MULTI-VALUED RANDOM DYNAMICS OF STOCHASTIC WAVE EQUATIONS WITH INFINITE DELAYS

JINGYU WANG

College of Applied Mathematics
Shanxi University of Finance and Economics
Taiyuan 030006, PR China

YEJUAN WANG*

School of Mathematics and Statistics
Gansu Key Laboratory of Applied Mathematics and Complex Systems
Lanzhou University, Lanzhou 730000, PR China

TOMÁS CARABALLO

Departamento de Ecuaciones Diferenciales y Análisis Numérico
Facultad de Matemáticas, Universidad de Sevilla
c/ Tarfia s/n, 41012-Sevilla, Spain

ABSTRACT. This paper is devoted to the asymptotic behavior of solutions to a non-autonomous stochastic wave equation with infinite delays. The nonlinear terms of the equation are not expected to be Lipschitz continuous, but only satisfy continuity assumptions along with growth conditions, under which the uniqueness of the solutions may not hold. Using the theory of multi-valued non-autonomous random dynamical systems, we prove the existence and measurability of a compact global pullback attractor.

1. **Introduction.** In this paper, we study the existence and measurability of pullback attractors for the following non-autonomous stochastic wave equation with infinite delays and additive white noise defined on a bounded domain $D \subset \mathbb{R}^n$ with smooth boundary:

2010 *Mathematics Subject Classification.* Primary: 35L05, 37L30, 37L55.

Key words and phrases. Random attractor, multi-valued non-autonomous random dynamical system, stochastic delay wave equation, nonlinear damping, infinite delay.

This work was supported by NSF of China (Grants No. 11801335, 41875084). The research of T. Caraballo has been partially supported by Ministerio de Ciencia Innovación y Universidades (Spain), FEDER (European Community) under grant PGC2018-096540-B-I00, and by FEDER and Junta de Andalucía (Consejería de Economía y Conocimiento) under projects US-1254251 and P18-FR-4509.

* Corresponding author.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + J\left(\dfrac{\partial u}{\partial t}\right) - \Delta u + \lambda u = f(x, u(t-\rho(t))) \\ + \displaystyle\int_{-\infty}^{0} F(x, s, u(t+s))ds + g(x,t) + \sum_{j=1}^{m} h_j(x)\dot{w}_j, \quad t > \tau,\ x \in D, \\ u(t,x) = 0,\ t > \tau,\ x \in \partial D, \\ u(t,x) = \phi(t-\tau, x),\ t \leqslant \tau,\ x \in D, \\ \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial \phi}{\partial t}(t-\tau, x),\ t \leqslant \tau,\ x \in D, \end{cases} \tag{1.1}$$

where $\lambda > 0$, $\tau \in \mathbb{R}$ is an initial time, ϕ is an initial datum on $(-\infty, 0]$, for each $j = 1, \dots, m$, $h_j(x) \in H_0^1(D)$, $\{w_j\}_{j=1}^m$ are independent two-sided real-valued Wiener processes on a probability space which will be specified below, and the other symbols satisfy the following conditions:

(**H1**) There exist two constants β_1, β_2 such that

$$J(0) = 0, \quad 0 < \beta_1 \leqslant J'(v) \leqslant \beta_2 < \infty,\ \forall v \in \mathbb{R}.$$

(**H2**) There exist a function $k_1 \in L^2(D)$ and a positive constant k_2 such that $f \in C(D \times \mathbb{R}; \mathbb{R})$ and $\rho \in C^1(\mathbb{R}; [0, h])$ satisfy

$$|f(x,\nu)|^2 \leqslant |k_1(x)|^2 + k_2^2|\nu|^2, \quad \forall x \in D,\ \nu \in \mathbb{R},$$
$$|\rho'(t)| \leqslant \rho_* < 1, \quad \forall t \in \mathbb{R},$$

where $h > 0$ is a given positive number.

(**H3**) There exist a positive scalar function $e^{-2\gamma\cdot} m_1(\cdot) \in L^1((-\infty, 0]; \mathbb{R})$ and a function $m_0 \in L^1((-\infty, 0]; L^2(D))$ such that the function $F \in C(D \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$ satisfies

$$|F(x, s, \nu)| \leqslant m_1(s)|\nu| + |m_0(s,x)|,\ \forall x \in D,\ s, \nu \in \mathbb{R}, \tag{1.2}$$

and we will denote

$$m_0 = \int_{-\infty}^{0} \|m_0(s,\cdot)\|_{L^2(D)} ds \quad \text{and} \quad m_1 = \int_{-\infty}^{0} e^{-2\gamma s} m_1(s) ds.$$

(**H4**) The external force $g \in C(\mathbb{R}; L^2(D))$ is such that

$$\int_{-\infty}^{t} \int_D e^{\left(\alpha - \frac{8m_1^2}{\beta_1 \delta^2}\right) r} |g(r,x)|^2 dx dr < \infty, \quad \forall t \in \mathbb{R},$$

where $\alpha > 0$ will be given in Lemma 11.

Wave equations with some delay terms are considered suitable models in analysis of oscillatory phenomena including aftereffects, times lags or hereditary characteristics [19, 22, 38, 44], as the deformation of viscoelastic materials [13, 14] or the retarded control of the dynamics of flexible structures [23, 26, 28]. Global attractors, uniform attractors or pullback attractors for deterministic autonomous or non-autonomous damped wave equations with deterministic have been studied by many authors, see [2, 3, 4, 5, 9, 11, 16, 18, 21, 27, 29, 31, 32, 41, 46, 47] and the references therein. The existence of random attractors for autonomous or non-autonomous stochastic wave equations has been considered in [12, 15, 20, 24, 30, 33, 35, 43, 45, 48].

The existence of pullback attractors for deterministic damped wave equations with bounded delays has been initially established in [6]. Very recently, the asymptotic behavior of non-autonomous damped wave equations with bounded delays and without the uniqueness of solutions has been investigated in [37, 42]. The goal of this paper is to study the multi-valued random dynamics of stochastic damped wave equations with inifinite delays. Since the non-Lipschitz continuity of the nonlinearities leads to the non-uniqueness of solutions of (1.1), comparing with the case of uniqueness the new difficulty appears in the proof of the measurability of solutions as well as the measurability of pullback attractors. In additions, the presence of both variable and distributed delays also make the analysis of the asymptotic compactness of solutions more complicated.

Therefore, in order to obtain the random attractor, we shall use the general theory of attractors for multi-valued random dynamical systems developed in [7], and then more generally for multi-valued non-autonomous random dynamical systems investigated in [36, 40]. For multi-valued random dynamical systems, the reader is referred to [8, 10, 39] for the existence and measurability of pullback attractors for parabolic equations with delays or first-order lattice systems.

This paper is organized as follows. In Section 2, we recall some basic concepts and results related to multi-valued non-autonomous random dynamical systems and global pullback attractors. In Section 3, we define a multi-valued non-autonomous random dynamical system for (1.1). Section 4 is devoted to the existence and uniqueness of the pullback attractor. The measurability of the pullback attractor is given in Section 5.

2. Preliminaries.

We recall some basic definitions for multi-valued non-autonomous random dynamical systems and some results ensuring the existence and measurability of pullback attractors for these systems. The reader is referred to [7, 34, 36, 40] for more details.

Let Q be a nonempty set, $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and (X, d) be a Polish space with Borel σ-algebra $\mathcal{B}(X)$. The Hausdorff semi-distance between two nonempty subsets A and B of X is defined by

$$d(A,\, B) = \sup\{d(a,\, B) : a \in A\},$$

where $d(a,\, B) = \inf\{d(a,\, b) : b \in B\}$. Denote by $\mathcal{N}_r(A)$ the open r-neighborhood $\{y \in X : d(y, A) < r\}$ of radius $r > 0$ of a subset A of X.

Let 2^X be the collection of all subsets of X. Assume that there are two groups $\{\sigma_t\}_{t\in\mathbb{R}}$ and $\{\theta_t\}_{t\in\mathbb{R}}$ acting on Q and Ω, respectively. Specifically, $\sigma : \mathbb{R} \times Q \to Q$ is a mapping such that σ_0 is the identity on Q, $\sigma_{t+\tau} = \sigma_t \circ \sigma_\tau$ for all $t, \tau \in \mathbb{R}$. Similarly, $\theta : \mathbb{R} \times \Omega \to \Omega$ is a $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable mapping such that θ_0 is the identity on Ω, $\theta_{t+\tau} = \theta_t \circ \theta_\tau$ for all $t, \tau \in \mathbb{R}$ and $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$. In the sequel, we will call both $(Q, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ parametric dynamical systems.

Definition 1. Let $(Q, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ be parametric dynamical systems. A multi-valued mapping $\Phi : \mathbb{R}^+ \times Q \times \Omega \times X \to 2^X$ with nonempty closed images is called a multi-valued cocycle on X over $(Q, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ if for all $q \in Q$, $\omega \in \Omega$ and $t, \tau \in \mathbb{R}^+$, the following conditions are satisfied:

(1) $\Phi(0, q, \omega, \cdot)$ is the identity on X; (2) $\Phi(t+\tau, q, \omega, \cdot) = \Phi(t, \sigma_\tau q, \theta_\tau\omega, \Phi(\tau, q, \omega, \cdot))$.

It is well known that the usual definition of multi-valued cocycle requires $\Phi(t + \tau, q, \omega, \cdot) \subset \Phi(t, \sigma_\tau q, \theta_\tau\omega, \Phi(\tau, q, \omega, \cdot))$, and the one in Definition 1 is referred to as strict multi-valued cocycle. As we will deal with strict multi-valued cocycles in our analysis, we will omit the word strict.

A multi-valued cocycle is called a random cocycle if the multi-valued mapping $(t,\omega,x)\to\Phi(t,q,\omega,x)$ is $\mathcal{B}(\mathbb{R}^+)\times\mathcal{F}\times\mathcal{B}(X)$ measurable, i.e., $\{(t,\omega,x):\Phi(t,q,\omega,x)\cap O\neq\emptyset\}\in\mathcal{B}(\mathbb{R}^+)\times\mathcal{F}\times\mathcal{B}(X)$ for every open set O of the topological space X.

For the above composition of multi-valued mappings, we use that for any nonempty set $V\subset X$, $\Phi(t,q,\omega,V)$ is defined by

$$\Phi(t,q,\omega,V)=\bigcup_{x_0\in V}\Phi(t,q,\omega,x_0).$$

Definition 2. (See [34].) A collection $\mathcal{D}$ of some families of nonempty subsets of X is said to be neighborhood closed if for each $D=\{D(q,\omega):q\in Q,\omega\in\Omega\}\in\mathcal{D}$, there exists a positive number ε depending on D such that the family

$$\{B(q,\omega):B(q,\omega)\text{ is a nonempty subset of }\mathcal{N}_\varepsilon(D(q,\omega)),\forall q\in Q,\forall\omega\in\Omega\}\tag{2.1}$$

also belongs to $\mathcal{D}$.

Note that the neighborhood closedness of $\mathcal{D}$ implies for each $D\in\mathcal{D}$,

$$\{\tilde{D}(q,\omega):\tilde{D}(q,\omega)\text{ is a nonempty subset of }D(q,\omega),\forall q\in Q,\forall\omega\in\Omega\}\in\mathcal{D}.\tag{2.2}$$

A collection $\mathcal{D}$ satisfying (2.2) is said to be inclusion-closed in the literature, see, e.g., [17].

Definition 3. (See [7, 34, 36, 40].)

(1) A set-valued mapping $K:Q\times\Omega\to 2^X$ is called measurable with respect to $\mathcal{F}$ in Ω if the value $K(q,\omega)$ is a closed nonempty subset of X for all $q\in Q$ and $\omega\in\Omega$, and the mapping $\omega\in\Omega\to d(x,\,K(q,\omega))$ is $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable for every fixed $x\in X$ and $q\in Q$.

(2) Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and $K=\{K(q,\omega):q\in Q,\omega\in\Omega\}\in\mathcal{D}$. Then K is called a $\mathcal{D}$-pullback absorbing set for Φ if for all $q\in Q$, $\omega\in\Omega$ and for every $B=\{B(q,\omega):q\in Q,\omega\in\Omega\}\in\mathcal{D}$, there exists $T=T(B,q,\omega)>0$ such that

$$\Phi(t,\sigma_{-t}q,\theta_{-t}\omega,B(\sigma_{-t}q,\theta_{-t}\omega))\subseteq K(q,\omega),\ \forall t\geqslant T.$$

If, in addition, for all $q\in Q$ and $\omega\in\Omega$, $K(q,\omega)$ is a closed nonempty subset of X and K is measurable with respect to the $\mathbb{P}$-completion of $\mathcal{F}$ in Ω, then we say K is a closed measurable $\mathcal{D}$-pullback absorbing set for Φ.

(3) Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X. Then Φ is said to be $\mathcal{D}$-pullback asymptotically upper-semicompact in X if for all $q\in Q$ and $\omega\in\Omega$, any sequence $y_n\in\Phi(T_n,\sigma_{-T_n}q,\theta_{-T_n}\omega,x_n)$ has a convergent subsequence in X whenever $T_n\to+\infty$ $(n\to\infty)$, $x_n\in B(\sigma_{-T_n}q,\theta_{-T_n}\omega)$ with $B=\{B(q,\omega):q\in Q,\omega\in\Omega\}\in\mathcal{D}$.

Definition 4. Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and $\mathcal{A}=\{\mathcal{A}(q,\omega):q\in Q,\omega\in\Omega\}\in\mathcal{D}$. Then $\mathcal{A}$ is called a $\mathcal{D}$-pullback attractor for Φ if it satisfies:

(1) $\mathcal{A}(q,\omega)$ is compact for all $q\in Q$ and $\omega\in\Omega$.

(2) $\mathcal{A}$ is invariant, that is, for every $q\in Q$ and $\omega\in\Omega$,

$$\Phi(t,q,\omega,\mathcal{A}(q,\omega))=\mathcal{A}(\sigma_tq,\theta_t\omega),\ \forall t\geqslant 0.$$

(3) $\mathcal{A}$ attracts every member of $\mathcal{D}$, that is, for every $B=\{B(q,\omega):q\in Q,\omega\in\Omega\}\in\mathcal{D}$ and for all $q\in Q$ and $\omega\in\Omega$,

$$\lim_{t\to+\infty}d(\Phi\,(t,\sigma_{-t}q,\theta_{-t}\omega,B(\sigma_{-t}q,\theta_{-t}\omega)),\,\mathcal{A}(q,\omega))=0.$$

Definition 5. Let $B = \{B(q,\omega) : q \in Q, \omega \in \Omega\}$ be a family of nonempty subsets of X. For every $q \in Q$ and $\omega \in \Omega$, let

$$\Theta(B,q,\omega) = \bigcap_{\tau \geqslant 0} \overline{\bigcup_{t \geqslant \tau} \Phi\left(t, \sigma_{-t}q, \theta_{-t}\omega, B(\sigma_{-t}q, \theta_{-t}\omega)\right)}.$$

Then the family $\{\Theta(B,q,\omega) : q \in Q, \omega \in \Omega\}$ is called the Θ-limit set of B and is denoted by $\Theta(B)$.

Theorem 6. (*See* [7, 34, 36, 40].) *Let $\mathcal{D}$ be a neighborhood closed collection of some families of nonempty subsets of X, and let Φ be a multi-valued cocycle on X over $(Q, \{\sigma_t\}_{t \in \mathbb{R}})$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ satisfying the norm-to-weak upper-semicontinuity on X, i.e., if $x_n \to x$ in X, then for any $y_n \in \Phi(t,q,\omega,x_n)$, there exist a subsequence y_{n_k} and a $y \in \Phi(t,q,\omega,x)$ such that $y_{n_k} \rightharpoonup y$ (weak convergence). Then Φ has a $\mathcal{D}$-pullback attractor $\mathcal{A}$ in $\mathcal{D}$ if and only if Φ is $\mathcal{D}$-pullback asymptotically upper-semicompact in X and Φ has a closed $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$. The $\mathcal{D}$-pullback attractor $\mathcal{A}$ is unique and is given by, for each $q \in Q$ and $\omega \in \Omega$,*

$$\mathcal{A}(q,\omega) = \Theta(K,q,\omega) = \bigcup_{B \in \mathcal{D}} \Theta(B,q,\omega). \tag{2.3}$$

Remark 7. The concept of neighborhood closedness of $\mathcal{D}$ is used to consider the necessary condition for the existence of a $\mathcal{D}$-pullback attractor. For the sufficient condition for the existence of such attractor, we only need the concept of inclusion-closedness of $\mathcal{D}$.

Theorem 8. (*See* [7, 34, 36, 40].) *Let Φ be a multi-valued random cocycle. Under the assumptions of Theorem 6, let $\omega \to \Phi(t,q,\omega,K(q,\omega))$ be measurable (w.r.t. the $\mathbb{P}$-completion of $\mathcal{F}$) for $t \geqslant 0$ and $q \in Q$, and let $\Phi(t,q,\omega,K(q,\omega))$ be closed for all $t \geqslant 0$, $q \in Q$ and $\omega \in \Omega$. Then for every fixed $q \in Q$, $\mathcal{A}(q,\cdot)$ defined by (2.3) is measurable with respect to the $\mathbb{P}$-completion of $\mathcal{F}$.*

Let $(X, \|\cdot\|_X)$ be a Banach space. The following result will be used to check the $\mathcal{D}$-pullback asymptotically upper-semicompactness of multi-valued cocycles.

Theorem 9. *Let $\mathcal{D}$ be a collection of same families of nonempty subsets of X, and let Φ be a multi-valued cocycle on a Banach space X. Suppose that Φ can be written as*

$$\Phi = \Phi_1 + \Phi_2$$

and for any fixed $q \in Q$ and $\omega \in \Omega$,

(1) $\lim\limits_{t \to +\infty} \|\Phi_2(t, \sigma_{-t}q, \theta_{-t}\omega, K(\sigma_{-t}q, \theta_{-t}\omega))\|_X = 0$;

(2) *for any fixed $t > 0$, every sequence $u_n \in \Phi_1(t, \sigma_{-t}q, \theta_{-t}\omega, K(\sigma_{-t}q, \theta_{-t}\omega))$ is a Cauchy sequence in X,*

where $K = \{K(q,\omega) : q \in Q, \omega \in \Omega\}$ is a $\mathcal{D}$-pullback absorbing set for Φ. Then Φ is $\mathcal{D}$-pullback asymptotically upper-semicompact in X.

Proof. Let $q \in Q$, $\omega \in \Omega$, $B = \{B(q,\omega) : q \in Q, \omega \in \Omega\} \in \mathcal{D}$, sequences $T_n \to +\infty$ $(n \to +\infty)$ and $u_n \in \Phi(T_n, \sigma_{-T_n}q, \theta_{-T_n}\omega, B(\sigma_{-T_n}q, \theta_{-T_n}\omega))$ be given arbitrarily. In order to prove the precompactness of u_n in X, it suffices to show that u_n is a Cauchy sequence in X.

By using the assumption (1), we can choose $t_1 > 0$ such that

$$\|\Phi_2(t_1, \sigma_{-t_1}q, \theta_{-t_1}\omega, K(\sigma_{-t_1}q, \theta_{-t_1}\omega))\|_X < \frac{\varepsilon}{3}. \tag{2.4}$$

On the other hand, we obtain that for sufficiently large n,

$$\begin{aligned}&\Phi(T_n,\sigma_{-T_n}q,\theta_{-T_n}\omega,B(\sigma_{-T_n}q,\theta_{-T_n}\omega))\\&=\Phi(t_1,\sigma_{-t_1}q,\theta_{-t_1}\omega,\Phi(T_n-t_1,\sigma_{-T_n}q,\theta_{-T_n}\omega,B(\sigma_{-T_n}q,\theta_{-T_n}\omega)))\\&\subset\Phi(t_1,\sigma_{-t_1}q,\theta_{-t_1}\omega,K(\sigma_{-t_1}q,\theta_{-t_1}\omega)).\end{aligned}$$

This implies that $u_n\in\Phi(t_1,\sigma_{-t_1}q,\theta_{-t_1}\omega,K(\sigma_{-t_1}q,\theta_{-t_1}\omega))$ for sufficiently large n. We write u_n as $u_n=y_n+x_n$, where $y_n\in\Phi_1(t_1,\sigma_{-t_1}q,\theta_{-t_1}\omega,K(\sigma_{-t_1}q,\theta_{-t_1}\omega))$, $x_n\in\Phi_2(t_1,\sigma_{-t_1}q,\ \theta_{-t_1}\omega,K(\sigma_{-t_1}q,\theta_{-t_1}\omega))$. It follow from assumption (2) that there exists an $N>0$ such that

$$\|y_n-y_m\|_X<\frac{\varepsilon}{3},\ \forall n,m>N. \tag{2.5}$$

By (2.4) and (2.5), we deduce that for all $n,m>N$,

$$\|u_n-u_m\|_X\leqslant\|y_n-y_m\|_X+\|x_n\|_X+\|x_m\|_X<\varepsilon, \tag{2.6}$$

which completes the proof. □

3. **The multi-valued cocycle associated to the model.** For the stochastic term in (1.1), we assume that for each $j=1,\dots,m$, $\{w_j\}_{j=1}^m$ are independent two-sided real-valued Wiener processes on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, where

$$\Omega=\{\omega=(\omega_1,\omega_2,\dots,\omega_m)\in C(\mathbb{R},\mathbb{R}^m):\omega(0)=0\},$$

$\mathcal{F}$ is the Borel σ-algebra generated by the compact-open topology of Ω, and $\mathbb{P}$ is the corresponding Wiener measure on $(\Omega,\mathcal{F})$. Then we will identify ω with $W(t)$, i.e.,

$$W(t,\omega)=(w_1(t),w_2(t),\dots,w_m(t))=\omega(t)\ \text{ for } t\in\mathbb{R}.$$

Define a group $\{\theta_t\}_{t\in\mathbb{R}}$ acting on $(\Omega,\mathcal{F},\mathbb{P})$ by

$$\theta_t\omega(\cdot)=\omega(\cdot+t)-\omega(t),\ \omega\in\Omega,\ t\in\mathbb{R}. \tag{3.1}$$

Then $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$ is a parametric dynamical system. Suppose $Q=\mathbb{R}$. Define a family $\{\sigma_t\}_{t\in\mathbb{R}}$ of shift operators by

$$\sigma_t(s)=s+t\ \text{ for all } t,s\in\mathbb{R}. \tag{3.2}$$

Given $j=1,\dots,m$, consider the one-dimensional Ornstein-Uhlenbeck equation

$$dz_j+\tilde{\alpha}z_jdt=dw_j(t). \tag{3.3}$$

One may easily verify that a solution to (3.3) is given by

$$z_j(t)=z_j(\theta_t\omega_j)\equiv-\tilde{\alpha}\int_{-\infty}^0e^{\tilde{\alpha}\tau}(\theta_t\omega_j)(\tau)d\tau,\ t\in\mathbb{R}.$$

It is known that there exists a θ_t-invariant set $\tilde{\Omega}\subseteq\Omega$ of full $\mathbb{P}$ measure such that $z_j(\theta_t\omega_j)$ is continuous in t for every $\omega\in\tilde{\Omega}$, and the random variable $|z_j(\omega_j)|$ is tempered. Hereafter, we will not distinguish $\tilde{\Omega}$ and Ω, and write $\tilde{\Omega}$ as Ω.

It follows from Proposition 4.3.3 in [1] that there exists a tempered function $r(\omega)>0$ such that

$$\sum_{j=1}^m|z_j(\omega_j)|^2\leqslant r(\omega), \tag{3.4}$$

where $r(\omega)$ satisfies, for every $\omega\in\Omega$,

$$r(\theta_t\omega)\leqslant e^{\frac{\tilde{\alpha}}{2}|t|}r(\omega),\ t\in\mathbb{R}. \tag{3.5}$$

Then (3.4) and (3.5) imply that, for every $\omega \in \Omega$,

$$\sum_{j=1}^{m} |z_j(\theta_t\omega_j)|^2 \leqslant e^{\frac{\tilde{\alpha}}{2}|t|} r(\omega),\ t \in \mathbb{R}. \tag{3.6}$$

Putting $z(\theta_t\omega) = \sum_{j=1}^{m} h_j z_j(\theta_t\omega_j)$, by (3.3) we have

$$dz + \tilde{\alpha} z dt = \sum_{j=1}^{m} h_j dw_j.$$

Let $\xi = \frac{\partial u}{\partial t} + \delta u$ where $\delta > 0$ will be fixed later. Then (1.1) can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} + \delta u = \xi, \\ \dfrac{\partial \xi}{\partial t} - \delta \xi + J(\xi - \delta u) - \Delta u + (\lambda + \delta^2) u = f(x, u(t - \rho(t))) \\ + \displaystyle\int_{-\infty}^{0} F(x, s, u(t+s)) ds + g(x,t) + \sum_{j=1}^{m} h_j(x) \dot{w}_j, \end{cases} \tag{3.7}$$

with bounded and initial conditions

$$\begin{cases} u(t,x) = \xi(t,x) = 0,\ t > \tau,\ x \in \partial D, \\ u(t,x) = u_\tau(t-\tau, x) = \phi(t-\tau, x),\ t \leqslant \tau,\ x \in D, \\ \xi(t,x) = \xi_\tau(t-\tau,x) = \dfrac{\partial \phi(t-\tau,x)}{\partial t} + \delta\phi(t-\tau,x),\ t \leqslant \tau,\ x \in D. \end{cases} \tag{3.8}$$

To convert the stochastic wave equation to a deterministic one with random parameters, let us consider a new variable given by $v(t,x) = \xi(t,x) - z(\theta_t\omega)$, where $z(\theta_t\omega) = \sum_{j=1}^{m} h_j z_j(\theta_t\omega_j)$. Then the system (3.7)-(3.8) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t} + \delta u = v + z(\theta_t\omega), \\ \dfrac{\partial v}{\partial t} - \delta v + J(v - \delta u + z(\theta_t\omega)) - \Delta u + (\lambda + \delta^2) u = f(x, u(t-\rho(t))) \\ + \displaystyle\int_{-\infty}^{0} F(x, s, u(t+s)) ds + g(x,t) + (\tilde{\alpha} + \delta) z(\theta_t\omega), \end{cases} \tag{3.9}$$

with bounded and initial conditions

$$\begin{cases} u(t,x) = v(t,x) = 0,\ t > \tau,\ x \in \partial D, \\ u(t,x) = u_\tau(t-\tau,x) = \phi(t-\tau,x),\ t \leqslant \tau,\ x \in D, \\ v(t,x) = v_\tau(t-\tau,x) = \dfrac{\partial\phi(t-\tau,x)}{\partial t} + \delta\phi(t-\tau,x) - z(\theta_t\omega),\ t \leqslant \tau,\ x \in D. \end{cases} \tag{3.10}$$

We denote by $C_{\gamma,X}$ the space

$$C_{\gamma,X} = \{\psi \in C((-\infty,0];X) \mid \lim_{s\to-\infty} \psi(s)e^{\gamma s} \text{ exists}\},$$

where the parameter $\gamma > 0$ will be determined later on, and set

$$\|\psi\|_{C_{\gamma,X}} := \sup_{s\in(-\infty,0]} e^{\gamma s}\|\psi(s)\|_X < \infty.$$

This is a separable Banach space. Given $T > \tau$ and $u : (-\infty, T) \to X$, for each $t \in [\tau, T)$ we denote by u_t the function defined on $(-\infty, 0]$ by the relation $u_t(s) = u(t+s)$, $s \in (-\infty, 0]$.

Let $H = L^2(D)$ with norm $|\cdot|$ and inner product $(\cdot,\cdot)$, and let $V = H_0^1(D)$ with norm $\|\cdot\|$. Set $E = C_{\gamma,V} \times C_{\gamma,H}$. In the sequel, C denotes an arbitrary positive constant, which may be different from line to line and even in the same line.

Thanks to Lemma 11, by the standard Galerkin approximation and compactness method, the following existence result of solutions follows immediately from the similar arguments of Theorem 3.1 in [42].

Theorem 10. *Suppose that* (**H1**)-(**H3**) *hold true and* $g \in L^2_{loc}(\mathbb{R}; H)$. *Then for each* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and for any* $(u_\tau, v_\tau) \in E$, *there exists a solution* $(u(t), v(t))$ *to problem* (3.9)-(3.10), *and*

$$u(\cdot, \tau, \omega, u_\tau) \in C((-\infty, T]; V),\ \ v(\cdot, \tau, \omega, v_\tau) \in C((-\infty, T]; H), \quad \forall T > \tau.$$

Now we define a multi-valued mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E \to E$ by

$$\Phi(t, \tau, \omega, (u_\tau, v_\tau)) = \Big\{ \big(u_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, u_\tau), v_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, v_\tau)\big) \big| \\ \big(u(\cdot), v(\cdot)\big) \text{ is a solution of (3.9)-(3.10) with } (u_\tau, v_\tau) \in E \Big\},$$

where $u_{t+\tau}$ and $v_{t+\tau}$ are defined for $\theta \in [-\infty, 0]$ as $u_{t+\tau}(\theta) = u(t + \tau + \theta)$ and $v_{t+\tau}(\theta) = v(t + \tau + \theta)$ respectively, and $v_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, v_\tau) = \xi_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, \xi_\tau) - z(\theta_{t+\cdot}\omega)$ with $v_\tau(\cdot) = \xi_\tau(\cdot) - z(\theta_\cdot\omega)$. By a standard way as in [7] (Lemma 5.1), we see that Φ satisfies conditions (1)-(2) in Definition 1. It suffices to show that for any $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $(u_\tau, v_\tau) \in E$, the set $\Phi(t, \tau, \omega, (u_\tau, v_\tau))$ is closed in E. Let $(u^n_{t+\tau}(\cdot), v^n_{t+\tau}(\cdot)) \in \Phi(t, \tau, \omega, (u_\tau, v_\tau))$ and $(u_{t+\tau}(\cdot), v_{t+\tau}(\cdot)) \in E$ such that

$$(u^n_{t+\tau}(\cdot), v^n_{t+\tau}(\cdot)) \to (u_{t+\tau}(\cdot), v_{t+\tau}(\cdot)) \quad \text{in } E. \tag{3.11}$$

By slightly modifying the proof of the existence of weak solutions, in view of Lemma 11, we find that there exists a solution $(\widetilde{u}, \widetilde{v})$ of problem (3.9)-(3.10) with initial condition (u_τ, v_τ) such that, up to a subsequence

$$u^n_{t+\tau}(\cdot) \rightharpoonup \widetilde{u}_{t+\tau}(\cdot) \qquad \text{in } L^2(-t, 0; V),$$

$$v^n_{t+\tau}(\cdot) \rightharpoonup \widetilde{v}_{t+\tau}(\cdot) \qquad \text{in } L^2(-t, 0; H).$$

Combining this with (3.11), we conclude that $(u_{t+\tau}(\cdot), v_{t+\tau}(\cdot)) = (\widetilde{u}_{t+\tau}(\cdot), \widetilde{v}_{t+\tau}(\cdot)) \in \Phi(t, \tau, \omega, (u_\tau, v_\tau))$. Hence, Φ is a multi-valued cocycle on E over $(\mathbb{R}, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

4. **Uniform estimates of solutions.** In this section, we derive uniform estimates on the solutions of (3.9)-(3.10) for the purpose of proving the existence of a pullback absorbing set of the multi-valued random dynamical system.

Assume that $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is a family of bounded nonempty subsets of E satisfying, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t \to -\infty} e^{\left(\alpha - \frac{8m_1^2}{\beta_1 \delta^2}\right)t} \sup_{(\varphi,\psi)\in D(\tau+t,\theta_t\omega)} \left(\|\varphi\|^2_{C_V} + \|\psi\|^2_{C_H} \right) = 0, \tag{4.1}$$

where α will be given in Lemma 11. Denote by $\mathcal{D}$ the collection of all families of bounded nonempty subsets of E which fulfill condition (4.1), i.e.,

$$\mathcal{D} = \{D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ satisfies } (4.1)\}.$$

Obviously $\mathcal{D}$ is neighborhood closed.

Lemma 11. *In addition to the assumptions* (**H1**)-(**H4**)*, assume that there exist positive constants* α*,* δ *and* γ *such that*

$$\frac{\tilde{\alpha}}{2} + \frac{8m_1^2}{\beta_1 \delta^2} < \alpha < 2\gamma, \tag{4.2}$$

$$\alpha < \frac{\beta_1}{4} - 2\delta, \tag{4.3}$$

$$\alpha < \delta, \tag{4.4}$$

and

$$\alpha(\lambda + \delta^2) < \delta(\lambda + \delta^2) - \frac{2\beta_2^2 \delta^2}{\beta_1} - \frac{4k_2^2 e^{\alpha h}}{\beta_1(1-\rho_*)}. \tag{4.5}$$

Then for every $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$*, the solution of* (3.9)-(3.10) *with* ω *replaced by* $\theta_{-\tau}\omega$ *satisfies for all* $t \geqslant 0$*,*

$$\begin{aligned}
&\|v_\tau\|^2_{C_{\gamma,H}} + \|u_\tau\|^2_{C_{\gamma,V}} + (\lambda+\delta^2)\|u_\tau\|^2_{C_{\gamma,H}} \\
&\leqslant Ce^{-\left(\alpha - \frac{8m_1^2}{\beta_1\delta^2}\right)t} \left(\|v_{\tau-t}\|^2_{C_{\gamma,H}} + \|u_{\tau-t}\|^2_{C_{\gamma,V}} + \|u_{\tau-t}\|^2_{C_{\gamma,H}}\right) \\
&+ Ce^{\left(\frac{8m_1^2}{\beta_1\delta^2} - \alpha\right)\tau} \int_{-\infty}^{\tau} e^{\left(\alpha - \frac{8m_1^2}{\beta_1\delta^2}\right)r} |g(r)|^2 dr \\
&+ C + C\int_{-\infty}^{0} e^{\left(\alpha - \frac{8m_1^2}{\beta_1\delta^2}\right)r} \sum_{j=1}^{m} |z_j(\theta_r\omega_j)|^2 dr.
\end{aligned}$$

where C *is a positive constant independent of* τ *and* ω*.*

Remark 12. We note that (4.2) holds for γ large, whereas (4.3) holds if δ is small enough. On the other hand, (4.2) and (4.5) are satisfied for λ large and m_1, k_2 small enough. These conditions can be read as: a combination of strong damping (λ large) and small effects of the delay (in terms of k_2 and m_1 small) ensure the existence of the attractor.

Proof. Taking the inner product in H of the second equation of (3.9) with v, we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|v|^2 - \delta|v|^2 + (J(v - \delta u + z(\theta_t\omega)), v) + (\lambda+\delta^2)(u,v) + (\nabla u, \nabla v) \\
&= (f(x, u(t-\rho(t))), v) + \left(\int_{-\infty}^{0} F(x,s,u(t+s)), v\right) + \big(g(t) + (\tilde{\alpha}+\delta)z(\theta_t\omega), v\big).
\end{aligned} \tag{4.6}$$

Recall that $v = \frac{\partial u}{\partial t} + \delta u - z(\theta_t\omega)$, hence by Young's inequality we have

$$(\lambda+\delta^2)(u,v) \geqslant \frac{(\lambda+\delta^2)}{2}\frac{d}{dt}|u|^2 + \frac{\delta(\lambda+\delta^2)}{2}|u|^2 - C|z(\theta_t\omega)|^2, \tag{4.7}$$

$$(\nabla u, \nabla v) \geqslant \frac{1}{2}\frac{d}{dt}\|u\|^2 + \frac{\delta}{2}\|u\|^2 - C\|z(\theta_t\omega)\|^2, \tag{4.8}$$

$$(g(t) + (\tilde{\alpha}+\delta)z(\theta_t\omega), v) \leqslant \frac{\beta_1}{8}|v|^2 + C|g(t)|^2 + C|z(\theta_t\omega)|^2, \tag{4.9}$$

and using (**H2**)-(**H3**), we find that

$$(f(x,u(t-\rho(t))), v) \leqslant \frac{\beta_1}{8}|v|^2 + \frac{2|k_1|^2}{\beta_1} + \frac{2k_2^2}{\beta_1}|u(t-\rho(t))|^2, \tag{4.10}$$

$$
\begin{aligned}
\left(\int_{-\infty}^{0} F(x,s,u(t+s))ds, v\right) &\leqslant \int_D \int_{-\infty}^{0} |m_0(x,s)||v(t)|dsdx \\
&+ \int_D \int_{-\infty}^{0} m_1(s)|u(t+s)||v(t)|dsdx \\
&\leqslant m_0|v(t)| + \int_{-\infty}^{0} m_1(s)|u(t+s)||v(t)|ds \\
&\leqslant \frac{\beta_1}{8}|v(t)|^2 + \frac{4m_0^2}{\beta_1} + \frac{4m_1^2}{\beta_1}\|u_t\|^2_{C_{\gamma,H}}.
\end{aligned} \tag{4.11}
$$

By Lagrange's mean value theorem and (**H1**), we obtain that

$$
\begin{aligned}
&(J(v-\delta u + z(\theta_t\omega)), v) = (J'(\zeta)(v-\delta u + z(\theta_t\omega)), v) \\
&\geqslant \beta_1|v|_2^2 - J'(\zeta)|(\delta u - z(\theta_t\omega), v)| \geqslant \frac{\beta_1}{2}|v|^2 - \frac{\beta_2^2\delta^2}{\beta_1}|u|^2 - C|z(\theta_t\omega)|^2,
\end{aligned} \tag{4.12}
$$

where ζ is between 0 and $v - \delta u + z(\theta_t\omega)$. Inserting (4.7)-(4.12) into (4.6) gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(|v|^2 + \|u\|^2 + (\lambda+\delta^2)|u|^2\right) + \left(\frac{\beta_1}{8} - \delta\right)|v|^2 + \frac{\delta}{2}\|u\|^2 + \left(\frac{\delta(\lambda+\delta^2)}{2} - \frac{\beta_2^2\delta^2}{\beta_1}\right)|u|^2 \\
&\leqslant \frac{2k_2^2}{\beta_1}|u(t-\rho(t))|^2 + \frac{4m_1^2}{\beta_1}\|u_t\|^2_{C_{\gamma,H}} + C|g(t)|^2 + C + C\|z(\theta_t\omega)\|^2.
\end{aligned} \tag{4.13}
$$

Then it follows from (4.13) that

$$
\begin{aligned}
&\frac{d}{dt}\left(e^{\alpha t}\left(|v|^2 + \|u\|^2 + (\lambda+\delta^2)|u|^2\right)\right) + \left(\frac{\beta_1}{4} - 2\delta - \alpha\right)e^{\alpha t}|v|^2 + (\delta-\alpha)e^{\alpha t}\|u\|^2 \\
&+ \left(\delta(\lambda+\delta^2) - \alpha(\lambda+\delta^2) - \frac{2\beta_2^2\delta^2}{\beta_1}\right)e^{\alpha t}|u|^2 \leqslant \frac{4k_2^2}{\beta_1}e^{\alpha t}|u(t-\rho(t))|^2 \\
&+ \frac{8m_1^2}{\beta_1}e^{\alpha t}\|u_t\|^2_{C_{\gamma,H}} + Ce^{\alpha t}|g(t)|^2 + Ce^{\alpha t} + Ce^{\alpha t}\|z(\theta_t\omega)\|^2.
\end{aligned} \tag{4.14}
$$

Given $t \geqslant 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\tau - t \leqslant T \leqslant \tau$, integrating (4.14) over $(\tau - t, T)$, we find that

$$
\begin{aligned}
&|v(T,\tau-t,\omega,v_{\tau-t})|^2 + \|u(T,\tau-t,\omega,u_{\tau-t})\|^2 + (\lambda+\delta^2)|u(T,\tau-t,\omega,u_{\tau-t})|^2 \\
&+ \left(\frac{\beta_1}{4} - 2\delta - \alpha\right)e^{-\alpha T}\int_{\tau-t}^{T} e^{\alpha r}|v(r,\tau-t,\omega,v_{\tau-t})|^2 dr \\
&+ (\delta-\alpha)\, e^{-\alpha T}\int_{\tau-t}^{T} e^{\alpha r}\|u(r,\tau-t,\omega,u_{\tau-t})\|^2 dr \\
&+ \left(\delta(\lambda+\delta^2) - \alpha(\lambda+\delta^2) - \frac{2\beta_2^2\delta^2}{\beta_1}\right)e^{-\alpha T}\int_{\tau-t}^{T} e^{\alpha r}|u(r,\tau-t,\omega,u_{\tau-t})|^2 dr \\
&\leqslant e^{-\alpha(T-\tau+t)}\left(|v(\tau-t,\tau-t,\omega,v_{\tau-t})|^2 + \|u(\tau-t,\tau-t,\omega,u_{\tau-t})\|^2\right) \\
&+ e^{-\alpha(T-\tau+t)}(\lambda+\delta^2)|u(\tau-t,\tau-t,\omega,u_{\tau-t})|^2 \\
&+ \frac{4k_2^2}{\beta_1}e^{-\alpha T}\int_{\tau-t}^{T} e^{\alpha r}|u(r-\rho(r),\tau-t,\omega,u_{\tau-t})|^2 dr + \frac{8m_1^2}{\beta_1}e^{-\alpha T}\int_{\tau-t}^{T} e^{\alpha r}\|u_r\|^2_{C_{\gamma,H}} dr \\
&+ Ce^{-\alpha T}\int_{\tau-t}^{T} e^{\alpha r}|g(r)|^2 dr + C + Ce^{-\alpha T}\int_{\tau-t}^{T} e^{\alpha r}\|z(\theta_r\omega)\|^2 dr.
\end{aligned} \tag{4.15}
$$

Let $r' = r - \rho(r)$, where $\rho(r) \in [0, h]$ and $\frac{1}{1-\rho'(r)} \leqslant \frac{1}{1-\rho_*}$ for all $r \in \mathbb{R}$. Therefore,

$$\begin{aligned}
&\int_{\tau-t}^{T} e^{\alpha r}|u(r-\rho(r),\tau-t,\omega,u_{\tau-t})|^2 dr \leqslant \frac{e^{\alpha h}}{1-\rho_*}\int_{\tau-t-h}^{T} e^{\alpha r'}|u(r',\tau-t,\omega,u_{\tau-t})|^2 dr' \\
&\leqslant \frac{e^{\alpha h}}{1-\rho_*}\Bigg(\int_{\tau-t-h}^{\tau-t} e^{\alpha r' \pm 2\gamma(r'-\tau+t)}|u(r',\tau-t,\omega,u_{\tau-t})|^2 dr' \\
&\qquad + \int_{\tau-t}^{T} e^{\alpha r'}|u(r',\tau-t,\omega,u_{\tau-t})|^2 dr'\Bigg) \\
&\leqslant \frac{e^{\alpha(\tau-t)}(e^{2\gamma h}-e^{\alpha h})}{(1-\rho_*)(2\gamma-\alpha)}\|\phi\|^2_{C_{\gamma,H}} + \frac{e^{\alpha h}}{1-\rho_*}\int_{\tau-t}^{T} e^{\alpha r'}|u(r',\tau-t,\omega,u_{\tau-t})|^2 dr'.
\end{aligned} \tag{4.16}$$

Inserting (4.16) into (4.15) and replacing ω by $\theta_{-\tau}\omega$, in view of (4.3)-(4.5), we deduce that

$$\begin{aligned}
&|v(T,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2 + \|u(T,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 + (\lambda+\delta^2)|u(T,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2 \\
&\leqslant e^{-\alpha(T-\tau+t)}\left(|v(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2 + \|u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\right) \\
&+ Ce^{-\alpha(T-\tau+t)}\|\phi\|^2_{C_H} + \frac{8m_1^2}{\beta_1}e^{-\alpha T}\int_{\tau-t}^{T} e^{\alpha r}\|u_r\|^2_{C_{\gamma,H}} dr \\
&+ Ce^{-\alpha T}\int_{\tau-t}^{T} e^{\alpha r}|g(r)|^2 dr + C + Ce^{-\alpha T}\int_{\tau-t}^{T} e^{\alpha r}\|z(\theta_{r-\tau}\omega)\|^2 dr.
\end{aligned} \tag{4.17}$$

By (4.2) we have $\alpha \leqslant 2\gamma$ and so $e^{(2\gamma-\alpha)T'} \leqslant 1$ for all $T' \leqslant 0$. Multiplying (4.17) by $e^{2\gamma T'}e^{-2\gamma T'}$ and replacing T by $T+T'$, it yields

$$\begin{aligned}
&\sup_{T'\in[\tau-t-T,0]} e^{2\gamma T'}|v(T+T',\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2 \\
&+ \sup_{T'\in[\tau-t-T,0]} e^{2\gamma T'}\|u(T+T',\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 \\
&+ (\lambda+\delta^2)\sup_{T'\in[\tau-t-T,0]} e^{2\gamma T'}|u(T+T',\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2 \\
&\leqslant e^{-\alpha(T-\tau+t)}\left(|v(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2 + \|u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\right) \\
&+ Ce^{-\alpha(T-\tau+t)}\|u_{\tau-t}\|^2_{C_H} + \frac{8m_1^2}{\beta_1}e^{-\alpha T}\int_{\tau-t}^{T} e^{\alpha r}\|u_r\|^2_{C_{\gamma,H}} dr \\
&+ Ce^{-\alpha T}\int_{\tau-t}^{T} e^{\alpha r}|g(r)|^2 dr + C + Ce^{-\alpha T}\int_{\tau-t}^{T} e^{\alpha r}\|z(\theta_{r-\tau}\omega)\|^2 dr.
\end{aligned} \tag{4.18}$$

On the other hand, in view of $\alpha \leqslant 2\gamma$, we have

$$\begin{aligned}
&e^{2\gamma T'}|v(T+T',\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2 + e^{2\gamma T'}\|u(T+T',\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 \\
&+ (\lambda+\delta^2)e^{2\gamma T'}|u(T+T',\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2 \\
&= e^{-2\gamma(T-\tau+t)}e^{2\gamma(T+T'-\tau+t)}|v(T+T',\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2 \\
&+ e^{-2\gamma(T-\tau+t)}e^{2\gamma(T+T'-\tau+t)}\|u(T+T',\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 \\
&+ (\lambda+\delta^2)e^{-2\gamma(T-\tau+t)}e^{2\gamma(T+T'-\tau+t)}|u(T+T',\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2 \\
&\leqslant e^{-\alpha(T-\tau+t)}\|v_{\tau-t}\|^2_{C_{\gamma,H}} + e^{-\alpha(T-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,V}}
\end{aligned}$$

$$+(\lambda+\delta^2)e^{-\alpha(T-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}},\ \forall T'\in(-\infty,\tau-t-T]. \tag{4.19}$$

Hence, we obtain that for all $t\geqslant 0$ and $T\in[\tau-t,\tau]$,

$$\begin{aligned}
&\|v_T\|^2_{C_{\gamma,H}}+\|u_T\|^2_{C_{\gamma,V}}+(\lambda+\delta^2)\|u_T\|^2_{C_{\gamma,H}}\\
&\leqslant e^{-\alpha(T-\tau+t)}\|v_{\tau-t}\|^2_{C_{\gamma,H}}+e^{-\alpha(T-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,V}}\\
&+Ce^{-\alpha(T-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}}+\frac{8m_1^2}{\beta_1}e^{-\alpha T}\int_{\tau-t}^{T}e^{\alpha r}\|u_r\|^2_{C_{\gamma,H}}dr\\
&+Ce^{-\alpha T}\int_{\tau-t}^{T}e^{\alpha r}|g(r)|^2dr+C+Ce^{-\alpha T}\int_{\tau-t}^{T}e^{\alpha r}\|z(\theta_{r-\tau}\omega)\|^2dr.
\end{aligned} \tag{4.20}$$

Applying Gronwall's lemma to (4.20), by Fubini's theorem we deduce that for all $t\geqslant 0$ and $T\in[\tau-t,\tau]$,

$$\begin{aligned}
&\|v_T\|^2_{C_{\gamma,H}}+\|u_T\|^2_{C_{\gamma,V}}+(\lambda+\delta^2)\|u_T\|^2_{C_{\gamma,V}}\\
&\leqslant e^{\left(\frac{8m_1^2}{\beta_1\delta^2}-\alpha\right)(T-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,V}}+Ce^{\left(\frac{8m_1^2}{\beta_1\delta^2}-\alpha\right)(T-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,V}}+C\\
&+Ce^{\left(\frac{8m_1^2}{\beta_1\delta^2}-\alpha\right)(T-\tau+t)}\|u_{\tau-t}\|^2_{C_{\gamma,H}}+Ce^{\left(\frac{8m_1^2}{\beta_1\delta^2}-\alpha\right)T}\int_{\tau-t}^{T}e^{\left(\alpha-\frac{8m_1^2}{\beta_1\delta^2}\right)r}|g(r)|^2dr\\
&+Ce^{\left(\frac{8m_1^2}{\beta_1\delta^2}-\alpha\right)T}\int_{\tau-t}^{T}e^{\left(\alpha-\frac{8m_1^2}{\beta_1\delta^2}\right)r}\|z(\theta_{r-\tau}\omega)\|^2dr.
\end{aligned} \tag{4.21}$$

In view of $z(\theta_t\omega)=\sum_{j=1}^m h_jz_j(\theta_t\omega_j)$ and $h_j\in H_0^1(D)$, we find that for all $t\geqslant 0$, $T\in[\tau-t,\tau]$ and every $\omega\in\Omega$,

$$\begin{aligned}
&Ce^{\left(\frac{8m_1^2}{\beta_1\delta^2}-\alpha\right)T}\int_{\tau-t}^{T}e^{\left(\alpha-\frac{8m_1^2}{\beta_1\delta^2}\right)r}\|z(\theta_{r-\tau}\omega)\|^2dr\\
&=Ce^{\left(\frac{8m_1^2}{\beta_1\delta^2}-\alpha\right)(T-\tau)}\int_{-t}^{T-\tau}e^{\left(\alpha-\frac{8m_1^2}{\beta_1\delta^2}\right)r}\|z(\theta_r\omega)\|^2dr\\
&\leqslant Ce^{\left(\frac{8m_1^2}{\beta_1\delta^2}-\alpha\right)(T-\tau)}\int_{0}^{-\infty}e^{\left(\alpha-\frac{8m_1^2}{\beta_1\delta^2}\right)r}\sum_{j=1}^{m}|z_j(\theta_r\omega_j)|^2dr.
\end{aligned} \tag{4.22}$$

Let $T=\tau$. Then by (4.21)-(4.22) we obtain that for all $t\geqslant 0$,

$$\begin{aligned}
&\|v_\tau\|^2_{C_{\gamma,H}}+\|u_\tau\|^2_{C_{\gamma,V}}+(\lambda+\delta^2)\|u_\tau\|^2_{C_{\gamma,V}}\\
&\leqslant Ce^{-\left(\alpha-\frac{8m_1^2}{\beta_1\delta^2}\right)t}\left(\|v_{\tau-t}\|^2_{C_{\gamma,H}}+\|u_{\tau-t}\|^2_{C_{\gamma,V}}+\|u_{\tau-t}\|^2_{C_{\gamma,H}}\right)\\
&+Ce^{\left(\frac{8m_1^2}{\beta_1\delta^2}-\alpha\right)\tau}\int_{-\infty}^{\tau}e^{\left(\alpha-\frac{8m_1^2}{\beta_1\delta^2}\right)r}|g(r)|^2dr\\
&+C+C\int_{-\infty}^{0}e^{\left(\alpha-\frac{8m_1^2}{\beta_1\delta^2}\right)r}\sum_{j=1}^{m}|z_j(\theta_r\omega_j)|^2dr.
\end{aligned} \tag{4.23}$$

This completes the proof. □

5. **Random pullback attractors.** This section is devoted to prove the existence of a pullback attractor for the multi-valued cocycle Φ associated with the system (3.9)-(3.10). First, we present the existence of a pullback absorbing set in $\mathcal{D}$.

Lemma 13. *Assume that the hypotheses in Lemma 11 hold. Then there exists $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ such that K is a closed measurable $\mathcal{D}$-pullback absorbing set for Φ, that is, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T_1 = T_1(\tau,\omega,B) > 0$ such that for all $t \geqslant T_1$,*

$$\Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau,\omega).$$

Proof. Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, let $K(\tau,\omega) = \big\{(\varphi,\psi) \in E : \|\varphi\|^2_{C_{\gamma,V}} + \|\psi\|^2_{C_{\gamma,H}} \leqslant L(\tau,\omega)\big\}$, where

$$\begin{aligned} L(\tau,\omega) = C + Ce^{\left(\frac{8m_1^2}{\beta_1\delta^2}-\alpha\right)\tau} \int_{-\infty}^{\tau} e^{\left(\alpha-\frac{8m_1^2}{\beta_1\delta^2}\right)r} |g(r)|^2 dr \\ + C\int_{-\infty}^{0} e^{\left(\alpha-\frac{8m_1^2}{\beta_1\delta^2}\right)r} \sum_{j=1}^{m} |z_j(\theta_r\omega_j)|^2 dr. \end{aligned} \tag{5.1}$$

Then for each $\tau \in \mathbb{R}$, $L(\tau,\cdot) : \Omega \to \mathbb{R}$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable, and

$$\lim_{r\to-\infty} e^{\left(\alpha-\frac{8m_1^2}{\beta_1\delta^2}\right)r} L(\tau + r, \theta_r\omega) = 0. \tag{5.2}$$

Therefore, $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ belongs to $\mathcal{D}$. By Lemma 11, $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is a closed measurable $\mathcal{D}$-pullback absorbing set in $\mathcal{D}$ for Φ. □

We are now ready to prove the existence of a pullback attractor for Φ in $\mathcal{D}$.

Theorem 14. *Assume that the hypotheses in Lemma 11 hold. Then the multi-valued cocycle Φ associated with problem (3.9)-(3.10) possesses a unique $\mathcal{D}$-pullback attractor $\mathcal{A} \in \mathcal{D}$ in E.*

Proof. We first show that Φ is $\mathcal{D}$-pullback asymptotically upper-semicompact in E. Let $\tau \in \mathbb{R}, \omega \in \Omega$ be given arbitrarily. By Lemmas 11 and 13, for any $T \geqslant \tau - t$ with $t \geqslant 0$, let

$$\begin{aligned} &\Phi(T - \tau + t, \tau - t, \theta_{-t}\omega, (u_{\tau-t}, v_{\tau-t})) \\ &\quad = \big\{(u_T(\cdot, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}), v_T(\cdot, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})) \,\big|\, (u(\cdot), v(\cdot)) \\ &\qquad \text{is a solution of (3.9)-(3.10) with } (u_{\tau-t}, v_{\tau-t}) \in K(\tau - t, \theta_{-t}\omega)\big\}, \end{aligned}$$

where $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is a closed measurable $\mathcal{D}$-pullback absorbing set for Φ in E.

Let $u = u_1 + u_2$ and $v = v_1 + v_2$, we decompose Eqs. (3.9)-(3.10) as follows:

$$\begin{cases} \dfrac{\partial u_2}{\partial T} + \delta u_2 = v_2, \\ \dfrac{\partial v_2}{\partial T} - \delta v_2 - \Delta u_2 + (\lambda + \delta^2)u_2 + J(v - \delta u + z(\theta_T\omega)) - J(v_1 - \delta u_1 + z(\theta_T\omega)) = 0, \end{cases} \tag{5.3}$$

with boundary and initial conditions

$$\begin{cases} u_2(T,x) = v_2(T,x) = 0,\ T > \tau - t, x \in \partial D, \\ u_2(T,x) = u_{2,\tau-t}\,(T-\tau+t,x) = \phi(T-\tau+t,x),\ T \leqslant \tau - t,\ x \in D, \\ v_2(T,x) = v_{2,\tau-t}(T-\tau+t,x) = \dfrac{\partial\phi(T-\tau+t,x)}{\partial T} \\ \ +\delta\phi(T-\tau+t,x) - z(\theta_T\omega),\ T \leqslant \tau - t,\ x \in D, \end{cases} \tag{5.4}$$

and the non-homogeneous equations

$$\begin{cases} \dfrac{\partial u_1}{\partial T} + \delta u_1 = v_1 + z(\theta_T\omega), \\ \dfrac{\partial v_1}{\partial T} - \delta v_1 + J\left(v_1 - \delta u_1 + z(\theta_T\omega)\right) - \Delta u_1 + (\lambda+\delta^2)u_1 = f(x, u(T-\rho(T))) \\ +g(x,T) + \displaystyle\int_{-\infty}^0 F(x,s,u(T+s))ds + (\tilde{\alpha}+\delta)z(\theta_T\omega), \end{cases} \tag{5.5}$$

with boundary and initial conditions

$$\begin{cases} u_1(T,x) = v_1(T,x) = 0,\ T > \tau - t, x \in \partial D, \\ u_1(T,x) = u_{1,\tau-t}(T-\tau+t,x) = 0, \quad T \leqslant \tau - t,\ x \in D, \\ v_1(T,x) = v_{1,\tau-t}(T-\tau+t,x) = 0, \quad T \leqslant \tau - t,\ x \in D, \end{cases} \tag{5.6}$$

Taking the inner product in H of the second equation of (5.3) with v_2, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dT}|v_2|^2 - \delta|v_2|^2 + \left(J\left(v - \delta u + z(\theta_T\omega)\right) - J\left(v_1 - \delta u_1 + z(\theta_T\omega)\right), v_2\right) \\ &+ (\nabla u_2, \nabla v_2) + (\lambda+\delta^2)(u_2, v_2) = 0. \end{aligned} \tag{5.7}$$

In a similar way as in (4.7)-(4.8) and (4.12), we deduce that

$$\begin{aligned} &\left(J\left(v - \delta u + z(\theta_T\omega)\right) - J\left(v_1 - \delta u_1 + z(\theta_T\omega)\right), v_2\right) \\ &= \left(J'(\tilde{\zeta})(v - v_1 - \delta(u - u_1)), v_2\right) \\ &= J'(\tilde{\zeta})(v_2 - \delta u_2, v_2) \geqslant \frac{\beta_1}{2}|v_2|^2 - \frac{\beta_2^2\delta^2}{2\beta_1}|u_2|^2, \end{aligned} \tag{5.8}$$

where $\tilde{\zeta}$ is between $v - \delta u + z(\theta_T\omega)$ and $v_1 - \delta u_1 + z(\theta_T\omega)$,

$$(\nabla u_2, \nabla v_2) = \left(\nabla u_2, \nabla\frac{\partial u_2}{\partial T} + \delta\nabla u_2\right) = \frac{1}{2}\frac{d}{dT}\|u_2\|^2 + \delta\|u_2\|^2, \tag{5.9}$$

and

$$(u_2, v_2) = \left(u_2, \frac{\partial u_2}{\partial T} + \delta u_2\right) = \frac{1}{2}\frac{d}{dT}|u_2|^2 + \delta|u_2|^2. \tag{5.10}$$

Then it follows from (5.7)-(5.10) that

$$\begin{aligned} &\frac{d}{dT}\left(e^{\alpha T}\left(|v_2|^2 + \|u_2\|^2 + (\lambda+\delta^2)|u_2|^2\right)\right) + (\beta_1 - 2\delta - \alpha)e^{\alpha T}|v_2|^2 \\ &+ (2\delta - \alpha)e^{\alpha T}\|u_2\|^2 + \left(2\delta(\lambda+\delta^2) - \alpha(\lambda+\delta^2) - \frac{\beta_2^2\delta^2}{\beta_1}\right)e^{\alpha T}|u_2|^2 \leqslant 0. \end{aligned} \tag{5.11}$$

Given $t \geqslant 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, inequality (5.11) over $(\tau - t, T)$, and replacing ω by $\theta_{-\tau}\omega$, in view of (4.3)-(4.5), we obtain that for all $T \in [\tau - t, \tau]$,

$$|v_2(T, \tau - t, \theta_{-\tau}\omega, v_{2,\tau-t})|^2 + \|u_2(T, \tau - t, \theta_{-\tau}\omega, u_{2,\tau-t})\|^2$$

$$+ (\lambda+\delta^2)|u_2(T,\tau-t,\theta_{-\tau}\omega,u_{2,\tau-t})|^2$$
$$\leqslant e^{-\alpha(T-\tau+t)}\left(|v_2(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{2,\tau-t})|_2^2+\|u_2(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{2,\tau-t})\|^2\right)$$
$$+ (\lambda+\delta^2)e^{-\alpha(T-\tau+t)}|u_2(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{2,\tau-t})|^2. \tag{5.12}$$

Arguing as in (4.18)-(4.19), we deduce that

$$\begin{aligned}&\|v_{2,\tau}\|^2_{C_{\gamma,H}}+\|u_{2,\tau}\|^2_{C_{\gamma,V}}+(\lambda+\delta^2)\|u_{2,\tau}\|^2_{C_{\gamma,H}}\\ &\leqslant Ce^{-\alpha t}\left(\|v_{2,\tau-t}\|^2_{C_{\gamma,H}}+\|u_{2,\tau-t}\|^2_{C_{\gamma,V}}+\delta^2\|u_{2,\tau-t}\|^2_{C_{\gamma,H}}\right),\end{aligned} \tag{5.13}$$

and thus condition (1) in Theorem 9 is proved.

Now, we consider a couple of solutions $(u^1(t),v^1(t))$ and $(u^2(t),v^2(t))$ of system (3.9) with initial data (u^1_τ,v^1_τ) and (u^2_τ,v^2_τ) respectively. Let $\overline{u}(T)=u^1_1(T)-u^2_1(T)$, $\overline{v}(T)=v^1_1(T)-v^2_1(T)$. Then it follows from Eqs. (5.5)-(5.6) that

$$\begin{cases}\dfrac{\partial\overline{u}}{\partial T}+\delta\overline{u}=\overline{v},\\ \dfrac{\partial\overline{v}}{\partial T}-\delta\overline{v}-\Delta\overline{u}+(\lambda+\delta^2)\overline{u}+J\left(v^1_1-\delta u^1_1+z(\theta_T\omega)\right)-J\left(v^2_1-\delta u^2_1+z(\theta_T\omega)\right)\\ =f(x,u^1(T-\rho(T)))-f(x,u^2(T-\rho(T)))\\ +\displaystyle\int_{-\infty}^0F(x,s,u^1(t+s))ds-\int_{-\infty}^0F(x,s,u^2(t+s))ds,\end{cases} \tag{5.14}$$

with boundary and initial data

$$\begin{cases}\overline{u}(T,x)=\overline{v}(T,x)=0,\quad T>\tau-t,\ x\in\partial D,\\ \overline{u}(T,x)=\overline{u}_{\tau-t}(T-\tau+t,x)=0,\quad T\geqslant\tau-t,\ x\in D,\\ \overline{v}(T,x)=\overline{v}_{\tau-t}(T-\tau+t,x)=0,\quad T\geqslant\tau-t,\ x\in D.\end{cases} \tag{5.15}$$

Taking the inner product in H of the second equation of (5.14) with $\overline{v}$, we have

$$\begin{aligned}&\frac12\frac{d}{dT}|\overline{v}|^2-\delta|\overline{v}|^2+(\nabla\overline{u},\nabla\overline{v})+(\lambda+\delta^2)(\overline{u},\overline{v})\\ &+\left(J\left(v^1_1-\delta u^1_1+z(\theta_T\omega)\right)-J\left(v^2_1-\delta u^2_1+z(\theta_T\omega)\right),\overline{v}\right)\\ &=\left(f(x,u^1(T-\rho(T)))-f(x,u^2(T-\rho(T))),\overline{v}\right)\\ &+\left(\int_{-\infty}^0F(x,s,u^1(t+s))ds-\int_{-\infty}^0F(x,s,u^2(t+s))ds,\overline{v}\right).\end{aligned} \tag{5.16}$$

By similar arguments as in (5.8)-(5.10), we obtain

$$\begin{aligned}&\frac{d}{dT}\left(|\overline{v}|^2+\|\overline{u}\|^2+(\lambda+\delta^2)|\overline{u}|^2\right)+\alpha_1\left(|\overline{v}|^2+\|\overline{u}\|^2+(\lambda+\delta^2)|\overline{u}|^2\right)\\ &\leqslant 2\left(f(x,u^1(T-\rho(T)))-f(x,u^2(T-\rho(T))),\overline{v}\right)\\ &+2\left(\int_{-\infty}^0F(x,s,u^1(t+s))ds-\int_{-\infty}^0F(x,s,u^2(t+s))ds,\overline{v}\right),\end{aligned} \tag{5.17}$$

where $\alpha_1:=\min\{\beta_1-2\delta,2\delta,2\delta-\frac{\beta_2^2\delta^2}{\beta_1(\lambda+\delta^2)}\}>0$. Given $\tau\in\mathbb{R}$, $\omega\in\Omega$, $t\geqslant 0$, integrating (5.17) over $(\tau-t,T)$ with $T\in[\tau-t,t]$ and replacing ω by $\theta_{-\tau}\omega$, we find that

$$|\overline{v}(T,\tau-t,\theta_{-\tau}\omega,\overline{v}_{\tau-t})|_2^2+\|\overline{u}(T,\tau-t,\theta_{-\tau}\omega,\overline{u}_{\tau-t})\|^2+|\overline{u}(T,\tau-t,\theta_{-\tau}\omega,\overline{u}_{\tau-t})|^2$$

$$
\begin{aligned}
&\leqslant C\int_{\tau-t}^{\tau}\left(f(x,u^1(r-\rho(r)))-f(x,u^2(r-\rho(r))),\overline{v}\right)dr\\
&+C\int_{\tau-t}^{\tau}\left(\int_{-\infty}^{0}F(x,s,u^1(r+s))ds-\int_{-\infty}^{0}F(x,s,u^2(r+s))ds,\overline{v}\right)dr\\
&\leqslant C\left\|f(x,u^1(r-\rho(r)))-f(x,u^2(r-\rho(r)))\right\|_{L^2(D\times[\tau-t,\tau])}\\
&\times\|\overline{v}(r,\tau-t,\theta_{-\tau}\omega,\overline{v}_{\tau-t})\|_{L^2(D\times[\tau-t,\tau])}\\
&+C\left\|\int_{-\infty}^{0}F(x,s,u^1(r+s))ds-\int_{-\infty}^{0}F(x,s,u^2(r+s))ds\right\|_{L^2(D\times[\tau-t,\tau])}\\
&\times\|\overline{v}(r,\tau-t,\theta_{-\tau}\omega,\overline{v}_{\tau-t})\|_{L^2(D\times[\tau-t,\tau])}.
\end{aligned}
\tag{5.18}
$$

Multiplying by (5.18) by $e^{2\gamma T'}e^{-2\gamma T'}$ and replacing T by $T+T'$, in view of the boundary condition (5.15), we deduce that, for all $T\in[\tau-t,t]$,

$$
\begin{aligned}
&\|\overline{v}_\tau\|^2_{C_{\gamma,H}}+\|\ \overline{u}_\tau\ \|^2_{C_{\gamma,V}}\\
&\leqslant C\left\|f(x,u^1(r-\rho(r)))-f(x,u^2(r-\rho(r)))\right\|_{L^2(D\times[\tau-t,\tau])}\\
&\times\|\overline{v}(r,\tau-t,\theta_{-\tau}\omega,\overline{v}_{\tau-t})\|_{L^2(D\times[\tau-t,\tau])}\\
&+C\left\|\int_{-\infty}^{0}F(x,s,u^1(r+s))ds-\int_{-\infty}^{0}F(x,s,u^2(r+s))ds\right\|_{L^2(D\times[\tau-t,\tau])}\\
&\times\|\overline{v}(r,\tau-t,\theta_{-\tau}\omega,\overline{v}_{\tau-t})\|_{L^2(D\times[\tau-t,\tau])}.
\end{aligned}
\tag{5.19}
$$

Let

$$
\left(u^n_r(\cdot,\tau-t,\theta_{-\tau}\omega,u^n_{\tau-t}),v^n_r(\cdot,\tau-t,\theta_{-\tau}\omega,v^n_{\tau-t})\right)\in\Phi\left(r-\tau+t,\tau-t,\theta_{-t}\omega,K(\tau-t,\theta_{-t}\omega)\right)
$$

with $(u^n_{\tau-t},v^n_{\tau-t}))\in K(\tau-t,\theta_{-t}\omega)$ be given arbitrarily. By (4.2) and (4.21)-(4.22) we obtain that

$$
\{(u^n_r,v^n_r)\}\text{ is uniformly (w.r.t. }r\in[\tau-t,\tau])\text{ bounded in }E. \tag{5.20}
$$

We recall that $e^{2\gamma\cdot}m_1(\cdot)\in L^1((-\infty,0];\mathbb{R}^+)$ and $m_0\in L^1((-\infty,0];L^2(D))$. Hence for any given $\varepsilon>0$, there exists $\widetilde{T}>h$ such that for any $n,m\in\mathbb{N}$,

$$
\begin{aligned}
&C\left\|\int_{-\infty}^{-\widetilde{T}}F(x,s,u^n(r+s))-F(x,s,u^m(r+s))ds\right\|^2_{L^2(D\times[\tau-t,\tau])}\\
&\leqslant C\int_{\tau-t}^{\tau}\left|\int_{-\infty}^{-\widetilde{T}}F(x,s,u^n(r+s))-F(x,s,u^m(r+s))ds\right|^2dr\\
&\leqslant C\int_{\tau-t}^{\tau}\left(\int_{-\infty}^{-\widetilde{T}}m_1(s)|u^n(r+s)|+2|m_0(s)|+m_1(s)|u^m(r+s)|ds\right)^2dr\\
&\leqslant C\int_{\tau-t}^{\tau}\left(\left(\|u^n_r\|^2_{C_{\gamma,H}}+\|u^n_r\|^2_{C_{\gamma,H}}\right)\left(\int_{-\infty}^{-\widetilde{T}}e^{2\gamma s}m_1(s)ds\right)^2+\left(\int_{-\infty}^{-\widetilde{T}}m_1(s)ds\right)^2\right)dr\\
&\leqslant Ct\sup_{r\in[\tau-t,\tau]}\left(\|u^n_r\|^2_{C_{\gamma,H}}+\|u^n_r\|^2_{C_{\gamma,H}}\right)\left(\int_{-\infty}^{-\widetilde{T}}e^{2\gamma s}m_1(s)ds\right)^2
\end{aligned}
$$

$$+Ct\left(\int_{-\infty}^{-\widetilde{T}} m_1(s)ds\right)^2 < \frac{\varepsilon}{4}, \tag{5.21}$$

thanks to (5.20). From (5.20), we find that

$$\{u^n(r,\tau-t,\theta_{-\tau}\omega,u^n_{\tau-t})\} \text{ is bounded in } L^\infty(\tau-t-\widetilde{T},\tau;V), \tag{5.22}$$

$$\{v^n(r,\tau-t,\theta_{-\tau}\omega,v^n_{\tau-t})\} \text{ is bounded in } L^\infty(\tau-t-\widetilde{T},\tau;H). \tag{5.23}$$

Since $\frac{\partial u}{\partial r} = v - \delta u + z(\theta_{r-\tau}\omega)$, in view of the continuity of $z(\theta_{r-\tau}\omega)$ in r, we obtain that for every $\omega \in \Omega$,

$$\left\{\frac{\partial u^n(r,\tau-t,\theta_{-\tau}\omega,u^n_{\tau-t})}{\partial r}\right\} \text{ is bounded in } L^\infty(\tau-t-\widetilde{T},\tau;H). \tag{5.24}$$

Hence, without loss of generality, we can assume that

$$u^n(r)\to u(r) \quad *\text{-weakly in } L^\infty(\tau-t-\widetilde{T},\tau;V),$$

and

$$\frac{\partial u^n(r)}{\partial r}\to\frac{\partial u(r)}{\partial r} \quad *\text{-weakly in } L^\infty(\tau-t-\widetilde{T},\tau;H).$$

Consequently,

$$u^n\to u \quad \text{in} \quad L^2(\tau-t-\widetilde{T},\tau;H),$$

and

$$u^n(r,x)\to u(r,x) \quad \text{as } n\to\infty$$

for almost every $(T,x)\in[\tau-t-\widetilde{T},\tau]\times D$. Since $f\in C(D\times\mathbb{R};\mathbb{R})$ and $F\in C(D\times\mathbb{R}\times\mathbb{R};\mathbb{R})$, we have

$$f(x,u^n(r))\to f(x,u(r)) \quad \text{as } n\to\infty,$$

and

$$F(x,s,u^n(r))\to F(x,s,u(r)) \quad \text{as } n\to\infty$$

for almost every $(r,x,s)\in[\tau-t-\widetilde{T},\tau]\times D\times\mathbb{R}$. Thanks to the Lebesgue convergence theorem, we deduce that

$$\lim_{n\to\infty}\lim_{m\to\infty}\|f(x,u^n(r-\rho(r))-f(x,u^m(r-\rho(r))\|_{L^2(D\times[\tau-t,\tau])}=0, \tag{5.25}$$

and by (5.21) we obtain that there exists N_1, such that for all $n,m\geqslant N_1$,

$$\begin{aligned} &C\left\|\int_{-\infty}^0 F(x,s,u^n(r+s))ds-\int_{-\infty}^0 F(x,s,u^m(r+s))ds\right\|^2_{L^2(D\times[\tau-t,\tau])}\\ &\leqslant\frac{\varepsilon}{2}+C\int_{\tau-t}^{\tau}\left|\int_{-\widetilde{T}}^0 F(x,s,u^n(r+s))-F(x,s,u^m(r+s))ds\right|^2dr \\ &\leqslant\frac{\varepsilon}{2}+C\widetilde{T}\int_{\tau-t}^{\tau}\int_D\int_{-\widetilde{T}}^0|F(x,s,u^n(r+s))-F(x,s,u^m(r+s))|^2dsdxdr<\varepsilon. \end{aligned} \tag{5.26}$$

Inserting (5.25)-(5.26) into (5.19), in view of (5.23), we have

$$\lim_{n\to\infty}\lim_{m\to\infty}\left(\|v^n_{1,\tau}-v^m_{1,\tau}\|^2_{C_{\gamma,H}}+\|\,u^n_{1,\tau}-u^m_{1,\tau}\,\|^2_{C_{\gamma,V}}\right)=0. \tag{5.27}$$

This implies that condition (2) in Theorem 9 holds true, and thus the $\mathcal{D}$-pullback asymptotically upper-semicompactness of Φ in E follows immediately.

Finally, by slightly modifying the proof of Lemma 15, we can show that Φ is norm-to-weak upper-semicontinuous. Then by using Lemma 13, the assertion of this theorem follows immediately from Theorem 6. $\square$

6. **The measurability of the pullback attractors.** In order to prove the measurability of the pullback attractor $\mathcal{A}(\tau,\omega)$ for each fixed $\tau\in\mathbb{R}$, we need to show that Φ is a multi-valued random cocycle and the mapping $\omega\to\Phi(t,\tau,\omega,K(\tau,\omega))$ is closed, where $K=\{K(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ is the closed measurable $\mathcal{D}$-pullback absorbing set given in Lemma 13.

For $N\in\mathbb{N}$, we consider the sets

$$\Omega_N=\left\{\omega\in\Omega:\sum_{j=1}^{m}|\omega_j(t)|\leqslant Ne^{\zeta|t|},\ \text{for } t\in\mathbb{R}\right\},\tag{6.1}$$

where $0<\zeta<\frac{\tilde{\alpha}}{4}$ and $\tilde{\alpha}$ is given in (3.2). Then, we can see that $\Omega=\bigcup_N\Omega_N$ and $\Omega_N\in\mathcal{F}$. It is shown in [10] that Ω_N is a Polish space for any N. By the similar arguments of Lemmas 6.1-6.2 in [42], we have the following properties:

(**P1**) For any $n\in N$, there exists $\beta=\beta(N)$ such that

$$\|z(\theta_{t+\cdot}\omega)\|_{C_{\gamma,V}}\leqslant\beta e^{\zeta|t|},\tag{6.2}$$

for all $\omega\in\Omega_N$.

(**P2**) The map $\mathbb{R}\times\Omega_N\ni(t,\omega)\to z(\theta_{t+\cdot}\omega)$ is continuous in the topology of $C_{\gamma,V}$.

Let $\mathcal{F}_{\Omega_N}$ be the trace σ-algebra of $\mathcal{F}$ with respect to Ω_N and let $B_{\Omega_N}(a,r)$, $a\in\Omega_N$, $r>0$ be a ball in Ω_N. These balls can be generated by $B_\Omega(a,r)\bigcap\Omega_N$, where $B_\Omega(a,r)$ is a ball in Ω. The same is true for all open sets in Ω_N. Therefore, $\mathcal{F}_{\Omega_N}$ is just the Borel σ-algebra of Ω_N. Moreover, since $\Omega_N\in\mathcal{F}$ we have $\mathcal{F}_{\Omega_N}\subset\mathcal{F}$. Also, let $\bar{\mathcal{F}}_{\Omega_N}$ be the completion of $\mathcal{F}_{\Omega_N}$ with respect to $\mathbb{P}_{\Omega_N}$, where $\mathbb{P}_{\Omega_N}(A):=\mathbb{P}(A)$, for $A\in\mathcal{F}_{\Omega_N}$, that is, $\mathbb{P}_{\Omega_N}$ is just the restriction of $\mathbb{P}$ to $\mathcal{F}_{\Omega_N}$.

The following facts are given in [10]:

(1) $\mathbb{P}_{\Omega_N}$ is a finite measure on $(\Omega_N,\mathcal{F}_{\Omega_N})$.
(2) If $A\in\bar{\mathcal{F}}_{\Omega_N}$, then $A\in\bar{\mathcal{F}}$.

Lemma 15. *Assume that the hypotheses in Lemma 11 hold. Then for each $\tau\in\mathbb{R}$, the mapping $[0,+\infty)\times\Omega_N\times E\ni(t,\omega,(u_\tau,v_\tau))\to\Phi(t,\tau,\omega,(u_\tau,v_\tau))$ is upper-semicontinuous in E.*

Proof. Suppose not. Then there exist (u_τ,v_τ), $t>0$, $\omega\in\Omega_N$, a neighborhood O of $\Phi(t,\tau,\omega,(u_\tau,v_\tau))$ and sequences $t^n\to t$, $\omega^n\to\omega$ in Ω_N, $(u^n_\tau,v^n_\tau)\to(u_\tau,v_\tau)$ in E, $\xi^n\in\Phi(t^n,\tau,\omega^n,(u^n_\tau,v^n_\tau))$ such that $\xi^n\notin O$. We shall prove that, up to a subsequence, $\xi^n\to\xi\in\Phi(t,\tau,\omega,(u_\tau,v_\tau))$, which is a contradiction.

Without loss of generality, we can assume that

$$0\leqslant t^n\leqslant 1+t \text{ and } \|v^n_\tau\|^2_{C_{\gamma,H}}+\|u^n_\tau\|^2_{C_{\gamma,V}}\leqslant 1+2\|v_\tau\|^2_{C_{\gamma,H}}+2\|u_\tau\|^2_{C_{\gamma,V}},\ \forall n\in\mathbb{N}.\tag{6.3}$$

Let $\big(u^n_{s+\tau}(\cdot,\tau,\theta_{-\tau}\omega^n,u^n_\tau),v^n_{s+\tau}(\cdot,\tau,\theta_{-\tau}\omega^n,v^n_\tau)\big)\in\Phi\big(s,\tau,\omega^n,(u^n_\tau,v^n_\tau)\big)$ be such that $\xi^n=\big(u^n_{t^n+\tau}(\cdot,\tau,\theta_{-\tau}\omega^n,u^n_\tau),v^n_{t^n+\tau}(\cdot,\tau,\theta_{-\tau}\omega^n,v^n_\tau)\big)$. Arguing as in the proof of Lemma 11, we deduce that for all $s\in[\tau,1+t+\tau]$ and $n\in\mathbb{N}$,

$$\|v^n_s\|^2_{C_{\gamma,H}}+\|u^n_s\|^2_{C_{\gamma,V}}\leqslant Ce^{\left(\frac{8m_1^2}{\beta_1\delta^2}-\alpha\right)(s-\tau)}\|v^n_\tau\|^2_{C_{\gamma,H}}+Ce^{\left(\frac{8m_1^2}{\beta_1\delta^2}-\alpha\right)(s-\tau)}\|u^n_\tau\|^2_{C_{\gamma,V}}$$

$$+ C + Ce^{\left(\frac{8m_1^2}{\beta_1\delta^2}-\alpha\right)s}\int_\tau^s e^{\left(\alpha-\frac{8m_1^2}{\beta_1\delta^2}\right)r}|g(r)|^2dr \tag{6.4}$$
$$+ Ce^{\left(\frac{8m_1^2}{\beta_1\delta^2}-\alpha\right)s}\int_\tau^s e^{\left(\alpha-\frac{8m_1^2}{\beta_1\delta^2}\right)r}\|z(\theta_{r-\tau}\omega^n)\|^2dr.$$

By using (6.2) we have

$$\begin{aligned}&\int_\tau^s e^{\left(\alpha-\frac{8m_1^2}{\beta_1\delta^2}\right)r}\|z(\theta_{r-\tau}\omega^n)\|^2dr\\ &\leqslant C\int_\tau^s e^{\left(\alpha-\frac{8m_1^2}{\beta_1\delta^2}\right)r}e^{2\zeta(r-\tau)}dr \leqslant Ce^{\left(\alpha-\frac{8m_1^2}{\beta_1\delta^2}\right)s}e^{2\zeta(s-\tau)}.\end{aligned} \tag{6.5}$$

Inserting (6.5) into (6.4) yields

$$\|v_s^n\|^2_{C_{\gamma,H}} + \|u_s^n\|^2_{C_{\gamma,V}} \leqslant Ce^{\left(\frac{8m_1^2}{\beta_1\delta^2}-\alpha\right)(s-\tau)}\|v_\tau^n\|^2_{C_{\gamma,V}} + Ce^{\left(\frac{8m_1^2}{\beta_1\delta^2}-\alpha\right)(s-\tau)}\|u_\tau^n\|^2_{C_{\gamma,V}}$$
$$+ C + Ce^{\left(\frac{8m_1^2}{\beta_1\delta^2}-\alpha\right)s}\int_\tau^s e^{\left(\alpha-\frac{8m_1^2}{\beta_1\delta^2}\right)r}|g(r)|^2dr + Ce^{2\zeta(s-\tau)} \tag{6.6}$$

for all $s \in [\tau, 1+t+\tau]$ and $n \in \mathbb{N}$. By the property (**P2**), we have $z(\theta_{t^n+\cdot}\omega^n) \to z(\theta_{t+\cdot\omega})$ in $C_{\gamma,V}$. Analogous to the proof of Theorem 3.1 in [11] Section XV.3 and the argument in [32] Sections II.4 and IV.4.4, in view of the continuity of the mappings J, F, f and $(t,\omega) \to z(\theta_t\omega)$, by a standard argument we deduce that there exists a solution $\big(u(\cdot,\tau,\theta_{-\tau}\omega,u_\tau), v(\cdot,\tau,\theta_{-\tau}\omega,v_\tau)\big) \in L^\infty(\tau,1+t+\tau;V)\times L^\infty(\tau,1+t+\tau;H)$ of problem (3.9)-(3.10) and a subsequence of $(u^n(\cdot),v^n(\cdot))$ (which we still denote (u^n,v^n)) such that

$$u^n(\cdot) \overset{*}{\rightharpoonup} u(\cdot) \text{ *-weakly in } L^\infty(\tau,1+t+\tau;V), \tag{6.7}$$
$$v^n(\cdot) \overset{*}{\rightharpoonup} v(\cdot) \text{ *-weakly in } L^\infty(\tau,1+t+\tau;H) \tag{6.8}$$

as $n\to\infty$.

Now we need to prove that there exists a subsequence $(u^{n_k}(\cdot),v^{n_k}(\cdot))$ such that $(u^{n_k}(\cdot),v^{n_k}(\cdot))$ converges to some function $(u'(\cdot),v'(\cdot))$ in $C([\tau,1+t+\tau];V)\times C([\tau,1+t+\tau];H)$. Indeed, if this holds true, then by (6.7) and (6.8) we have $u'(\cdot)=u(\cdot)$, $v'(\cdot)=v(\cdot)$ and

$$u^{n_k}(\cdot,\tau,\theta_{-\tau}\omega^{n_k},u_\tau^{n_k}) \to u(\cdot,\tau,\theta_{-\tau}\omega,u_\tau) \text{ in } C([\tau,1+t+\tau];V), \tag{6.9}$$
$$v^{n_k}(\cdot,\tau,\theta_{-\tau}\omega^{n_k},v_\tau^{n_k}) \to v(\cdot,\tau,\theta_{-\tau}\omega,v_\tau) \text{ in } C([\tau,1+t+\tau];H), \tag{6.10}$$

where $(u(\cdot),v(\cdot))$ is a solution of problem (3.9). Let $\varepsilon>0$ be given arbitrarily. Then we have

$$\sup_{s'\in[-1-t,0]} e^{\gamma s'}|v^{n_k}(\tau+t^{n_k}+s',\tau,\theta_{-\tau}\omega^{n_k},v_\tau^{n_k}) - v(\tau+t+s',\tau,\theta_{-\tau}\omega,v_\tau)|$$
$$+\sup_{s'\in[-1-t,0]} e^{\gamma s'}\|u^{n_k}(\tau+t^{n_k}+s',\tau,\theta_{-\tau}\omega^{n_k},u_\tau^{n_k}) - u(\tau+t+s',\tau,\theta_{-\tau}\omega,u_\tau)\|$$
$$\leqslant \sup_{s'\in[-1-t,0]} e^{\gamma s'}|v^{n_k}(\tau+t^{n_k}+s',\tau,\theta_{-\tau}\omega^{n_k},v_\tau^{n_k}) - v(\tau+t^{n_k}+s',\tau,\theta_{-\tau}\omega,v_\tau)|$$
$$+\sup_{s'\in[-1-t,0]} e^{\gamma s'}|v(\tau+t^{n_k}+s',\tau,\theta_{-\tau}\omega,v_\tau) - v(\tau+t+s',\tau,\theta_{-\tau}\omega,v_\tau)| \tag{6.11}$$

$$
\begin{aligned}
&+\sup_{s'\in[-1-t,0]} e^{\gamma s'}\|u^{n_k}(\tau+t^{n_k}+s',\tau,\theta_{-\tau}\omega^{n_k},u_\tau^{n_k})-u(\tau+t^{n_k}+s',\tau,\theta_{-\tau}\omega,u_\tau)\|\\
&+\sup_{s'\in[-1-t,0]} e^{\gamma s'}\|u(\tau+t^{n_k}+s',\tau,\theta_{-\tau}\omega,u_\tau)-u(\tau+t+s',\tau,\theta_{-\tau}\omega,u_\tau)\|\leqslant\frac{\varepsilon}{2}
\end{aligned}
$$

for k sufficiently large. On the other hand, since $(u_\tau^n, v_\tau^n)\to(u_\tau,v_\tau)$ in E, in view of $\lim\limits_{s'\to-\infty} u_\tau(s')e^{\gamma s'}=u\in V$ and $\lim\limits_{s'\to-\infty} v_\tau(s')e^{\gamma s'}=v\in H$, we can choose $T>1+t$ and k sufficiently large such that

$$
\begin{aligned}
&\sup_{s'\leqslant-1-t} e^{\gamma s'}|v^{n_k}(\tau+t^{n_k}+s')-v(\tau+t+s')|\\
&\leqslant\sup_{s'\leqslant-1-t} e^{\gamma s'}|v_\tau^{n_k}(t^{n_k}+s')-v_\tau(t^{n_k}+s')|\\
&+\sup_{s'\in[-T,-1-t]} e^{\gamma s'}|v_\tau(t^{n_k}+s')-v_\tau(t+s')| \qquad (6.12)\\
&+\sup_{s'\leqslant-T}\left(e^{\gamma s'}|v_\tau(t^{n_k}+s')-v|+e^{\gamma s'}|v-v_\tau(t+s')|\right)\leqslant\frac{\varepsilon}{4}.
\end{aligned}
$$

and in a similar way,

$$
\sup_{s'\leqslant-1-t} e^{\gamma s'}\|u^{n_k}(\tau+t^{n_k}+s')-u(\tau+t+s')\|\leqslant\frac{\varepsilon}{4}. \qquad (6.13)
$$

Combining (6.11)-(6.13) together, we have

$$
\begin{aligned}
&\left(u^{n_k}_{t^{n_k}+\tau}(\cdot,\tau,\theta_{-\tau}\omega^{n_k},u_\tau^{n_k}),v^{n_k}_{t^{n_k}+\tau}(\cdot,\tau,\theta_{-\tau}\omega^{n_k},v_\tau^{n_k})\right)\\
&\to\left(u_{t+\tau}(\cdot,\tau,\theta_{-\tau}\omega,u_\tau),v_{t+\tau}(\cdot,\tau,\theta_{-\tau}\omega,v_\tau)\right)
\end{aligned}
$$

in E, and consequently,

$$
\begin{aligned}
\xi^{n_k}&=\left(u^{n_k}_{t^{n_k}+\tau}(\cdot,\tau,\theta_{-\tau}\omega^{n_k},u_\tau^{n_k}),v^{n_k}_{t^{n_k}+\tau}(\cdot,\tau,\theta_{-\tau}\omega^{n_k},v_\tau^{n_k})\right)\\
&\to\xi=\left(u_{t+\tau}(\cdot,\tau,\theta_{-\tau}\omega,u_\tau),v_{t+\tau}(\cdot,\tau,\theta_{-\tau}\omega,v_\tau)\right)\\
&\in\Phi(t,\tau,\omega,(u_\tau,v_\tau))\subset O
\end{aligned}
$$

as $k\to\infty$. This is in contradiction with $\xi^{n_k}\notin O$ for all k. Therefore, it only remains to show that there exists a subsequence (u^{n_k},v^{n_k}) of (u^n,v^n) such that

$$
(u^{n_k}(\cdot),v^{n_k}(\cdot))\to(u'(\cdot),v'(\cdot)) \text{ in } C([\tau,1+t+\tau];V)\times C([\tau,1+t+\tau];H) \qquad (6.14)
$$

for some function $(u'(\cdot),v'(\cdot))$ as $k\to\infty$.

Let $u=u_1+u_2$, $v=v_1+v_2$, we decompose Eqs. (3.9)-(3.10) as follows:

$$
\begin{cases}
\dfrac{\partial u_2}{\partial t}+\delta u_2=v_2,\\
\dfrac{\partial v_2}{\partial t}-\delta v_2-\Delta u_2+(\lambda+\delta^2)u_2+J\left(v-\delta u+z(\theta_t\omega)\right)-J\left(v_1-\delta u_1+z(\theta_t\omega)\right)=0,
\end{cases}
\qquad (6.15)
$$

with boundary and initial conditions

$$
\begin{cases}
u_2(t,x)=v_2(t,x)=0,\ t>\tau, x\in\partial D,\\
u_2(t,x)=u_{2,\tau}(t-\tau,x)=\phi(t-\tau,x),\ t\leqslant\tau,\ x\in D,\\
v_2(t,x)=v_{2,\tau}(t-\tau,x)=\dfrac{\partial\phi(t-\tau,x)}{\partial t}+\delta\phi(t-\tau,x)-z(\theta_t\omega),\\
\ t\leqslant\tau,\ x\in D,
\end{cases}
\qquad (6.16)
$$

and the non-homogeneous equations

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + \delta u_1 = v_1 + z(\theta_t\omega), \\ \dfrac{\partial v_1}{\partial t} - \delta v_1 + J\left(v_1 - \delta u_1 + z(\theta_t\omega)\right) - \Delta u_1 + (\lambda+\delta^2)u_1 = f(x, u(t-\rho(t))) \\ +g(x,t) + \displaystyle\int_{-\infty}^{0} F(x,s,u(t+s))ds + (\tilde{\alpha}+\delta)z(\theta_t\omega), \end{cases} \tag{6.17}$$

with boundary and initial conditions

$$\begin{cases} u_1(t,x) = v_1(t,x) = 0,\ t > \tau, x \in \partial D, \\ u_1(t,x) = u_{1,\tau}(t-\tau,x) = 0, \quad t \leqslant \tau,\ x \in D, \\ v_1(t,x) = v_{1,\tau}(t-\tau,x) = 0, \quad t \leqslant \tau,\ x \in D. \end{cases} \tag{6.18}$$

We divide the proof into two steps.

Step 1. Since $(-\Delta)^{-1}$ is a continuous compact operator in H, by the classical spectral theory, there exist a sequence $\{\lambda_j\}_{j=1}^{\infty}$,

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots,\ \lambda_j \to +\infty, \quad \text{as} \quad j \to \infty,$$

and a family of elements $\{e_j\}_{j=1}^{\infty}$ of $D(-\Delta)$ which are orthogonal in H and V but orthonormal in H such that

$$-\Delta e_j = \lambda_j e_j \quad \text{for all } j \in \mathbb{N}.$$

Let H_m=span$\{e_1, e_2, \cdots, e_m\}$, P_m be the canonical projector on H_m and I be the identity. Then for any $u_2 \in V$ and $v_2 \in H$, $u_2 = \hat{u}_2 + \tilde{u}_2$, $v_2 = \hat{v}_2 + \tilde{v}_2$, where $\hat{u}_2 = P_m u_2$, $\tilde{u}_2 = (I - P_m)u_2$, $\hat{v}_2 = P_m v_2$, $\tilde{v}_2 = (I - P_m)v_2$.

We decompose (6.15)-(6.16) as follows:

$$\begin{cases} \dfrac{\partial \hat{u}_2}{\partial t} + \delta \hat{u}_2 = \hat{v}_2, \\ \dfrac{\partial \hat{v}_2}{\partial t} - \delta \hat{v}_2 - \Delta \hat{u}_2 + (\lambda+\delta^2)\hat{u}_2 + P_m J\left(v - \delta u + z(\theta_t\omega)\right) \\ -P_m J\left(v_1 - \delta u_1 + z(\theta_t\omega)\right) = 0, \end{cases} \tag{6.19}$$

with boundary and initial conditions

$$\begin{cases} \hat{u}_2(t,x) = \hat{v}_2(t,x) = 0,\ t > \tau, x \in \partial D, \\ \hat{u}_2(t,x) = \hat{u}_{2,\tau}\left(t-\tau,x\right) = P_m\phi(t-\tau,x),\ t \leqslant \tau,\ x \in D, \\ \hat{v}_2(t,x) = \hat{v}_{2,\tau}(t-\tau,x) = P_m \dfrac{\partial \phi(t-\tau,x)}{\partial t} + \delta P_m \phi(t-\tau,x) - P_m z(\theta_t\omega), \\ \ t \leqslant \tau,\ x \in D, \end{cases} \tag{6.20}$$

and

$$\begin{cases} \dfrac{\partial \tilde{u}_2}{\partial t} + \delta \tilde{u}_2 = \tilde{v}_2, \\ \dfrac{\partial \tilde{v}_2}{\partial t} - \delta \tilde{v}_2 - \Delta \tilde{u}_2 + (\lambda+\delta^2)\tilde{u}_2 + (I-P_m)J\left(v - \delta u + z(\theta_t\omega)\right) \\ -(I-P_m)J\left(v_1 - \delta u_1 + z(\theta_t\omega)\right) = 0, \end{cases} \tag{6.21}$$

with boundary and initial conditions

$$\begin{cases} \tilde{u}_2(t,x)=\tilde{v}_2(t,x)=0,\ t>\tau, x\in\partial D, \\ \tilde{u}_2(t,x)=\tilde{u}_{2,\tau}\,(t-\tau,x)=(I-P_m)\phi(t-\tau,x),\ t\leqslant\tau,\ x\in D, \\ \tilde{v}_2(t,x)=\tilde{v}_{2,\tau}(t-\tau,x)=(I-P_m)\dfrac{\partial\phi(t-\tau,x)}{\partial t}+\delta(I-P_m)\phi(t-\tau,x) \\ \ -(I-P_m)z(\theta_t\omega),\ t\leqslant\tau,\ x\in D, \end{cases} \tag{6.22}$$

Taking the inner product in H of the second equation of (6.21) with $\tilde{v}_2$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\tilde{v}_2|^2-\delta|\tilde{v}_2|^2+(\nabla\tilde{u}_2,\nabla\tilde{v}_2)+\bigg((I-P_m)J(v-\delta u+z(\theta_t\omega)) \\ &\quad-(I-P_m)J(v_1-\delta u_1+z(\theta_t\omega)),\tilde{v}_2\bigg)+(\lambda+\delta^2)(\tilde{u}_2,\tilde{v}_2)=0. \end{aligned} \tag{6.23}$$

Arguing as in (5.8)-(5.11) we deduce that

$$\begin{aligned} &\frac{d}{dt}\big(e^{\alpha t}\left(|\tilde{v}_2|^2+\|\tilde{u}_2\|^2+(\lambda+\delta^2)|\tilde{u}_2|^2\right)\big)+(\beta_1-2\delta-\alpha)e^{\alpha t}|\tilde{v}_2|^2 \\ &\quad+(2\delta-\alpha)e^{\alpha t}\|\tilde{u}_2\|^2+\left(2\delta(\lambda+\delta^2)-\alpha(\lambda+\delta^2)-\frac{\beta_2^2\delta^2}{\beta_1}\right)e^{\alpha t}|\tilde{u}_2|^2\leqslant 0. \end{aligned} \tag{6.24}$$

Integrating (6.24) over (τ,s) with $s\in[\tau,\tau+t+1]$, and replacing ω by $\theta_{-\tau}\omega^n$, in view of (4.3)-(4.5), we obtain that for all $s\in[\tau,\tau+t+1]$ and $n\in\mathbb{N}$,

$$\begin{aligned} &|\tilde{v}_2^n(s,\tau,\theta_{-\tau}\omega^n,\tilde{v}_{2,\tau}^n)|^2+\|\tilde{u}_2^n(s,\tau,\theta_{-\tau}\omega^n,\tilde{u}_{2,\tau}^n)\|^2 \\ &\quad\leqslant Ce^{-\alpha(s-\tau)}\left(|\tilde{v}_2^n(\tau,\tau,\theta_{-\tau}\omega^n,\tilde{v}_{2,\tau}^n)|^2+\|\tilde{u}_2^n(\tau,\tau,\theta_{-\tau}\omega^n,\tilde{u}_{2,\tau}^n)\|^2\right). \end{aligned} \tag{6.25}$$

Recall that $(u_\tau^n,v_\tau^n)\to(u_\tau,v_\tau)$ in E, hence for any $\varepsilon>0$, there exist $\widetilde{N}$, such that for all $n\geqslant\widetilde{N}$,

$$\|v_\tau^n-v_\tau\|_{C_{\gamma,H}}^2+\|u_\tau^n-u_\tau\|_{C_{\gamma,V}}^2\leqslant\varepsilon. \tag{6.26}$$

Since $(u_\tau,v_\tau)\in E$, we can choose m sufficiently large such that

$$|(I-P_m)v(\tau)|^2+\|(I-P_m)u(\tau)\|^2\leqslant\varepsilon, \tag{6.27}$$

By using (6.25)-(6.27), we can take m sufficiently large such that for all $s\in[\tau,\tau+t+1]$ and $n\geqslant\widetilde{N}$,

$$\begin{aligned} &|\tilde{v}_2^n(s,\tau,\theta_{-\tau}\omega^n,\tilde{v}_{2,\tau}^n)|_2^2+\|\tilde{u}_2^n(s,\tau,\theta_{-\tau}\omega^n,\tilde{u}_{2,\tau}^n)\|^2\leqslant C|v^n(\tau)-v(\tau)|^2 \\ &\quad+C\|u^n(\tau)-u(\tau)\|^2+C|(I-P_m)v(\tau)|^2+C\|(I-P_m)u(\tau)\|^2\leqslant C\varepsilon. \end{aligned} \tag{6.28}$$

Now we consider the finite-dimensional system (6.19). Without loss of generality, we assume that s_1, $s_2\in[\tau,\tau+t+1]$ with $0<s_2-s_1<1$. In view of $|-\Delta\hat{u}_2^n|^2\leqslant\lambda_m\|\hat{u}_2^n\|^2\leqslant\lambda_m^2|\hat{u}_2^n|^2$, then by the second equation of (6.19) we have

$$\begin{aligned} &|\hat{v}_2^n(s_2,\tau,\theta_{-\tau}\omega^n,\hat{v}_{2,\tau}^n)-\hat{v}_2^n(s_1,\tau,\theta_{-\tau}\omega^n,\hat{v}_{2,\tau}^n)|\leqslant\int_{s_1}^{s_2}\left|\frac{d\hat{v}_2^n(s',\tau,\theta_{-\tau}\omega^n,\hat{v}_{2,\tau}^n)}{ds'}\right|ds' \\ &=\int_{s_1}^{s_2}\Big|\delta\hat{v}_2^n(s',\tau,\theta_{-\tau}\omega^n,\hat{v}_{2,\tau}^n)-P_mJ(v^n-\delta u^n+z(\theta_t\omega^n))+P_mJ(v_1^n-\delta u_1^n+z(\theta_t\omega^n)) \\ &\quad+\Delta\hat{u}_2^n(s',\tau,\theta_{-\tau}\omega^n,\hat{u}_{2,\tau}^n)-(\lambda+\delta^2)\hat{u}_2^n(s',\tau,\theta_{-\tau}\omega^n,\hat{u}_{2,\tau}^n)\Big|ds' \end{aligned}$$

$$\begin{aligned}
&\leqslant \int_{s_1}^{s_2} \left(\delta|\hat{v}_2^n(s',\tau,\theta_{-\tau}\omega^n,\hat{v}_{2,\tau}^n)| + (\lambda+\delta^2)|\hat{u}_2^n(s',\tau,\theta_{-\tau}\omega^n,\hat{u}_{2,\tau}^n)|\right) ds' \\
&+ \int_{s_1}^{s_2} J'(\zeta_1)\left(|\hat{v}_2^n(s',\tau,\theta_{-\tau}\omega^n,\hat{v}_{2,\tau}^n)| + \delta|\hat{u}_2^n(s',\tau,\theta_{-\tau}\omega^n,\hat{u}_{2,\tau}^n)|\right) ds' \\
&+ \int_{s_1}^{s_2} |\Delta\hat{u}_2^n(s',\tau,\theta_{-\tau}\omega^n,\hat{u}_{2,\tau}^n)| ds' \\
&\leqslant (\delta+\beta_2)\int_{s_1}^{s_2} |\hat{v}_2^n(s',\tau,\theta_{-\tau}\omega^n,\hat{v}_{2,\tau}^n)| ds' \\
&+ (\lambda_m+\beta_2\delta+(\lambda+\delta^2))\int_{s_1}^{s_2} |\hat{u}_2^n(s',\tau,\theta_{-\tau}\omega^n,\hat{u}_{2,\tau}^n)| ds',
\end{aligned} \tag{6.29}$$

where ζ_1 is between $v^n - \delta u^n + z(\theta_t\omega^n)$ and $v_1^n - \delta u_1^n + z(\theta_t\omega^n)$. By the similar arguments as in (6.24) and (6.25), in view of (6.3), we deduce that for all $s \in [\tau, \tau+t+1]$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&|\hat{v}_2^n(s,\tau,\theta_{-\tau}\omega^n,\hat{v}_{2,\tau}^n)|_2^2 + \|\hat{u}_2^n(s,\tau,\theta_{-\tau}\omega^n,\hat{u}_{2,\tau}^n)\|^2 \\
&\leqslant Ce^{-\alpha(s-\tau)}\left(|\hat{v}_2^n(\tau,\tau,\theta_{-\tau}\omega^n,\hat{v}_{2,\tau}^n)|_2^2 + \|\hat{u}_2^n(\tau,\tau,\theta_{-\tau}\omega^n,\hat{u}_{2,\tau}^n)\|^2\right) \\
&\leqslant Ce^{-\alpha(s-\tau)}.
\end{aligned} \tag{6.30}$$

Inserting (6.30) into (6.29) gives

$$|\hat{v}_2^n(s_2,\tau,\theta_{-\tau}\omega^n,\hat{v}_{2,\tau}^n) - \hat{v}_2^n(s_1,\tau,\theta_{-\tau}\omega^n,\hat{v}_{2,\tau}^n)| \leqslant C\left(e^{-\frac{\alpha}{2}s_1} - e^{-\frac{\alpha}{2}s_2}\right), \tag{6.31}$$

for all $n \in \mathbb{N}$ and $s_1,\ s_2 \in [\tau, \tau+t+1]$ with $0 < s_2 - s_1 < 1$. By the first equation of (6.19) and (6.30), we obtain that for all $n \in \mathbb{N}$ and $s_1,\ s_2 \in [\tau,\tau+t+1]$ with $0 < s_2 - s_1 < 1$,

$$\begin{aligned}
&\|\hat{u}_2^n(s_2,\tau,\theta_{-\tau}\omega^n,\hat{u}_{2,\tau}^n) - \hat{u}_2^n(s_1,\tau,\theta_{-\tau}\omega^n,\hat{u}_{2,\tau}^n)\| \\
&\leqslant \sqrt{\lambda_m}|\hat{u}_2^n(s_2,\tau,\theta_{-\tau}\omega^n,\hat{u}_{2,\tau}^n) - \hat{u}_2^n(s_1,\tau,\theta_{-\tau}\omega^n,\hat{u}_{2,\tau}^n)| \\
&\leqslant \sqrt{\lambda_m}\int_{s_1}^{s_2} \left|\frac{d\hat{u}_2^n(s',\tau,\theta_{-\tau}\omega^n,\hat{u}_{2,\tau}^n)}{ds'}\right| ds' \\
&\leqslant \sqrt{\lambda_m}\int_{s_1}^{s_2} (|\hat{v}_2^n(s',\tau,\theta_{-\tau}\omega^n,\hat{v}_{2,\tau}^n)| + \delta|\hat{u}_2^n(s',\tau,\theta_{-\tau}\omega^n,\hat{u}_{2,\tau}^n)|) ds' \\
&\leqslant C\left(e^{-\frac{\alpha}{2}s_1} - e^{-\frac{\alpha}{2}s_2}\right).
\end{aligned} \tag{6.32}$$

Combining (6.28), (6.31) and (6.32) together, we find that

$$\{(u_2^n(\cdot,\tau,\theta_{-\tau}\omega^n,u_{2,\tau}^n), v_2^n(\cdot,\tau,\theta_{-\tau}\omega^n,v_{2,\tau}^n))\}_{n=1}^\infty$$

is a Cauchy sequence in $C([\tau,1+t+\tau];V) \times C([\tau,1+t+\tau];H)$.

Step 2. Arguing as in (5.18), (5.25) and (5.26), in view of (6.6) and the property (**P2**), we deduce that for all $s \in [\tau, 1+t+\tau]$,

$$\begin{aligned}
&|v_1^n(s,\tau,\theta_{-\tau}\omega^n,v_{1,\tau}^n) - v_1^m(s,\tau,\theta_{-\tau}\omega^m,v_{2,\tau}^m)|^2 \\
&+ \|u_1^n(s,\tau,\theta_{-\tau}\omega^n,u_{1,\tau}^n) - u_1^m(s,\tau,\theta_{-\tau}\omega^m,u_{1,\tau}^m)\|^2 \\
&\leqslant C\,\|f(x,u^n(r-\rho(r))) - f(x,u^m(r-\rho(r)))\|_{L^2(D\times[\tau,\tau+t+1])} \\
&\times \left\|v_1^n(r,\tau,\theta_{-\tau}\omega^n,v_{1,\tau}^n) - v_1^m(r,\tau,\theta_{-\tau}\omega^m,v_{1,\tau}^m)\right\|_{L^2(D\times[\tau,\tau+t+1])}
\end{aligned} \tag{6.33}$$

$$+ C \left\| \int_{-\infty}^{0} F(x,s,u^n(r+s))ds - \int_{-\infty}^{0} F(x,s,u^m(r+s))ds \right\|_{L^2(D\times[\tau,\tau+t+1])}$$
$$\times \left\| v_1^n(r,\tau,\theta_{-\tau}\omega^n, v_{1,\tau}^n) - v_1^m(r,\tau,\theta_{-\tau}\omega^m, v_{1,\tau}^m) \right\|_{L^2(D\times[\tau,\tau+t+1])}$$
$$+ C\|z(\theta_{r-\tau}\omega^n) - z(\theta_{r-\tau}\omega^m)\|_{L^2(D\times[\tau,\tau+t+1])}$$
$$\times \left\| v_1^n(r,\tau,\theta_{-\tau}\omega^n, v_{1,\tau}^n) - v_1^m(r,\tau,\theta_{-\tau}\omega^m, v_{1,\tau}^m) \right\|_{L^2(D\times[\tau,\tau+t+1])} \to 0 \text{ as } n,m\to\infty.$$

Therefore, $\{(u^n(\cdot,\tau,\theta_{-\tau}\omega^n,u_\tau^n), v^n(\cdot,\tau,\theta_{-\tau}\omega^n,v_\tau^n))\}_{n=1}^{\infty}$ is a Cauchy sequence in $C([\tau,1+t+\tau];V)\times C([\tau,1+t+\tau];H)$. The proof of this lemma is complete. □

Remark 16. It is worth mentioning that if we reduce the nonlinear term J in (3.9) to the linear case, i.e., $J=\beta_1 v$ for all $v\in\mathbb{R}$, the proof of *Step* 2 in Lemma 15 can be simplified. In fact, by a similar way as in (6.24)-(6.25), we obtain that for all $s\in[\tau,1+t+\tau]$,

$$\begin{aligned}
&|v_2^n(s,\tau,\theta_{-\tau}\omega^n,v_{2,\tau}^n) - v_2^m(s,\tau,\theta_{-\tau}\omega^m,v_{2,\tau}^m)|^2 \\
&+\|u_2^n(s,\tau,\theta_{-\tau}\omega^n,u_{2,\tau}^n) - u_2^m(s,\tau,\theta_{-\tau}\omega^m,u_{2,\tau}^m)\|^2 \\
&\leqslant Ce^{-\alpha(s-\tau)}\left(\|v_\tau^n - v_\tau^m\|_{C_{\gamma,H}}^2 + \|u_\tau^n - u_\tau^m\|_{C_{\gamma,V}}^2\right) \to 0,
\end{aligned}$$

as $n,m\to\infty$, since $(u_\tau^n,v_\tau^n)\to(u_\tau,v_\tau)$ in E.

By slightly modifying the proofs of Lemma 6.5 and Theorem 6.1 in [39], we have

Theorem 17. *Assume that the hypotheses in Lemma 11 hold. Let $\mathcal{A}$ be the $\mathcal{D}$-pullback attractor given in Theorem 14. Then for every fixed $\tau\in\mathbb{R}$, $\mathcal{A}(\tau,\cdot)$ is measurable with respect to the $\mathbb{P}$-completion of $\mathcal{F}$.*

REFERENCES

[1] L. Arnold, Random Dynamical Systems, Springer-Verlag, New York, Berlin, 1998.

[2] J. M. Arrieta, A. N. Carvalho, J. K. Hale, A damped hyperbolic equation with critical exponent, Comm. Partial Differential Equations, 17 (1992), 841-866.

[3] A. V. Babin, M. I. Vishik, Attractors of Evolution Equations, North-Holland, Amsterdam, 1992.

[4] J. M. Ball, Global attractors for damped semilinear wave equations, Discrete Contin. Dyn. Syst., 10 (2004), 31-52.

[5] S. Borini, V. Pata, Uniform attractors for a strongly damped wave equation with linear memory, Asymptot. Anal., 20 (1999), 263-277.

[6] T. Caraballo, P. Kloeden, J. Real, Pullback and forward attractors for a damped wave equation with delays, Stoch. Dyn., 4 (2004), 1-19.

[7] T. Caraballo, M. J. Garrido-Atienza, B. Schmalfuß, J. Valero, Nonautonomous and random attractors for delay random semilinear equations without uniqueness, Discrete Contin. Dyn. Syst., 21 (2008), 415-443.

[8] T. Caraballo, M. J. Garrido-Atienza, B. Schmalfuß, J. Valero, Asymptotic behaviour of a stochastic semilinear dissipative functional equation without uniqueness of solutions, Discrete Contin. Dyn. Syst., 14 (2010), 439-455.

[9] T. Caraballo, A. N. Carvalho, J. A. Langa, F. Rivero, A non-autonomous strongly damped wave equation: existence and continuity of the pullback attractor, Nonlinear Anal., 74 (2011), 2272-2283.

[10] T. Caraballo, F. Morillas, J. Valero, Random attractors for stochastic lattice systems with non-Lipschitz nonlinearity, J. Difference Equ. Appl., 17 (2011), 161-184.

[11] V. V. Chepyzhov, M. I. Vishik, Attractors for Equations of Mathematical Physics, Amer. Math. Soc. Colloq. Publ., vol. 49, American Mathematical Society, Providence, RI, 2002.

[12] H. Crauel, A. Debussche, F. Flandoli, Random attractors, J. Dynam. Differential Equations, 9 (1997), 307-341.
[13] C. M. Dafermos, Asymptotic stability in viscoelasticity, Arch. Ration. Mech. Anal., 37 (1970), 297-308.
[14] M. Fabrizio, A. Morro, Mathematical Problems in Linear Viscoelasticity, SIAM Studies in Applied Mathematics, vol. 12, SIAM, Philadelphia, 1992.
[15] X. Fan, Random attractors for damped stochastic wave equations with multiplicative noise, Internat. J. Math., 19 (2008), 421-437.
[16] E. Feireisl, Attractors for semilinear damped wave equations on $\mathbb{R}^3$, Nonlinear Anal., 23 (1994), 187-195.
[17] F. Flandoli, B. Schmalfuß, Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative noise, Stoch. Stoch. Rep., 59 (1996), 21-45.
[18] J. K. Hale, Asymptotic Behavior of Dissipative Systems, American Mathematical Society, Providence, RI, 1988.
[19] M. He, A. Liu, The oscillation of hyperbolic functional differential equations, Appl. Math. Comput., 142 (2003), 205-224.
[20] R. Jones, B. Wang, Asymptotic behavior of a class of stochastic nonlinear wave equations with dispersive and dissipative terms, Nonlinear Anal. Real World Appl., 14 (2013), 1308-1322.
[21] A. Kh. Khanmamedov, Global attractors for wave equations with nonlinear interior damping and critical exponents, J. Differential Equations, 230 (2006), 702-719.
[22] V. Kolmanovskii, A. Myshkis, Introduction to the Theory and Applications of Functional Differential Equations, Kluwer Academic Publishers, Dordrecht, 1999.
[23] I. Kucuk, I. Sadek, Y. Yilmaz, Active control of a smart beam with time delay by Legendre wavelets, Appl. Math. Comput., 218 (2012), 8968-8977.
[24] H. Li, Y. You, J. Tu, Random attractors and averaging for non-autonomous stochastic wave equations with nonlinear damping, J. Differential Equations, 258 (2015), 148-190.
[25] F. Morillas, J. Valero, Attractors for reaction-diffusion equations in $\mathbb{R}^N$ with continuous nonlinearity, Asymptot. Anal., 44 (2005), 111-130.
[26] S. Nicaise, C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, SIAM J. Control Optim., 45 (2006), 1561-1585.
[27] V. Pata, Attractors for a damped wave equation on $\mathbb{R}^3$ with linear memory, Math. Methods Appl. Sci., 23 (2000), no. 7, 633-653.
[28] N. Raskin, Y. Halevi, Control of flexible structures governed by the wave equation, in: Proceedings of the American Control Conference, Arlington, VA, 2001, pp. 2486-2491.
[29] R. Sell, Y. You, Dynamics of Evolutionary Equations, Springer-Verlag, New York, 2002.
[30] Z. Shen, S. Zhou, W. Shen, One-dimensional random attractor and rotation number of the stochastic damped sine-Gordon equation, J. Differential Equations, 248 (2010), 1432-1457.
[31] C. Sun, M. Yang, C. Zhong, Global attractors for the wave equation with nonlinear damping, J. Differential Equations, 227 (2006), 427-443.
[32] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, New York, 1997.
[33] B. Wang, Asymptotic behavior of stochastic wave equations with critical exponents on $\mathbb{R}^3$, Trans. Amer. Math. Soc., 363 (2011), 3639-3663.
[34] B. Wang, Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, J. Differential Equations, 253 (2012), 1544-1583.
[35] B. Wang, Random attractors for non-autonomous stochastic wave equations with multiplicative noise, Discrete Contin. Dyn. Syst., 34 (2014), 269-300.
[36] B. Wang, Multivalued non-autonomous random dynamical systems for wave equations without uniqueness, Discrete Contin. Dyn. Syst., Ser. B 22 (2017), 2011-2051.
[37] Y. J. Wang, Y. Z. Qin, J. Y. Wang, Pullback attractors for a strongly damped delay wave equation in $\mathbb{R}^n$, Stoch. Dyn., Vol. 18, No. 3 (2018), 1850016, (24 pages).
[38] J. Wang, F. Meng, S. Liu, Integral average method for oscillation of second order partial differential equations with delays, Appl. Math. Comput., 187 (2007), 815-823.
[39] J. Y. Wang, Y. J. Wang, D. Zhao, Pullback attractors for multi-valued non-compact random dynamical systems generated by semi-linear degenerate parabolic equations with unbounded delays, Stoch. Dyn., 17 (2016), 1750001, 49 pp.

[40] Y. J. Wang, J. Y. Wang, Pullback attractors for multi-valued non-compact random dynamical systems generated by reaction-diffusion equations on an unbounded domain, J. Differential Equations, 259 (2015), 728-776.
[41] Y. J. Wang, S. F. Zhou, Kernel sections and uniform attractors of multi-valued semiprocesses, J. Differential Equations, 232 (2007), 573-622.
[42] Y. J. Wang, Pullback attractors of a damped wave equation with delays, Stoch. Dyn., 15 (2015), 1550003, 21 pp.
[43] Z. Wang, S. Zhou, A. Gu, Random attractor for a stochastic damped wave equation with multiplicative noise on unbounded domains, Nonlinear Anal. Real World Appl., 12 (2011), 3468-3482.
[44] J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer-Verlag, New York, 1996.
[45] M. Yang, J. Duan, P. Kloeden, Asymptotic behavior of solutions for random wave equations with nonlinear damping and white noise, Nonlinear Anal. Real World Appl., 12 (2011), 464-478.
[46] Y. You, Global dynamics of nonlinear wave equations with cubic non-monotone damping, Dyn. Partial Differ. Equ., 1 (2004), 65-86.
[47] S. Zelik, Asymptotic regularity of solutions of a non-autonomous damped wave equation with a critical growth exponent, Commun. Pure Appl. Anal., 3 (2004), 921-934.
[48] S. Zhou, M. Zhao, Random attractors for damped non-autonomous wave equations with memory and white noise, Nonlinear Anal., 120 (2015), 202-226.

E-mail address: wangjy@sxufe.edu.cn
E-mail address: wangyj@lzu.edu.cn
E-mail address: caraball@us.es