ASYMPTOTIC BEHAVIOR OF NONLOCAL PARTIAL DIFFERENTIAL EQUATIONS WITH LONG TIME MEMORY

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Dedicated to Georg Hetzer on occasion of his 75th birthday

Abstract. In this paper, it is first addressed the well-posedness of weak solutions to a nonlocal partial differential equation with long time memory, which is carried out by exploiting the nowadays well-known technique used by Dafermos in the early 70's. Thanks to this Dafermos transformation, the original problem with memory is transformed into a nondelay one for which the standard theory of autonomous dynamical system can be applied. Thus, some results about the existence of global attractors to the transformed problem are provided. Particularly, when the initial values have higher regularity, the solutions of both problems (the original and the transformed ones) are equivalent. Nevertheless, the equivalence of global attractors for both problems is still unsolved due to the lack of enough regularity of solutions in the transformed problem, it is therefore proved the existence of global attractors of the transformed problem. Eventually, it is highlighted how to proceed to obtain meaningful results about the original problem, without performing any transformation, but working directly with the original delay problem.

Keywords: Nonlocal partial differential equations, Long time memory, Dafermos transformation, Global attractors
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1. **Introduction.** Many physical phenomena are properly described by (systems of) partial differential equations whose dynamics is influenced by the past history of one or more variables (see [1, 6, 2, 7, 11, 13] and the references therein). Also, in recent decades, nonlocal problems have been investigated with great interest due to their wide applications in the real world (see, [3, 4, 5, 6] and the references therein).

There is a significant amount of literature studying partial differential equations with long time memory. For example, the authors in [6] introduced the following semilinear problem describing the heat flow in a rigid, isotropic, homogeneous heat conductor with linear memory,

$$\begin{cases} c_0\partial_t\theta - k_0\Delta\theta - \displaystyle\int_{-\infty}^{t} k(t-s)\Delta\theta(s)ds + g(\theta) = h, & \text{in } \Omega\times\mathbb{R}^+, \\ \theta(x,t) = 0, & \text{on } \partial\Omega\times\mathbb{R}^+, \\ \theta(x,t) = \theta_0(x,t), & \text{in } \Omega\times(-\infty,0], \end{cases} \tag{1}$$

where $\Omega\subset\mathbb{R}^N$ is a fixed bounded domain with regular boundary occupied by a rigid heat conductor, $\theta:\Omega\times\mathbb{R}\to\mathbb{R}$ is the temperature variation field relative to the equilibrium reference value, $k:\mathbb{R}^+\to\mathbb{R}$ is the heat flux memory kernel, the constants c_0 and k_0 denote the specific heat and the instantaneous conductivity, respectively ($\mathbb{R}^+$ denotes the open interval $(0,+\infty)$). To handle (1) successfully, the authors considered this problem as a non-delay one by making the past history of θ from $-\infty$ to 0^- be part of the forcing term given by the causal function f, which is defined by,

$$f(x,t) = h(x,t) + \int_{-\infty}^{0} k(t-s)\Delta\theta(x,s)ds, \qquad x\in\Omega,\ t\geq 0.$$

In this way, (1) becomes an initial value problem without delay or memory,

$$\begin{cases} c_0\partial_t\theta - k_0\Delta\theta - \displaystyle\int_{0}^{t} k(t-s)\Delta\theta(s)ds + g(\theta) = f, & \text{in } \Omega\times\mathbb{R}^+, \\ \theta(x,t) = 0, & \text{on } \partial\Omega\times\mathbb{R}^+, \\ \theta(x,0) = \theta_0(x,0), & \text{in } \Omega. \end{cases} \tag{2}$$

Amongst the many notable results regarding the nonlocal differential equations, it is worth mentioning the pioneer work [9], in which the authors examined one model of single-species dynamics incorporating nonlocal effects, comparing with the standard approach to model a single-species domain Ω of "Kolmogorov" type,

$$\partial_t u = \Delta u + \lambda u g(u), \qquad \text{in} \quad \Omega,\ t>0.$$

If we take into account the following backgrounds: (i) a population in which individuals compete for a shared rapidly equilibrate resource; (ii) a population in which individuals communicate either visually or by chemical means, then the most straightforward way of introducing nonlocal effects is to consider, instead of $g(u)$, a "crowding" effect of the form $g(u,\bar{u})$, where

$$\bar{u}(x,t) = \int_\Omega G(x,y)u(y,t)dy,$$

and $G(x,y)$ is some reasonable kernel. Heuristically, Chipot et al. studied in [5] the behavior of a population of bacteria with nonlocal term $a(\int_\Omega u)$ in a container. Later, Chipot et al. extended this term to a general nonlocal operator $a(l(u))$ (cf. [3, 4, 5]), where $l\in\mathcal{L}(L^2(\Omega);\mathbb{R})$, for instance, if $g\in L^2(\Omega)$,

$$l(u) = l_g(u) = \int_\Omega g(x)u(x)dx.$$

Inspired by the above work, the dynamics of the following non-autonomous nonlocal partial differential equations with delay and memory was investigated in [17] by using the Galerkin method and energy estimations,

$$\begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + h(t,u_t), & \text{in } \Omega \times (\tau,\infty), \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}, \\ u_\tau(x,t) = \varphi(x,t), & \text{in } \Omega \times (-\rho, 0], \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $\tau \in \mathbb{R}$, function $a \in C(\mathbb{R};\mathbb{R}^+)$ is locally Lipschitz, $f \in C(\mathbb{R})$ and h contains hereditary characteristics involving delays, and $u_t : (-\infty, 0] \to \mathbb{R}$ is a segment of the solution given by $u_t(x,s) = u(x,t+s)$, $s \le 0$, which essentially is the history of the solution up to time t. Moreover, $0 < \rho \le \infty$, which implies that the authors considered both cases, bounded and unbounded delays, for this model. However, the technique applied in this paper to deal with memory terms is limited, since only exponential kernels can be contained in the delay terms. For instance, $h(t,u_t) = \int_{-\infty}^t k(t-s)\Delta u(x,s)ds$, where $k(\cdot)$ is nonsingular and of sub-exponential type (e.g., $k(t) = k_1 e^{-d_0 t}$, $k_1 \in \mathbb{R}, d_0 > 0$), for more details, see [1]. Whereas, this technique fails to deal with various important models with memory, whose kernels have singularities.

Consequently, it is not surprising that very recently a new discussion has started concerning the following long time memory differential equation associated with nonlocal diffusion,

$$\begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u - \displaystyle\int_{-\infty}^t k(t-s)\Delta u(s)ds + f(u) = g, & \text{in } \Omega \times (\tau,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}, \\ u(t+\tau) = u_0(t), & \text{in } \Omega \times (-\infty, 0], \end{cases} \tag{4}$$

where $\Omega \subset \mathbb{R}^N$ is a fixed bounded domain with regular boundary, $a \in C(\mathbb{R};\mathbb{R}^+)$ satisfies

$$0 < m \le a(r), \qquad \forall r \in \mathbb{R}, \tag{5}$$

$k : \mathbb{R}^+ \to \mathbb{R}$ is the memory kernel, with or without singularities, whose properties will be specified later, $g \in L^2(\Omega)$ which is independent of time. Notice that, thanks to a change of variable, the long time memory term in problem (4) can be interpreted as an infinite delay term,

$$\int_{-\infty}^t k(t-s)\Delta u(x,s)ds = \int_{-\infty}^0 k(-s)\Delta u(x,t+s)ds = \int_{-\infty}^0 k(-s)\Delta u_t(x,s)ds := h(u_t). \tag{6}$$

In this way, it is obvious that we are working on an autonomous nonlocal partial differential equation.

Problem (4) was analyzed in [1] when $a(\cdot)$ is a constant function (local problem) and the kernel k is of special type mentioned above. It is proved that the problem generates a dynamical system in the phase space $L^2_{H_0^1}$ given by the measurable functions $\varphi : (-\infty, 0] \to H_0^1(\Omega)$, such that $\int_{-\infty}^0 e^{\gamma s}\|\varphi(s)\|^2_{H_0^1} ds < +\infty$. Under the construction of this phase space, there exists a global attractor to this problem (in fact, the problem in [1] is non-autonomous and the attractor is of pullback type). Notice that, for this kind of delay problems, in which the initial value at zero may not be related to the values for negative times, the standard phase space to construct a dynamical system is the cartesian product $L^2(\Omega) \times L^2_{H_0^1}$. In such a way, for any initial values $u_0 \in L^2(\Omega)$ and

$\varphi \in L^2_{H^1_0}$, there exists a unique solution to the following problem,

$$\begin{cases} \frac{\partial u}{\partial t} - a\Delta u - \displaystyle\int_{-\infty}^{t} k(t-s)\Delta u(s)ds + f(u) = g, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}, \\ u(x,0) = u_0(x), & \text{in } \Omega, \\ u(x,t) = \varphi(x,t), & \text{in } \Omega \times (-\infty, 0). \end{cases} \tag{7}$$

According to the regularity of solutions to the above equation, one can define a dynamical system $S(t) : L^2(\Omega) \times L^2_{H^1_0} \to L^2(\Omega) \times L^2_{H^1_0}$ by the relation

$$S(t)(u_0, \varphi) := (u(t; 0, u_0, \varphi), u_t(\cdot; u_0, \varphi)),$$

where $u(\cdot; 0, u_0, \varphi)$ denotes the solution of problem (7) (see [2] for more details on this set-up). We emphasize that the two components of the dynamical system are the current state of the solution and the past history up to present, respectively, what is sensible in a problem with delays or memory. By using this framework, the method in [1] can be successfully applied to prove the existence of attractors to problem (7) when k is of exponential type. As for much more realistic cases, an alternative method based on the Dafermos transformation can provide us some ideas to study the long time behavior of the problem.

The idea to bypass problem (4) is to redefine g as

$$g + \int_{-\infty}^{\tau} k(t-s)\Delta u(x,s)ds,$$

hence, the new external term g includes the memory or history of the problem. Then, a way to associate a semigroup with such equation is to view the past history of u (in integrated form) as a new variable of the system, which will be ruled by a supplementary equation, $\eta^t_t = -\eta^t_s + u$. This idea goes back to the seventies, and it was first introduced by Dafermos [8] for linear viscoelasticity. We define the new variables,

$$u^t(x,s) = u(x, t-s), \qquad s \geq 0, \ t \geq \tau,$$

and

$$\eta^t(x,s) = \int_0^s u^t(x,r)dr = \int_{t-s}^{t} u(x,r)dr, \qquad s \geq 0, \ t \geq \tau. \tag{8}$$

Assuming $k(\infty) = 0$, a change of variable and a formal integration by parts yield

$$\int_{-\infty}^{t} k(t-s)\Delta u(s)ds = -\int_0^\infty k'(s)\Delta\eta^t(s)ds.$$

Here and in the sequel, the prime denotes derivation with respect to the variable s. Setting

$$\mu(s) = -k'(s),$$

the original equation (4) turns into the following autonomous system without delay,

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(l(u))\Delta u - \displaystyle\int_0^\infty \mu(s)\Delta\eta^t(s)ds + f(u) = g, & \text{in } \Omega \times (\tau,\infty), \\ \eta^t_t(s) = -\eta^t_s(s) + u(t), & \text{in } \Omega \times (\tau,\infty) \times \mathbb{R}^+, \\ u(x,t) = \eta^t(x,s) = 0, & \text{on } \partial\Omega \times \mathbb{R} \times \mathbb{R}^+ \\ u(x,\tau) = u_0(0), & \text{in } \Omega, \\ \eta^\tau(x,s) = \eta_0(s), & \text{in } \Omega \times \mathbb{R}^+. \end{cases} \tag{9}$$

where, η_s^t denotes the distributional derivative of $\eta^t(s)$ with respect to the internal variable s. It follows from the definition of $\eta^t(x,s)$ (see (8)) that

$$\eta_0(s) = \int_{\tau-s}^{\tau} u(r)dr = \int_{\tau-s}^{\tau} u_0(r-\tau)dr = \int_{-s}^{0} u_0(r)dr, \tag{10}$$

which is the initial integrated past history of u with vanishing boundary. Consequently, any solution to (4) is solution to (9) for the corresponding initial values $(u_0(0),\eta_0)$ given by (10). It is worth emphasizing that problem (9) can be solved for arbitrary initial values (u_0,η_0) in a proper phase space $L^2(\Omega)\times L^2_\mu(\mathbb{R}^+;H^1_0(\Omega))$ (see Section 2), i.e., the second component η^t does not need to depend on u_0. This allows us to construct a dynamical system in this phase space and prove the existence of global attractors. However, the transformed equation (9) is in fact a generalization of problem (4), which means, not every solution to equation (9) possesses a corresponding one to (4). In fact, both problems are equivalent if and only if the initial value η_0 belongs to a proper subset of $L^2_\mu(\mathbb{R}^+;H^1_0(\Omega))$, which coincides with the domain of the distributional derivative with respecto to s, denoted by $D(\mathbf{T})$ (for more details, see [11]). In this paper, when we work on (9) assuming $\eta_0\in D(\mathbf{T})$, we will prove existence and uniqueness of solution to problem (9), and this also provides us the existence of solution to problem (4). However, when η_0 does not necessarily belong to $D(\mathbf{T})$, we may not have any corresponding solution to the original problem. Once we construct the dynamical system in the phase space $L^2(\Omega)\times L^2_\mu(\mathbb{R}^+;H^1_0(\Omega))$, the existence of absorbing sets and an asymptotic compactness property ensuring the existence of global attractor for the problem (9) can be shown, but we are not able to prove that such an attractor exists in a more regular phase space (related to $D(\mathbf{T})$ for the second component of the dynamical system), so that we could have the complete equivalent information for the initial problem (4). This is also the case in the papers dealing with local problems (see [6, 7, 11]).

The content of this paper is as follows. In Section 2, we recall some preliminaries and notations which are necessary for our analysis. Section 3 is devoted to proving the existence and uniqueness of solution to problem (9) as well as some necessary regularity results and continuous dependence on the initial values. The main techniques are the Faedo-Galerkin scheme and some compactness results. The nonlocal term introduces some technical difficulties which are successfully overcome. Next, in Section 4, it is first proved the existence of absorbing sets in two phase spaces and the asymptotic compactness of the dynamical system, which ensure the existence of global attractors. Eventually, in Section 5, some conclusions and further comments for future investigations are included.

2. **Preliminaries.**

The purpose of this section is to state assumptions on μ and f, which allow us to prove properly the well-posedness and asymptotic behavior of problem (9). In addition, some notations will be introduced so that we can study our problem in suitable phase spaces. At last, we will establish several lemmas and corollaries for the supplementary equation.

2.1. **Assumptions.**

Suppose the nonlinear term, $f:\mathbb{R}\to\mathbb{R}$, is a polynomial of odd degree with positive leading coefficient,

$$f(u) = \sum_{k=1}^{2p} f_{2p-k}u^{k-1},\ p\in\mathbb{N}. \tag{11}$$

Remark 1.

(a) In fact, to prove the main results of this paper, we can consider, more generally, a continuously differentiable function f on $\mathbb{R}$ satisfying
   (i) $|f(u)| \leq k_1(1+|u|^{\beta})$,
   (ii) $u \cdot f(u) \geq -k_2 + k_3|u|^{\beta+1}$,
   (iii) $f'(u) \geq -k_4$,

   for some $\beta > 0$ and $k_j \geq 0$, $j = 1, 2, 3, 4$. The proofs of main results in this paper do not have significant changes with assumptions (i)-(iii) comparing to (11).

(b) Comparing with assumptions (2.3)-(2.4) to nonlinear term $f(u)$ in Section 2 in [17], here, (11) is the quite weaker condition which has less restrictions on $f(u)$. We also want to emphasize if condition (11) were adopted in [17] instead of (2.3)-(2.4), the method to prove main results of [17] can not be used successfully.

In view of the evolution problem (9), the variable μ is required to verify the following hypotheses,

(h_1) $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $\quad \mu(s) \geq 0$, $\quad \mu'(s) \leq 0$, $\quad \forall s \in \mathbb{R}^+$;

(h_2) $\mu'(s) + \delta\mu(s) \leq 0$, $\quad \forall s \in \mathbb{R}^+$, for some $\delta > 0$.

Remark 2. *(i) Restriction (h_1) is equivalent to assuming $k(s)$ is a bounded, positive, non-increasing, convex function of class C^2 vanishing at infinity. Moreover, from (h_1) it easily follows that*

$$k(0) = \int_0^{\infty} \mu(s)ds \qquad \text{is finite and nonegative.}$$

(ii) Observe that, by the Gronwall inequality, (h_2) implies $\mu(s)$ decays exponentially, i.e.,

$$\mu(s) \leq \mu(s_0)e^{-\delta(s-s_0)}, \qquad \forall s \geq s_0 > 0. \tag{12}$$

2.2. **Notations.** Throughout this manuscript, we make use of several notations introduced in what follows. Let $|\cdot|_p$ denotes the L^p-norm for $p \geq 1$, let $(\cdot,\cdot)$ and $|\cdot|$ denote the L^2-inner product and L^2-norm, $((\cdot,\cdot))$ and $\|\cdot\|$ denote the H_0^1-inner product and H_0^1-norm, respectively. Recall that for every $v \in H_0^1(\Omega)$, the Poincaré inequality

$$\lambda_1(\Omega)|v|^2 \leq \|v\|^2$$

holds. If $v \in H^2(\Omega) \cap H_0^1(\Omega)$, the Poincaré and Young inequalities yield

$$\lambda_1(\Omega)|\nabla v|^2 \leq |\Delta v|^2,$$

where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ with zero Dirichlet boundary conditions on Ω. In the sequel, unless otherwise specified, we write λ_1 instead of $\lambda_1(\Omega)$ for simplicity. To simplify the presentation, from now on, the space $L^2(\Omega)$ is denoted by H, $H_0^1(\Omega)$ is denoted by V and $H_0^1(\Omega) \cap H^2(\Omega)$ is denoted by $D(A)$. As usual, V^* is the dual space of V, whose norm is denoted by $\|\cdot\|_*$.

In view of system (9) and (h_1), it is necessary to introduce suitable Banach spaces, which aims at capturing the essence of the problem. Let $L^2_\mu(\mathbb{R}^+; H)$ be the Hilbert space of functions $w : \mathbb{R}^+ \to H$ endowed with the inner product,

$$(w_1, w_2)_\mu = \int_0^{\infty} \mu(s)(w_1(s), w_2(s))ds,$$

and let $|\cdot|_\mu$ denote the corresponding norm. In a similar way, we introduce the inner products $((\cdot,\cdot))_\mu$, $(((\cdot,\cdot)))_\mu$ and corresponding norms $\|\cdot\|_\mu$, $|||\cdot|||_\mu$ on $L^2_\mu(\mathbb{R}^+; V)$, $L^2_\mu(\mathbb{R}^+; D(A))$, respectively. It follows then

$$((\cdot,\cdot))_\mu = (\nabla\cdot, \nabla\cdot)_\mu \quad \text{and} \quad (((\cdot,\cdot)))_\mu = (\Delta\cdot, \Delta\cdot)_\mu.$$

We also define the Hilbert spaces,

$$\mathcal{H} = H \times L^2_\mu(\mathbb{R}^+; V) \quad \text{and} \quad \mathcal{V} = V \times L^2_\mu(\mathbb{R}^+; D(A)),$$

which are respectively endowed with inner products,

$$(w_1, w_2)_{\mathcal{H}} = (w_1, w_2) + ((w_1, w_2))_\mu,$$

and

$$(w_1, w_2)_{\mathcal{V}} = ((w_1, w_2)) + (((w_1, w_2)))_\mu,$$

where $w_i \in \mathcal{H}$ or $\mathcal{V}$ $(i = 1, 2)$. The norms induced on $\mathcal{H}$ and $\mathcal{V}$ are the so-called energy norm, which read,

$$\|(w_1, w_2)\|^2_{\mathcal{H}} = |w_1|^2 + \int_0^\infty \mu(s)\|w_2(s)\|^2 ds,$$

and

$$\|(w_1, w_2)\|^2_{\mathcal{V}} = \|w_1\|^2 + \int_0^\infty \mu(s)\|\nabla w_2(s)\|^2 ds.$$

Let T be the linear operator with domain

$$D(\mathbf{T}) = \left\{\eta(\cdot) \in L^2_\mu(\mathbb{R}^+; V) \mid \eta_s(\cdot) \in L^2_\mu(\mathbb{R}^+; V),\ \eta(0) = 0\right\},$$

defined by

$$\mathbf{T}\eta = -\eta_s, \qquad \eta \in D(\mathbf{T}).$$

Indeed, $\mathbf{T}$ is the infinitesimal generator of the right-translation semigroup on $L^2_\mu(\mathbb{R}^+; V)$, see [11].

Eventually, we introduce the following space

$$\mathcal{E} = \{\eta(\cdot) \in L^2_\mu(\mathbb{R}^+; D(A)) \mid \eta(\cdot) \in D(\mathbf{T}),\ \sup_{r\geq 1} r\mathcal{L}_\eta(r) < \infty\}.$$

Here, $\mathcal{L}_\eta$ is the tail function of $\eta(\cdot)$, which is given by

$$\mathcal{L}_\eta(r) = \int_{(0,1/r)\cup(r,\infty)} \mu(s)\|\eta(s)\|^2 ds, \quad r \geq 1.$$

It is straightforward to see (cf. [10]) that $\mathcal{E}$ is a Banach space endowed with the norm

$$\|\eta\|^2_{\mathcal{E}} = \|\eta\|^2_{L^2_\mu(\mathbb{R}^+;D(A))} + \|\mathbf{T}\eta\|^2_{L^2_\mu(\mathbb{R}^+;V)} + \sup_{r\geq 1} r\mathcal{L}_\eta(r).$$

At this point, an immediate generalization of a compactness result is presented in [15, Lemma 3.1].

Lemma 3. *Let $\mathcal{K} \subset L^2_\mu(\mathbb{R}^+; V)$, satisfying*

$$\sup_{\eta\in\mathcal{K}} \left[\|\eta\|_{L^2_\mu(\mathbb{R}^+;D(A))} + \|\eta_s\|_{L^2_\mu(\mathbb{R}^+;V^*)}\right] < \infty \quad \text{and} \quad \lim_{r\to\infty}\left[\sup_{\eta\in\mathcal{K}} \mathcal{L}_\eta(r)\right] = 0.$$

Then $\mathcal{K}$ is relatively compact in $L^2_\mu(\mathbb{R}^+; V)$.

At last, with standard notations, $\mathcal{D}_c(I; X)$ is the space of infinitely differentiable X-valued function with compact support in $I \subset \mathbb{R}$, whose dual space is the distribution space on I with values in X^* (dual of X), denoted by $\mathcal{D}'(I; X^*)$.

2.3. **The representation formula and some useful lemmas.** We recall the following definitions and properties of the supplementary equation, for more details, see [7, 11] and references therein. Assume that u is a given function belonging to $L^1(\tau,T;V)$ for every $T>\tau$. Then, for every $\eta_0\in L^2_\mu(\mathbb{R}^+;V)$, the Cauchy problem

$$\begin{cases} \frac{d}{dt}\eta^t(s)=\mathbf{T}\eta^t(s)+u, \qquad t\geq\tau,\\ \eta^0=\eta_0,\ t<\tau,\ s\geq 0,\end{cases}$$

has a unique solution $\eta^\cdot\in C(\tau,\infty;L^2_\mu(\mathbb{R}^+;V))$, which has the explicit expression formula (see [11]),

$$\eta^t(s)=\begin{cases}\displaystyle\int_{t-s}^t u(r)dr, \qquad 0<s\leq t,\\ \displaystyle\eta_0(s-t)+\int_0^t u(t-r)dr, \quad s>t.\end{cases} \tag{13}$$

We next show estimates for the functions η^t belonging to $\mathcal{E}$, since they play an important role in the proof of the asymptotically compact theorem.

Lemma 4. *Let $\eta_0\in D(\mathbf{T})$, assume that $\|u(t)\|\leq\rho$, for some $\rho>0$ and every $t\geq\tau$. Then*

$$\|\mathbf{T}\eta^t\|^2_{L^2_\mu(\mathbb{R}^+;V)}\leq e^{-\delta t}\|\mathbf{T}\eta_0\|^2_{L^2_\mu(\mathbb{R}^+;V)}+\rho^2\|\mu\|_{L^1}.$$

Lemma 5. *Let $\eta_0\in D(\mathbf{T})$, assume that $\|u(t)\|\leq\rho$, for some $\rho>0$ and $t\geq\tau$. Then*

$$\sup_{r\geq1} r\pounds_{\eta^t}(r)\leq\sup_{r\geq1} r\pounds_{\eta_0}(r)\Psi(t)+\Pi\rho^2,$$

where $\Psi(t)=2(t+2)e^{-\delta t}$ and $\Pi=2\int_0^\infty s\mu(s)ds+2\int_1^\infty s^3\mu(s)ds$, which is finite.

Obviously, if we only require that $\|u(t)\|\leq\rho$ for every $t\in[\tau,T]$, then the results above are true on $[\tau,T]$. Therefore, a straightforward consequence of Lemmas 4 and 5 is:

Corollary 6. *If $\eta_0\in D(\mathbf{T})$ and $u\in L^\infty(\tau,T;V)$ for every $T>\tau$. Then $\eta^t\in D(\mathbf{T})$ for all $t>\tau$.*

Lemma 7. ([7, Lemma 3.7]) *Let φ be a nonnegative absolutely continuous function on $[0,\infty)$ satisfying, for some $\nu>0$ and $0\leq\sigma<1$, the differential inequality*

$$\frac{d}{dt}\varphi+\nu\varphi\leq g(1+\varphi^\sigma),$$

where g is a nonnegative function fulling

$$\sup_{t\geq0}\int_t^{t+1} g(y)dy<\infty.$$

Then there exists a constant $C=C(\sigma,\nu,g)$ such that,

$$\varphi(t)\leq\frac{1}{1-\sigma}\varphi(0)e^{-\nu t}+C, \qquad \forall t\geq0.$$

3. **Existence of solutions to nonlocal differential equations with memory.** As we did in the previous sections, in order to set up equation (4) in the framework of dynamical systems, we actually study transformed equation (9) by Dafermos. One natural question might be whether this procedure is consistent, namely, whether there is a link between equations (4) and (9). Indeed, it turns out that they are equivalent, or, to be more precise, the transformed equation (9) is in fact a generalization of the original equation (4), for more explanation, see [11]. Especially, when $\eta_0 \in D(\mathbf{T})$, problems (4) and (9) are the same.

By denoting

$$z(t) = (u(t), \eta^t), \qquad \text{and} \qquad z_0 = (u_0, \eta_0),$$

setting

$$\mathcal{L}z = \left(a(l(u))\Delta u + \int_0^\infty \mu(s)\Delta\eta(s)ds,\ u + \mathbf{T}\eta \right),$$

and

$$\mathcal{G}(z) = (-f(u) + g,\ 0).$$

Then problem (9) can be written in the following compact form,

$$\begin{cases} z_t = \mathcal{L}z + \mathcal{G}(z), & \text{in } \Omega \times (\tau, \infty), \\ z(x,t) = 0, & \text{on } \partial\Omega \times (\tau, \infty), \\ z(x,\tau) = z_0, & \text{in } \Omega. \end{cases} \tag{14}$$

We are now ready to state the main result in this section.

Theorem 8. *Suppose* (5), (11) *and* (h_1)-(h_2) *hold true, let* $g \in H$. *In addition, assume that* $a(\cdot)$ *is locally Lipschitz, and there exists a positive constant* $\tilde{m}$ *such that,*

$$a(s) \le \tilde{m}, \qquad \forall s \in \mathbb{R}. \tag{15}$$

Then:

(i) *For any* $z_0 \in \mathcal{H}$, *there exists a unique solution* $z = (u, \eta)$ *to problem* (14) *which satisfies*

$$u \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^{2p}(\tau, T; L^{2p}(\Omega)), \qquad \forall T > \tau,$$
$$\eta \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; V)), \qquad \forall T > \tau.$$

Furthermore, $z \in C(\tau, T; \mathcal{H})$ *for every* $T > \tau$, *and the mapping* $F : z_0 \in \mathcal{H} \to z(t) \in \mathcal{H}$ *is continuous for every* $t \in [\tau, T]$;

(ii) *For any* $z_0 \in \mathcal{V}$, *the unique solution* $z = (u, \eta)$ *to problem* (14) *satisfies*

$$u \in L^\infty(\tau, T; V) \cap L^2(\tau, T; D(A)), \qquad \forall T > \tau,$$
$$\eta \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; D(A))), \qquad \forall T > \tau.$$

In addtion, $z \in C(\tau, T; \mathcal{V})$ *for every* $T > \tau$.

Proof. (i) The proof is divided into 5 steps.

Step 1. (Faedo-Galerkin method) Recall that there exists a smooth orthonormal basis $\{w_j\}_{j=1}^\infty$ in H which is also orthogonal in V. In fact, it is a complete set of normalized eigenfunctions for $-\Delta$ in V such that $-\Delta w_j = \lambda_j w_j$, where λ_j is the eigenvalue corresponding to w_j. Meanwhile, we choose an orthonormal basis $\{\zeta_j\}_{j=1}^\infty$ of $L^2_\mu(\mathbb{R}^+; V)$ which also belongs to $\mathcal{D}(\mathbb{R}^+; V)$.

Then, for every fixed $T > \tau$ and a given integer n, we look for a function $z_n(\cdot) = (u_n(\cdot), \eta_n^{\cdot})$ of the form

$$u_n(t) = \sum_{j=1}^{n} b_j(t) w_j \qquad \text{and} \qquad \eta_n^t(s) = \sum_{j=1}^{n} c_j(t) \zeta_j(s),$$

satisfying

$$\begin{cases} (\partial_t z_n, (w_k, \zeta_j))_{\mathcal{H}} = (\mathcal{L} z_n, (w_k, \zeta_j)) + (\mathcal{G}(z), (w_k, \zeta_j)), & k, j = 0, \cdots, n, \\ z_n|_{t=0} = (P_n u_0, Q_n \eta_0), \end{cases} \tag{16}$$

for a.e. $\tau \le t \le T$, where P_n and Q_n denote the projections on the subspaces V and $L^2_\mu(\mathbb{R}^+; V)$, respectively, w_0 and ζ_0 are the zero vectors in each subspace. Taking (w_k, ζ_0) and (w_0, ζ_k) in (16), applying the divergence theorem, we derive a system of ODE in the variables

$$\begin{cases} \frac{d}{dt} b_k(t) = -\lambda_k a(l(\sum_{j=1}^n b_i(t) w_j)) b_k - \sum_{j=1}^n c_j((\zeta_j, w_k))_\mu - (f(\sum_{j=1}^n b_j(t) w_j), w_k) + (g, w_k), \\ \frac{d}{dt} c_k(t) = \sum_{j=1}^n b_j((w_j, \zeta_k))_\mu - \sum_{j=1}^n c_j((\zeta_j', \zeta_k))_\mu, \end{cases} \tag{17}$$

fulfilling the initial values,

$$b_k(\tau) = (u_0, w_k), \qquad c_k(\tau) = ((\eta_0, \zeta_k))_\mu. \tag{18}$$

According to standard existence theory for ODE, there exists a continuous solution of (17)-(18) on some interval (τ, T_n). Moreover, a priori estimation implies $T_n = \infty$, for more details, see [17].

Step 2. (Energy estimation) Multiplying the first equation of (17) by b_k and the second one by c_k, summing over k ($k = 1, 2, \cdots, n$) and adding the results, we have

$$\frac{1}{2} \frac{d}{dt} \|z_n\|^2_{\mathcal{H}} = (\mathcal{L} z_n, z_n)_{\mathcal{H}} + (\mathcal{G}(z_n), z_n)_{\mathcal{H}}. \tag{19}$$

By the divergence theorem,

$$\left(\int_0^\infty \mu(s) \Delta \eta_n^t(s) ds, u_n \right) = -\int_0^\infty \mu(s) \int_\Omega \nabla \eta_n^t(s) \cdot \nabla u_n(s) dx ds = -((u_n, \eta_n^t))_\mu,$$

therefore,

$$(\mathcal{L} z_n, z_n)_{\mathcal{H}} = -a(l(u_n)) |\nabla u_n|^2 - (((\eta_n^t)', \eta_n^t))_\mu. \tag{20}$$

On the other hand, using the Young inequality, from (11) we know there exists a positive constant a_0, such that

$$f(u) u \ge \frac{1}{2} f_0 u^{2p} - a_0,$$

hence,

$$(\mathcal{G}(z_n), z_n)_{\mathcal{H}} = (-f(u_n) + g, u_n) \le -\frac{1}{2} f_0 |u_n|^{2p}_{2p} + a_0 |\Omega| + (g, u_n). \tag{21}$$

From (5), (19)-(21) and the Young and Poincaré inequalities, it follows

$$\frac{d}{dt} \|z_n\|^2_{\mathcal{H}} + 2m |\nabla u_n|^2 + 2(((\eta_n^t)', \eta_n^t))_\mu + f_0 |u_n|^{2p}_{2p} \le 2 a_0 |\Omega| + \frac{1}{m \lambda_1} |g|^2 + m |\nabla u_n|^2. \tag{22}$$

Integration by parts and (h_1) yield,

$$2(((\eta_n^t)', \eta_n^t))_\mu = -\int_0^\infty \mu'(s) |\nabla \eta_n^t(s)|^2 ds \ge 0, \tag{23}$$

thus the third term of the right hand side of (22) can be neglected. We obtain

$$\frac{d}{dt}\|z_n\|_{\mathcal{H}}^2 + m|\nabla u_n|^2 + f_0|u_n|_{2p}^{2p} \leq 2a_0|\Omega| + \frac{1}{m\lambda_1}|g|^2.$$

Integrating the above inequality between τ and t, $t \in (\tau, T]$, we have

$$\|z_n(t)\|_{\mathcal{H}}^2 + \int_\tau^t \Big[m\|u_n\|^2 + f_0|u_n|_{2p}^{2p}\Big]\, ds \leq \|z_0\|_{\mathcal{H}}^2 + \Lambda(T-\tau), \tag{24}$$

where we have used the notation $\Lambda := 2a_0|\Omega| + \frac{1}{m\lambda_1}|g|^2$. Therefore, it arrives at

$$u_n \quad \text{is bounded in} \quad L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^{2p}(\tau, T; L^{2p}(\Omega)),$$
$$\eta_n \quad \text{is bounded in} \quad L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; V)).$$

Passing to a subsequence, there exists a function $z = (u, \eta)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{weak-star in} & L^\infty(\tau, T; H); \\ u_n \rightharpoonup u & \text{weakly in} & L^2(\tau, T; V); \\ u_n \rightharpoonup u & \text{weakly in} & L^{2p}(\tau, T; L^{2p}(\Omega)); \\ \eta_n \rightharpoonup \eta & \text{weak-star in} & L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; V)). \end{cases} \tag{25}$$

Step 3. (Pass to limit) For a fixed integer m, choose a function

$$v = (\sigma, \pi) \in \mathcal{D}((\tau, T); V \cap L^{2p}(\Omega)) \times \mathcal{D}((\tau, T); \mathcal{D}(\mathbb{R}^+; V))$$

of the form

$$\sigma(t) = \sum_{j=1}^m \tilde{b}_j(t) w_j \qquad \text{and} \qquad \pi^t(s) = \sum_{j=1}^m \tilde{c}_j(t)\zeta_j(s),$$

where $\{\tilde{b}_j\}_{j=1}^m$ and $\{\tilde{c}_j\}_{j=1}^m$ are given functions in $\mathcal{D}(\tau, T)$. Obviously, (16) holds with (σ, π) in place of (ω_k, ζ_j).

We aim at proving problem (14) has a solution in the weak sense. To this end, let us pick up $v = (\sigma, \pi) \in \mathcal{D}(\tau, T)$ as a test function, so that the following equality

$$\begin{aligned} \int_\tau^t (\partial_s z_n, v)_{\mathcal{H}} ds = \int_\tau^t \Big[& -a(l(u_n))(\nabla u_n, \nabla\sigma) - ((\eta_n^t, \sigma))_\mu - (f(u_n), \sigma) \\ & + (g, \sigma) + ((u_n, \pi^t))_\mu - \ll (\eta_n^t)', \pi^t \gg \Big] ds, \end{aligned} \tag{26}$$

holds in the sense of $\mathcal{D}'(\tau, T)$. Here, denoting by $\ll \cdot, \cdot \gg$ the duality map between $H^1_\mu(\mathbb{R}^+; V)$ and its dual space.

Firstly, using the same argument as [17, Theorem 2.7] and $(25)_2$, we know

$$\int_\tau^t a(l(u_n))(\nabla u_n, \nabla\sigma) ds \to \int_\tau^t a(l(u))(\nabla u, \nabla\sigma) ds \qquad \text{as} \quad n \to \infty.$$

Analogously, by means of $(25)_4$ and $(25)_2$, we have

$$\int_\tau^t ((\eta_n^t, \sigma))_\mu ds \to \int_\tau^t ((\eta^t, \sigma))_\mu ds \qquad \text{as} \quad n \to \infty,$$

and

$$\int_\tau^t ((u_n, \pi^t))_\mu ds \to \int_\tau^t ((u, \pi^t))_\mu ds \qquad \text{as} \quad n \to \infty,$$

respectively.

Secondly, we now show that

$$\lim_{n\to\infty} \ll (\eta_n^t)', \pi^t \gg = \ll (\eta^t)', \pi^t \gg .$$

Indeed, for every $\upsilon \in L^2_\mu(\mathbb{R}^+; V)$, making use of integration by parts, we derive

$$\ll \upsilon', \pi^t \gg = -\int_0^\infty \mu'(s)(\nabla\upsilon(s), \nabla\pi^t(s))ds - \int_0^\infty \mu(s)(\nabla\upsilon(s), \nabla(\pi^t)'(s))ds. \tag{27}$$

Replacing υ by η_n in (27), together with $(25)_4$, it is clear the right hand side of (27) converges to $\ll \eta', \pi^t \gg$ as $n\to\infty$.

Thirdly, we are going to prove that

$$\lim_{n\to\infty}\int_\tau^T\int_\Omega |f(u_n)\sigma|dxdt = \int_\tau^T\int_\Omega |f(u)\sigma|dxdt.$$

Based on the dominated convergence theorem, it is sufficient to show

$$f(u_n(t,x)) \to f(u(t,x)) \qquad \text{for a.e. } (t,x)\in(\tau,T)\times\Omega,$$

and

$$|f(u_n)|_{L^q((\tau,T)\times\Omega)} \le C,$$

where $q = \frac{2p}{2p-1} \in (1,2)$, which is the conjugate exponent of $2p$ and the constant C is independent of n. Observe that

$$\begin{aligned}\|\partial_t u_n\|_{L^2(\tau,T;V^*)+L^q(\tau,T;L^q(\Omega))} \le \|a(l(u_n))\Delta u_n\|_{L^2(\tau,T;V^*)} + \left\|\int_0^\infty \mu(s)\Delta\eta_n^t(s)ds\right\|_{L^2(\tau,T;V^*)} \\ + \|g\|_{V^*} + \|f(u_n)\|_{L^q(\tau,T;L^q(\Omega))}.\end{aligned} \tag{28}$$

It follows from (11), there exists a constant $K>0$ such that

$$|f(u_n)|^q \le K(1+|u_n|^{2p}). \tag{29}$$

Together with (15), (25) and the assumption $g\in H$, we know that $\{\partial_t u_n\}$ is bounded in $L^2(\tau,T;V^*)+L^q(\tau,T;L^q(\Omega))$. Thus, up to a subsequence, we infer

$$\partial_t u_n \rightharpoonup \tilde{u} \qquad \text{weakly in} \quad L^2(\tau,T;V^*)+L^q(\tau,T;L^q(\Omega)). \tag{30}$$

By a standard argument we infer that $\tilde{u} = u_t$. Since

$$L^2(\tau,T;V^*)+L^q(\tau,T;L^q(\Omega)) \subset L^q(\tau,T;V^*+L^q(\Omega))$$

and

$$L^2(\tau,T;V) \subset L^q(\tau,T;V),$$

by (25) and (30), we deduce

$$u_n \rightharpoonup u \qquad \text{weakly in} \quad W^{1,q}(\tau,T;V+L^q(\Omega))\cap L^q(\tau,T;V). \tag{31}$$

Applying a compactness argument [14], it derives the injection

$$W^{1,q}(\tau,T;V^*+L^q(\Omega))\cap L^q(\tau,T;V) \hookrightarrow L^q(\tau,T;L^q(\Omega))$$

is compact. Therefore, (31) implies that

$$u_n \to u \qquad \text{strongly in} \quad L^q(\tau,T;L^q(\Omega)).$$

By the continuity of f we obtain that (up to a subsequence),

$$f(u_n(t,x)) \to f(u(t,x)) \qquad \text{for a.e. } (t,x) \in (\tau,T)\times\Omega. \tag{32}$$

By (29) we obtain

$$|f(u_n)|^q_{L^q((\tau,T)\times\Omega)} = \int_\tau^T \int_\Omega |f(u_n)|^q dxdt \le K|\Omega|(T-\tau) + K\int_\tau^T |u_n|^{2p}_{2p}dt,$$

which is bounded uniformly with respect to n.

Eventually, by a standard argument, we derive

$$\partial_t z_n \to z_t \quad \text{in} \quad \mathcal{D}'(\tau,T;V\cap L^{2p}) \times \mathcal{D}'(\tau,T;\mathcal{D}(\mathbb{R}^+;V)).$$

Pass to the limit in both sides of equality (26), combine with the previous statements, it is proved $z(t) = (u(t),\eta^t)$ is a weak solution of problem (14).

Step 4. (Continuity of solution) By (27) and (28), it is immediate to see that $z_t = (u_t,\eta_t)$ fulfills

$$\begin{aligned} &u_t \in L^2(\tau,T;V^*) + L^q(\tau,T;L^q(\Omega));\\ &\eta_t \in L^2(\tau,T;H^{-1}_\mu(\mathbb{R}^+;V)),\end{aligned}$$

where $L^2(\tau,T;V^*) + L^q(\tau,T;L^q(\Omega))$ is the dual space of $L^2(\tau,T,V)\cap L^{2p}(\tau,T;L^{2p}(\Omega))$. Using a slightly modified version of [16, Lemma III.1.2], together with (25), we infer that $u \in C(\tau,T;H)$.

As for the second component, by means of the same argument as [6, Theorem, Section 2], we obtain that $\eta^t \in C(\tau,T;L^2_\mu(\mathbb{R}^+;V))$. Thus, $z(\tau)$ makes sense, and the equality $z(\tau) = z_0$ follows from the fact that $(P_n u_0, Q_n\eta_0)$ converges to z_0 strongly.

Step 5. (Continuity with respect to the initial value and uniqueness) Let $z_1 = (u_1,\eta_1)$ and $z_2 = (u_2,\eta_2)$ be the two solutions of (14) with initial data z_{10} and z_{20}, respectively. Due to the a priori estimates on the first component of solutions u (see (24)) together with the fact that $u \in C(\tau,T;H)$, we can ensure that there exists a bounded set $S \subset H$, such that $u_i(t) \in S$ for all $t \in [\tau,T]$ and $i = 1,2$. In addition, taking into account that $l \in \mathcal{L}(H;\mathbb{R})$, we have $\{l(u_i(t))\}_{t\in[\tau,T]} \subset [-R,R]$ for $i = 1,2$, for some $R > 0$. Therefore, let $\bar{z} = z_1 - z_2 = (\bar{u},\bar{\eta}) = (u_1-u_2,\eta_1-\eta_2)$ and $\bar{z}_0 = z_{10} - z_{20}$, thanks to (5), the locally Lipschitz continuity of function a with Lipschitz constant $L_a(R)$ and the Poincaré inequality, we have

$$\begin{aligned}\frac{d}{dt}\|\bar{z}\|^2_{\mathcal{H}} &\le 2a(l(u_1))|\nabla\bar{u}|^2 + 2L_a(R)|l||\bar{u}||\nabla u_2||\nabla\bar{u}|\\
&\quad - 2 < f(u_1) - f(u_2), \bar{u} >_{L^{p,q}} -2(((\bar{\eta})',\bar{\eta}))_\mu\\
&\le -2m|\nabla\bar{u}|^2 + 2L_a(R)|l||\bar{u}||\nabla u_2||\nabla\bar{u}|\\
&\quad - 2 < f(u_1) - f(u_2), \bar{u} >_{L^{p,q}} -2(((\bar{\eta})',\bar{\eta}))_\mu\\
&\le -2m|\nabla\bar{u}|^2 + 2m|\nabla\bar{u}|^2 + \frac{1}{2m}L_a^2(R)|l|^2|\bar{u}|^2\|u_2\|^2\\
&\quad - 2 < f(u_1) - f(u_2), \bar{u} >_{L^{p,q}} -2(((\bar{\eta})',\bar{\eta}))_\mu\\
&\le \frac{1}{2m}L_a^2(R)|l|^2\|\bar{z}\|^2_{\mathcal{H}}\|u_2\|^2 - 2 < f(u_1) - f(u_2), \bar{u} >_{L^{p,q}} -2(((\bar{\eta})',\bar{\eta}))_\mu,\end{aligned} \tag{33}$$

where $< \cdot,\cdot >_{L^{p,q}}$ is the duality between L^{2p} and L^q. The previous calculation is obtained formally taking the product in $\mathcal{H}$ between $\bar{z}$ and the difference of (14) with z_1 and z_2 in place of z, and it can be made rigorous with the use of mollifier, see [6, Theorem, Section 2]. In fact, integrating by parts and by the fact that $\mu' < 0$, we have

$$2(((\bar{\eta})', \bar{\eta}))_\mu = -\lim_{s\to 0} \mu(s)|\nabla\bar{\eta}^t(s)|^2 - \int_0^\infty \mu'(s)|\nabla\bar{\eta}^t(s)|^2 ds \geq 0.$$

Hence, the last term of the right hand side of (33) can be neglected.

At last, from (11) we know that $f(u)$ is increasing for $|u| \geq M$, for some $M > 0$. Fix $t \in (\tau, T]$, and let

$$\Omega_1 := \{x \in \Omega : |u_1(t,x)| \leq M \text{ and } |u_2(t,x)| \leq M\},$$

and

$$N = 2 \sup_{|s|\leq M} |f'(s)|.$$

Let $x \in \Omega_1$, then we have

$$2|f(u_1(x)) - f(u_2(x))| \leq N|\bar{u}(x)|.$$

Then, by the monotonicity of $f(u)$ for $|u| \geq M$ and the Poincaré inequality, it follows that

$$\begin{aligned} -2 < f(u_1) - f(u_2), \bar{u} >_{L^{p,q}} &\leq -2\int_{\Omega_1} (f(u_1(x)) - f(u_2(x)))\bar{u}(x)dx \\ &\leq \int_{\Omega_1} N|\bar{u}(x)|^2 dx \\ &\leq N\|\bar{z}\|^2_{\mathcal{H}}. \end{aligned} \tag{34}$$

(33)-(34) imply that

$$\frac{d}{dt}\|\bar{z}\|^2_{\mathcal{H}} \leq \left(\frac{1}{2m}L_a^2|l|^2\|u_2\|^2 + N\right)\|\bar{z}\|^2_{\mathcal{H}}.$$

The uniqueness and continuous dependence on initial date of solution to problem (14) follow from the Gronwall inequality. Till now, we finish the proof of the first assertion.

(ii) We are going to study further regularity of (u, η). To this end, multiplying $(9)_1$ by $-\Delta u$ with respect to the inner product of H, the Laplacian of $(9)_2$ by η with respect to the inner product of $L^2_\mu(\mathbb{R}^+; D(A))$, and adding the two terms, we obtain

$$\frac{d}{dt}\|z\|^2_{\mathcal{V}} + 2a(l(u))|\Delta u|^2 + 2(((\eta^t, (\eta^t)')))_\mu = 2(-f(u) + g, \Delta u). \tag{35}$$

Since f is a polynomial of odd degree, there exists a constant $d_0 > 0$, such that

$$f'(u) \geq -\frac{d_0}{2}, \qquad \forall u \in \mathbb{R}. \tag{36}$$

Then, it follows from the above inequality, (11), the Green formula and the Young inequality that

$$\begin{aligned} 2(f(u), \Delta u) &= 2\int_\Omega f_{2p-1}\Delta u dx - 2\int_\Omega f'(u)\nabla u \cdot \nabla u dx \\ &\leq \frac{2}{m}f^2_{2p-1}|\Omega| + \frac{m}{2}|\Delta u|^2 + d_0|\nabla u|^2. \end{aligned}$$

Again by the Young inequality, we have

$$2(g, \Delta u) \leq \frac{m}{2}|\Delta u|^2 + \frac{2}{m}|g|^2.$$

Together with (5), (35) becomes

$$\frac{d}{dt}\|z\|_{\mathcal{V}}^2 + m|\Delta u|^2 + 2(((\eta^t, (\eta^t)')))_\mu \leq \Theta, \tag{37}$$

where we have used the notation $\Theta = \frac{2}{m} f_{2p-1}^2 |\Omega| + d_0 |\nabla u|^2 + \frac{2}{m}|g|^2$, which belongs to $\in L^1(\tau, T)$. Under the suitable spatial regularity assumptions on η, integration by parts in time, and using (h_1), we obtain

$$(((\eta^t, (\eta^t)')))_\mu = -\int_0^\infty \mu'(s)|\Delta\eta^t(s)|^2 ds \geq 0.$$

Therefore, the term $2(((\eta^t, (\eta^t)')))_\mu$ in (37) can be neglected, we integrate (37) between τ and t, where $t \in (\tau, T)$, which leads to

$$\|z(t)\|_{\mathcal{V}}^2 + m\int_\tau^t |\Delta u(s)|^2 ds \leq \|z(\tau)\|_{\mathcal{V}}^2 + \int_\tau^t \Theta(s) ds. \tag{38}$$

From the above estimation, we conclude that

$$u \in L^\infty(\tau, T, V) \cap L^2(\tau, T; D(A))$$
$$\eta \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; D(A))).$$

Concerning the assertion (ii) of this theorem, the continuity of u follows again using a slightly modified version of [16, Lemma III.1.2]. The continuity of η can be proved mimicking the idea of the proof of Step 4 of (i), with $D(A)$ in place of V. The proof of this theorem is complete. □

Remark 9. *The upper bound $a(r) \leq \tilde{m}$ can be removed to obtain Theorem 8. Indeed, consider the function a substituted by*

$$\begin{cases} a(\widetilde{M}) & \text{if} \quad s \geq \widetilde{M} \\ a(s) & \text{if} \quad |s| \leq \widetilde{M} \\ a(-\widetilde{M}) & \text{if} \quad s \leq -\widetilde{M}, \end{cases}$$

where $\widetilde{M} := |l|\sqrt{(\|z_0\|_{\mathcal{H}}^2 + \Lambda T)}$, thanks to the a priori estimation in Step 2 of Theorem 8, cf. (24).

4. **Existence of attractor.** In this section, we will study the long time behavior of problem (14). Notice that, Theorem 8 ensures the solution to problem (14) exists globally in time. This fact entitles us to construct an autonomous dynamical system $S(t)$,

$$S(t) : H \times L^2_\mu(\mathbb{R}^+; V) \to H \times L^2_\mu(\mathbb{R}^+; V),$$

defined by,

$$S(t)(u_0, \eta_0) = (u(t; 0, (u_0, \eta_0)), \eta^t(\cdot; 0, (u_0, \eta_0))).$$

According to the construction of problem (14), the above semigroup can be rewritten equivalently,

$$S(t) : \mathcal{H} \to \mathcal{H},$$

given by,

$$S(t)z_0 = z(t; 0, z_0),$$

where $z(\cdot) = z(\cdot; 0, z_0)$ is the solution of (14) with initial value $z_0 = (u_0, \eta_0)$ and initial time $\tau = 0$. Observe that, on the one hand, $S(t)$ is well-defined as we have proved the solution of (14) $z(\cdot) \in C(0, T; \mathcal{H})$, for all $T > 0$ (cf. Theorem 8 (i)). On the other hand, $S(t) : \mathcal{V} \to \mathcal{V}$ is also well-defined based on the result of Theorem 8 (ii).

In the sequel, we will take $\tau = 0$ in problem (14), and we assume it generates an autonomous dynamical system $S(t)$ on $\mathcal{H}$ and $\mathcal{V}$.

4.1. **Existence of absorbing sets in $\mathcal{H}$ and $\mathcal{V}$.** We prove the existence of absorbing sets in both phase spaces, $\mathcal{H}$ and $\mathcal{V}$, in this subsection.

Lemma 10. *Under assumptions of Theorem 8, it holds:*

(i) *If the initial value $z_0 \in \mathcal{H}$, then the solution to problem* (14) *in the weak sense satisfies,*

$$\|z(t)\|_{\mathcal{H}}^2 \leq \|z_0\|_{\mathcal{H}}^2 e^{-c_1 t} + \frac{\Lambda}{c_1}, \qquad \forall t \geq 0,$$

where Λ is given in Theorem 8 and $c_1 = \min\{m\lambda_1, \delta\}$;

(ii) *If the initial value $z_0 \in \mathcal{V}$ and $c_1 > d_0$ (cf.* (36)*), then*

$$\|z(t)\|_{\mathcal{V}}^2 \leq \|z_0\|_{\mathcal{V}}^2 e^{-(c_1-d_0)t} + \frac{2}{m(c_1-d_0)}\left(f_{2p-1}^2|\Omega| + |g|^2\right), \qquad \forall t \geq 0.$$

Proof. We begin by proving (i). By (23) and (h_1)-(h_2), we obtain

$$2(((\eta^t)', \eta^t))_\mu = -\int_0^\infty \mu'(s)|\nabla\eta^t(s)|^2 ds \geq \delta\int_0^\infty \mu(s)|\nabla\eta^t(s)|^2 ds = \delta\|\eta^t\|_\mu^2.$$

Thus, from (22) and the above estimation, together with the Poincaré inequality, we arrive at

$$\frac{d}{dt}\|z\|_{\mathcal{H}}^2 + m\lambda_1|u|^2 + \delta\|\eta^t\|_\mu^2 + f_0|u|_{2p}^{2p} \leq 2a_0|\Omega| + \frac{1}{m\lambda_1}|g|^2 = \Lambda.$$

Since $c_1 = \min\{m\lambda_1, \delta\}$, we have

$$\frac{d}{dt}\|z\|_{\mathcal{H}}^2 + c_1\|z\|_{\mathcal{H}}^2 \leq \Lambda.$$

By the Gronwall lemma, we obtain the desired result.

(ii) In order to achieve uniform estimation involving the existence of a bounded absorbing set in $\mathcal{V}$, let us proceed like in the previous proof. Integration by parts and (h_1)-(h_3) yield,

$$2(((\eta^t, (\eta^t)')))_\mu = -\int_0^\infty \mu'(s)|\Delta\eta^t(s)|^2 ds \geq \delta\int_0^\infty \mu(s)|\Delta\eta^t(s)|^2 ds = \delta|||\eta^t|||_\mu^2.$$

Therefore, by means of the Poincaré inequality, (37) can be written as

$$\frac{d}{dt}\|z\|_{\mathcal{V}}^2 + m\lambda_1\|u\|^2 + \delta|||\eta^t|||_\mu^2 \leq \frac{2}{m}f_{2p-1}^2|\Omega| + d_0\|u\|^2 + \frac{2}{m}|g|^2.$$

Since $c_1 > d_0$, the above inequality becomes

$$\frac{d}{dt}\|z\|_{\mathcal{V}}^2 + (c_1 - d_0)\|z\|_{\mathcal{V}}^2 \leq \frac{2}{m}f_{2p-1}^2|\Omega| + \frac{2}{m}|g|^2.$$

The proof of assertion (ii) is complete thanks to the Gronwall lemma. □

4.2. **Asymptotic compactness.** Throughout this subsection, we will focus on the asymptotic compactness of the semigroup $S(t)$ in phase space $\mathcal{H}$. Let us start with the following lemma which is an immediate consequence of Lemma 10(i).

Theorem 11. *Let assumptions of Theorem 8 hold true. Then, there exists a positive constant $R_{\mathcal{H}}$, such that the ball $B_{\mathcal{H}} := B(0, R_{\mathcal{H}})$ is an absorbing set for $S(t)$ on $\mathcal{H}$. Namely, for any given bounded set $B := B(0, R) \subset \mathcal{H}$, there exists $t_{\mathcal{H}} = t_{\mathcal{H}}(B)$ such that,*

$$\|S(t)z_0\|_{\mathcal{H}}^2 \leq Re^{-c_1 t} + \frac{\Lambda}{c_1}, \qquad \forall t \geq t_{\mathcal{H}}.$$

Our main purpose is to prove for any $z_0^n \in B \subset \mathcal{H}$, for any sequence $t_n \to \infty$ as $n \to \infty$, $z^n(t_n) := z(t_n; 0, z_0^n) = S(t_n)z_0^n$ is relatively compact in $\mathcal{H}$. To this end, by the well-known result that the embedding $V \hookrightarrow H$ is compact, together with the fact in Lemma 3 that $\mathcal{E}$ is relatively compact in $L^2_\mu(\mathbb{R}^+; V)$, it is enough to prove the sequence $z^n(t_n)$ is bounded in $V \times \mathcal{E}$.

Theorem 12. *Suppose the conditions of Theorem 8 are true. Then there exists a positive constant $R_\mathcal{E} > 0$, such that the set $B_\mathcal{E} := B(0, R_\mathcal{E})$ is an absorbing set for $S(t)$ on $V \times \mathcal{E}$.*

Proof. The proof of this theorem is based on the subsequent lemma. □

Lemma 13. *Under assumptions of Theorem 8. Then, for any given constants $R_0^\mathcal{E} > 0$ and $R_0^\mathcal{H} > 0$, there exist a positive constant C and a positive function Ψ_1 vanishing at infinity, such that for any initial value $z_0 = (u_0, \eta_0)$ satisfying $\|z_0\|^2_{V\times\mathcal{E}} \leq R_0^\mathcal{E}$ and $\|z_0\|^2_\mathcal{H} \leq R_0^\mathcal{H}$, there holds:*

$$\|S(t)z_0\|^2_{V\times\mathcal{E}} \leq R_0^\mathcal{E}\Psi_1 + C, \qquad \forall t \geq 0.$$

Proof. Thanks to Lemma 10(ii), we have

$$\|S(t)z_0\|^2_\mathcal{V} \leq R_0^\mathcal{E} e^{-(c_1-d_0)t} + \frac{2}{m(c_1-d_0)}\left(f^2_{2p-1}|\Omega| + |g|^2\right), \qquad \forall t \geq 0, \tag{39}$$

here we used the fact that $\|z_0\|^2_\mathcal{V} \leq \|z_0\|^2_{V\times\mathcal{E}} \leq R_0^\mathcal{E}$.

It remains to show the required control on the last two terms of the norm of $\eta^t(s)$ on $\mathcal{E}$. From (39), we know there exsits $t_1 = t(R_0^\mathcal{E}) > 0$, such that $\|u(t)\|^2 \leq c := 1 + \frac{2}{m(c_1-d_0)}\left(f^2_{2p-1}|\Omega| + |g|^2\right)$, for all $t \geq t_1 > 0$. Thus, by lemmas 4 and 5, we have

$$\|\mathbf{T}\eta^t\|^2_{L^2_\mu(\mathbb{R}^+;V)} + \sup_{x\geq 1} x\mathcal{L}_{\eta^t}(x) \leq \begin{cases} R_0^\mathcal{E}\left(e^{-\delta t} + \Psi(t)\right) + c, & \forall t \geq t_1, \\ cR_0^\mathcal{E}, & \forall t \in [0, t_1]. \end{cases} \tag{40}$$

With the help of Corollary 6, we finish the proof of this lemma. □

In what follows, we need to prove for any bounded set $B \subset \mathcal{H}$, $\mathrm{dist}_\mathcal{H}(S(t)B, B_\mathcal{E}) \to 0$ holds as $t \to \infty$.

Theorem 14. *Suppose the conditions of Theorem 8 hold. Then, there exists an increasing positive function Φ, such that, up to (possibly) enlarging the radius $R_\mathcal{E}$,*

$$\mathrm{dist}_\mathcal{H}(S(t)B, B_\mathcal{E}) \leq \Phi(R)e^{-kt}, \qquad \forall t \geq 0,$$

for every bounded set $B := B(0, R) \subset \mathcal{H}$, where $k = \min\{2m\lambda_1, \delta\}$.

In view of Theorem 11, it is enough to show that

$$\mathrm{dist}_\mathcal{H}(S(t)B_\mathcal{H}, B_\mathcal{E}) \leq R_\mathcal{H}e^{-kt}, \qquad \forall t \geq 0.$$

The proof of this fact is based on a suitable decomposition of the solutions $S(t)z_0$. Recalling assumption (11) on f, noticing Remark 1(iii), we set

$$f_0(u) = f(u) + k_4u, \qquad \forall u \in \mathbb{R}.$$

Obviously, $f_0'(u) \geq 0$ for every $u \in \mathbb{R}$, then, for initial value $z_0 = (u_0, \eta_0)$ of problem (4), we write $S(t)z_0$ as the sum

$$S(t)z_0 = L(t)z_0 + K(t)z_0,$$

where $L(t)z_0 = (v(t), \xi^t)$ and $K(t)z_0 = (w(t), \zeta^t)$ solves the following problems,

$$\begin{cases} v_t - a(l(u))\Delta v - \int_0^\infty \mu(s)\Delta\xi^t(s)ds + f_0(u) - f_0(w) = 0, & \text{in } \Omega\times(0,\infty),\\ \xi^t_t = -\xi^t_s + v, & \text{in } \Omega\times(0,\infty)\times\mathbb{R}^+,\\ (v(0),\xi^0) = (u_0,\eta_0), & \text{in } \Omega, \end{cases} \tag{41}$$

and

$$\begin{cases} w_t - a(l(u))\Delta w - \int_0^\infty \mu(s)\Delta\zeta^t(s)ds + f_0(w) - k_4 u = g, & \text{in } \Omega\times(0,\infty),\\ \zeta^t_t = -\zeta^t_s + w, & \text{in } \Omega\times(0,\infty)\times\mathbb{R}^+,\\ (w(0),\zeta^0) = (0,0), & \text{in } \Omega. \end{cases} \tag{42}$$

Remark 15. *Notice that, in general, $L(t)$ and $K(t)$ are not semigroups.*

The following lemmas furnish some properties of these two mappings, L and K, respectively.

Lemma 16. *Under assumptions of Theorem 8, let $L(t)z_0$ satisfy equation (41). Then, there exists a constant $k = \min\{2m\lambda_1, \delta\}$, such that,*

$$\sup_{z_0\in B} \|L(t)z_0\|^2_{\mathcal{H}} \le R_{\mathcal{H}} e^{-kt}, \qquad \forall t \ge 0.$$

Proof. Multiplying the first and second equations of problem (41) by v and ξ^t in spaces H and $L^2_\mu(\mathbb{R}^+;V)$, respectively. Taking into account (5) and (h_2), adding the results and using the Poincaré inequality, it yields,

$$\frac{d}{dt}\left(|v|^2 + \|\xi^t\|^2_\mu\right) + 2m\lambda_1|v|^2 + \delta\|\xi^t\|^2_\mu + 2(f_0(u) - f_0(w), u - w) = 0.$$

Since $f_0'(u) \ge 0$, for all $u \in \mathbb{R}$, we have

$$\frac{d}{dt}\|L(t)z_0\|^2_{\mathcal{H}} + k\|L(t)z_0\|^2_{\mathcal{H}} \le 0.$$

By the Gronwall lemma, we finish the proof of this lemma. □

Lemma 17. *Under assumptions of Theorem 8, let $K(t)z_0$ satisfy equation (42). Then, there holds,*

$$\sup_{t\ge 0}\sup_{z_0\in B} \|K(t)z_0\|^2_{V\times\mathcal{E}} \le C,$$

for some constant $C = C(R_{\mathcal{H}})$.

Proof. We proceed similarly as in the proof of Lemma 16. Multiplying the first and second equations of problem (42) by $-\Delta w$ and ζ^t in spaces V and $L^2_\mu(\mathbb{R}^+;D(A))$, respectively. Taking into account (5) and (h_2) and adding the results, we have,

$$\frac{d}{dt}\left(\|w\|^2 + |||\zeta^t|||^2_\mu\right) + 2m|\Delta w|^2 + \delta|||\zeta^t|||^2_\mu + 2(\nabla(f_0(w) - k_4 u), \nabla w) = 2(\nabla g, \nabla w). \tag{43}$$

Since $f_0'(w) \ge 0$ for all $w \in \mathbb{R}$, integration by parts implies,

$$2(\nabla f_0(w), \nabla w) = 2(f_0'(w)\nabla w, \nabla w) \ge 0.$$

By the Young inequality, we derive

$$2(\nabla g, \nabla w) \le \frac{2}{m}|g|^2 + \frac{m}{2}|\Delta w|^2, \qquad 2k_4(u, \Delta w) \le \frac{m}{2}|\Delta w|^2 + \frac{2}{m}k_4^2|u|^2.$$

Therefore, neglecting the fourth term of left hand side of (43), making use of the above estimations and the Poincaré inequality, (43) can be written as,

$$\frac{d}{dt}\|K(t)z_0\|_{\mathcal{V}}^2 + k\|K(t)z_0\|_{\mathcal{V}}^2 \leq \frac{2}{m}(k_4^2|u|^2 + |g|^2).$$

Because $u(t)$ is uniformly bounded on H (cf. Lemma 16), together with the initial value $(w(\tau), \zeta^\tau) = (0,0)$, by using the uniform Gronwall lemma (see, Lemma 7), we have

$$\sup_{t\geq 0}\sup_{z_0\in B}\|K(t)z_0\|_{\mathcal{V}}^2 \leq C.$$

The proof of this lemma is finished by applying Lemmas 4 and 5, combining with Corollary 6. □

Proof of Theorem 14. By Lemmas 16 and 17, we see at once the desired inequality,

$$\begin{aligned}\mathrm{dist}_{\mathcal{H}}\left(S(t)B, B_{\mathcal{E}}\right) &= \sup_{z_0\in B}\inf_{y\in B_{\mathcal{E}}}\|L(t)z_0 + K(t)z_0 - y\|_{\mathcal{H}} \\ &\leq \sup_{z_0\in B}\|L(t)z_0\|_{\mathcal{H}} + \mathrm{dist}_{\mathcal{H}}\left(K(t)B, B_{\mathcal{E}}\right) \\ &\leq R_{\mathcal{H}}e^{-kt}.\end{aligned}$$

The result holds true since we choose $R_{\mathcal{E}}$ bigger than $C(R_{\mathcal{H}})$. □

Theorem 18. *Assume the conditions of Theorem 8 hold true. Then the semigroup $S(t)$ defined in $H \times L_\mu^2(\mathbb{R}^+; V)$, i.e., $\mathcal{H}$, associated to problem (14) has a global attractor $\mathcal{A} \subset \mathcal{H}$.*

5. **Conclusions and further comments.** We have analyzed the dynamics of a nonlocal partial differential equation with long time memory (4) by performing a transformation, first used by Dafermos, which allows us to rewrite the original equation (4) with memory as a system (14) without delay. Next, all the results related to problem (14) are shown assuming the initial values $(u_0, \eta_0) \in H \times L_\mu^2(\mathbb{R}^+; V)$. In particular, we proved the existence of a global attractor in this phase space, which in principle does not have a counterpart for the dynamical system generated by the original problem (4) on the phase space $H \times L_V^2$. Notice that when we refer to our orginal problem (4), the initial function u_0 and the corresponding η_0 are related via $\eta_0(s) := \int_{-s}^0 u_0(r)dr$ (cf. (10)), thanks to the Dafermos transformation. When we assume that $\eta_0 \in D(\mathbf{T})$, then equations (4) and (14) are proved to be equivalent, but with the current regularity of solutions, we are not able to prove the dynamical system defined in the phase space $H \times D(\mathbf{T})$ possesses a global attractor. This motivates the interest to design a new technique to analyze the original problem (4) using a similar phase space $H \times L_V^2$, where it is only necessary to know the information of the initial function $u_0(s)$ for all $s \in (-\infty, 0]$. This problem will be explored in a forthcoming paper.

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