STABILITY OF DELAY EVOLUTION EQUATIONS WITH FADING STOCHASTIC PERTURBATIONS

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Abstract. Stability of nonlinear delay evolution equation with stochastic perturbations is considered. It is shown that if the level of stochastic perturbations fades on the infinity, for instance, if it is given by square integrable function, then an exponentially stable deterministic system remains to be exponentially stable (in mean square). Applications of the obtained results to stochastic reaction-diffusion equations and stochastic 2D Navier-Stokes model are shown.

1. Introduction

It is well known that an asymptotically stable deterministic system remains asymptotically stable (in mean square) under stochastic perturbations if the level of stochastic perturbations is small enough (Caraballo & Shaikhet, 2014; Shaikhet, 2013). A new idea is that stochastic perturbations fading at infinity quickly enough do not violate the asymptotic stability of the initial deterministic system, regardless of the maximum value of the level of stochastic perturbations. This idea has already been checked for linear stochastic differential equations and linear stochastic difference equations (Shaikhet, 2019a, 2019b, 2020). Here a similar statement is proved for a nonlinear stochastic evolution equation with quickly enough fading stochastic perturbations. More exactly: if the level of the stochastic perturbation is given by a continuous and square integrable function, then the zero solution of the considered exponentially stable deterministic system remains exponentially mean square stable independently on the maximum magnitude of the stochastic perturbation. The obtained abstract results are later applied to some important and interesting applications: stochastic reaction-diffusion equations and stochastic 2D Navier-Stokes models (Caraballo, Real, & Shaikhet, 2007; Caraballo & Shaikhet, 2014).

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1.1. **Basic notations.** Let U, H, K be real, separable Hilbert spaces such that $U \subset H \equiv H^* \subset U^*$, where U^* is the dual of U and the injections are continuous and dense. It is assumed also that U and U^* are uniformly convex.

Let $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$ denote the norms in U, H and U^*, respectively; $\langle\cdot,\cdot\rangle$ denote the duality product between U^*, U, and $(\cdot,\cdot)$ the scalar product in H.

Let β be the constant satisfying the inequality

$$|u| \leq \beta\|u\|, \qquad u \in U. \tag{1}$$

Let $\mathfrak{L}(K,H)$ be the space of all bounded linear operators from K into H and $\|G\|_2$ be the Hilbert-Schmidt norm of an operator $G \in \mathfrak{L}(K,H)$.

Let $W(t)$ be a Q-valued Wiener process on a complete probability space $(\Omega, \mathfrak{F}, P)$ which takes values in the separable Hilbert space K, where $Q \in \mathfrak{L}(K,K)$ is a symmetric nonnegative operator, $\mathbf{E}W(t) = 0$, $\mathbf{Cov}(W(t)) = tQ$ (for more details, see Caraballo & Shaikhet, 2014; Da Prato & Zabczyk, 1992).

We denote by $\mathfrak{F}_t, t \geq 0$ the σ-algebra generated by $\{W(s), 0 \leq s \leq t\}$ which make $W(t)$ be a martingale relative to the family $(\mathfrak{F}_t)_{t\geq 0}$.

Given $h \geq 0$, and $T > 0$, we denote by $I^p(-h,T;U)$, $p > 0$, the space of all U–valued processes $(x(t))_{t\in[-h,T]}$ (we will write $x(t)$ for short) measurable (from $[-h,T]\times\Omega$ into U), and satisfying:

1. $x(t)$ is $\mathfrak{F}_t$-measurable almost surely in t, where we set $\mathfrak{F}_t = \mathfrak{F}_0$ for $t \leq 0$;
2. $\int_{-h}^T \mathbf{E}\|x(t)\|^p\,dt < +\infty$.

It is not difficult to check that the space $I^p(-h,T;U)$ is a closed subspace of $L^p(\Omega\times[-h,T], \mathfrak{F}\otimes\mathfrak{B}([-h,T]), dP\otimes dt; U)$, where $\mathfrak{B}([-h,T])$ denotes the Borel σ–algebra on $[-h,T]$. We also write $L^2(\Omega; C(-h,T;H))$ instead of $L^2(\Omega,\mathfrak{F},dP; C(-h,T;H))$, where $C(-h,T;H)$ denotes the space of all continuous functions from $[-h,T]$ into H.

Let $C_H = C([-h,0],H)$ be the space of all continuous functions from $[-h,0]$ into H with sup-norm $\|\psi\|_C = \sup_{-h\leq s\leq 0}|\psi(s)|$, $\psi \in C_H$, $L_H^2 = L^2([-h,0];H)$. Given a stochastic process $u(t) \in I^2(-h,T;U)\cap L^2(\Omega; C(-h,T;H))$, we associate it with an $L_U^2 \cap C_H$-valued stochastic process $u_t : \Omega \to L_U^2\cap C_H$, $t \geq 0$, by setting $u_t(s)(\omega) = u(t+s)(\omega)$, $s \in [-h,0]$.

The aim of this paper is to analyze the stability properties of nonlinear stochastic evolution equation

$$\begin{aligned} du(t) &= \left(A(t,u(t)) + \sum_{i=1}^m F_i(t,u(t-h_i(t)))\right)dt + B(t,u(t-\tau))dW(t), \\ h_i(t) &\in [0,h_{0i}],\ h_0 = \max_i h_{0i},\ u(s) = \psi(s),\ s\in[-h,0],\ h = \max(h_0,\tau). \end{aligned} \tag{2}$$

Here $A(t,\cdot), F_i(t,\cdot) : U \to U^*$ are appropriate partial differential operators, $B(t,\cdot) : H \to \mathfrak{L}(K,H)$, $W(t)$ is a Q-Wiener process, $h_i(\cdot)$ are continuously differentiable functions, and ψ is an appropriate initial datum.

The analysis of the existence and uniqueness of solutions for this model has already been carried out, for instance, by Caraballo, Garrido-Atienza, & Real (2003), Caraballo, Liu, & Truman (2000), and we will not insist in this point here. However, we will explain now which is the concept of solution to be used in our stability analysis.

For a fixed $T > 0$, given an initial value $\psi \in I^2(-h,0;U)\cap L^2(\Omega;C_H)$, a (variational) solution of (2) is a process $u(\cdot) = u(\cdot;\psi) \in I^2(-h,T;U)\cap L^2(\Omega;C(-h,T;H))$

such that

$$
\begin{aligned}
u(t) =&\psi(0) + \int_0^t [A(s,u(s)) + f(s,u_s)]\, ds \\
&\qquad + \int_0^t B(s,u_s)\, dW(s) \quad P- \text{ a.s.}, \quad \forall t \in [0,T], \\
u(t) =&\psi(t), \quad P-\text{a.}s., \quad \forall t \in [-h,0], \\
f(s,u_s) =& \sum_{i=1}^m F_i(s,u(s-h_i(s))), \quad B(s,u_s) = B(s,u(s-\tau)),
\end{aligned} \tag{3}
$$

where the first equality is understood in U^*.

From now on, as we will be interested in the long-time behavior of solutions to (2), we will assume that (3) possesses solutions for all $T > 0$.

1.2. **Definitions and auxiliary statements.** As we mentioned above, let $u(t) = u(t,\psi)$ denote the solution of equation (2) corresponding to the initial condition ψ. We first establish the definitions of stability that we will use in our analysis.

Definition 1.1. *The zero solution of equation (2) is said to be mean square stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathbf{E}|u(t;\psi)|^2 < \varepsilon$ for all $t \geq 0$ if $\|\psi\|^2_{C_H} = \sup_{s\in[-h,0]} \mathbf{E}|\psi(s)|^2 < \delta$.*

Definition 1.2. *The zero solution of equation (2) is said to be exponentially mean square stable if it is stable and there exists a positive constant λ such that for any $\psi \in C(-h,0,U)$ there exists C (which may depend on ψ) such that $\mathbf{E}|u(t;\psi)|^2 \leq Ce^{-\lambda t}$ for $t > 0$.*

Consider a functional $V(\cdot,\cdot) : [0,\infty) \times C_H \to \mathbf{R}_+$ that can be represented in the form $V(t,\varphi) = V(t,\varphi(0),\varphi(\theta))$, $\theta < 0$ and for $\varphi(\theta) = u(t+\theta) = u_t$ put

$$
\begin{aligned}
V_\varphi(t,u) &= V(t,\varphi) = V(t,u,\varphi(\theta)), \\
u &= \varphi(0) = u(t), \qquad \theta < 0.
\end{aligned} \tag{4}
$$

Denote by D the set of functionals, for which the function $V_\varphi(t,u)$ defined by (4) has a continuous derivative with respect to t and two continuous derivatives with respect to u.

To calculate the stochastic differential of the process $\eta(t) = V(t,u_t)$, where $V(\cdot,\cdot) \in D$ and $u(t)$ is a solution of equation (2), the Itô formula is used (see [**?**, **?**])

$$
d\eta(t) = LV(t,u_t)dt + \langle V'_{\varphi u}(t,u(t)), B(t,u(t-\tau))dW(t)\rangle, \tag{5}
$$

where the generator L of equation (2) has the form

$$
\begin{aligned}
LV(t,u_t) =&V'_{\varphi t}(t,u(t)) + \left\langle V'_{\varphi u}(t,u(t)), A(t,u(t)) + \sum_{i=1}^m F_i(t,u(t-h_i(t))) \right\rangle \\
&+ \frac{1}{2} Tr[V''_{\varphi uu}(t,u(t)) B(t,u(t-\tau)) Q B^*(t,u(t-\tau))].
\end{aligned} \tag{6}
$$

The following theorem provides a sufficient condition for the exponential mean square stability in terms of a functional from D (Caraballo & Shaikhet, 2014; Shaikhet, 2013).

Theorem 1.1. *Assume that there exists a functional $V(\cdot,\cdot)\in D$ and some positive numbers c_1, c_2, λ, such that the following conditions hold for any solution $u(\cdot)=u(\cdot,\psi)$ of equation (2)*

$$\begin{aligned}\mathbf{E}V(t,u_t) &\geq c_1 e^{\lambda t}\mathbf{E}|u(t)|^2, \quad t\geq 0\\ \mathbf{E}V(0,u_0) &\leq c_2\|\psi\|^2_{C_H},\\ \mathbf{E}LV(t,u_t) &\leq 0, \quad t\geq 0.\end{aligned} \tag{7}$$

Then the zero solution of equation (2) is exponentially mean square stable.

2. Main result

We can now state and prove our main stability result in this paper. The technique is the method of Lyapunov functionals construction (see Caraballo et al., 2007; Caraballo & Shaikhet, 2014; Shaikhet, 2013) which has proven to be a very powerful technique over the past years.

Theorem 2.1. *Assume that operators in equation (2) satisfy the conditions*

$$\begin{aligned}&\langle A(t,u),u\rangle \leq -\gamma\|u\|^2, \quad \gamma>0, \quad u\in U,\\ &F_i: U\to U^*, \quad \|F_i(t,u)\|_* \leq \alpha_i\|u\|, \quad u\in U,\\ &\|B(t,u)\|_2 \leq \sigma(t)|u|, \quad u\in H, \quad t\geq 0,\end{aligned} \tag{8}$$

where $\sigma(t)$ is a continuous, bounded, square-integrable function, i.e.,

$$\int_0^\infty \sigma^2(s)ds<\infty, \tag{9}$$

and such that

$$\sigma^2(t+\tau)\leq\sigma^2(t)\leq C. \tag{10}$$

If

$$0\leq \dot h_i(t)\leq h_{1i}<1, \quad \gamma>\sum_{i=1}^m \frac{\alpha_i}{\sqrt{1-h_{1i}}}, \tag{11}$$

then the zero solution of equation (2) is exponentially mean square stable.

Proof. Following the procedure of Lyapunov functionals construction ([**?**, **?**, **?**])Caraballo et al., 2007; Caraballo & Shaikhet, 2014; Shaikhet, 2013), we will construct a Lyapunov functional V for equation (2) in the form $V=V_1+V_2+V_3$, where V_1 is defined as

$$V_1(t,u_t)=e^{\lambda t-\int_0^t\sigma^2(s)ds}|u_t(0)|^2=e^{\lambda t-\int_0^t\sigma^2(s)ds}|u(t)|^2, \tag{12}$$

and the additional functionals V_2, V_3 will be defined below. The parameter $\lambda>0$ in V_1 will also be chosen below.

Notice that via (9), the functional V_1 satisfies the first condition from (7) with

$$c_1=e^{-\int_0^\infty\sigma^2(s)ds}>0.$$

Thanks to (6), for the functional (12) and equation (2), using (8) and (1), we have

$$\begin{aligned}LV_1(t,u_t) &= e^{\lambda t-\int_0^t\sigma^2(s)ds}(\lambda-\sigma^2(t))|u(t)|^2+e^{\lambda t-\int_0^t\sigma^2(s)ds}\|B(t,u(t-\tau))\|_2^2\\ &\quad +2e^{\lambda t-\int_0^t\sigma^2(s)ds}\left\langle A(t,u(t))+\sum_{i=1}^m F_i(t,u(t-h_i(t))),u(t)\right\rangle\end{aligned}$$

$$
\begin{aligned}
&\le e^{\lambda t-\int_0^t \sigma^2(s)ds}\Bigg[(\lambda-\sigma^2(t))|u(t)|^2+\sigma^2(t)|u(t-\tau)|^2\\
&\quad +2\left\langle A(t,u(t))+\sum_{i=1}^m F_i(t,u(t-h_i(t))),u(t)\right\rangle\Bigg]\\
&\le e^{\lambda t-\int_0^t \sigma^2(s)ds}\Bigg[\lambda\beta^2\|u(t)\|^2+\sigma^2(t)(|u(t-\tau)|^2-|u(t)|^2)\\
&\quad +2\left(-\gamma\|u(t)\|^2+\sum_{i=1}^m\alpha_i\|u(t-h_i(t))\|\|u(t)\|\right)\Bigg].
\end{aligned}
$$

Using the simple inequality $2ab\le\varepsilon a^2+\varepsilon^{-1}b^2$, where a,b,ε are positive parameters, we obtain

$$
\begin{aligned}
LV_1(t,u_t)\le & e^{\lambda t-\int_0^t \sigma^2(s)ds}\Bigg[\left(\lambda\beta^2-2\gamma+\sum_{i=1}^m\frac{\alpha_i}{\varepsilon_i}\right)\|u(t)\|^2\\
&+\sum_{i=1}^m\alpha_i\varepsilon_i\|u(t-h_i(t))\|^2+\sigma^2(t)(|u(t-\tau)|^2-|u(t)|^2)\Bigg].
\end{aligned}
\tag{13}
$$

For the additional functional

$$
\begin{aligned}
V_2(t,u_t)=&\sum_{i=1}^m r_i\int_{-h_i(t)}^0 e^{\lambda(s+t+h_{0i})-\int_0^{s+t+h_i(t)}\sigma^2(\tau)d\tau}\|u_t(s)\|^2ds\\
=&\sum_{i=1}^m r_i\int_{t-h_i(t)}^t e^{\lambda(s+h_{0i})-\int_0^{s+h_i(t)}\sigma^2(\tau)d\tau}\|u(s)\|^2ds,\quad r_i>0,
\end{aligned}
$$

we have

$$
\begin{aligned}
LV_2(t,u_t)=&\sum_{i=1}^m r_i\Bigg[e^{\lambda(t+h_{0i})-\int_0^{t+h_i(t)}\sigma^2(\tau)d\tau}\|u(t)\|^2\\
&-(1-\dot h_i(t))e^{\lambda(t-h_i(t)+h_{0i})\int_0^t\sigma^2(\tau)d\tau}\|u(t-h_i(t))\|^2\\
&-\dot h_i(t)\int_{t-h_i(t)}^t\sigma^2(s+h_i(t))e^{\lambda(s+h_{0i})-\int_0^{s+h_i(t)}\sigma^2(\tau)d\tau}\|u(s)\|^2ds\Bigg].
\end{aligned}
$$

From here, thanks to (11), it follows

$$
\begin{aligned}
LV_2(t,u_t)&\le\sum_{i=1}^m r_i\Big[e^{\lambda(t+h_{0i})-\int_0^{t+h_i(t)}\sigma^2(\tau)d\tau}\|u(t)\|^2\\
&\quad -(1-\dot h_i(t))e^{\lambda(t-h_i(t)+h_{0i})-\int_0^t\sigma^2(\tau)d\tau}\|u(t-h_i(t))\|^2\Big]\\
&= e^{\lambda t-\int_0^t\sigma^2(\tau)d\tau}\sum_{i=1}^m r_i\Bigg[e^{\lambda h_{0i}-\int_t^{t+h_i(t)}\sigma^2(\tau)d\tau}\|u(t)\|^2\\
&\quad -(1-\dot h_i(t))e^{\lambda(h_{0i}-h_i(t))}\|u(t-h_i(t))\|^2\Bigg]\\
&\le e^{\lambda t-\int_0^t\sigma^2(\tau)d\tau}\sum_{i=1}^m r_i\big[e^{\lambda h_{0i}}\|u(t)\|^2-(1-h_{1i})\|u(t-h_i(t))\|^2\big].
\end{aligned}
\tag{14}
$$

For the next additional functional

$$\begin{aligned} V_3(t,u_t) &= \int_{-\tau}^{0} e^{\lambda(s+t+\tau)-\int_0^{s+t+\tau}\sigma^2(\nu)d\nu}\sigma^2(s+t+\tau)|u_t(s)|^2 ds \\ &= \int_{t-\tau}^{t} e^{\lambda(s+\tau)-\int_0^{s+\tau}\sigma^2(\nu)d\nu}\sigma^2(s+\tau)|u(s)|^2 ds \end{aligned}$$

we have

$$\begin{aligned} LV_3(t,u_t) &= e^{\lambda(t+\tau)-\int_0^{t+\tau}\sigma^2(\nu)d\nu}\sigma^2(t+\tau)|u(t)|^2 \\ &\quad -e^{\lambda t-\int_0^{t}\sigma^2(\nu)d\nu}\sigma^2(t)|u(t-\tau)|^2 \\ &\leq e^{\lambda t-\int_0^{t}\sigma^2(\nu)d\nu}\left[e^{\lambda\tau}\sigma^2(t+\tau)|u(t)|^2-\sigma^2(t)|u(t-\tau)|^2\right]. \end{aligned} \tag{15}$$

From (13), (14) and (15) we deduce for the functional $V = V_1 + V_2 + V_3$

$$\begin{aligned} LV(t,u_t) &\leq e^{\lambda t-\int_0^t\sigma^2(s)ds}\Bigg[\left(\lambda\beta^2-2\gamma+\sum_{i=1}^m\frac{\alpha_i}{\varepsilon_i}\right)\|u(t)\|^2 \\ &\quad +\sum_{i=1}^m\alpha_i\varepsilon_i\|u(t-h_i(t))\|^2 \\ &\quad +\sum_{i=1}^m r_i\left[e^{\lambda h_{0i}}\|u(t)\|^2-(1-h_{1i})\|u(t-h_i(t))\|^2\right] \\ &\quad +e^{\lambda\tau}\sigma^2(t+\tau)|u(t)|^2-\sigma^2(t)|u(t)|^2\Bigg] \\ &= e^{\lambda t-\int_0^t\sigma^2(s)ds}\Bigg[\Bigg(\lambda\beta^2-2\gamma+\sum_{i=1}^m\left(\frac{\alpha_i}{\varepsilon_i}+r_i\right) \\ &\quad +\sum_{i=1}^m r_i\left(e^{\lambda h_{0i}}-1\right)\Bigg)\|u(t)\|^2 \\ &\quad +\sum_{i=1}^m(\alpha_i\varepsilon_i-r_i(1-h_{1i}))\|u(t-h_i(t))\|^2 \\ &\quad +(e^{\lambda\tau}-1)\sigma^2(t+\tau)|u(t)|^2+(\sigma^2(t+\tau)-\sigma^2(t))|u(t)|^2\Bigg]. \end{aligned} \tag{16}$$

To get rid of the term with $\|u(t-h_i(t))\|^2$, we put $r_i = \alpha_i\varepsilon_i(1-h_{1i})^{-1}$ and then, to minimise $\alpha_i\varepsilon_i^{-1}+\alpha_i\varepsilon_i(1-h_{1i})^{-1}$, we choose $\varepsilon_i=\sqrt{1-h_{1i}}$. Besides of this, using conditions (1) and (10), we obtain

$$\begin{aligned} LV(t,u_t) &\leq e^{\lambda t-\int_0^t\sigma^2(s)ds}\Bigg[\lambda\beta^2-2\gamma+2\sum_{i=1}^m\frac{\alpha_i}{\sqrt{1-h_{1i}}} \\ &\quad +\sum_{i=1}^m r_i(e^{\lambda h_{0i}}-1)+C(e^{\lambda\tau}-1)\beta^2\Bigg]\|u(t)\|^2 \\ &= -e^{\lambda t-\int_0^t\sigma^2(s)ds}\left[2\left(\gamma-\sum_{i=1}^m\frac{\alpha_i}{\sqrt{1-h_{1i}}}\right)-\rho(\lambda)\right]\|u(t)\|^2, \end{aligned} \tag{17}$$

where $\rho(\lambda) = (\lambda + C(e^{\lambda\tau} - 1))\beta^2 + \sum_{i=1}^{m} r_i(e^{\lambda h_{0i}} - 1) \geq 0$. Since $\rho(\cdot)$ is a continuous function and $\rho(0) = 0$, then by condition (11), there exists $\lambda > 0$, small enough, such that $2\left(\gamma - \sum_{i=1}^{m} \frac{\alpha_i}{\sqrt{1-h_{1i}}}\right) \geq \rho(\lambda)$.

From here and (17) it follows that $\mathbf{E}LV(t, u_t) \leq 0$. Therefore, the functional $V(t, u_t)$ constructed above satisfies conditions (7) of Theorem 1.1, which implies that the zero solution of equation (2) is exponentially mean square stable. The proof is complete. □

Remark 2.1. *If, in particular, $h_i(t) \equiv h_{0i}$ then $h_{1i} = 0$ and thus condition (11) takes the form $\gamma > \sum_{i=1}^{m} \alpha_i$.*

Remark 2.2. *If the stochastic term does not contain any delay, i.e., $\tau = 0$, then condition (10) trivially holds true and the functional V_3 is not necessary for the proof.*

Remark 2.3. *It is possible to weaken assumption (10) but imposing a more restrictive condition in (11). More precisely, the thesis of Theorem 2.1 holds true if we replace assumption (10) by*

$$\sigma^2(t) \leq C, \tag{18}$$

and assumption (11) by

$$0 \leq \dot{h}_i(t) \leq h_{1i} < 1, \quad 2\left(\gamma - \sum_{i=1}^{m} \frac{\alpha_i}{\sqrt{1-h_{1i}}}\right) > C\beta^2. \tag{19}$$

Indeed, the proof follows in a similar way but taking into account the following computations. Eq. (16) becomes

$$\begin{aligned}
LV(t, u_t) \leq\ & e^{\lambda t - \int_0^t \sigma^2(s)ds}\Bigg[\Bigg(\lambda\beta^2 - 2\gamma + \sum_{i=1}^{m}\left(\frac{\alpha_i}{\varepsilon_i} + r_i\right) \\
& + \sum_{i=1}^{m} r_i\left(e^{\lambda h_{0i}} - 1\right)\Bigg)\|u(t)\|^2 \\
& + \sum_{i=1}^{m}(\alpha_i\varepsilon_i - r_i(1-h_{1i}))\|u(t-h_i(t))\|^2 \\
& + (e^{\lambda\tau} - 1)\sigma^2(t+\tau)|u(t)|^2 + (\sigma^2(t+\tau) - \sigma^2(t))|u(t)|^2\Bigg] \\
\leq\ & e^{\lambda t - \int_0^t \sigma^2(s)ds}\Bigg[\Bigg(\lambda\beta^2 - 2\gamma + \sum_{i=1}^{m}\left(\frac{\alpha_i}{\varepsilon_i} + r_i\right) \\
& + \sum_{i=1}^{m} r_i\left(e^{\lambda h_{0i}} - 1\right)\Bigg)\|u(t)\|^2 \\
& + \sum_{i=1}^{m}(\alpha_i\varepsilon_i - r_i(1-h_{1i}))\|u(t-h_i(t))\|^2 + e^{\lambda\tau}C\beta^2\|u(t)\|^2\Bigg].
\end{aligned} \tag{20}$$

From here we deduce a similar expression for Eq. (17), where now the function $\rho(\lambda)$ is given by

$$\rho(\lambda) = (\lambda + Ce^{\lambda\tau})\beta^2 + \sum_{i=1}^{m} r_i(e^{\lambda h_{0i}} - 1) \geq 0.$$

As $\rho(0) = C\beta^2$, and ρ is continuous and increasing ($\rho'(\lambda) > 0$ for $\lambda \geq 0$), thanks to Eq. (20), we deduce that for a positive value of λ (small enough)

$$2\left(\gamma - \sum_{i=1}^{m} \frac{\alpha_i}{\sqrt{1-h_{1i}}}\right) \geq \rho(\lambda),$$

and the proof is finished.

3. Some applications

In this section we will show some interesting applications to illustrate the obtained above result.

3.1. **Application to some stochastic reaction-diffusion equations.**

In this subsection we will consider several linear reaction-diffusion equations to analyze their asymptotic behaviors with our theory. We can extend these examples to nonlinear equations in a straightforward way but we prefer to illustrate our results in a simple but important case.

Let us then consider the following three problems:

$$\begin{aligned} du(t,x) &= \left(\nu \frac{\partial^2 u(t,x)}{\partial x^2} + \sum_{i=1}^{m} \mu_i \frac{\partial^2 u(t-h_i(t),x)}{\partial x^2}\right) dt \\ &\quad + \sigma(t) u(t-\tau,x) dW(t), \end{aligned} \tag{21}$$

$$\begin{aligned} du(t,x) &= \left(\nu \frac{\partial^2 u(t,x)}{\partial x^2} + \sum_{i=1}^{m} \mu_i \frac{\partial u(t-h_i(t),x)}{\partial x}\right) dt \\ &\quad + \sigma(t) u(t-\tau,x) dW(t), \end{aligned} \tag{22}$$

$$\begin{aligned} du(t,x) &= \left(\nu \frac{\partial^2 u(t,x)}{\partial x^2} + \sum_{i=1}^{m} \mu_i \, u(t-h_i(t),x)\right) dt \\ &\quad + \sigma(t) u(t-\tau,x) dW(t), \end{aligned} \tag{23}$$

with the conditions

$$\begin{aligned} &t \geq 0, \quad x \in [a,b], \quad u(t,a) = u(t,b) = 0, \\ &h_i(t) \in C^1(\mathbf{R}), \quad h_i(t) \in [0, h_{0i}], \quad \dot{h}_i(t) \leq h_{1i} < 1, \\ &\sigma(t) = \frac{1}{(1+t)^\alpha}, \quad \alpha > \frac{1}{2}, \quad t \geq 0, \end{aligned}$$

where $\nu > 0$ μ_i, $i = 1, ..., m$, are arbitrary constants and $\tau > 0$ is a constant delay. For simplicity, we consider $W(t)$ as a scalar Wiener process over a probability space $(\Omega, \mathfrak{F}, P)$. Note that in all of these situations we can consider $U = H_0^1([a,b])$ and $H = L^2([a,b])$. The constant β for the injection $U \subset H$ becomes $\beta = \lambda_1^{-1/2}$, where $\lambda_1 = \pi^2(b-a)^{-2}$ is the first eigenvalue of the operator $-\frac{\partial^2}{\partial x^2}$ with Dirichlet boundary conditions. By straightforward computations, it is clear that $\gamma = \nu$, and denoting by $F_i(t,u) = \mu_i \frac{\partial^2 u}{\partial x^2}$ for $u \in U$, we obtain that $\|F_i(t,u)\|_* \leq |\mu_i| \|u\|$.

Similarly with $F_i(t,u) = \mu_i \dfrac{\partial u}{\partial x}$ for $u \in U$, it follows $\|F_i(t,u)\|_* \le |\mu_i|\lambda_1^{-1/2}\|u\|$, and also if $F_i(t,u) = \mu_i u$ for $u \in U$, it follows $\|F_i(t,u)\|_* \le |\mu_i|\lambda_1^{-1}\|u\|$. Therefore, we can apply Theorem 2.1 to all these examples yielding the following sufficient conditions (expressed by the corresponding condition (11)) for exponential mean square stability of the zero solution:
for equation (21)

$$\nu > \sum_{i=1}^{m} \frac{|\mu_i|}{\sqrt{1-h_{1i}}},$$

for equation (22)

$$\nu > \sum_{i=1}^{m} \frac{|\mu_i|}{\sqrt{\lambda_1(1-h_{1i})}},$$

and for equation (23)

$$\nu > \sum_{i=1}^{m} \frac{|\mu_i|}{\lambda_1\sqrt{1-h_{1i}}}.$$

Note that in the particular case $[a,b] = [0,\pi]$ it holds $\lambda_1 = 1$ and these three conditions in Theorem 2.1 are the same. It is also remarkable that these results hold independently of the length of the constant delay τ in the stochastic term, and also of the fading function $\sigma(t)$ provided it satisfies the conditions (9), (10), which confirms that stochastic perturbations fading at infinity do not affect the stability of the deterministic problem.

3.2. **Application to a stochastic 2D Navier-Stokes model.** We consider a stochastic 2D Navier-Stokes model with delay. The deterministic version of this problem has already been analyzed in details in Caraballo et al. (2007). The stochastic situation has also been considered in Chen (2012), Wei & Zhang (2009) when the delay function is the same in the diffusion and driving terms. We will analyze the case of different delays in both terms.

Let $\mathcal{O} \subset \mathbf{R}^2$ be an open and bounded set with regular boundary Γ, $T > 0$ given but arbitrary, and consider the following functional Navier-Stokes problem:

$$\begin{aligned}
&du + \left(-\nu\Delta u + \sum_{i=1}^{2} u_i \frac{\partial u}{\partial x_i}\right) dt \\
&\qquad\qquad = \left(-\nabla p + f(t,u_t)\right) dt + \Phi(t,u_t)dW(t) \text{ in } (0,T)\times\mathcal{O}, \\
&div\ u = 0 \quad \text{in} \quad (0,T)\times\mathcal{O}, \\
&u = 0 \quad \text{on} \quad (0,T)\times\Gamma, \\
&u(0,x) = u_0(x), \quad x\in\mathcal{O}, \\
&u(t,x) = \psi(t,x), \quad t\in(-h,0), \quad x\in\mathcal{O},
\end{aligned} \tag{24}$$

where we assume that $\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p the pressure, u_0 the initial velocity field, $f(t,u_t)$ is an external force containing some hereditary characteristics, $\Phi(t,u_t)dW(t)$ represents a stochastic term, where $W(t)$ is the standard Wiener process as we considered in the previous sections, and ψ the initial datum in the interval of time $[-h,0]$, where h is a positive fixed number.

To begin with, we consider the following usual abstract spaces

$$\mathcal{U} = \left\{ u \in \left(C_0^\infty(\mathcal{O})\right)^2 : div\ u = 0 \right\},$$

$H =$ the closure of $\mathcal{U}$ in $(L^2(\mathcal{O}))^2$ with the norm $|\cdot|$, and inner product $(\cdot,\cdot)$, where for $u, v \in (L^2(\mathcal{O}))^2$,

$$(u,v) = \sum_{j=1}^{2} \int_{\mathcal{O}} u_j(x) v_j(x) dx,$$

$U =$ the closure of $\mathcal{U}$ in $(H_0^1(\mathcal{O}))^2$ with the norm $\|\cdot\|$, and associated scalar product $((\cdot,\cdot))$, where for $u, v \in (H_0^1(\mathcal{O}))^2$,

$$((u,v)) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx.$$

It follows that $U \subset H \equiv H^* \subset U^*$, where the injections are dense and compact. Now we denote $a(u,v) = ((u,v))$, and define the trilinear form b on $U \times U \times U$ by

$$b(u,v,w) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} w_j dx \quad \forall\, u,v,w \in U.$$

Assume that the delay terms are given by

$$f(t,u_t) = \sum_{i=1}^{n} F_i u(t-h_i(t)), \qquad \Phi(t,u_t) = \sigma(t)\hat{\Phi} u(t-\tau),$$

where $F_i \in \mathfrak{L}(U,U^*)$, $i = 1, 2, \cdots, n$, is self-adjoint, $\hat{\Phi} \in \mathfrak{L}(H,H)$, the functions $h_i(t)$ are continuously differentiable with $0 \le \dot{h}_i(t) \le h_{1i} < 1$, and $\sigma(t)$ satisfy assumptions in the previous reaction diffusion-equations. Then, problem (24) can be set in the variational formulation:

$$\begin{aligned} d(u(t),v) + (\nu a(u(t),v) + b(u(t),u(t),v))dt &= \sum_{i=1}^{n} (F_i u(t-h_i(t)),v)\, dt \\ &\quad + (\sigma(t)\hat{\Phi}(u(t-\tau))dW(t),v), \\ u(0) = u_0, \;\; u(t) = \psi(t), \;\; t \in (-\max(h_1,h_2,\cdots,h_n,\tau),0), &\quad h_i(t) \le h_i, \end{aligned} \tag{25}$$

where equation (25) must be understood in the distributional sense of $\mathfrak{D}'(0,T)$.

Observe that equation (25) can be rewritten as equation (2) by denoting $A(t,\cdot), F_i : U \to U^*$ the operators defined as

$$A(t,u) = -\nu a(u,\cdot) - b(u,u,\cdot), \quad F_i(t,u) = F_i u, \quad B(t,u) = \sigma(t)\hat{\Phi} u, \quad u \in U.$$

In the present situation, i.e., for operators $F_i \in \mathfrak{L}(U,U^*)$, $\hat{\Phi} \in \mathfrak{L}(H,H)$ and functions $f(t,u_t) = \sum_{i=1}^{n} F_i u(t-h_i(t))$, $\Phi(t,u_t) = \sigma(t)\hat{\Phi} u(t-\tau(t))$ defined above, we have that $\gamma = \nu$, $\alpha = \|G\|_{\mathfrak{L}(U,U^*)}$, $\beta = \lambda_1^{-1/2}$ (λ_1 is the first eigenvalue of the Stokes operator) and the assumptions in Theorem 2.1 hold assuming that

$$\nu > \sum_{i=1}^{n} \frac{\|F_i\|_{\mathfrak{L}(U,U^*)}}{\sqrt{1-h_{1i}}}. \tag{26}$$

Observe that if $F_i \in \mathfrak{L}(H,H)$, then $F_i \in \mathfrak{L}(U,U^*)$. In addition, we have that

$$\|F_i\|_{\mathfrak{L}(U,U^*)} \le \lambda_1^{-1} \|F_i\|_{\mathfrak{L}(H,H)}.$$

Therefore, if we assume that

$$\nu\lambda_1 > \sum_{i=1}^{n} \frac{\|F_i\|_{\mathfrak{L}(H,H)}}{\sqrt{1-h_{1i}}},$$

it also follows (26) and, consequently, we have the exponential stability of the zero solution of equation (25).

Notice that a typical example for the forcing term with variable delay can be given by the Laplacian operator, i.e., we can consider that $f(t,u_t) = F_1(u(t-h_1(t))) = \Delta u(t-h_1(t))$. In this case, $F_1 : U \to U^*$ and it is straightforward to check that $\|F_1\|_{\mathfrak{L}(U,U^*)} = 1$. For the situation in which $F_i \in \mathfrak{L}(H,H)$, we can consider the forcing term $f(t,u_t) = F_2(u(t-h_1(t))) = \delta u(t-h_1(t))$, where $\delta > 0$ is a constant, which provides us with the norm $\|F_2\|_{\mathfrak{L}(H,H)} = \delta$.

4. Conclusions

It is shown that fading stochastic perturbations do not violate stability of an asymptotically stable deterministic nonlinear evolution equation by assumption that the level of stochastic perturbations is given by continuous, square integrable function. Earlier this idea was implemented for stochastic linear delay differential equations and for stochastic linear difference equations (Shaikhet, 2019a, 2019b, 2020). It is shown how the obtained results can be applied to known mathematical models that are very popular in researches. Consideration of other types of fading stochastic perturbations (for instance, if the function $\sigma(t)$ converges to zero at infinity but is not square integrable) is an open problem for future investigations.

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