}}{E_{ij}}\right)\frac{\overline{R}_j}{R_j}\right\},$$

and

$$\sigma = \min\Big\{e^{-\widehat{r}_i\omega},\ i=1,2,\dots,n\Big\}.$$

Let K be the cone in C_ω defined by

$$K = \Big\{x(\cdot)=(x_1,x_2,...,x_n)^T\in C_\omega : x_i(t)\geq\sigma|x_i|_0, i=1,2,...,n,\forall t\in\mathbb{R}\Big\}.$$

Use (3.1) to define the operator $\Phi : C_\omega\to C_\omega$ by

$$(\Phi x)(t) := [(\Phi_1 x)(t),(\Phi_2 x)(t),...,(\Phi_n x)(t)]^T,$$

where

$$\begin{aligned}(\Phi_i x)(t) = & \int_t^{t+\omega}G_i(t,s)\,x_i(s)\Bigg\{\sum_{j=1}^{n}a_{ij}(s)h_j(x_j(s)) \\ & +\sum_{j=1}^{n}b_{ij}(s)f_j(x_j(s-\tau_{ij}(s))) \\ & +\sum_{j=1}^{n}c_{ij}(s)\int_{-\infty}^{s}D_{ij}(s,z)g_j(x_j(z)\,dz\Bigg\}ds \\ & +\sum_{t\leq t_k<t+\omega}G_i(t,t_k)\,I_{ik}(t_k,x_i(t_k)),\end{aligned} \tag{3.8}$$

$$G_i(t,s)=\frac{e^{-\int_t^s r_i(\xi)d\xi}}{1-e^{-\int_0^\omega r_i(\xi)d\xi}},\ s\in[t,t+\omega],\ i=1,2,...,n.$$

It is clear that $G_i(t+\omega,s+\omega)=G_i(t,s)$, $(\partial G_i(t,s)/\partial t)=r_i(t)G_i(t,s)$, $G_i(t,t+\omega)-G_i(t,t)=-1$, and

$$A_i:=\frac{\delta_i}{1-\delta_i}\le G_i(t,s)\le\frac{1}{1-\delta_i}=:B_i,\ t,s\in\mathbb{R},\ i=1,2,...,n. \tag{3.9}$$

By (2.6), it is easy to check that $x\in C_\omega$ is an $\omega-$periodic solution of equation (1.4) provided x is a fixed point of Φ.

Lemma 3.2. *Assume that* $(H_1)-(H_5)$ *hold, then* $\Phi:K\to K$ *defined by Equation* (3.8) *is well defined, namely,* $\Phi K\subset K$.

Proof. From (3.8) it is easy to verify that $(\Phi x)(\cdot)$ is continuous in (t_k,t_{k+1}), and $(\Phi x)(t_k^+)$ and $(\Phi x)(t_k^-)$ exist, and $(\Phi x)(t_k^-)=(\Phi x)(t_k)$ for $k\in\mathbb{Z}^+$. Moreover, for any $x\in K$,

$$\begin{aligned}
&(\Phi_i x)(t+\omega)\\
&=\int_{t+\omega}^{t+2\omega}G_i(t+\omega,s)x_i(s)\left\{\sum_{j=1}^n a_{ij}(s)h_j(x_j(s))\right.\\
&\quad+\sum_{j=1}^n b_{ij}(s)f_j(x_j(s-\tau_{ij}(s)))\\
&\left.+\sum_{j=1}^n c_{ij}(s)\int_{-\infty}^s D_{ij}(s,z)g_j(x_j(z)\,dz\right\}ds\\
&\quad+\sum_{t+\omega\le t_k<t+2\omega}G_i(t,t_k)I_{ik}(t_k,x_i(t_k))\\
&=\int_t^{t+\omega}G_i(t+\omega,s+\omega)x_i(s+\omega)\times\\
&\quad\times\left\{\sum_{j=1}^n a_{ij}(s+\omega)h_j(x_j(s+\omega))+\sum_{j=1}^n b_{ij}(s+\omega)f_j(x_j(s+\omega))\right.\\
&\quad\left.+\sum_{j=1}^n c_{ij}(s+\omega)\int_{-\infty}^{s+\omega}D_{ij}(s+\omega,z)g_j(u_j(z)\,dz\right\}ds\\
&\quad+\sum_{t\le t_k<t+\omega}G_i(t,t_k)I_{ik}(t_k,x_i(t_k))
\end{aligned}$$

$$
\begin{aligned}
&= \int_t^{t+\omega} G_i\left(t,s\right) x_i(s) \left\{ \sum_{j=1}^n a_{ij}(s) h_j\left(x_j(s)\right) + \sum_{j=1}^n b_{ij}(s) f_j\left(x_j(s-\tau_{ij}\left(s\right))\right) \right. \\
&\quad \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\infty}^s D_{ij}(s,z) g_j\left(x_j(z\right) dz \right\} ds \\
&\quad + \sum_{t \le t_k < t+\omega} G_i\left(t,t_k\right) I_{ik}\left(t_k, x_i\left(t_k\right)\right) \\
&= \left(\Phi_i x\right)(t)\,,\ \ i = 1,2,...,n.
\end{aligned}
$$

That is $(\Phi_i x)(t+\omega) = (\Phi_i x)(t)\,,\ t \in [0,\omega]\,.$ Thus $\Phi x \in C_\omega$. Moreover, from (3.8) and (3.9), we have for $x \in K$

$$
\begin{aligned}
\left|(\Phi_i x)\right|_0 \quad \le \quad & \frac{1}{1-\delta_i} \int_0^\omega x_i(s) \left\{ \left[\sum_{j=1}^n a_{ij}(s) h_j\left(x_j(s)\right) \right.\right. \\
& + \sum_{j=1}^n b_{ij}(s) f_j\left(x_j(s-\tau_{ij}\left(s\right))\right) \\
& \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\infty}^s D_{ij}(s,z) g_j\left(x_j(z\right) dz \right] ds \\
& \left. + \sum_{t \le t_k < t+\omega} I_{ik}\left(t_k, x_i\left(t_k\right)\right) \right\},
\end{aligned}
$$

and

$$
\begin{aligned}
(\Phi_i x) \quad \ge \quad & \frac{\delta_i}{1-\delta_i} \int_0^\omega x_i(s) \left\{ \left[\sum_{j=1}^n a_{ij}(s) h_j\left(x_j(s)\right) + \sum_{j=1}^n b_{ij}(s) f_j\left(x_j(s-\tau_{ij}\left(s\right))\right) \right.\right. \\
& \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\infty}^s D_{ij}(s,z) g_j\left(x_j(z\right) dz \right] ds \\
& \left. + \sum_{t \le t_k < t+\omega} I_{ik}\left(t_k, x_i\left(t_k\right)\right) \right\} \\
\ge \quad & \frac{A_i}{B_i} \left|(\Phi_i x)\right|_0 \\
\ge \quad & \sigma \left|(\Phi_i x)\right|_0\,,\ \ i = 1,2,...,n.
\end{aligned}
$$

Hence, $\Phi K \subset K$. This completes the proof of Lemma 3.2. ■

3.1 The case of subquadratic impulses.

In this section we consider subquadratic impulse functions.

Lemma 3.3. *In addition to conditions* $(H_1)-(H_5)$, *we further assume the following one:*

(H_6) *There exist nonnegative functions* $\overline{\lambda}_{ik}, \lambda_{ik} \in C(\mathbb{R}^+, \mathbb{R}^+)$ *such that for all* $x \in \mathbb{R}^+$,

$$\overline{\lambda}_{ik}(t)\, x^2 \leq I_{ik}(t,x) \leq \lambda_{ik}(t)\, x^2, i = 1,2,...,n, k = 1,2,...$$

Then $\Phi : K \to K$ *defined by equation* (3.8) *is completely continuous.*

Proof. Set

$$\begin{aligned}\Gamma_i(t,x)(t) = x_i(t) &\left[\sum_{j=1}^{n} a_{ij}(t) h_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t) f_j(x_j(t-\tau_{ij}(t)))\right.\\ &\left. + \sum_{j=1}^{n} c_{ij}(t) \int_{-\infty}^{t} D_{ij}(t,s) g_j(x_j(s)) ds\right], \ t \in \mathbb{R}.\end{aligned} \tag{3.10}$$

We first show that Φ is continuous. Since h, f, g and I are continuous in x, it follows that, for any $L_0 > 0$ and $\varepsilon > 0$, there exists $\mu_1 > 0$ such that for $\|x\| \leq L_0$, $\|y\| \leq L_0$, and $\|x-y\| < \mu_1$ it follows

$$|\Gamma_i(s,x)(s) - \Gamma_i(s,y)(s)| < \frac{\varepsilon}{2nB\omega}, \ s \in \mathbb{R}^+, \ i = 1,2,...,n, \tag{3.11}$$

where $B = \max_{1\leq i\leq n} B_i$. For any $L_0 > 0$ and $\varepsilon > 0$, there exists $\mu_2 > 0$ such that for $\|x\| \leq L_0$, $\|y\| \leq L_0$, and $\|x-y\| < \mu_2$

$$|I_{ik}(t_k, x_i(t_k)) - I_{ik}(t_k, y_i(t_k))| < \frac{\varepsilon}{2qBn}, \ q \in \mathbb{Z}^+ \ i = 1,2,...,n. \tag{3.12}$$

Therefore, if $x, y \in C_\omega$ with $\|x\| \leq L_0, \|y\| \leq L_0$, and $\|x-y\| \leq \mu$, where $\mu = \min(\mu_1, \mu_2)$ then, from $(3.8), (3.9), (3.11)$ and (3.12),

$$\begin{aligned}|(\Phi_i x) - (\Phi_i y)|_0 &\leq B\int_t^{t+\omega} |\Gamma_i(s,x)(s) - \Gamma_i(s,y)(s)| ds\\ &\quad + \sum_{t\leq t_k<t+\omega} |G_i(t,t_k)| \, |I_{ik}(t_k,x_i(t_k)) - I_{ik}(t_k,y_i(t_k))|\\ &\leq B\frac{\omega\varepsilon}{2nB\omega} + Bq\frac{\varepsilon}{2qBn}\\ &< \frac{\varepsilon}{n}, \ i = 1,2,...,n.\end{aligned}$$

This yields

$$\|\Phi x - \Phi y\| = \sum_{i=1}^{n} |(\Phi_i x) - (\Phi_i y)|_0 < \varepsilon,$$

which implies that Φ is continuous on K.
We let

$$S=\{x(\cdot)=(x_1(\cdot),x_2(\cdot),...,x_n(\cdot))^T\in C_\omega:\ ||x||\leq L\ \},$$

where L is a non-negative constant. For any $x\in S$, it follows from (3.8) and (3.9) that

$$\begin{aligned}(\Phi_i x)(t) &= \int_t^{t+\omega} G_i(t,s)\,x_i(s)\left\{\sum_{j=1}^n a_{ij}(s)h_j(x_j(s))+\sum_{j=1}^n b_{ij}(s)f_j(x_j(s-\tau_{ij}(s)))\right.\\ &\left.+\sum_{j=1}^n c_{ij}(s)\int_{-\infty}^s D_{ij}(s,z)g_j(x_j(z)\,dz\right\}ds\\ &+\sum_{t\leq t_k<t+\omega} G_i(t,t_k)\,I_{ik}(t_k,x_i(t_k))\\ &\leq \frac{L^2}{1-\delta_i}\int_0^\omega\left[\sum_{j=1}^n A_j a_{ij}(s)+\sum_{j=1}^n F_j b_{ij}(s)+\sum_{j=1}^n R_j E_{ij}c_{ij}(s)\right]ds\\ &+\frac{L^2}{1-\delta_i}\sum_{t\leq t_k<t+\omega}\lambda_{ik}(t_k)\\ :=\ & B_i^*,\quad i=1,2,...,n,\end{aligned}$$

and, consequently,

$$\|\Phi x\|=\sum_{i=1}^n|(\Phi_i x)|_0\leq\sum_{i=1}^n B_i^*,\quad \forall x\in S.$$

This shows that $\Phi(S)$ is uniformly bounded.

To show that $\Phi(S)$ is equicontinuous, let $x\in S$, we calculate $\dfrac{d}{dt}(\Phi_i x)(t)$ and show that it is uniformly bounded. Indeed, by taking derivative in (3.8) we have

$$\begin{aligned}\left|(\Phi_i x)'(t)\right| &\leq \left|r_i(t)(\Phi_i x)(t)-x_i(t)\left[\sum_{j=1}^n a_{ij}(t)h_j(x_j(t))\right.\right.\\ &+\sum_{j=1}^n b_{ij}(t)f_j(x_j(t-\tau_{ij}(t)))\\ &\left.\left.+\sum_{j=1}^n c_{ij}(t)\int_{-\infty}^t D_{ij}(t,s)g_j(x_j(s)\,ds\right]\right|\\ &\leq r_i^M B_i^*+L^2\sum_{j=1}^n\left(A_j a_{ij}^M+F_j b_{ij}^M+E_{ij}R_j c_{ij}^M\right),\\ i &= 1,2,...,n,\end{aligned}$$

and

$$\| (\Phi x)' \| \leq \sum_{j=1}^{n} \left[r_i^M B_i^* + L^2 \sum_{j=1}^{n} \left(A_j a_{ij}^M + F_j b_{ij}^M + E_{ij} R_j c_{ij}^M \right) \right].$$

Hence, $\Phi S \subset C_\omega$ is a family of uniformly bounded and equi-continuous functions. By the Ascoli-Arzelà Theorem, the operator Φ is compact, and therefore completely continuous. The proof is complete. ■

We can now state and prove our main result in this paper.

Theorem 3.1. *Assume hypotheses $(H_1) - (H_6)$ and the next one as well:*
(H_7) The linear system

$$\sum_{j=1}^{n} \left(\widehat{a}_{ij} + \widehat{b}_{ij} + \widehat{c}_{ij} \right) x_j + \widehat{\beta}_{ik} x_i = \widehat{r}_i, i = 1, 2, ..., n, \ k = 1, 2, ...$$

possesses a unique positive solution. Then, system (1.4) possesses at least one positive $\omega-$periodic solution.

Proof. Let

$$x^* = (x_1^*, x_2^*, ..., x_n^*)^T$$

with $x_i^* > 0, i = 1, 2, ..., n$, be a positive solution of (1.12). Set

$$m_0 = \min_{1 \leq i \leq n} \{\widehat{r}_i A_i\},$$
$$M_0 = \max_{1 \leq i \leq n} \{\widehat{r}_i B_i\}.$$

Then $0 < m_0 < M_0 < +\infty$. Choose a constant $M \geq M_0$ such that $\dfrac{1}{M\omega} < 1$. Let $\alpha_1 = \dfrac{1}{M\omega}$ and

$$\Omega_1 = \left\{ x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in C_\omega : |x_i|_0 < \alpha_1 x_i^*, i = 1, 2, ..., n \right\}.$$

If $x \in K \cap \partial\Omega_1$, then

$$\sigma |x_i|_0 \leq x_i(t) \leq |x_i|_0 = \alpha_1 x_i^*, i = 1, 2, ..., n,$$

and

$$
\begin{aligned}
(\Phi_i x)(t) &\leq B_i \int_0^{\omega} \left\{ \left[x_i(s) \sum_{j=1}^{n} a_{ij}(s) h_j(x_j(s)) \right.\right. \\
&\quad + x_i(s) \sum_{j=1}^{n} b_{ij}(s) f_j(x_j(s-\tau_{ij}(s))) \\
&\quad \left. + x_i(s) \sum_{j=1}^{n} c_{ij}(s) \int_{-\infty}^{s} D_{ij}(s,z) g_j(x_j(z)\, dz \right] ds \\
&\quad \left. + \sum_{t \leq t_k < t+\omega} I_{ik}(t_k, x_i(t_k)) \right\} \\
&\leq B_i \int_0^{\omega} |x_i|_0 \sum_{j=1}^{n} T_j a_{ij}(s) |x_j|_0 \, ds + B_i \int_0^{\omega} |x_i|_0 \sum_{j=1}^{n} F_j b_{ij}(s) |x_j|_0 \, ds \\
&\quad + B_i \int_0^{\omega} |x_i|_0 \sum_{j=1}^{n} R_j E_{ij} c_{ij}(s) |x_j|_0 \, ds + B_i |x_i|_0 \sum_{t \leq t_k < t+\omega} \lambda_i(t_k) |x_i|_0 \\
&\leq \alpha_1 B_i \omega |x_i|_0 \sum_{j=1}^{n} \widehat{a}_{ij} x_j^* + \alpha_1 B_i \omega |x_i|_0 \sum_{j=1}^{n} \widehat{b}_{ij} x_j^* \\
&\quad + \alpha_1 B_i \omega |x_i|_0 \sum_{j=1}^{n} \widehat{c}_{ij} x_j^* + \alpha_1 B_i \omega |x_i|_0 \widehat{\beta}_{ik} x_i^* \\
&= \alpha_1 B_i \omega |x_i|_0 \left[\sum_{j=1}^{n} \left(\widehat{a}_{ij} + \widehat{b}_{ij} + \widehat{c}_{ij} \right) x_j^* + \widehat{\beta}_{ik} x_i^* \right] \\
&= (B_i \widehat{r}_i)\, \alpha_1 \omega |x_i|_0 \\
&\leq \alpha_1 M_0 \omega |x_i|_0 \\
&\leq |x_i|_0, \quad i = 1, 2, ..., n.
\end{aligned}
$$

Hence for any $x \in K \cap \partial\Omega_1$

$$
\|\Phi x\| = \sum_{i=1}^{n} |(\Phi_i x_i)|_0 \leq \sum_{i=1}^{n} |x_i|_0 = \|x\|.
$$

On the other hand, choose $0 < m \leq m_0$ such that $\dfrac{1}{\sigma^2 m\theta\omega} > 1$. Let $\alpha_2 = \dfrac{1}{\sigma^2 m\theta\omega}$ and

$$
\Omega_2 = \{x \in C_\omega : |x_i|_0 < \alpha_2 x_i^*, i = 1, 2, ..., n\}.
$$

If $x \in K \cap \partial\Omega_2$, then $\sigma |x_i|_0 \leq x_i(t) \leq |x_i|_0 = \alpha_2 x_i^*, i = 1, 2, ..., n$, and, consequently

$$
\begin{aligned}
\left(\Phi_i x\right)(t) \geq\ & A_i \int_0^\omega x_i(s)\left\{\sum_{j=1}^n a_{ij}(s) h_j\left(x_j(s)\right)+\sum_{j=1}^n b_{ij}(s) f_j\left(x_j\left(s-\tau_{ij}(s)\right)\right)\right. \\
& \left.+\sum_{j=1}^n c_{ij}(s) \int_{-\infty}^s D_{ij}(s, z) g_j\left(x_j(z)\, dz\right\} ds \\
& +\sum_{t \leq t_k<t+\omega} G_i\left(t, t_k\right) I_{ik}\left(t_k, x_i\left(t_k\right)\right) \\
\geq\ & \sigma^2 A_i\left|x_i\right|_0 \sum_{j=1}^n \int_0^\omega a_{ij}(s) T_j\left[\min_{j=\overline{1, n}}\left(\frac{\overline{T}_j}{T_j}\right)\right]\left|x_j\right|_0 ds \\
& +\sigma^2 A_i\left|x_i\right|_0 \sum_{j=1}^n \int_0^\omega F_j\left[\min_{i=\overline{1, n}}\left(\frac{\overline{F}_j}{F_j}\right)\right] b_{ij}(s)\left|x_j\right|_0 ds \\
& +\sigma^2 A_i\left|x_i\right|_0 \sum_{j=1}^n \int_0^\omega E_{ij} R_j\left\{\left[\min_{i=\overline{1, n}}\left(\frac{\overline{E}_{ij}}{E_{ij}}\right)\right] \times\left(\frac{\overline{R}_j}{R_j}\right)\right\} c_{ij}(s)\left|x_j\right|_0 ds \\
& +\sigma^2 A_i\left|x_i\right|_0 \sum_{t \leq t_k<t+\omega} \lambda_{ik}\left(t_k\right)\left|x_i\left(t_k\right)\right|_0 \\
\geq\ & \theta_1 \times \sigma^2 A_i \omega \alpha_2\left|x_i\right|_0 \sum_{j=1}^n \widehat{a}_{ij} x_j^*+\theta_2 \times \sigma^2 A_i \omega \alpha_2\left|x_i\right|_0 \sum_{j=1}^n \widehat{b}_{ij} x_j^* \\
& +\theta_3 \times \sigma^2 A_i \omega \alpha_2\left|x_i\right|_0 \sum_{j=1}^n \widehat{c}_{ij} x_j^*+1 \times \sigma^2 A_i \omega \alpha_2\left|x_i\right|_0 \widehat{\beta}_{ik} x_i^* \\
\geq\ & \theta \times \sigma^2 A_i \omega \alpha_2\left|x_i\right|_0 \sum_{j=1}^n \widehat{a}_{ij} x_j^*+\theta \times \sigma^2 A_i \omega \alpha_2\left|x_i\right|_0 \sum_{j=1}^n \widehat{b}_{ij} x_j^* \\
& +\theta \times \sigma^2 A_i \omega \alpha_2\left|x_i\right|_0 \sum_{j=1}^n \widehat{c}_{ij} x_j^* \\
& +\theta \times \sigma^2 A_i \omega \alpha_2\left|x_i\right|_0 \widehat{\beta}_{ik} x_i^* \\
=\ & \theta A_i \omega \sigma^2 \alpha_2\left|x_i\right|_0\left(\sum_{j=1}^n\left(\widehat{a}_{ij}+\widehat{b}_{ij}+\widehat{c}_{ij}\right) x_j^*+\widehat{\beta}_{ik} x_i^*\right) \\
=\ & \left(A_i \widehat{r}_i\right) \theta \omega \sigma^2 \alpha_2\left|x_i\right|_0 \\
\geq\ & m_0 \theta \omega \sigma^2 \alpha_2\left|x_i\right|_0 \\
\geq\ & \left|x_i\right|_0, i=1,2, \ldots, n,
\end{aligned}
$$

and thus

$$\|\Phi x\| = \sum_{i=1}^{n} |(\Phi_i x_i)|_0 \geq \sum_{i=1}^{n} |x_i|_0 = \|x\|, \forall x \in K \cap \partial\Omega_2.$$

Obviously, Ω_1 and Ω_2 are open bounded subsets of C_ω with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Hence, $\Phi : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ is a completely continuous operator and satisfies condition (a) in Theorem 2.1. By Krasnoselskii's Theorem, there exists a fixed point $x^*(\cdot) = (x_1^*(\cdot), x_2^*(\cdot), ..., x_n^*(\cdot))^T \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$ such that $x^*(\cdot) = (\Phi x^*)(\cdot)$, i.e., x^* is a positive $\omega-$ periodic solution of system (1.4). The proof is completed. ■

3.2 The case of sublinear impulse functions

In the case of sublinear impulses we can prove similar results for system (1.4).

Theorem 3.2. *Assume that* $(H_1) - (H_5)$ *hold, assume further that:*
(H_8) *There exist nonnegative functions* $\overline{\lambda}_{ik}, \lambda_{ik} \in C(\mathbb{R}^+, \mathbb{R}^+)$ *such that for all* $x \in \mathbb{R}^+$,

$$\overline{\lambda}_{ik}(t)\, x \leq I_{ik}(t, x) \leq \lambda_{ik}(t)\, x, i = 1, 2, ..., n, k = 1, 2, ...$$

(H_9) *The linear system*

$$\sum_{j=1}^{n} \left(\widehat{a}_{ij} + \widehat{b}_{ij} + \widehat{c}_{ij}\right) x_j = \widehat{r}_i, i = 1, 2, ..., n,\ k = 1, 2, ...$$

possesses a unique positive solution. Then, system (1.4) *possesses at least one positive* $\omega-$*periodic solution.*

Proof. To prove that $\Phi : K \to K$ is completely continuous is similar to the corresponding proof in Lemma 3.2. We only need to prove (a) in Theorem 2.1. Let

$$x^* = (x_1^*, x_2^*, ..., x_n^*)^T$$

with $x_i^* > 0, i = 1, 2, ..., n$, be a positive solution of (1.12). Set

$$\begin{aligned} \widetilde{m}_0 &= \min\left\{\min_{1\leq i\leq n}\{\widehat{r}_i A_i\}, \min_{1\leq i\leq n}\left\{A_i \widehat{\beta}_{ik}\right\}\right\}, \\ \widetilde{M}_0 &= \max\left\{\max_{1\leq i\leq n}\{\widehat{r}_i B_i\}, \max_{1\leq i\leq n}\left\{B_i \widehat{\beta}_{ik}\right\}\right\}. \end{aligned}$$

Choose a constant $\widetilde{M} \geq \widetilde{M}_0$ such that $0 < \dfrac{1 - \widetilde{M}\omega}{\widetilde{M}\omega} < 1$ where $0 < \widetilde{M}\omega < 1$.

Let $\eta_1 = \dfrac{1 - \widetilde{M}\omega}{\widetilde{M}\omega}$ and

$$\widehat{\Omega}_1 = \left\{x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in C_\omega : |x_i|_0 < \eta_1 x_i^*, i = 1, 2, ..., n\right\}.$$

If $x \in K \cap \partial\widehat{\Omega}_1$, then

$$\sigma \left|x_i\right|_0 \leq x_i\left(t\right) \leq \left|x_i\right|_0 = \eta_1 x_i^*, i = 1, 2, ..., n,$$

and

$$\begin{aligned}
\left(\Phi_i x\right)(t) &\leq B_i \int_0^{\omega} \left\{ \left[x_i(s) \sum_{j=1}^{n} a_{ij}(s) h_j\left(x_j(s)\right) \right.\right. \\
&\quad + x_i(s) \sum_{j=1}^{n} b_{ij}(s) f_j\left(x_j\left(s - \tau_{ij}(s)\right)\right) \\
&\quad \left. + x_i(s) \sum_{j=1}^{n} c_{ij}(s) \int_{-\infty}^{s} D_{ij}(s,z) g_j\left(x_j(z)\, dz\right) \right] ds \\
&\quad \left. + \sum_{t \leq t_k < t+\omega} I_{ik}\left(t_k, x_i\left(t_k\right)\right) \right\} \\
&\leq B_i \int_0^{\omega} \left|x_i\right|_0 \sum_{j=1}^{n} T_j a_{ij}(s) \left|x_j\right|_0 ds + B_i \int_0^{\omega} \left|x_i\right|_0 \sum_{j=1}^{n} F_j b_{ij}(s) \left|x_j\right|_0 ds \\
&\quad + B_i \int_0^{\omega} \left|x_i\right|_0 \sum_{j=1}^{n} R_j E_{ij} c_{ij}(s) \left|x_j\right|_0 ds + B_i \left|x_i\right|_0 \sum_{t \leq t_k < t+\omega} \lambda_i\left(t_k\right) \left|x_i\right|_0 \\
&\leq \eta_1 B_i \omega \left|x_i\right|_0 \sum_{j=1}^{n} \widehat{a}_{ij} x_j^* + \eta_1 B_i \omega \left|x_i\right|_0 \sum_{j=1}^{n} \widehat{b}_{ij} x_j^* \\
&\quad + \eta_1 B_i \omega \left|x_i\right|_0 \sum_{j=1}^{n} \widehat{c}_{ij} x_j^* + B_i \omega \widehat{\beta}_{ik} \left|x_i\right|_0 \\
&= \eta_1 B_i \omega \left|x_i\right|_0 \left\{ \sum_{j=1}^{n} \left(\widehat{a}_{ij} + \widehat{b}_{ij} + \widehat{c}_{ij}\right) x_j^* \right\} + B_i \omega \widehat{\beta}_{ik} \left|x_i\right|_0 \\
&= \omega\left(B_i \widehat{r}_i\right) \eta_1 \left|x_i\right|_0 + \omega\left(B_i \widehat{\beta}_{ik}\right) \left|x_i\right|_0 \\
&\leq \eta_1 \widetilde{M}_0 \omega \left|x_i\right|_0 + \widetilde{M}_0 \omega \left|x_i\right|_0 \\
&\leq \left|x_i\right|_0, \quad i = 1, 2, ..., n.
\end{aligned}$$

Hence for any $x \in K \cap \partial\widehat{\Omega}_1$

$$\left\|\Phi x\right\| = \sum_{i=1}^{n} \left|\left(\Phi_i x\right)\right|_0 \leq \sum_{i=1}^{n} \left|x_i\right|_0 = \left\|x\right\|.$$

On the other hand, choose $0 < \widetilde{m} \leq \widetilde{m}_0$ such that $\dfrac{1 - \sigma\widetilde{m}\theta\omega}{\sigma^2\widetilde{m}\theta\omega} > 1$ where

$0 < \sigma\widetilde{m}\theta\omega < 1$. Let $\eta_2 = \dfrac{1-\sigma\widetilde{m}\theta\omega}{\sigma^2\widetilde{m}\theta\omega}$ and

$$\widehat{\Omega}_2 = \{x \in C_\omega : |x_i|_0 < \eta_2 x_i^*, i = 1, 2, ..., n\}.$$

If $x \in K \cap \partial\widehat{\Omega}_2$, then $\sigma |x_i|_0 \leq x_i(t) \leq |x_i|_0 = \eta_2 x_i^*, i = 1, 2, ..., n$, and, consequently

$$\begin{aligned}
(\Phi_i x)(t) \quad \geq \quad & A_i \int_0^\omega x_i(s) \left\{ \sum_{j=1}^n a_{ij}(s) h_j(x_j(s)) + \sum_{j=1}^n b_{ij}(s) f_j(x_j(s-\tau_{ij}(s))) \right. \\
& \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\infty}^s D_{ij}(s,z) g_j(x_j(z)\, dz \right\} ds \\
& + \sum_{t \leq t_k < t+\omega} G_i(t, t_k) I_{ik}(t_k, x_i(t_k)) \\
\geq \quad & \sigma^2 A_i |x_i|_0 \sum_{j=1}^n \int\limits_0^\omega a_{ij}(s) T_j \left[\min_{j=\overline{1,n}} \left(\frac{\overline{T}_j}{T_j} \right) \right] |x_j|_0 \, ds \\
& + \sigma^2 A_i |x_i|_0 \sum_{j=1}^n \int\limits_0^\omega F_j \left[\min_{i=\overline{1,n}} \left(\frac{\overline{F}_j}{F_j} \right) \right] b_{ij}(s) |x_j|_0 \, ds \\
& + \sigma^2 A_i |x_i|_0 \sum_{j=1}^n \int_0^\omega E_{ij} R_j \left\{ \left[\min_{i=\overline{1,n}} \left(\frac{\overline{E}_{ij}}{E_{ij}} \right) \right] \times \left(\frac{\overline{R}_j}{R_j} \right) \right\} c_{ij}(s) |x_j|_0 \, ds \\
& + \sigma A_i \sum_{t \leq t_k < t+\omega} \lambda_{ik}(t_k) |x_i(t_k)|_0 \\
\geq \quad & \theta_1 \times \sigma^2 A_i \omega \eta_2 |x_i|_0 \sum_{j=1}^n \widehat{a}_{ij} x_j^* + \theta_2 \times \sigma^2 A_i \omega \eta_2 |x_i|_0 \sum_{j=1}^n \widehat{b}_{ij} x_j^* \\
& + \theta_3 \times \sigma^2 A_i \omega \eta_2 |x_i|_0 \sum_{j=1}^n \widehat{c}_{ij} x_j^* + 1 \times \sigma A_i \omega |x_i|_0 \widehat{\beta}_{ik}
\end{aligned}$$

$$\begin{aligned}
&\geq \theta\times\sigma^2 A_i\omega\eta_2\left|x_i\right|_0\sum_{j=1}^{n}\widehat{a}_{ij}x_j^*+\theta\times\sigma^2 A_i\omega\eta_2\left|x_i\right|_0\sum_{j=1}^{n}\widehat{b}_{ij}x_j^* \\
&\quad +\theta\times\sigma^2 A_i\omega\eta_2\left|x_i\right|_0\sum_{j=1}^{n}\widehat{c}_{ij}x_j^* \\
&\quad +\theta\times\sigma A_i\omega\left|x_i\right|_0\widehat{\beta}_{ik} \\
&= \theta A_i\omega\sigma^2\eta_2\left|x_i\right|_0\sum_{j=1}^{n}\left(\widehat{a}_{ij}+\widehat{b}_{ij}+\widehat{c}_{ij}\right)x_j^* \\
&\quad +\theta\times\sigma\left(A_i\widehat{\beta}_{ik}\right)\omega\left|x_i\right|_0 \\
&= \left(A_i\widehat{r}_i\right)\theta\omega\sigma^2\eta_2\left|x_i\right|_0+\left(A_i\widehat{\beta}_{ik}\right)\omega\theta\sigma\left|x_i\right|_0 \\
&\geq \widetilde{m}_0\theta\omega\sigma^2\eta_2\left|x_i\right|_0+\widetilde{m}_0\theta\omega\sigma\left|x_i\right|_0 \\
&\geq \left|x_i\right|_0, i=1,2,...,n,
\end{aligned}$$

and therefore

$$\|\Phi x\|=\sum_{i=1}^{n}\left|(\Phi_i x)\right|_0\geq\sum_{i=1}^{n}\left|x_i\right|_0=\|x\|, \forall x\in K\cap\partial\widehat{\Omega}_2.$$

Hence, $\Phi : K\cap(\overline{\widehat{\Omega}}_2\backslash\widehat{\Omega}_1)\to K$ is a completely continuous operator and satisfies condition (a) in Theorem 2.1. Consequently, there exists a fixed point $x^*(\cdot)=(x_1^*(\cdot),x_2^*(\cdot),...,x_n^*(\cdot))^T\in K\cap(\overline{\widehat{\Omega}}_2\backslash\widehat{\Omega}_1)$ such that $x^*(\cdot)=(\Phi x^*)(\cdot)$. Therefore, system (1.4) has a positive ω-periodic solution. The proof is completed. ■

Remark 3.1: The method applied in this paper can be used to treat a more general nonlinear impulse function. For instance, assuming that $I_{ik}(\cdot,x)$ satisfies

(H_{10}) There exist nonnegative functions $\overline{\lambda}_{ik},\lambda_{ik}\in C(\mathbb{R}^+,\mathbb{R}^+)$ and $F\in C\left(\mathbb{R}_+^n,\mathbb{R}_+^n\right)$ such that for all $x\in\mathbb{R}_+^n$,

$$\overline{\lambda}_{ik}(t)F(\|x\|)\leq I_{ik}(t,x)\leq\lambda_{ik}(t)F(\|x\|), i=1,2,...,n, k=1,2,...$$

Note that (H_6) and (H_8) are special cases of condition (H_{10}) which has been used in [6, 7, 8].

Remark 3.2: Notice that when $a_{ij}=0$ in the second term on the right hand side of (1.4), $I_{ik}(t_k,x(t_k))=0$, $f_j(x_j)=0$, and $g_j(x_j)=x_j$, we can easily derive the corresponding results in [25]. Therefore, the results presented in this paper improve and extend the main results in Ref. [25].

4 An example

In this section, we analyze an example to show the effectiveness of our result.

Example 4.1. Let us consider the following system:

$$\begin{aligned} x_i'(t) &= x_i(t)\left\{ r_i(t) - \sum_{j=1}^{2} b_{ij}(t) f_j\left(x_j(t-\tau_{ij}(t))\right) \right. \\ &\qquad \left. - \sum_{j=1}^{2} c_{ij}(t) \int_{-\infty}^{t} D_{ij}(t,s) g_j\left(x_j(s)\, ds\right) \right\}, \quad (4.1) \\ t &\neq t_k = \frac{1}{2}k\pi, k \in \mathbb{Z}^+, t > 0, \\ x_i(t_k^+) - x_i(t_k^-) &= I_{ik}\left(t_k, x_i(t_k)\right) = \frac{x_i^2(t_k)}{\cos(2t_k)}\left(|\sin x_i(t_k)| + 1\right), \quad (4.2) \end{aligned}$$

for $i = 1, 2$. This model corresponds to system (1.4) when $n = 2, \omega = 2\pi$. Let

$$r_1(t) = \frac{1}{2}(1 + \sin 2t), r_2(t) = \frac{1}{3}(1 + \cos 2t),$$

and $\tau_{ij} \in (\mathbb{R}^+, \mathbb{R}^+)$ be arbitrary continuous functions which satisfy $\tau_{ij}(t+\omega) = \tau_{ij}(t), i = 1, 2$.
We then have

$$\begin{aligned} \widehat{r}_1 &= \frac{1}{\omega}\int_0^{\omega} r_1(t)dt = \frac{1}{4\pi}\int_0^{2\pi}(1 + \sin 2t)\, dt = \frac{1}{2}, \\ \widehat{r}_2 &= \frac{1}{\omega}\int_0^{\omega} r_2(t)dt = \frac{1}{6\pi}\int_0^{2\pi}(1 + \cos 2t)\, dt = \frac{1}{3}, \end{aligned}$$

and it is straightforward to check that $A_i \leq G_i(t,s) \leq B_i$, for $i = 1, 2$, where

$$G_i(t,s) = \frac{1}{1 - e^{-\widehat{r}_i\omega}} \exp\left(-\int_t^s r_i(\xi)\, d\xi\right), \ i = 1, 2,$$

and

$$\begin{aligned} A_1 &:= \frac{e^{-\widehat{r}_1\omega}}{1 - e^{-\widehat{r}_1\omega}} = \frac{e^{-\pi}}{1 - e^{-\pi}}, \qquad A_2 := \frac{e^{-\widehat{r}_2\omega}}{1 - e^{-\widehat{r}_2\omega}} = \frac{e^{-\frac{2\pi}{3}}}{1 - e^{-\frac{2\pi}{3}}}, \\ B_1 &:= \frac{1}{1 - e^{-\widehat{r}_1\omega}} = \frac{1}{1 - e^{-\pi}}, \qquad B_2 := \frac{1}{1 - e^{-\widehat{r}_2\omega}} = \frac{1}{1 - e^{-\frac{2\pi}{3}}}. \end{aligned}$$

Let

$$\begin{aligned} f_j(x) &= \frac{x}{2}e^{(\sin x)+1}, \qquad g_j(x) = \frac{x}{3}\left(e^{|\cos x|} + 1\right), \\ I_i(t,x) &= \frac{\pi x^2}{\cos 2t}\left(|\sin x| + 1\right), \qquad i, j = 1, 2, \end{aligned}$$

and

$$\begin{aligned} D_{ij}(t,s) &= e^{s-t}(\cos t + 2) \times \frac{3}{2}, \qquad i \neq j = 1, 2, \\ D_{ij}(t,s) &= e^{s-t}(\sin t + 4) \times \frac{1}{3}, \qquad i = j = 1, 2. \end{aligned}$$

Since $|\sin x| \leq 1$ and $|\cos x| \leq 1$ for $x \in \mathbb{R}^+$, we have

$$\begin{aligned}\overline{F}_j x &\leq f_j(x) \leq F_j x,\\ \overline{R}_j x &\leq g_j(x) \leq R_j x,\end{aligned}$$

and

$$\begin{aligned}\frac{3}{2} &= \overline{E}_{ij} \leq \int_{-\infty}^{t} D_{ij}(t,s)\,ds \leq E_{ij} = \frac{9}{2}, i \neq j,\\ 1 &= \overline{E}_{ij} \leq \int_{-\infty}^{t} D_{ij}(t,s)\,ds \leq E_{ij} = \frac{5}{3}, i = j,\end{aligned}$$

and

$$\overline{\lambda}_i(t)\,x^2 \leq I_i(t,x) \leq \lambda_i(t)\,x^2,$$

where $F_j = \frac{e^2}{2}, \overline{F}_j = \frac{1}{2}$, $R_j = \frac{e+1}{3}, \overline{R}_j = \frac{2}{3}$, $\overline{\lambda}_i(t) = \frac{\pi}{\cos 2t}, \lambda_i(t) = \frac{2\pi}{\cos 2t}, i, j = 1, 2.$

We can choose $b_{11}(t) = \frac{(1+\cos 2t)}{3F_1}, b_{12}(t) = \frac{(1+\sin 2t)}{2F_2}, b_{21}(t) = 0, b_{22}(t) = \cos(4t)$, which implies

$$\begin{aligned}\widehat{b}_{11} &= \frac{F_1}{\omega}\int_0^{\omega} b_{11}(s)ds = \frac{1}{3}, \widehat{b}_{12} = \frac{F_2}{\omega}\int_0^{\omega} b_{12}(s)ds = \frac{1}{2},\\ \widehat{b}_{21} &= \frac{F_1}{\omega}\int_0^{\omega} b_{21}(s)ds = 0,\ \widehat{b}_{22} = \frac{F_2}{\omega}\int_0^{\omega} b_{22}(s)ds = 0,\end{aligned}$$

and also choose $c_{11}(t) = 0, c_{12}(t) = \frac{2(1+\cos 4t)}{E_1 R_{12}}, c_{21}(t) = \frac{(1+\sin 2t)}{E_2 R_{21}}, c_{22}(t) = \frac{(1+\sin 2t)}{2E_2 R_{22}}$, obtaining

$$\begin{aligned}\widehat{c}_{11} &= \frac{E_1 R_{11}}{\omega}\int_0^{\omega} c_{11}(s)ds = 0, \widehat{c}_{12} = \frac{E_1 R_{12}}{\omega}\int_0^{\omega} c_{12}(s)ds = 2,\\ \widehat{c}_{21} &= \frac{E_2 R_{21}}{\omega}\int_0^{\omega} c_{21}(s)ds = 1, \widehat{c}_{22} = \frac{E_2 R_{22}}{\omega}\int_0^{\omega} c_{22}(s)ds = \frac{1}{2}.\end{aligned}$$

Choosing $q = 8$, we have

$$\begin{aligned}\widehat{\beta}_{ik} &= \frac{1}{\omega}\sum_{0 \leq t_k < \omega} \lambda_{ik}(t_k) = \frac{1}{2\pi}\sum_{k=1}^{q} \frac{2\pi}{\cos 2t_k}\\ &= \frac{1}{2}\sum_{k=1}^{8} \frac{2}{\cos 2\left(\frac{1}{2}k\pi\right)} = 1, i = 1, 2.\end{aligned}$$

Moreover, it is easy to verify that the corresponding system of nonlinear equation (4.1) ,

$$\begin{cases} \sum_{j=1}^{2} \left(\widehat{b}_{1j} + \widehat{c}_{1j}\right) x_j + \widehat{\beta}_{1k} x_1 = \widehat{r}_1 \\ \sum_{j=1}^{2} \left(\widehat{b}_{2j} + \widehat{c}_{2j}\right) x_j + \widehat{\beta}_{2k} x_2 = \widehat{r}_2, \end{cases}$$

has a unique positive solution $x = (x_1, x_2) = \left(\frac{1}{6}, \frac{1}{9}\right)$. It is straightforward to show that all conditions of Theorem 3.1 are fulfilled. Hence, we conclude that this system possesses at least one positive $2\pi-$periodic solution.

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