# A new result on stabilization analysis for stochastic nonlinear affine systems via Gamidov's inequality

Tomás Caraballo[a*] Faten Ezzine[b]
Mohamed Ali Hammami[c]

[a]Universidad de Sevilla,
Depto. Ecuaciones Diferenciales y Análisis Numérico,
Facultad de Matemáticas, c/ Tarfia s/n, 41012-Sevilla (Spain)
[{b,c}]University of Sfax, Faculty of Sciences of Sfax, Department of Mathematics, Tunisia
b: E.mail: ezzinefaten94fss@gmail.com
c: E.mail: MohamedAli.Hammami@fss.rnu.tn

**Abstract**

The Lyapunov approach is one of the most effective and efficient methods for the investigation of the stability of stochastic systems. Several authors analyzed the stability and stabilization of stochastic differential equations via Lyapunov techniques. Nevertheless, few results are concerned with the stability of stochastic systems based on the knowledge of the solution of the system explicitly. The originality of our work is to investigate the problem of stabilization of stochastic perturbed control–bilinear systems based on the explicit solution of the system by using the integral inequalities of the Gronwall type in particular Gamidov's inequality. Namely, under some restrictions on the perturbed term, and based on the method of integral inequalities, we prove that the stochastic system can be stabilized by constant feedback. Further, we study the problem of stabilization of stochastic perturbed control affine systems based on the use of bilinear approximation. Different examples are provided to verify the effectiveness of the proposed results.

**Keywords:** Affine systems, Bilinear systems, Stochastic systems, Integral inequalities, Gamidov inequalities, Stabilization, Practical stability.
**Mathematics Subject Classification** Primary 93E03 Secondary 60H10.

*The research of Tomás Caraballo has been partially supported by the Spanish Ministerio de Ciencia, Innovación y Universidades (MCIU), Agencia Estatal de Investigación (AEI) and Fondo Europeo de Desarrollo Regional (FEDER) under the project PGC2018-096540-B-I00, and by Junta de Andalucía (Consejería de Economía y Conocimiento) and FEDER under projects US-1254251 and P18-FR-4509.

# 1 Introduction

Gronwall's inequality was first proposed and proved as its differential form in 1911 by the Swedish mathematician Thomas Hacon Gronwall [19]. Later on, the integral form was proved in 1943 by the American mathematician Bellmen [8]. Since then, many authors introduced several generalizations of this inequality, see [6, 7, 15, 20, 22].

Gronwall Lemmas are very useful in many analysis problems. In particular, it is an essential tool in the analysis of the problem of boundedness, uniqueness, and other aspects of the qualitative behavior of solutions of differential and stochastic equations.

As it is well known, environmental noise exists and cannot neglect it in many dynamical systems. Indeed, it is essential to analyze whether the presence of some random terms in the equations of the models may produce a very different behavior of their solutions. Although there exists a wide literature on this topic, see [1, 21], [23]-[25].

The result is related to the relation between a perturbed stochastic system and the associated unperturbed one. Given two solutions to the perturbed stochastic system and the associated unperturbed one with initial conditions that are close at the same value of time, these solutions will remain close over the entire time interval and not just at the initial time.

Different intrinsic variants to Lyapunov's original concepts of practical stability were proposed in [4, 16]. In the case that the origin is not an equilibrium point, we can investigate the stability of the SDEs in a small neighborhood of the origin in terms of convergence of solution in probability to a small ball. This property is defined as "Practical Stability". The practical stability, in the sense introduced in [2, 5, 9, 17, 18, 26]. In fact, it is very important and very useful for analyzing the stability or for designing practical controllers of dynamical systems since controlling system to an idealized point are either expensive or impossible in the presence of disruptions and the best which we can hope in such situations is to use practical stability. In practice, we may only need to stabilize a system into the region of phase space in which the implementation is still acceptable. It is well known that asymptotic stability is more important than stability. Also, the desired system may oscillate near the origin. Thus, the notion of practical stability is more suitable in several situations than asymptotic stability. In this case, all state trajectories are bounded and approach a sufficiently small neighborhood of the origin. One also desires that the state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner especially in presence of perturbations. In general, we know some information on the upper bound of the term of perturbation whose size influences the size of the ball.

The Lyapunov method is one of the most effective ones for the investigation of the stability of stochastic systems, without knowing the explicit solution form of the system. The Lyapunov stability has attracted the attention of many authors. There exists a huge amount of work on this subject, see [10, 11, 12, 13, 14]. However, the construction of a suitable Lyapunov function is still a difficult task. The novelty of our work is to develop the problem of stability and stabilization of stochastic perturbed control-bilinear systems based on the explicit solution formed through integral inequalities of the Gronwall type, in particular Gamidov's inequality.

The qualitative behavior of the solutions of perturbed stochastic systems is usually studied by considering a Lyapunov function candidate for the unperturbed system and using it as an appropriate Lyapunov function candidate for the stochastic perturbed system. Nevertheless, unlike the linear case, the construction of a suitable Lyapunov function is still a difficult task for nonlinear stochastic differential equations. This motivates us to investigate the problem of stability of stochastic perturbed systems by using integral inequalities of Gronwall type under some restrictions on the perturbation term. The usual property of the solutions that can be deduced for such systems is ultimate boundedness. That means that the solutions remain in some neighborhoods of the origin after a sufficiently large time, (see [3], [10]-[13]). In different cases, the linearized system is independent of the control. Therefore, we can study the stabilization problem for such systems via a bilinear system.

The organization of this paper is as follows: In Section 2, we investigate the stability of linear time–invariant stochastic perturbed systems via non linear–integral inequalities. In Section 3, we analyze the problem of stabilization of stochastic perturbed control–bilinear systems under some restrictions on the bound of perturbations. In Section 4, we prove that the problem of stabilization of stochastic perturbed affine system can be performed by considering a bilinear approximation. Further, we display some illustrative examples to exhibit the applicability of our abstract theory. In Section 5, some conclusions are included.

## 2 Linear time–invariant stochastic perturbed systems

Consider the following linear time–invariant system:

$$dx(t) = Ax(t)dt, \tag{2.1}$$

where $x \in \mathbb{R}^n$, $A$ is a constant matrix $(n \times n)$ .

Assume that some parameters are excited or perturbed by Brownian motion, and the linear time-invariant stochastic perturbed system is expressed by the following form:

$$dx(t) = Ax(t)dt + \phi(t, x(t))dB_t, \tag{2.2}$$

where $\phi : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m}$, $B_t = (B_1(t), ..., B_m(t))^T$ is an $m$-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. In the sequel, $||\cdot||$ represents the Euclidean norm.

We will study the asymptotic behaviors of the solutions of the stochastic perturbed system (2.2) in the sense that all state trajectories are bounded and approach a sufficiently small neighborhood of the origin. In this objective, we recall the following definitions, see [10, 11, 12, 13].

Let's consider the following stochastic system :

$$dx(t) = F(t, x(t))dt + G(t, x(t))dB_t, \quad t \geq 0, \tag{2.3}$$

where $x \in \mathbb{R}^n,\ F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n,\ G : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $B_t$ is an $m$-dimensional Brownian motion.

We assume that both functions $F$ and $G$ satisfy the following conditions:

$$||F(t,x)||^2 + ||G(t,x)||^2 \leq \alpha_1\big(1 + ||x||^2\big),\ \text{for all } t \geq 0,\ x \in \mathbb{R}^n,$$

$$||F(t,x) - F(t,y)|| \vee ||G(t,x) - G(t,y)|| \leq \alpha_2 ||x - y||,\ \text{for all } t \geq 0,\ x, y \in \mathbb{R}^n,$$

where $\alpha_1$ and $\alpha_2$ are given positive real constant, then under the precedent assumptions, there exists a unique global solution $x(t)$ corresponding to the initial condition $x_0 \in \mathbb{R}^n$ (see Mao [25]).

We assume that there exists $t$ such that $F(t,0) \neq 0$ or $G(t,0) \neq 0$, i.e., the stochastic differential equation (2.3) does not have the trivial solution $x \equiv 0$.

The study of the exponential stability of the solutions of the stochastic system (2.3) leads to analyze the stability behavior of a ball centered at the origin:

$$\mathbb{B}_r := \{x \in \mathbb{R}^n : ||x|| \leq r\},\ r > 0.$$

**Definition 2.1.**

i) The ball $\mathbb{B}_r$ is said to be almost surely globally uniformly exponentially stable, if there exists a pair of positive constants $\mu_1$ and $\mu_2$, such that for all $x_0 \in \mathbb{R}^n$, the following inequalities are satisfied:

$$||x(t)|| \leq \mu_1 ||x_0|| e^{-\mu_2 t} + r, \quad \text{a.s.}, \quad \forall t \geq 0. \tag{2.4}$$

ii) The stochastic system (2.3) is said to be almost surely globally practically uniformly exponentially stable, if there exists $r > 0$ such that $\mathbb{B}_r$ is almost surely globally uniformly exponentially stable.

Eq.(2.4) implies that $x(t)$ will be bounded by a small bound $r > 0$, that is $||x(t)||$ will be small for sufficiently large $t$. It means that the solution given in (2.4) will be uniformly ultimately bounded for sufficiently large $t$. This means that solution given in (2.4) will be uniformly ultimately bounded for sufficiently large $t$. The factor $\mu_2$ in Eq.(2.4) is called the convergence speed, whereas the factor $\mu_1$ is called the transient estimate.

It is also worth noticing that, in the previous definition, if we take $r = 0$, then we recover the standard concept of the global exponential stability of the origin considered as an equilibrium point. Further, we out to investigate the asymptotic behavior of a small ball centered at the origin for $0 \leq ||x(t)|| - r,\ \forall t \geq 0$, so that the initial conditions are taken outside the ball $\mathbb{B}_r$. If $r$ is small enough, the trajectories tend to the origin when $t$ goes to infinity.

Our main objective in this section is to state sufficient conditions ensuring the almost sure practical uniform exponential stability of the linear time–invariant stochastic system (2.2). If we suppose that the perturbation term is bounded, then the origin is not necessarily an equilibrium point of the stochastic perturbed system (2.2). That's why, we will study the convergence of the solutions toward a neighborhood of origin.

**Remark 2.1.** Different authors tackle the problem of practical stability of stochastic differential equations via Lyapunov functions, see [10, 11, 12, 13, 14]. The construction of appropriate Lyapunov functions is not always possible, which motivates us to look for another method. Our approach in this paper is to analyze the stability and stabilization of stochastic perturbed control bilinear systems by using the explicit solution form and it is based on integral inequalities of the Gronwall type.

We suppose that the matrix $A$ is asymptotically stable. A basic result in systems theory is that

$$\sigma(A) \subset \mathbb{C}^-,$$

where $\sigma(A)$ denotes the set of eigenvalues of a the matrix $A$. With this condition, we have $\mathcal{R}e\lambda(A) < 0$, where $\mathcal{R}e\lambda(A)$ denotes the real parts of the eigenvalues of matrix $A$. A simple result for the asymptotic stability is that the eigenvalues of the matrix (all of which are real) remain strictly in the left-half complex plane: $\mathcal{R}e\lambda(A) < 0$.

First, we suppose the following assumption required for the stability purposes.

$(\mathcal{H}_1)$ We assume that, $\mathcal{R}e\lambda(A) < 0$. Note that, the assumption $(\mathcal{H}_1)$ implies that

$$||e^{tA}|| \leq ke^{-\gamma t}, \quad \forall t \geq 0,$$

for a certain $k > 0$ and

$$\gamma \leq \min_{1\leq i\leq n} |\mathcal{R}e\lambda_i(A)|.$$

$(\mathcal{H}_2)$ There exists a continuous nonnegative known function $\xi(t)$, such that

$$||\phi(t,x)|| \leq \xi(t), \quad \forall x \in \mathbb{R}^n, \quad \forall t \geq 0,$$

where $\lim_{t\to+\infty} \xi(t) = 0$.

**Theorem 2.2.** *Under assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ the linear time–invariant stochastic perturbed system* (2.2) *is almost surely globally practically uniformly exponentially stable.*

In order to prove Theorem 2.2, we need to recall the following lemma.

**Lemma 2.3.** *[25] Let $G = (G_1, ...., G_n) \in L^2(\mathbb{R}_+, \mathbb{R}^n)$, $T$, $a$, $b$ be any positive numbers. Then,*

$$\mathbb{P}\left(\sup_{0\leq t\leq T}\left[\int_0^t G(s)dB_s - \frac{a}{2}\int_0^t ||G(s)||^2 ds\right] > b\right) \leq e^{-ab}.$$

**Proof of Theorem 2.2.** The solution with initial condition $x_0$ of the linear time–invariant stochastic perturbed system is the following:

$$x(t) = e^{tA}\, x_0 + \int_0^t e^{(t-s)A}\, \phi(s, x(s))dB_s.$$

Assign $\varepsilon \in ]0, 1[$ arbitrarily, and let $n = 1, 2, \cdots$ . Then, using Lemma 2.3, we obtain

$$\mathbb{P}\left\{ \sup_{0\le t\le n} \left[ \int_0^t e^{(t-s)A}\, \phi(s, x(s))dB_s - \frac{\varepsilon}{2}\int_0^t ||e^{(t-s)A}||^2\, ||\phi(s, x(s))||^2 ds \right] > \frac{2}{\varepsilon}\frac{\ln(n)}{n} \right\} \le \frac{e^{\frac{1}{n}}}{n^2}.$$

By an application of the well known Borel-Cantelli lemma, we see that for almost all $\omega \in \Omega$, there exists an integer $n_0 = n_0(\omega)$, such that if $n \ge n_0$, it yields that

$$\int_0^t e^{(t-s)A}\, \phi(s, x(s))dB_s \le \frac{2}{\varepsilon}\frac{\ln(n)}{n} + \frac{\varepsilon}{2}\int_0^t ||e^{(t-s)A}||^2\, ||\phi(s, x(s))||^2 ds, \quad \text{for all } 0 \le t \le n. \tag{2.5}$$

Based on inequality (2.5), one obtains

$$||x(t)|| \le ||e^{tA}||\, ||x_0|| + \frac{\varepsilon}{2}\int_0^t ||e^{(t-s)A}||^2\, ||\phi(s, x(s))||^2 ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n}, \quad \text{for all } 0 \le t \le n,\ n \ge n_0.$$

Taking into account assumption $(\mathcal{H}_1)$ and inequality (3.6), one deduce

$$||x(t)|| \le k||x_0||e^{-\gamma t} + \frac{\varepsilon}{2}\int_0^t k^2 e^{-2\gamma(t-s)}||\phi(s, x(s))||^2 ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n}, \quad \text{for all } 0 \le t \le n,\ n \ge n_0.$$

Using assumption $(\mathcal{A}_2)$, one obtains

$$||x(t)|| \le k||x_0||e^{-\gamma t} + \frac{\varepsilon}{2}k^2 e^{-2\gamma t}\int_0^t e^{2\gamma s}\xi^2(s)ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n}, \quad \text{for all } 0 \le t \le n,\ n \ge n_0.$$

Since $\xi(\cdot) \to 0$ as $t \to +\infty$, then there exists $\bar{\xi} > 0$, such that

$$\xi(t) \le \bar{\xi}, \quad \forall t \ge 0.$$

$$\begin{aligned}
||x(t)|| &\le k||x_0||e^{-\gamma t} + \frac{\varepsilon}{2}k^2 e^{-2\gamma t}\int_0^t e^{2\gamma s}\bar{\xi}^2 ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n} \\
&\le k||x_0||e^{-\gamma t} + \frac{\varepsilon}{2}k^2 e^{-2\gamma t}\int_0^t e^{2\gamma s}\bar{\xi}^2 ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n} \\
&\le k||x_0||e^{-\gamma t} + \frac{\varepsilon}{2}\frac{\bar{\xi}^2}{2\gamma}k^2 e^{-2\gamma t}(e^{2\gamma t} - 1) + \frac{2}{\varepsilon}\frac{\ln(n)}{n} \\
&\le k||x_0||e^{-\gamma t} + \frac{\varepsilon}{2}\frac{\bar{\xi}^2}{2\gamma}k^2 + \frac{2}{\varepsilon}\frac{\ln(n)}{n} \quad \text{for all } 0 \le t \le n,\ n \ge n_0.
\end{aligned}$$

Further, sine $\dfrac{\ln(n)}{n} \to 0$ as $n \to +\infty$, then there exists $\alpha > 0$, such that

$$\frac{\ln(n)}{n} \leq \alpha, \quad \forall n \geq n_0. \tag{2.6}$$

Then, one obtains

$$||x(t)|| \leq k||x_0||e^{-\gamma t} + \frac{\bar{\xi}^2}{4\gamma}\varepsilon k^2 + \frac{2}{\varepsilon}\alpha, \quad \text{for all } 0 \leq t \leq n,\ n \geq n_0.$$

Letting $\varepsilon \to 1$, we deduce that

$$||x(t)|| \leq k||x_0||e^{-\gamma t} + \frac{\bar{\xi}^2 k^2}{4\gamma} + 2\alpha, \quad \text{a.s.}$$

Thus, the ball $\mathbb{B}_r$ with $r = \dfrac{\bar{\xi}^2 k^2}{4\gamma} + 2\alpha$ is almost surely globally uniformly exponentially stable. Then, the stochastic perturbed system (2.2) is almost sure globally practically uniformly exponentially stable. □

A simple extension can be done, if we replace the assumption $(\mathcal{H}_2)$ by the following assumption on the perturbed term.

$(\mathcal{H}_2')$ There exist a continuous positive functions $\lambda_1(t)$ and $\lambda_2(t)$, such that

$$||\phi(t,x)||^2 \leq e^{-\mu_2 t}\left(\lambda_1(t)||x||^q + \lambda_2(t)\right), \quad 0 < q < 1,\ \forall x \in \mathbb{R}^n,\ t \geq 0,$$

where $\lambda_1$ satisfies

$$\lambda_1(t) \leq m, \quad \forall t \geq 0, \tag{2.7}$$

and the function $\lambda_2$ satisfies the following

$$\int_0^{+\infty} e^{\mu_2 s}\lambda_2(s)ds = \bar{\lambda}_2 < +\infty. \tag{2.8}$$

**Theorem 2.4.** *Under assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2')$, the stochastic perturbed system* (2.2) *is almost surely globally practically uniformly exponentially stable.*

The proof of Theorem 2.4 is based on the generalized integral inequality of the Gamidov type [20].

**Lemma 2.5.** *If*

$$V(t) \leq \epsilon(t) + C\int_0^t \Phi(s)V^q(s)ds,$$

*where all functions are continuous and nonnegative on* $[0,T)$, $0<q<1$ *and* $T,C>0$. *Then, there exists a constant* $\rho>0$, *such that*

$$V(t)\leq \epsilon(t)+C\rho^q\left(\int_0^t \Phi^{\frac{1}{1-q}}(s)ds\right)^{1-q}.$$

Taking into account the above integral inequality, we can prove the almost sure global practical uniform exponential stability of the stochastic perturbed system (2.2).

**Proof of Theorem 2.4** The solution with initial condition $x_0$ of the stochastic perturbed system (2.2) is:

$$x(t)=e^{tA}\,x_0+\int_0^t e^{(t-s)A}\,\phi(s,x(s))dB_s. \tag{2.9}$$

Using assumption $(\mathcal{H}_1)$ and Lemma 2.3, it yields that

$$||x(t)||\leq k||x_0||e^{-\gamma t}+\frac{\varepsilon}{2}\int_0^t k^2e^{-2\gamma(t-s)}||\phi(s,x(s))||^2ds+\frac{2}{\varepsilon}\frac{\ln(n)}{n},$$

for all $0\leq t\leq n,\ n\geq n_0$.

Based on $(\mathcal{H}_2')$, we obtain

$$||x(t)||\leq k||x_0||e^{-\gamma t}+\frac{\varepsilon}{2}\int_0^t k^2e^{-2\gamma(t-s)}e^{-\gamma s}\left(\lambda_1(s)||x(s)||^q+\lambda_2(s)\right)ds+\frac{2}{\varepsilon}\frac{\ln(n)}{n},$$

for all $0\leq t\leq n,\ n\geq n_0$.

Thus, one obtains

$$||x(t)||\leq k||x_0||e^{-\gamma t}+\frac{\varepsilon}{2}k^2e^{-2\gamma t}\int_0^t e^{\gamma s}\left(\lambda_1(s)||x(s)||^q+\lambda_2(s)\right)ds+\frac{2}{\varepsilon}\frac{\ln(n)}{n},$$

for all $0\leq t\leq n,\ n\geq n_0$.

Multiplying both sides by $e^{\gamma t}$, it yields that

$$\begin{aligned}||x(t)||e^{\gamma t}&\leq k||x_0||+\frac{\varepsilon}{2}k^2e^{-\gamma t}\int_0^t e^{\gamma s}\left(\lambda_1(s)||x(s)||^q+\lambda_2(s)\right)ds+\frac{2}{\varepsilon}\frac{\ln(n)}{n}e^{\gamma t}\\&\leq k||x_0||+\frac{\varepsilon}{2}k^2\int_0^t e^{\gamma s}\left(\lambda_1(s)||x(s)||^q+\lambda_2(s)\right)ds+\frac{2}{\varepsilon}\frac{\ln(n)}{n}e^{\gamma t},\end{aligned}$$

for all $0\leq t\leq n,\ n\geq n_0$.

Using (2.6) and (2.7), we obtain

$$\begin{aligned}||x(t)||e^{\gamma t} &\leq k||x_0|| + \frac{\varepsilon}{2}k^2 e^{-\gamma t}\int_0^t e^{\gamma s}\left(m||x(s)||^q + \lambda_2(s)\right)ds + \frac{2}{\varepsilon}\alpha e^{\gamma t}\\ &\leq k||x_0|| + \frac{\varepsilon}{2}k^2 m\int_0^t e^{\gamma s}||x(s)||^q ds + \frac{\varepsilon}{2}k^2\int_0^{+\infty} e^{\gamma s}\lambda_2(s)ds + \frac{2}{\varepsilon}\alpha e^{\gamma t},\end{aligned}$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Let $V(t) = e^{\gamma t}||x(t)||$, and using (4.10), we have

$$V(t) \leq k||x_0|| + \frac{2}{\varepsilon}\alpha e^{\gamma t} + \frac{\varepsilon}{2}k^2\bar{\lambda}_2 + \frac{\varepsilon}{2}k^2 m\int_0^t e^{(1-q)\gamma s}V^q(s)ds,$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Thus,

$$V(t) \leq \epsilon(t) + C\int_0^t e^{(1-q)\gamma s}V^q(s)ds,$$

where $\epsilon(t) = k||x_0|| + \frac{2}{\varepsilon}\alpha e^{\gamma t} + \frac{\varepsilon}{2}k^2\bar{\lambda}_2,\ \ C = \frac{\varepsilon}{2}k^2 m.$

By the application of the Gamidov inequality (Lemma 2.5), it follows that

$$\begin{aligned}V(t) &\leq \epsilon(t) + C\rho^q\left(\int_0^t \exp\left(\frac{1-q}{1-q}\gamma s\right)ds\right)^{1-q}\\ &\leq \epsilon(t) + C\rho^q\left(\int_0^t e^{\gamma s}ds\right)^{1-q}\\ &= \epsilon(t) + C\rho^q\left(\frac{1}{\gamma}\right)^{1-q}e^{\gamma(1-q)t}.\end{aligned}$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Consequently, we obtain

$$||x(t)|| \leq k||x_0||e^{-\gamma t} + \frac{2}{\varepsilon}\alpha + \frac{\varepsilon}{2}k^2\bar{\lambda}_2 + \frac{\varepsilon}{2}k^2 m\rho^q\left(\frac{1}{\gamma}\right)^{1-q},$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Letting $\varepsilon \to 1$, we deduce that

$$||x(t)|| \leq k||x_0||e^{-\gamma t} + 2\alpha + \frac{1}{2}k^2\bar{\lambda}_2 + \frac{1}{2}k^2 m\rho^q\left(\frac{1}{\gamma}\right)^{1-q}.$$

As a consequence, the ball $\mathbb{B}_r$, with $r = 2\alpha + \frac{1}{2}k^2\bar{\lambda}_2 + \frac{1}{2}k^2 m\rho^q \left(\frac{1}{\gamma}\right)^{1-q}$, is almost surely globally uniformly exponentially stable, which in turn gives the linear time–invariant stochastic perturbed system (2.2) is almost sure globally uniformly practically exponentially stable. □

# 3 Application to stochastic perturbed bilinear systems

Let us consider the following bilinear system:

$$dx(t) = (\mathcal{A}x(t) + u\mathcal{B}x(t))\,dt, \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control input vector, $\mathcal{A},\ \mathcal{B} \in \mathbb{R}^{n\times n}$ are constant matrices.

Assume that some parameters are excited or perturbed by Brownian motion, and the perturbed stochastic bilinear system is expressed by the following form:

$$dx(t) = (\mathcal{A}x(t) + u\mathcal{B}x(t))\,dt + \mathcal{G}(t, x(t))dB_t, \tag{3.2}$$

where $\mathcal{G} : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m},\ B_t = (B_1(t), ..., B_m(t))^T$ is an $m$-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

The associated closed–loop stochastic system with a constant feedback $u = \bar{u}$ is given by:

$$dx(t) = (\mathcal{A}x(t) + \bar{u}\mathcal{B}x(t))\,dt + \mathcal{G}(t, x(t))dB_t. \tag{3.3}$$

Our main objective in this section is to state sufficient conditions ensuring the almost sure practical uniform exponential stability of the stochastic perturbed bilinear system (3.2). If we suppose that the perturbation term is bounded, then the origin is not necessarily an equilibrium point of the stochastic perturbed bilinear system (3.2). That's why, we will study the convergence of the solutions toward a neighborhood of origin.

Let us now state some assumptions, which we will impose it later on:

$(\mathcal{A}_1)$ There exists a stabilizing constant feedback $\bar{u}$, such that $\mathcal{R}e\lambda\,(\mathcal{A} + \bar{u}\mathcal{B}) < 0$. ($\mathcal{R}e\lambda(\mathcal{A})$ denotes the real parts of the eigenvalues of matrix $\mathcal{A}$).

$(\mathcal{A}_2)$ There exists a continuous nonnegative known function $\varphi(t)$, such that

$$||\mathcal{G}(t, x)|| \leq \varphi(t), \quad \forall x \in \mathbb{R}^n, \quad \forall t \geq 0.$$

The bounds of the nonlinearities must be in general related to the dynamic of the nominal system, and in our situation, they should be small enough. A restriction on the function $\varphi(t)$ will be imposed to study the asymptotic behavior of the stochastic perturbed bilinear system (3.2).

$(\mathcal{A}_3)$ The continuous nonnegative function $\varphi(t)$ satisfies

$$\int_0^{+\infty} \varphi^2(s)ds < +\infty,$$

or

$$\varphi(t) \leq \widetilde{\varphi} < +\infty, \quad \forall t \geq 0.$$

**Theorem 3.1.** *Under assumptions* $(\mathcal{A}_1) - (\mathcal{A}_3)$ *the stochastic perturbed bilinear system* (3.3) *is almost surely globally practically uniformly exponentially stable.*

**Proof of Theorem 3.1.** Let $\bar{\mathcal{A}} = \mathcal{A} + \bar{u}\mathcal{B}$, then the solution with initial condition $x_0$ of the closed–loop stochastic system (3.3) expressed by the following:

$$x(t) = e^{t\bar{\mathcal{A}}}\, x_0 + \int_0^t e^{(t-s)\bar{\mathcal{A}}}\, \mathcal{G}(s, x(s))dB_s. \tag{3.4}$$

Assign $\varepsilon \in ]0, 1[$ arbitrarily, and let $n = 1, 2, \cdots$ . Then, using Lemma 2.3, we obtain

$$\mathbb{P}\left\{ \sup_{0\leq t\leq n} \left[ \int_0^t e^{(t-s)\bar{\mathcal{A}}}\, \mathcal{G}(s, x(s))dB_s - \frac{\varepsilon}{2} \int_0^t ||e^{(t-s)\bar{\mathcal{A}}}||^2\, ||\mathcal{G}(s, x(s))||^2 ds \right] > \frac{2}{\varepsilon}\frac{\ln(n)}{n} \right\} \leq \frac{e^{\frac{1}{n}}}{n^2}.$$

By an application of the well known Borel-Cantelli lemma, we see that for almost all $\omega \in \Omega$, there exists an integer $n_0 = n_0(\omega)$, such that if $n \geq n_0$, it yields that

$$\int_0^t e^{(t-s)\bar{\mathcal{A}}}\, \mathcal{G}(s, x(s))dB_s \leq \frac{2}{\varepsilon}\frac{\ln(n)}{n} + \frac{\varepsilon}{2} \int_0^t ||e^{(t-s)\bar{\mathcal{A}}}||^2\, ||\mathcal{G}(s, x(s))||^2 ds, \quad \text{for all } 0 \leq t \leq n. \tag{3.5}$$

Based on assumption $(\mathcal{A}_1)$, since $\bar{\mathcal{A}} + \bar{u}\mathcal{B}$ the state feedback stabilization problem reduces to designing $\bar{u}$ to assign the eigenvalues of $\bar{\mathcal{A}} + \bar{u}\mathcal{B}$ in the open left-half complex plane. This yields that there exist two nonnegative constants $\mu_1$ and $\mu_2$, such that

$$||e^{\bar{\mathcal{A}}t}|| = ||e^{(\mathcal{A}+\bar{u}\mathcal{B})t}|| \leq \mu_1 e^{-\mu_2 t}, \tag{3.6}$$

where $\mu_2 \leq \min\limits_{1\leq i\leq n} |\mathcal{R}e\,(\lambda_i(\mathcal{A} + \bar{u}\mathcal{B}))\,|$.

Using inequality (3.5), we have

$$||x(t)|| \leq ||e^{t\bar{\mathcal{A}}}||\, ||x_0|| + \frac{\varepsilon}{2} \int_0^t ||e^{(t-s)\bar{\mathcal{A}}}||^2\, ||\mathcal{G}(s, x(s))||^2 ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n}, \quad \text{for all } 0 \leq t \leq n,\ n \geq n_0.$$

This together with (3.6), implies

$$||x(t)|| \leq \mu_1||x_0||e^{-\mu_2 t} + \frac{\varepsilon}{2}\int_0^t \mu_1^2 e^{-2\mu_2(t-s)}||\mathcal{G}(s,x(s))||^2 ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n}, \quad \text{for all } 0 \leq t \leq n,\ n \geq n_0.$$

Using assumption $(\mathcal{A}_2)$, one obtains

$$||x(t)|| \leq \mu_1||x_0||e^{-\mu_2 t} + \frac{\varepsilon}{2}\mu_1^2 e^{-2\mu_2 t}\int_0^t e^{2\mu_2 s}\varphi^2(s)ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n}, \quad \text{for all } 0 \leq t \leq n,\ n \geq n_0.$$

Since the nonnegative continuous function $\varphi(t)$ satisfies condition $(\mathcal{A}_3)$, then there exists $m > 0$, such that

$$e^{-2\mu_2 t}\int_0^t e^{2\mu_2 s}\varphi^2(s)ds \leq m, \quad \forall t \geq 0,$$

where $m = \min\left(\dfrac{\widetilde{\varphi}^2}{2\mu_2}, ||\varphi||_2^2\right)$. Using (2.6), one obtains

$$||x(t)|| \leq \mu_1||x_0||e^{-\mu_2 t} + \frac{\varepsilon}{2}\mu_1^2 m + \frac{2}{\varepsilon}\alpha, \quad \text{for all } 0 \leq t \leq n,\ n \geq n_0.$$

Letting $\varepsilon \to 1$, we deduce that

$$||x(t)|| \leq \mu_1||x_0||e^{-\mu_2 t} + \frac{\mu_1^2 m}{2} + 2\alpha, \quad \text{a.s.}$$

Finally, the ball $\mathbb{B}_r$ with $r = \dfrac{\mu_1^2 m}{2} + 2\alpha$ is almost surely globally uniformly exponentially stable. That is, the stochastic perturbed bilinear system (3.2) is almost sure globally practically uniformly exponentially stable. □

A simple extension can be done, if we replace the assumption $(\mathcal{A}_2)$ by the following assumption on the perturbed term.

$(\mathcal{A}_2')$ There exist a nonnegative $\mu$ and a continuous positive function $\bar{\varphi}(t)$, such that

$$||\mathcal{G}(t,x)||^2 \leq e^{-\mu_2 t}\left(\mu||x||^q + \bar{\varphi}(t)\right), \quad 0 < q < 1,\ \forall x \in \mathbb{R}^n,\ t \geq 0,$$

where $\bar{\varphi}$ satisfies

$$\int_0^{+\infty} e^{\mu_2 s}\bar{\varphi}(s)ds = \varrho < +\infty. \tag{3.7}$$

**Theorem 3.2.** *Under assumptions* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2')$, *the stochastic perturbed bilinear system* (3.2) *is almost surely globally practically uniformly exponentially stable.*

**Proof of Theorem 3.2.** The solution with initial condition $x_0$ of the closed–loop stochastic system (3.3) expressed by the following:

$$x(t) = e^{t\widetilde{A}}x_0 + \int_0^t e^{(t-s)\widetilde{A}}\mathcal{G}(s, x(s))dB_s.$$

Similar to the proof of Theorem 3.1, under assumption $(\mathcal{A}_1)$ and Lemma 2.3, we obtain

$$||x(t)|| \leq \mu_1||x_0||e^{-\mu_2 t} + \frac{\varepsilon}{2}\int_0^t \mu_1^2 e^{-2\mu_2(t-s)}||\mathcal{G}(s, x(s))||^2 ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n},$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Taking into account assumption $(\mathcal{A}_2')$, it yields that

$$||x(t)|| \leq \mu_1||x_0||e^{-\mu_2 t} + \frac{\varepsilon}{2}\int_0^t \mu_1^2 e^{-2\mu_2(t-s)}e^{-\mu_2 s}\left(\mu||x(s)||^q + \bar{\varphi}(s)\right)ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n},$$

for all $0 \leq t \leq n,\ n \geq n_0$.

That is,

$$||x(t)|| \leq \mu_1||x_0||e^{-\mu_2 t} + \frac{\varepsilon}{2}\mu_1^2 e^{-2\mu_2 t}\int_0^t e^{2\mu_2 s}e^{-\mu_2 s}\left(\mu||x(s)||^q + \bar{\varphi}(s)\right)ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n},$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Hence, it yields that

$$||x(t)|| \leq \mu_1||x_0||e^{-\mu_2 t} + \frac{\varepsilon}{2}\mu_1^2 e^{-2\mu_2 t}\int_0^t e^{\mu_2 s}\left(\mu||x(s)||^q + \bar{\varphi}(s)\right)ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n},$$

for all $\quad 0 \leq t \leq n,\ n \geq n_0$.

Multiplying both sides by $e^{\mu_2 t}$, it follows that

$$\begin{aligned}
||x(t)||e^{\mu_2 t} &\leq \mu_1||x_0|| + \frac{\varepsilon}{2}\mu_1^2 e^{-\mu_2 t}\int_0^t e^{\mu_2 s}\left(\mu||x(s)||^q + \bar{\varphi}(s)\right)ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n}e^{\mu_2 t} \\
&\leq \mu_1||x_0|| + \frac{\varepsilon}{2}\mu_1^2\int_0^t e^{\mu_2 s}\left(\mu||x(s)||^q + \bar{\varphi}(s)\right)ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n}e^{\mu_2 t},
\end{aligned}$$

for all $\quad 0 \leq t \leq n,\ n \geq n_0$.

Using (2.6), it comes that

$$\begin{aligned}||x(t)||e^{\mu_2 t} &\leq \mu_1||x_0|| + \frac{\varepsilon}{2}\mu_1^2\mu\int_0^t e^{\mu_2 s}||x(s)||^q ds + \frac{\varepsilon}{2}\mu_1^2\int_0^t e^{\mu_2 s}\bar{\varphi}(s)ds + \frac{2}{\varepsilon}\alpha e^{\mu_2 t}\\ &\leq \mu_1||x_0|| + \frac{\varepsilon}{2}\mu_1^2\mu\int_0^t e^{\mu_2 s}||x(s)||^q ds + \frac{\varepsilon}{2}\mu_1^2\int_0^{+\infty} e^{\mu_2 s}\bar{\varphi}(s)ds + \frac{2}{\varepsilon}\alpha e^{\mu_2 t},\end{aligned}$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Using (3.7), it follows that

$$||x(t)||e^{\mu_2 t} \leq \mu_1||x_0|| + \frac{2}{\varepsilon}\alpha e^{\mu_2 t} + \frac{\varepsilon}{2}\mu_1^2\varrho + \frac{\varepsilon}{2}\mu_1^2\mu\int_0^t e^{\mu_2 s}||x(s)||^q ds,$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Setting $V(t) = e^{\mu_2 t}||x(t)||$, we see that

$$V(t) \leq \mu_1||x_0|| + \frac{2}{\varepsilon}\alpha e^{\mu_2 t} + \frac{\varepsilon}{2}\mu_1^2\varrho + \frac{\varepsilon}{2}\mu_1^2\mu\int_0^t e^{(1-q)\mu_2 s}V^q(s)ds.$$

That is,

$$V(t) \leq \epsilon(t) + C\int_0^t e^{(1-q)\mu_2 s}V^q(s)ds,$$

where $\epsilon(t) = \mu_1||x_0|| + \frac{2}{\varepsilon}\alpha e^{\mu_2 t} + \frac{\varepsilon}{2}\mu_1^2\varrho,\ \ C = \frac{\varepsilon}{2}\mu_1^2\mu.$

Applying the Gamidov inequality (Lemma 2.5), it yields that

$$\begin{aligned}V(t) &\leq \epsilon(t) + C\rho^q\left(\int_0^t \exp\left(\frac{1-q}{1-q}\mu_2 s\right)ds\right)^{1-q}\\ &\leq \epsilon(t) + C\rho^q\left(\int_0^t e^{\mu_2 s}ds\right)^{1-q}\\ &= \epsilon(t) + C\rho^q\left(\frac{1}{\mu_2}\right)^{1-q}e^{\mu_2(1-q)t},\end{aligned}$$

for all $0 \leq t \leq n,\ n \geq n_0$.

That is, one obtains

$$e^{\mu_2 t}||x(t)|| \leq \epsilon(t) + C\rho^q\left(\frac{1}{\mu_2}\right)^{1-q}e^{\mu_2(1-q)t},$$

for all $0 \leq t \leq n,\ n \geq n_0$.

As a consequence, it yields that

$$||x(t)|| \leq \epsilon(t)e^{-\mu_2 t} + C\rho^q \left(\frac{1}{\mu_2}\right)^{1-q} e^{-q\mu_2 t} \leq \epsilon(t)e^{-\mu_2 t} + C\rho^q \left(\frac{1}{\mu_2}\right)^{1-q},$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Hence, we see that

$$\begin{aligned} ||x(t)|| &\leq \mu_1||x_0||e^{-\mu_2 t} + \frac{2}{\varepsilon}\alpha + \frac{\varepsilon}{2}\mu_1^2\varrho e^{-\mu_2 t} + \frac{\varepsilon}{2}\mu_1^2\mu\rho^q \left(\frac{1}{\mu_2}\right)^{1-q} e^{-\mu_2 t} \\ &\leq \mu_1||x_0||e^{-\mu_2 t} + \frac{2}{\varepsilon}\alpha + \frac{\varepsilon}{2}\mu_1^2\varrho + \frac{\varepsilon}{2}\mu_1^2\mu\rho^q \left(\frac{1}{\mu_2}\right)^{1-q}, \end{aligned}$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Letting $\varepsilon \to 1$, we deduce that

$$||x(t)|| \leq \mu_1||x_0||e^{-\mu_2 t} + 2\alpha + \frac{\mu_1^2\varrho}{2} + \frac{\mu_1^2\mu\rho^q}{2}\left(\frac{1}{\mu_2}\right)^{1-q}, \quad \text{a.s.}$$

As a consequence, the ball $\mathbb{B}_r$, with $r = 2\alpha + \frac{\mu_1^2\varrho}{2} + \frac{\mu_1^2\mu\rho^q}{2}\left(\frac{1}{\mu_2}\right)^{1-q}$, is almost sure globally uniformly exponentially stable, which in turn gives the stochastic perturbed bilinear system (3.3) is almost sure globally uniformly practically exponentially stable. □

Now, we will impose another class of the stochastic perturbed bilinear system (3.3) that can be stabilizable by constant feedback.

($\mathcal{A}_3'$) There exists a continuous nonnegative known function $\zeta(t)$, such that

$$||\mathcal{G}(t,x)||^2 \leq e^{-\mu_2 t}\zeta(t)||x||, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{3.8}$$

where $\zeta(t)$ satisfies the following condition:

$$\int_0^{+\infty} \zeta(t) \leq \Theta. \tag{3.9}$$

**Theorem 3.3.** *Under assumptions* ($\mathcal{A}_1$) *and* ($\mathcal{A}_3'$), *the stochastic perturbed bilinear system* (3.2) *is almost surely globally practically uniformly exponentially stable.*

In order to prove Theorem 3.3 we need to recall the following integral inequality.

**Lemma 3.4.** *[15] Let $a(t), b(t), c(t), u(t)$ be continuous functions for $t \geq 0$, and $b(t)$ be nonnegative for $t \geq 0$, suppose that*

$$u(t) \leq a(t) + \int_0^t [b(s)u(s) + c(s)]\, ds.$$

*Then,*

$$u(t) \leq a(t) + \int_0^t [a(s)b(s) + c(s)] \exp\left(\int_s^t b(\tau)d\tau\right) ds.$$

**Corollary 3.5.** *[15] For $a(t) \equiv a$, we have*

$$u(t) \leq a \exp\left(\int_0^t b(\tau)d\tau\right) + \int_0^t c(s) \exp\left(\int_s^t b(\tau)d\tau\right) ds.$$

**Proof of Theorem 3.3.** The solution with initial condition $x_0$ of the closed–loop stochastic system (3.3) expressed by the following:

$$x(t) = e^{t\bar{A}}\, x_0 + \int_0^t e^{(t-s)\bar{A}}\, \mathcal{G}(s, x(s))dB_s. \tag{3.10}$$

Using Lemma 2.3, it follows that

$$||x(t)|| \leq ||e^{t\bar{A}}||\, ||x_0|| + \frac{\varepsilon}{2}\int_0^t ||e^{(t-s)\bar{A}}||^2\, ||\mathcal{G}(s, x(s))||^2 ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n}, \quad \text{for all } 0 \leq t \leq n,\ n \geq n_0.$$

Using assumption $(\mathcal{A}_1)$, one obtains

$$||e^{\bar{A}t}|| = ||e^{(\mathcal{A}+\bar{u}\mathcal{B})t}|| \leq \mu_1 e^{-\mu_2 t}, \tag{3.11}$$

where $\mu_2 \leq \min\limits_{1\leq i\leq n} |\mathcal{R}e\,(\lambda_i(\mathcal{A}+\bar{u}\mathcal{B}))\,|$.

Based on (3.11) and assumption $(\mathcal{A}_3')$, one obtains

$$||x(t)|| \leq \mu_1||x_0||e^{-\mu_2 t} + \frac{\varepsilon}{2}\mu_1^2 e^{-2\mu_2 t}\int_0^t e^{\mu_2 s}\zeta(s)||x(s)||ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n}, \quad \text{for all } 0 \leq t \leq n,\ n \geq n_0.$$

Multiplying both sides by $e^{\mu_2 t}$, it comes that

$$\begin{aligned} ||x(t)||e^{\mu_2 t} &\leq \mu_1||x_0|| + \frac{\varepsilon}{2}\mu_1^2 e^{-\mu_2 t}\int_0^t e^{\mu_2 s}\zeta(s)||x(s)||ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n}e^{\mu_2 t}, \\ &\leq \mu_1||x_0|| + \frac{\varepsilon}{2}\mu_1^2\int_0^t e^{\mu_2 s}\zeta(s)||x(s)||ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n}e^{\mu_2 t}, \quad \text{for all } 0 \leq t \leq n,\ n \geq n_0. \end{aligned}$$

Let $U(t) = e^{\mu_2 t}||x(t)||$, then we see that

$$U(t) \leq \mu_1||x_0|| + \frac{\varepsilon}{2}\mu_1^2 \int_0^t \zeta(s)U(s)ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n}e^{\mu_2 t}, \quad \text{for all } 0 \leq t \leq n,\ n \geq n_0.$$

Using (2.6), it yields that

$$\begin{aligned} U(t) &\leq \mu_1||x_0|| + \frac{\varepsilon}{2}\mu_1^2 \int_0^t \zeta(s)U(s)ds + \frac{2}{\varepsilon}\alpha e^{\mu_2 t}, \\ &= \mu_1||x_0|| + \frac{\varepsilon}{2}\mu_1^2 \int_0^t \zeta(s)U(s)ds + \frac{2}{\varepsilon}\frac{\alpha}{\mu_2}\int_0^t e^{\mu_2 s}ds, \quad \text{for all } 0 \leq t \leq n,\ n \geq n_0. \end{aligned}$$

Applying the Gronwall inequality (Corollary 3.5), it yields that

$$\begin{aligned} U(t) &\leq \mu_1||x_0|| \exp\left(\int_0^t \frac{\varepsilon}{2}\mu_1^2\zeta(s)ds\right) + \int_0^t \frac{2}{\varepsilon}\frac{\alpha}{\mu_2}e^{\mu_2 s}\exp\left(\int_s^t \frac{\varepsilon}{2}\mu_1^2\zeta(\tau)d\tau\right)ds \\ &\leq \mu_1||x_0|| \exp\left(\int_0^{+\infty} \frac{\varepsilon}{2}\mu_1^2\zeta(s)ds\right) + \int_0^t \frac{2}{\varepsilon}\frac{\alpha}{\mu_2}e^{\mu_2 s}\exp\left(\int_0^{+\infty} \frac{\varepsilon}{2}\mu_1^2\zeta(\tau)d\tau\right)ds, \end{aligned}$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Taking into account (3.9), it follows that

$$\begin{aligned} U(t) &\leq \mu_1||x_0|| \exp\left(\frac{\varepsilon}{2}\mu_1^2\Theta\right) + \int_0^t \frac{2}{\varepsilon}\frac{\alpha}{\mu_2}e^{\mu_2 s}\exp\left(\frac{\varepsilon}{2}\mu_1^2\Theta\right)ds \\ &\leq \mu_1||x_0|| \exp\left(\frac{\varepsilon}{2}\mu_1^2\Theta\right) + \frac{2}{\varepsilon}\frac{\alpha}{\mu_2^2}e^{\mu_2 t}\exp\left(\frac{\varepsilon}{2}\mu_1^2\Theta\right). \end{aligned}$$

Thus, one obtains

$$e^{\mu_2 t}||x(t)|| \leq \mu_1||x_0|| \exp\left(\frac{\varepsilon}{2}\mu_1^2\Theta\right) + \frac{2}{\varepsilon}\frac{\alpha}{\mu_2^2}e^{\mu_2 t}\exp\left(\frac{\varepsilon}{2}\mu_1^2\Theta\right),$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Then, it follows that

$$||x(t)|| \leq \mu_1 \exp\left(\frac{\varepsilon}{2}\mu_1^2\Theta\right)e^{-\mu_2 t}||x_0|| + \frac{2}{\varepsilon}\frac{\alpha}{\mu_2^2}\exp\left(\frac{\varepsilon}{2}\mu_1^2\Theta\right),$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Letting $\varepsilon \to 1$, it yields that

$$||x(t)|| \leq \mu_1 \exp\left(\frac{\mu_1^2\Theta}{2}\right)e^{-\mu_2 t}||x_0|| + \frac{2\alpha}{\mu_2^2}\exp\left(\frac{\mu_1^2\Theta}{2}\right), \text{ a.s.}$$

Finally, the ball $\mathbb{B}_r$ with $r = \dfrac{2\alpha}{\mu_2^2}\exp\left(\dfrac{\mu_1^2\Theta}{2}\right)$, is almost sure globally uniformly exponentially stable, that is the stochastic perturbed bilinear system (3.3) is almost sure globally practically uniformly exponentially stable. □

**Example 3.6.** *Let consider the following stochastic perturbed bilinear system:*

$$dx(t) = (\mathcal{A}x(t) + u\mathcal{B}x(t))dt + \mathcal{G}(t, x(t))dB_t, \tag{3.12}$$

*where* $x = (x_1, x_2) \in \mathbb{R}^2$.

$$\mathcal{A} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{G}(t,x) = \begin{pmatrix} \mathcal{G}_1(t,x) \\ \mathcal{G}_2(t,x) \end{pmatrix},$$

*with*

$$\begin{cases} \mathcal{G}_1(t,x) = \sin(x_2)e^{-\varsigma t} \\ \mathcal{G}_2(t,x) = e^{-\varsigma t}, \quad \varsigma > 0. \end{cases}$$

*The stochastic system (3.12) can be regarded as a bilinear perturbed system of:*

$$dx(t) = (\mathcal{A}x(t) + u\mathcal{B}x(t))dt. \tag{3.13}$$

*The unperturbed nominal system is globally exponentially stabilizable by the constant feedback* $\bar{u}(x) = \sigma$, $\sigma > 2$, *since the closed–loop system* $dx(t) = (\mathcal{A} + \bar{u}\mathcal{B})\,xdt$ *satisfies* $\mathcal{R}e\lambda(\widetilde{\mathcal{A}}) < 0$, *as we can see in the following Fig.1, for* $\sigma = 3$.

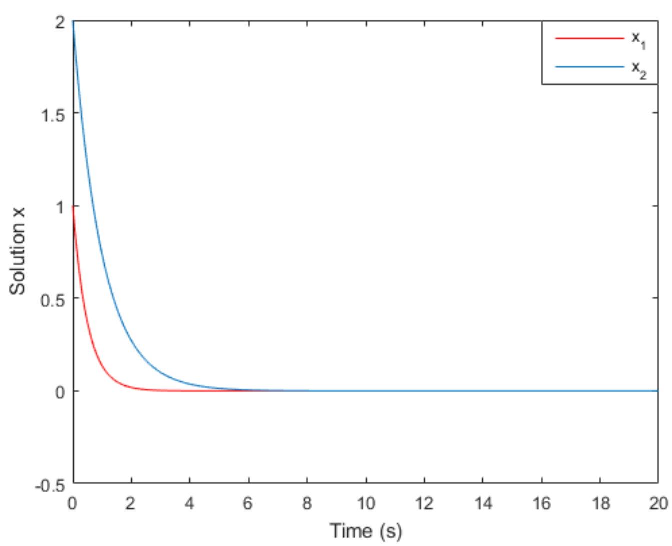


Figure 1: Time evolution of the states $x_1(t)$ and $x_2(t)$ of the bilinear system (3.13)

*On the other side, we have*

$$||\mathcal{G}(t,x)||^2 = \mathcal{G}_1^2(t,x) + \mathcal{G}_2^2(t,x).$$

*That is, we have*

$$||\mathcal{G}(t,x)||^2 = \sin^2(x_2)e^{-2\varsigma t} + e^{-2\varsigma t} \leq 2e^{-2\varsigma t}.$$

*It is clear, that assumption* $(\mathcal{A}_3)$ *is satisfied with* $\varphi(t) = \sqrt{2}e^{-\varsigma t}$.
*Hence, all conditions of Theorem* (3.1) *are satisfied. Thus, the stochastic perturbed bilinear system* (3.12) *is almost sure globally practically uniformly exponentially stable, as we can see Fig.*2 *and Fig.*3*, for* $\varsigma = 1$.

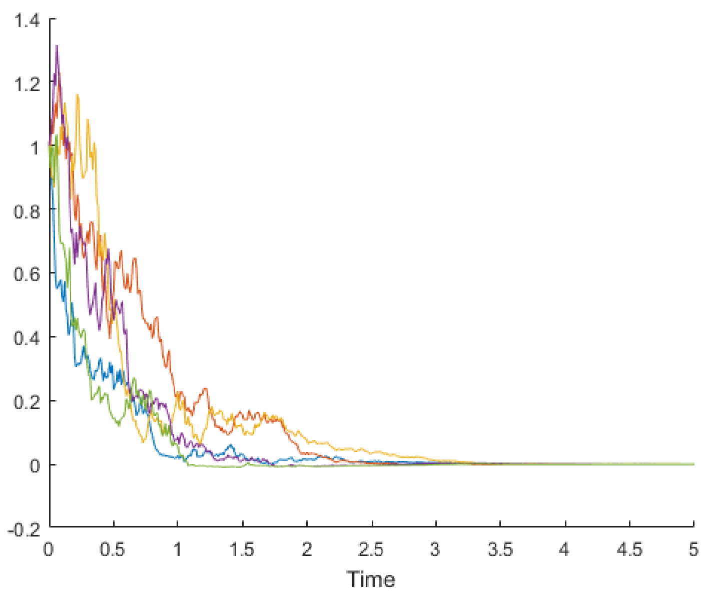


Figure 2: Time evolution of the state $x_1(t)$ of the stochastic perturbed bilinear system (3.12), with five different Brownian motions

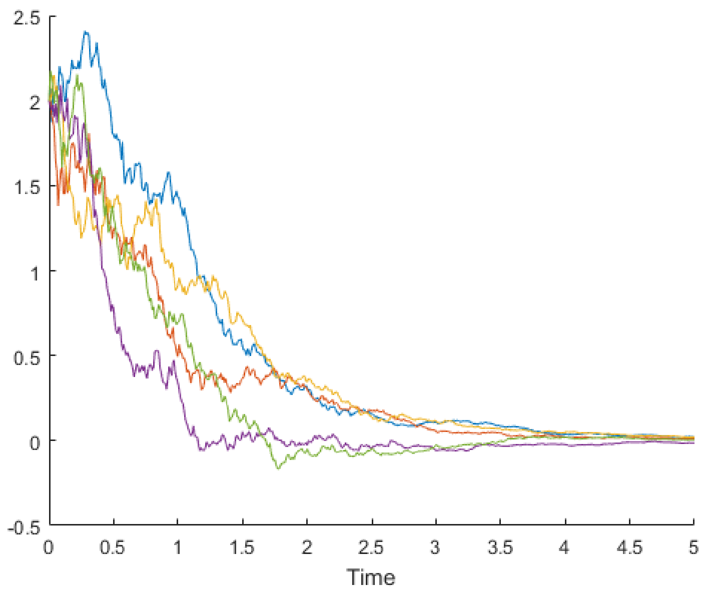


Figure 3: Time evolution of the state $x_2(t)$ of the the stochastic perturbed bilinear system (3.12), with five different Brownian motions

# 4 Stabilization within a bilinear approximation

In this section, we prove the almost sure local uniform exponential stability to a small ball for stochastic perturbed affine system. We will investigate the asymptotic behavior of the solutions in the sense that the trajectories converge to a small ball centered at the origin $\mathbb{B}_r(0,r)$, $r>0$ small enough in such away $\mathbb{B}(0,r)\subset\mathbb{B}(0,\eta)$ and for all solutions starting from $\mathbb{B}(0,\eta)\setminus\mathbb{B}(0,r)$, will approach exponentially to $\mathbb{B}(0,r)$ for $t$ large enough. We consider the stochastic perturbed affine system in closed–loop with the constant feedback $\bar{u}$ for $t$ large enough.

Let's consider the following affine system:

$$dx = (f(x) + ug(x))\,dt, \tag{4.1}$$

where $x\in\mathbb{R}^n$ is the state vector, $u\in\mathbb{R}^n$ is the control input vector, $f$, $g$ are two smooth functions defined on $\mathbb{R}^n$, with $f(0)=0$, $g(0)=0$.

Assume that some parameters are excited or perturbed by Brownian motion, and the perturbed stochastic system system is given by the following form:

$$dx(t) = (f(x) + ug(x))\,dt + \mathcal{G}(t,x(t))dB_t. \tag{4.2}$$

where $\mathcal{G} : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m}$, $B_t = (B_1(t), ..., B_m(t))^T$ is an $m$-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We assume that there exist $t$ such that $\mathcal{G}(t, 0) \neq 0$.

The associated closed–loop stochastic system with a constant feedback $u = \bar{u}$ is expressed by the following:

$$dx(t) = (f(x) + \bar{u}g(x))\, dt + \mathcal{G}(t, x(t))dB_t. \tag{4.3}$$

Since $f$ and $g$ are smooth functions, locally on a certain neighborhood of the origin $\mathcal{V}(0)$, then we can write $f(x)$ and $g(x)$ as the following:

$$f(x) = \mathcal{A}x + \chi_1(x), \quad \text{and} \quad g(x) = \mathcal{B}x + \chi_2(x),$$

with

$$\lim_{x\to 0} \frac{||\chi_1(x)||}{||x||} = 0, \quad \lim_{x\to 0} \frac{||\chi_2(x)||}{||x||} = 0.$$

Then, the closed–loop stochastic system (4.3) can be written as the following:

$$dx(t) = (\mathcal{A}x + \bar{u}\mathcal{B}x)\, dt + (\chi_1(x) + \bar{u}\chi_2(x))dt + \mathcal{G}(t, x(t))dB_t, \quad \forall x \in \mathcal{V}(0). \tag{4.4}$$

Let $\chi(x) = \chi_1(x) + \bar{u}\chi_2(x)$, then the stochastic system (4.4) can be regarded as follow:

$$dx = (\mathcal{A}x + \bar{u}\mathcal{B}x + \chi(x))\, dt + \mathcal{G}(t, x(t))dB_t. \tag{4.5}$$

$(\mathcal{A}_4)$ There exist two continuous positive functions $\varphi_1(t)$ and $\varphi_2(t)$, such that

$$||\mathcal{G}(t, x)||^2 \leq e^{-\mu_2 t} (\varphi_1(t)||x|| + \varphi_2(t)), \quad \forall x \in \mathbb{R}^n,\ t \geq 0,$$

where $\varphi_1$ and $\varphi_2$ satisfy, the following condition:

$$\int_0^{+\infty} \varphi_1(s)ds = \bar{\varphi}_1 < +\infty, \tag{4.6}$$

and

$$\int_0^{+\infty} e^{\mu_2 s}\varphi_2(s)ds = \bar{\varphi}_2 < +\infty. \tag{4.7}$$

**Theorem 4.1.** *Under assumptions* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_4)$ *the stochastic perturbed affine system* (4.3) *is almost surely practically uniformly exponentially stable.*

**Proof.** The solution of the stochastic system (4.5) with initial condition $x_0$ is given by the following:

$$x(t) = e^{t\bar{A}}\, x_0 + \int_0^t e^{(t-s)\bar{A}}\chi(x(s))ds + \int_0^t e^{(t-s)\bar{A}}\, \mathcal{G}(s, x(s))dB_s. \tag{4.8}$$

Similar to the proof of Theorem (3.1), under assumption $(\mathcal{A}_1)$ and Lemma 2.3, we obtain

$$||x(t)|| \leq \mu_1||x_0||e^{-\mu_2 t} + \int_0^t \mu_1 e^{-\mu_2(t-s)}\chi(x(s))ds + \frac{\varepsilon}{2}\int_0^t \mu_1^2 e^{-2\mu_2(t-s)}||\mathcal{G}(s,x(s))||^2 ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n},$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Since, $\lim_{x\to 0}\dfrac{||\chi(x)||}{||x||} = 0$, then for a given constant $\delta > 0$, there exists $\eta_0 > 0$, such that $\forall x \in \mathbb{B}(0,\eta_0) \subset \mathbb{B}(0,\eta)$, for all $t \geq 0$, one obtains

$$||\chi(x)|| \leq \delta||x||.$$

For $\delta = \dfrac{\theta\mu_2}{\mu_1}$, it yields that

$$||\chi(x)|| \leq \frac{\theta\mu_2}{\mu_1}, \quad 0 < \theta < 1.$$

Taking into account assumption $(\mathcal{A}_4)$, we have

$$\begin{aligned}
||x(t)|| &\leq \mu_1||x_0||e^{-\mu_2 t} + \mu_1\int_0^t e^{-\mu_2(t-s)}\frac{\theta\mu_2}{\mu_1}||x(s)||ds + \frac{\varepsilon}{2}\int_0^t \mu_1^2 e^{-2\mu_2(t-s)} \\
&\quad \times e^{-\mu_2 s}\left(\varphi_1(s)||x(s)|| + \varphi_2(s)\right)ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n} \\
&= \mu_1||x_0||e^{-\mu_2 t} + \theta\mu_2 e^{-\mu_2 t}\int_0^t e^{\mu_2 s}||x(s)||ds \\
&\quad + \frac{\varepsilon}{2}\mu_1^2 e^{-2\mu_2 t}\int_0^t e^{\mu_2 s}\left(\varphi_1(s)||x(s)|| + \varphi_2(s)\right)ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n},
\end{aligned}$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Multiplying both sides by $e^{\mu_2 t}$, one obtains

$$\begin{aligned}
e^{\mu_2 t}||x(t)|| &\leq \mu_1||x_0|| + \theta\mu_2\int_0^t e^{\mu_2 s}||x(s)||ds + \frac{\varepsilon}{2}\mu_1^2 e^{-\mu_2 t}\int_0^t e^{\mu_2 s}\left(\varphi_1(s)||x(s)|| + \varphi_2(s)\right)ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n}e^{\mu_2 t} \\
&\leq \mu_1||x_0|| + \theta\mu_2\int_0^t e^{\mu_2 s}||x(s)||ds + \frac{\varepsilon}{2}\mu_1^2\int_0^t e^{\mu_2 s}\left(\varphi_1(s)||x(s)|| + \varphi_2(s)\right)ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n}e^{\mu_2 t},
\end{aligned}$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Let $\widetilde{V}(t) = e^{\mu_2 t}||x(t)||$, we see that

$$\widetilde{V}(t) \leq \mu_1||x_0|| + \theta\mu_2\int_0^t \widetilde{V}(s)ds + \frac{\varepsilon}{2}\mu_1^2\int_0^t \varphi_1(s)\widetilde{V}(s)ds + \frac{\varepsilon}{2}\mu_1^2\int_0^t e^{\mu_2 s}\varphi_2(s)ds + \frac{2}{\varepsilon}\frac{\ln(n)}{n}e^{\mu_2 t},$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Using (2.6), we obtain

$$\begin{aligned}\bar{V}(t) &\leq \mu_1||x_0|| + \int_0^t \left(\theta\mu_2 + \frac{\varepsilon}{2}\mu_1^2\varphi_1(s)\right)\widetilde{V}(s)ds + \frac{\varepsilon}{2}\mu_1^2\int_0^t e^{\mu_2 s}\varphi_2(s)ds + \frac{2}{\varepsilon}\alpha e^{\mu_2 t}\\ &\leq \mu_1||x_0|| + \int_0^t \left(\theta\mu_2 + \frac{\varepsilon}{2}\mu_1^2\varphi_1(s)\right)\widetilde{V}(s)ds + \frac{\varepsilon}{2}\mu_1^2\int_0^{+\infty} e^{\mu_2 s}\varphi_2(s)ds + \frac{2}{\varepsilon}\alpha e^{\mu_2 t},\end{aligned}$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Now, using condition (4.7), it yields that

$$V(t) \leq \left(\mu_1||x_0|| + \frac{\varepsilon}{2}\mu_1^2\bar{\varphi}_2\right) + \int_0^t \left(\theta\mu_2 + \frac{\varepsilon}{2}\mu_1^2\varphi_1(s)\right)\widetilde{V}(s)ds + \frac{2}{\varepsilon}\frac{\alpha}{\mu_2}\int_0^t e^{\mu_2 s}ds,$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Applying, the Gronwall lemma 3.5, one obtains

$$\begin{aligned}\bar{V}(t) &\leq \left(\mu_1||x_0|| + \frac{\varepsilon}{2}\mu_1^2\bar{\varphi}_2\right)\exp\left(\int_0^t \theta\mu_2 + \frac{\varepsilon}{2}\mu_1^2\varphi_1(s)ds\right)\\ &\quad + \int_0^t \frac{2}{\varepsilon}\frac{\alpha}{\mu_2}e^{\mu_2 s}\exp\left(\int_s^t \theta\mu_2 + \frac{\epsilon}{2}\mu_1^2\varphi_1(\tau)d\tau\right)ds,\end{aligned}$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Thus, we have

$$\begin{aligned}\bar{V}(t) &\leq \left(\mu_1||x_0|| + \frac{\varepsilon}{2}\mu_1^2\bar{\varphi}_2\right)\exp(\theta\mu_2 t)\exp\left(\frac{\epsilon}{2}\mu_1^2\int_0^{+\infty}\varphi_1(s)ds\right)\\ &\quad + \int_0^t \frac{2}{\varepsilon}\frac{\alpha}{\mu_2}e^{\mu_2 s}e^{(t-s)\theta\mu_2}\exp\left(\frac{\varepsilon}{2}\mu_1^2\bar{\varphi}_1\right)ds,\end{aligned}$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Based on condition (4.6), it yields that

$$\begin{aligned}\bar{V}(t) &\leq \left(\mu_1||x_0|| + \frac{\varepsilon}{2}\mu_1^2\bar{\varphi}_2\right)\exp(\theta\mu_2 t)\exp\left(\frac{\epsilon}{2}\mu_1^2\bar{\varphi}_1\right)\\ &\quad + \frac{2}{\varepsilon}\frac{\alpha}{\mu_2}\exp\left(\frac{\varepsilon}{2}\mu_1^2\bar{\varphi}_1\right)e^{\theta\mu_2 t}\int_0^t e^{\mu_2(1-\theta)s}ds\\ &\leq \left(\mu_1||x_0|| + \frac{\varepsilon}{2}\mu_1^2\bar{\varphi}_2\right)\exp(\theta\mu_2 t)\exp\left(\frac{\epsilon}{2}\mu_1^2\bar{\varphi}_1\right)\\ &\quad + \frac{2}{\varepsilon}\frac{\alpha}{\mu_2}\exp\left(\frac{\varepsilon}{2}\mu_1^2\bar{\varphi}_1\right)\frac{1}{\mu_2(1-\theta)}e^{\theta\mu_2 t}e^{\mu_2(1-\theta)t},\end{aligned}$$

for all $0 \leq t \leq n,\ n \geq n_0$.

That is, one obtains

$$\bar{V}(t) \leq \left(\mu_1||x_0|| + \frac{\varepsilon}{2}\mu_1^2\bar{\varphi}_2\right)\exp(\theta\mu_2 t)\exp\left(\frac{\epsilon}{2}\mu_1^2\bar{\varphi}_1\right)$$
$$+\frac{2}{\varepsilon}\frac{\alpha}{\mu_2}\exp\left(\frac{\varepsilon}{2}\mu_1^2\bar{\varphi}_1\right)\frac{1}{\mu_2(1-\theta)}e^{\mu_2 t},$$

for all $0 \leq t \leq n,\ n \geq n_0$.

As a consequence, we deduce

$$||x(t)|| \leq \mu_1\exp\left(\frac{\epsilon}{2}\mu_1^2\bar{\varphi}_1\right)||x_0||e^{-\mu_2(1-\theta)t} + \frac{\varepsilon}{2}\mu_1^2\bar{\varphi}_2\exp\left(\frac{\epsilon}{2}\mu_1^2\bar{\varphi}_1\right)e^{-\mu_2(1-\theta)t}$$
$$+\frac{\varepsilon}{2}\frac{\alpha}{\mu_2}\frac{1}{\mu_2(1-\theta)}\exp\left(\frac{2}{\varepsilon}\mu_1^2\bar{\varphi}_1\right),$$

for all $0 \leq t \leq n,\ n \geq n_0$.

That is,

$$||x(t)|| \leq \mu_1\exp\left(\frac{\epsilon}{2}\mu_1^2\bar{\varphi}_1\right)||x_0||e^{-\mu_2(1-\theta)t} + \frac{\varepsilon}{2}\mu_1^2\bar{\varphi}_2\exp\left(\frac{\epsilon}{2}\mu_1^2\bar{\varphi}_1\right) + \frac{2}{\varepsilon}\frac{\alpha}{\mu_2}\frac{1}{\mu_2(1-\theta)}\exp\left(\frac{\epsilon}{2}\mu_1^2\bar{\varphi}_1\right),$$

for all $0 \leq t \leq n,\ n \geq n_0$.

Letting, $\varepsilon \to 1$, it follows that

$$||x(t)|| \leq \mu_1\exp\left(\frac{1}{2}\mu_1^2\bar{\varphi}_1\right)||x_0||e^{-\mu_2(1-\theta)t} + \frac{1}{2}\mu_1^2\bar{\varphi}_2\exp\left(\frac{1}{2}\mu_1^2\bar{\varphi}_1\right) + \frac{2\alpha}{\mu_2^2(1-\theta)}\exp\left(\frac{1}{2}\mu_1^2\bar{\varphi}_1\right), \text{ a.s.}$$

Finally, the ball $\mathbb{B}_r$ with $r = \frac{1}{2}\mu_1^2\bar{\varphi}_2\exp\left(\frac{1}{2}\mu_1^2\bar{\varphi}_1\right) + \dfrac{2\alpha}{\mu_2^2(1-\theta)}\exp\left(\frac{1}{2}\mu_1^2\bar{\varphi}_1\right)$, is almost sure uniformly exponentially stable, thus the stochastic perturbed affine system (4.3) is almost surely practically uniformly exponentially stable. □

**Example 4.2.** *Let consider the following stochastic perturbed affine system:*

$$\begin{cases} dx_1(t) = (x_1\cos(x_1) - u\sin(x_1))\,dt + \left(\dfrac{e^{-t/2}}{\sqrt{\mathrm{ch}}(t)}\sqrt{|x_1|} + e^{-\vartheta/2t}\right)dB_t \\ dx_2(t) = (-x_1 - \sin(x_2)\cos(x_1))\,dt, \quad \vartheta > 2. \end{cases} \tag{4.9}$$

*The previous system can be written as:*

$$dx(t) = (f(x(t)) + ug(x(t)))dt + \mathcal{G}(t, x(t))dB_t, \tag{4.10}$$

*with* $x = (x_1, x_2) \in \mathbb{R}^2$ *is the state of the system,* $u \in \mathbb{R}$ *is the input,*

$$f(x) = \begin{cases} x_1 \cos(x_1) \\ \\ -x_1 - \sin(x_2)\cos(x_1). \end{cases} , g(x) = \begin{cases} -2\sin(x_1) \\ \\ 0. \end{cases} , \mathcal{G}(t,x) = \begin{cases} \dfrac{e^{-t/2}}{\sqrt{\mathrm{ch}}(t)}\sqrt{|x_1|} + e^{-\vartheta/2t} \\ \\ 0. \end{cases}$$

*The stochastic system* (4.10) *can be regarded as a perturbed affine system of the following:*

$$dx(t) = (f(x) + ug(x))dt. \tag{4.11}$$

*The bilinear approximation for the associated nominal system of* (4.11) *is the following:*

$$dx(t) = (\mathcal{A}x + u\mathcal{B}x)\, dt, \tag{4.12}$$

*where*

$$\mathcal{A} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

*Notice that, the previous approximation can be done in a small neighborhood of the origin. It means that there exists* $\eta > 0$*, such that for all* $x \in \mathbb{B}(0, \eta)$ *the passage from affine system to bilinear system is possible. The objective is to seek a constant* $r > 0$ *small enough in such away* $\mathbb{B}(0, r) \subset \mathbb{B}(0, \eta)$ *and all solutions starting from* $\mathbb{B}(0, \eta) \setminus \mathbb{B}(0, r)$*, will approach exponentially to* $\mathbb{B}(0, r)$ *for* $t$ *large enough. The unperturbed nominal bilinear system* (4.12) *is globally uniformly exponentially stabilizable by the constant feedback* $\bar{u}(x) = u_0$*,* $u_0 > 1$*, since the closed–loop system* $dx(t) = (\mathcal{A} + \bar{u}\mathcal{B})\, xdt$ *satisfies* $\mathcal{R}e\lambda(\widetilde{\mathcal{A}})$*, as we can see in Fig.*4 *(for* $u_0 = 2$*). That is, one has the following estimation:*

$$||e^{t(\mathcal{A}+\bar{u}\mathcal{B})}|| \leq \mu_1 e^{-\mu_2 t}.$$

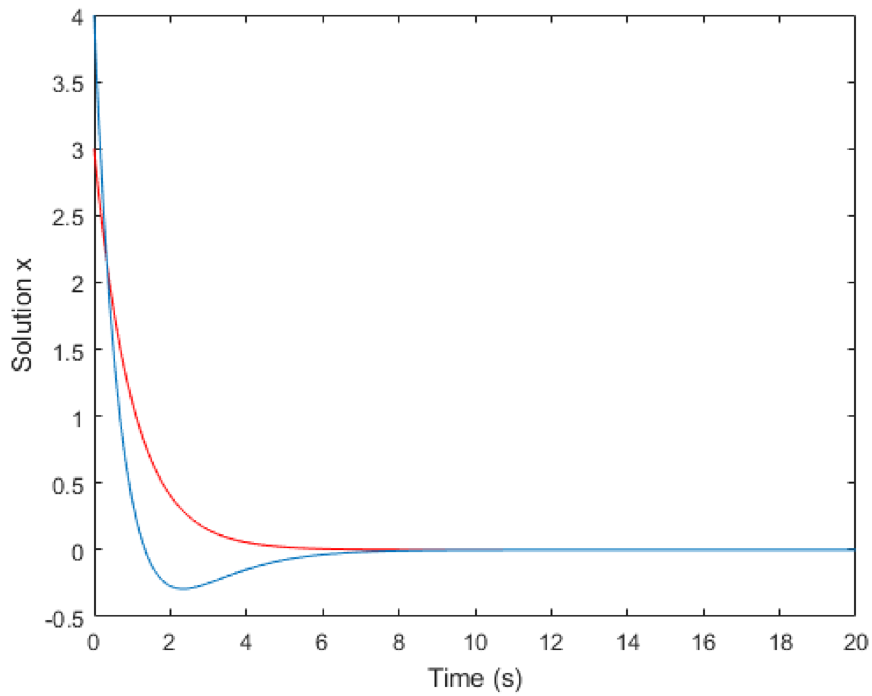


Figure 4: Time evolution of the state $x_1(t)$ and $x_2(t)$ of the bilinear system (4.12)

*In the sequel, we will choose* $\mu_2 = 1$. *On the other side, based on the fact that,* $(a+b)^n \leq 2^{n-1}(a^n + b^n)$, *for all* $a, b \geq 0$, $n \geq 1$, *one obtains:*

$$\begin{aligned} ||G(t,x)||^2 &\leq 2\frac{e^{-t}}{\mathrm{ch(t)}}|x_1| + 2e^{-\vartheta t} \\ &\leq e^{-t}\left(\frac{2}{\mathrm{ch}(t)}||x|| + 2e^{-(\vartheta-1)t}\right) \\ &= e^{-t}\left(\varphi_1(t)||x|| + \varphi_2(t)\right), \end{aligned}$$

*where* $\varphi_1(t) = \dfrac{2}{\mathrm{ch}(t)}$ *and* $\varphi_2(t) = 2e^{-(\vartheta-1)t}$, *which satisfy both conditions* (4.6) *and* (4.7) *of Theorem 4.1. That is, all conditions of Theorem 4.1 are fulfilled and then stochastic perturbed affine system* (4.9) *is almost surely practically uniformly exponentially stable, as shown in Fig.*4 *and Fig.*5, *for* $\vartheta = 4$.

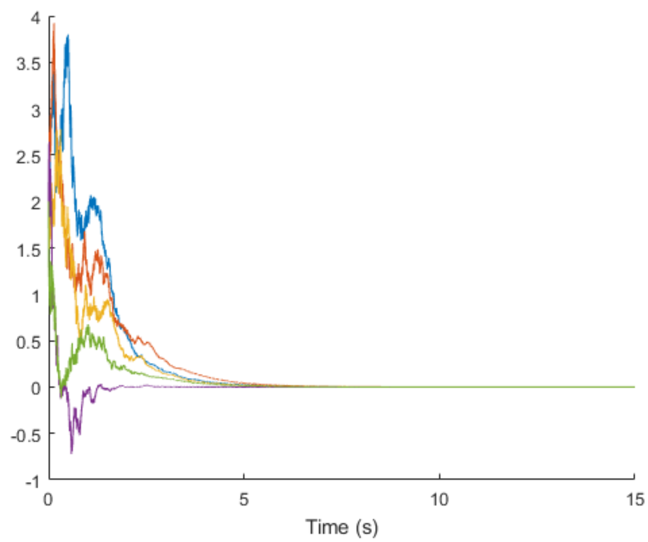


Figure 5: Time evolution of the state $x_1(t)$ of the stochastic affine system (4.9), with five different Brownian motions

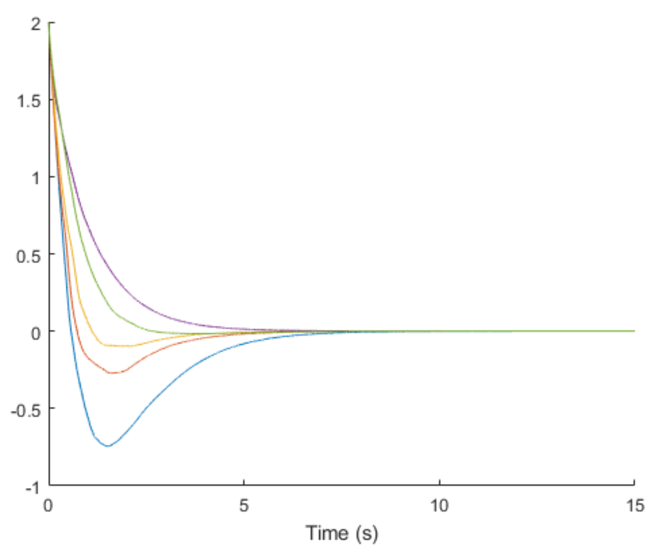


Figure 6: Time evolution of the state $x_2(t)$ of the stochastic affine system (4.9), with five different Brownian motions

# 5 Conclusion

In this paper, we investigate the problem of stabilization of stochastic perturbed control–bilinear systems under some restrictions on the bound of perturbations. Also, we prove that the problem of stabilization of stochastic perturbed affine system can be performed by considering a bilinear approximation. The principal technical tool for deriving stabilization results is generalized Gronwall inequalities. Further, we provide different examples to validate the developed methods.

# References


[1] Syed Abbas, S. Dhama, Existence and stability of square-mean almost automorphic solution for neutral stochastic evolution equations with Stepanov-like terms on time scales Revista de la Real Academia de Ciencias Exactas,*Físicas y Naturales, Serie A. Matemticas,* 113 (2019), 1231-1250.

[2] M. Ate, C. Tunç, Stability and boundedness results for solutions of certain third order nonlinear vector differential equations, *Nonlinear dynamics*, 45 (2005), 273–281.

[3] A. Ben Makhlouf, M. A. Hammami, A nonlinear inequality and application to global asymptotic stability of perturbed systems, *Mathematical Methods in the Applied Sciences*, 38 (2015), 2496–2505.

[4] A. BenAbdallah, I. Ellouze, M. A. Hammami, Practical stability of nonlinear time-varying cascade systems, *Journal of Dynamical and Control Systems*, 15 (2009), 45–62.

[5] B. Ben Hamed, I. Ellouze, M. A. Hammami, Practical uniform stability of nonlinear differential delay equations, *Mediterranean Journal of Mathematics,* 8 (2011), 603–616.

[6] D. Bainov, P. Simenov, Integral inequalities and applications, *Kluwer Academic Publishers, Dordrecht,* 1992.

[7] I. Bihari, A generalization of a lemma of Bellman and its applications to uniqueness problems of differential equations, *Acta Mathematica Academiae Scientiarum Hungarica*, 7 (1956), 71–94.

[8] R. Bellman, The stability of solutions of linear differential equations, *Duke Math Journal,* 10 (1943), 643–647.

[9] M. Corless, Guaranteed rates of exponential convergence for uncertain systems, *Journal of Optimization Theory and Applications*, 64 (1990), 481–494.

[10] T. Caraballo, F. Ezzine, M. Hammami, L. Mchiri, Practical stability with respect to a part of variables of stochastic differential equations, *Stochastics an International Journal of Probability and Stochastic Processes,* 6 (2020), 1–18.

[11] T. Caraballo, F. Ezzine, M. Hammami, On the exponential stability of stochastic perturbed singular systems in mean square, *Applied Mathematics and Optimization,* 84 (2021), 1–23.

[12] T. Caraballo, F. Ezzine, M. Hammami, New stability criteria for stochastic perturbed singular systems in mean square, *Nonlinear Dynamics,* 105 (2021), 241–256.

[13] T. Caraballo, F. Ezzine, M. Hammami, Partial stability analysis of stochastic differential equations with a general decay rate, *Journal of Engineering Mathematics,* 130 (2021), 1–17.

[14] T. Caraballo, F. Ezzine, M. Hammami, Practical stability with respect to a part of the variables of stochastic differential equations driven by G-Brownian motion, *Journal of Dynamical and Control Systems*, 33 (2022), 1–19.

[15] S. S. Dragomir, Some Gronwall type inequalities and applications, *Nova Science Publishers,* 2003.

[16] F. Ezzine, M. Hammami, Growth conditions for the stability of a class of time-varying perturbed singular systems, *Kybernetika*, 58 (2022), 1–24.

[17] E. Sontag, Input to State Stability: Basic Concepts and Results, DOI: $10.1007/978-3-540-77653-6_3$. In book: Nonlinear and optimal control theory.

[18] E. Sontag, Y. Wang, on characterizations of input-to-state stability with respect to compact sets, *IFAC Nonlinear Control Systems Design, Tahoe City, California, USA,* 1995.

[19] T. H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, *Annals of Mathematics*, 40 (1919), 292–296.

[20] Sh. G. Gamidov, Certain integral inequalities for boundary value problems of differential equations, *D*ifferentsialnye Uravneniya, 5 (1969), 463–472.

[21] R. Z.Has'minskii, Stochastic Stability of Differential Equations, *Sijthoff Noordhoff*, 1980.

[22] M. Hammi, M. A. Hammami, Non–linear integral inequalities and applications to asymptotic stability, *Ima journal of mathematical control and information*, 32 (2015), 717–735.

[23] K. Liu, Stochastic Stability of Differential Equations in Abstract Spaces, *Cambridge University Press,* (2019).

[24] X. Li, S. Song, Impulsive Systems with Delays: Stability and Control, *Science Press Beijing Springer,* (2021).

[25] X. Mao, Stochastic differential equations and applications, *Ellis Horwood, Chichester, U. K,* 1997.

[26] O. Tunç, On the behaviors of solutions of systems of non–linear diferential equations with multiple constant delays, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas,* 164 (2021) 115–164.