Reference Dependent Invariant Sets: Sum of Squares Based Computation and Applications in Constrained Control ⋆

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Abstract

The goal of this paper is to present a systematic method to compute reference dependent positively invariant sets for systems subject to constraints. To this end, we first characterize these sets as level sets of reference dependent Lyapunov functions. Based on this characterization and using Sum of Squares theory, we provide a polynomial certificate for the existence of such sets. Subsequently, through some algebraic manipulations, we express this certificate in terms of a Semi-Definite Programming problem which maximizes the size of the resulting reference dependent invariant sets. We then present some results implementing the proposed method to an example and propose some variants that may help in reducing possible numerical issues. Finally, the proposed approach is employed in the Model Predictive Control for Tracking scheme to compute the terminal set, and in the Explicit Reference Governor framework to compute the so-called Dynamic Safety Margin. The effectiveness of the proposed method in each of the schemes is demonstrated through simulation studies.

Key words: Invariance, Control of Constrained Systems, Sum of Squares, Tracking, Reference Dependence.

1 Introduction

The relevance of positively invariant sets lies in their numerous applications [1]. Given an autonomous dynamical system, a subset of the state space is said to be *positively* invariant if, assuming it contains the state of the system at some time, it will also contain it in the future. Invariance is particularly important in the control of dynamical systems subject to constraints. In particular, in Model Predictive Control (MPC) [2], the use of an invariant set as a terminal constraint is typically used to ensure stability and recursive feasibility. Additionally, invariant sets are at the basis of Reference Governor (RG) [3] approaches, of the recently introduced Explicit Reference Governors (ERG) [4], and of multimode regulators for constrained control [5].

The computation of positively invariant sets has been the object of many research works. In [6], the authors compute polyhedral invariant sets for switched linear systems. In [7], a method to compute a polyhedral invariant set for linear systems with polytopic uncertainty is presented; for what concerns input saturated systems, in [8] a novel concept of invariance for saturated systems is introduced. For an extensive survey on research in invariant sets and their usage in control, the reader is referred to [9]. The majority of the work present in the literature focuses on finding domains of attraction for a single point of equilibrium. However, in the realm

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of reference tracking, the required invariant sets need to be centered around any admissible point of equilibrium, and thus become parameterized in the reference. These sets are much less straightforward to be computed explicitly.

To the best of our knowledge, the first instance of this concept dates to the Maximal Output Admissible Sets [10]. These sets are defined as the sets of pairs of initial states and references such that the trajectories of the system fulfills the constraints when said reference is kept constant. For discrete time systems, polyhedral representations of these sets can be computed for linear [11] and nonlinear systems [12], or characterized using Lyapunov arguments [13]. Due to the link between the Maximal Output Admissible Sets and the Dynamic Safety Margin (DSM) in the ERG framework, this line of research has seen a recent increase in activity. In [14], a closed form solution of the optimal DSM for linear systems subject to linear constraints is provided. In [15], the authors present a method to estimate online the DSM in the case of convex Lyapunov functions and concave constraints. A method able to work with unions and intersections of concave constraints has been presented in [16].

A promising tool to tackle the computation of reference dependent invariant sets is Sum of Squares (SOS) programming [17]. In recent years the SOS framework has been used extensively to tackle invariance-related problems [18–21].

In this paper we propose to employ the SOS framework to compute reference dependent invariant sets for constrained systems. The main idea is to fix a polynomial Lyapunov function parameterized in the reference, and through SOS arguments compute the largest level set fully contained in the constraints. The effectiveness of the proposed method is showcased with two applications: a discrete time system controlled with an MPC for Tracking, and a continuous time system controlled with an ERG.

Notation: The set of polynomials with variables $x_1, \ldots, x_n$, whose coefficients belong to $\mathbb{R}$, is denoted by $\mathbb{R}[x_1, \ldots, x_n]$. We denote the set of all non-negative reals, and the set of all non-negative integers as $\mathbb{R}_{\geq 0}$ and $\mathbb{Z}_{\geq 0}$, respectively. For polynomials p_j, $j = 1, \ldots, N$, we will use $\{p_j\}_{i=1}^N$ to denote the set $\{p_1, \ldots, p_N\}$. The set of all Sum of Squares polynomials with variables $x_1, x_2, \ldots, x_n$ is denoted by $\Sigma[x_1, x_2, \ldots, x_n]$. We denote the degree of a polynomial p by ∂p. The n-dimensional identity matrix is denoted by I_n. The Jacobian matrix of a vector valued function f is denoted by ∇f. The gradient of a scalar function V is denoted as ∇V. For two sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$, $\mathcal{A} \ominus \mathcal{B}$ denotes the Pontryagin difference. We denote the n-dimensional ball of radius ε by $\mathcal{B}_n(\varepsilon) \triangleq \{x \in \mathbb{R}^n : x^{\mathrm{T}} x \leq \varepsilon^2\}$. For two matrices A and B, their Kronecker product is denoted by $A \otimes B$. For a vector u and a matrix Q we denote $u^{\mathrm{T}} Q u$ as $\|u\|_Q^2$.

2 Problem Statement

Consider the system

$$\delta x = \phi(x, u), \tag{1}$$

where δx is the successor state in discrete time or the time derivative in continuous time, $x \in \mathbb{R}^n$ is the state, and $u \in \mathbb{R}^m$ is the input. The system is subject to constraints in the following form:

$$x \in \mathcal{X},\ u \in \mathcal{U},$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ are simply connected sets with nonempty interiors.

As often done in reference tracking problems, it is assumed that a control law $u = \kappa(x, r)$ is used to stabilize [1] the system, where $r \in \mathbb{R}^p$ is the reference signal. System (1) then becomes

$$\delta x = f(x, r), \tag{2}$$

where $f(x, r) \triangleq \phi(x, \kappa(x, r))$ and the pair (x, r) is constrained in the set $\mathcal{D} \triangleq \{(x, r) : x \in \mathcal{X},\ \kappa(x, r) \in \mathcal{U}\}$. Typically, this set can be expressed through a set of inequalities as

$$\mathcal{D} = \{(x, r) : c_i(x, r) \geq 0,\ i = 1, \ldots, n_c\}. \tag{3}$$

In this paper we assume that these inequalities are polynomial, *i.e.* $c_i(x, r) \in \mathbb{R}[x, r],\ i = 1, \ldots, n_c$ (see Fig. 1). We also define the set of admissible references

$$\mathcal{R} \triangleq \{r \in \mathbb{R}^p : \overline{c}_i(r) \geq 0,\ i = 1, \ldots, n_c\},$$

where $\overline{c}_i(r) \triangleq c_i(\overline{x}_r, r)$, with $\overline{x}_r$ as the equilibrium point of (2) associated with a constant reference signal r , *i.e.*, $f(\overline{x}_r, r) = 0$ in continuous time, and $f(\overline{x}_r, r) = \overline{x}_r$ in discrete time. Furthermore, $\mathcal{R}$ is assumed to be connected.

The purpose of this paper is to solve the following problem:

Problem 1 *(Safe Reference Dependent Positively Invariant Sets) Consider system* (2) *where the pair* (x, r) *is constrained in the set* $\mathcal{D}$ *as in* (3). *Compute a family of sets parameterized in the reference* $\mathcal{S}(r)$ *such that*

[1] The stabilization of unconstrained nonlinear systems is the subject of an extensive literature (*e.g.*, [22,23]), and can be approached using a variety of available control tools.

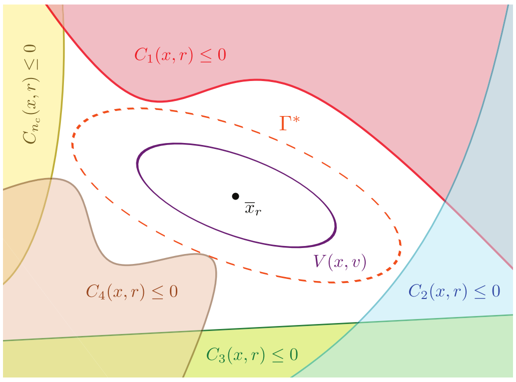

Fig. 1. Geometric illustration of the class of constraints under study.

- $\mathcal{S}(r)$ *is invariant for system* (2) *for any constant* $r \in \mathcal{R}$, *i.e., every trajectory of* (2) *with initial condition* $x(t_0) \in \mathcal{S}(r)$ *and a constant reference* $r \in \mathcal{R}$ *is such that* $x(t) \in \mathcal{S}(r)$ *for all* $t \geq t_0$,
- $\mathcal{S}(r)$ *is fully contained in the constraint set* $\mathcal{D}$ *for every* $r \in \mathcal{R}$.

Remark 2 *Contrary to regulation problems where a single invariant set centered around the origin is computed, in reference tracking we need to compute invariant sets centered around all possible points of equilibrium* $\overline{x}_r$ *associated with every* $r \in \mathcal{R}$.

It is well known that one way to determine invariant sets is using Lyapunov level sets. In this paper we will assume, without loss of generality, that for each reference r, the corresponding equilibrium point of system (2) denoted by $\overline{x}_r$ is globally asymptotically stable (see Remark 3 for local stability). We will also assume that a polynomial Reference Dependent Lyapunov Function (RDLF) $V(x,r) : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}_{\geq 0}$ is known such that:

$$\begin{array}{l} V(x,r) > 0 \;\forall x \in \mathbb{R}^n \setminus \{\overline{x}_r\},\; V(x,r) = 0 \Leftrightarrow x = \overline{x}_r \\ \delta V(x,r) < 0 \;\forall x \in \mathbb{R}^n \setminus \{\overline{x}_r\},\; \delta V(x,r) = 0 \Leftrightarrow x = \overline{x}_r. \end{array} \tag{4}$$

where $\delta V(x,r)$ denotes $\nabla V(x,r) f(x,r)$ in the continuous time case, and $V(f(x,r),r) - V(x,r)$ in the discrete time case. It is easy to see that given $\Gamma(r) : \mathbb{R}^p \rightarrow \mathbb{R}$ with $\Gamma(r) > 0 \;\forall r \in \mathcal{R}$ if the level set $\mathcal{S}_\Gamma(r) = \{x \in \mathbb{R}^n : V(x,r) \leq \Gamma(r),\, \Gamma(r) > 0\},\; \forall r \in \mathcal{R}$ is fully contained in $\mathcal{D}$, then $\mathcal{S}_\Gamma(r)$ is a safe positively invariant set.

Typically it is useful to determine the largest possible safe invariant set. In the case of invariant sets based on Lyapunov functions, this corresponds to finding the largest bound $\Gamma^*(r)$ corresponding to the solution of the following optimization problem:

$$\Gamma^*(r) = \left\{ \begin{array}{ll} & \max \Gamma \\ \text{s.t.} & \\ & \{x \in \mathbb{R}^n : V(x,r) \leq \Gamma\} \subseteq \mathcal{D} \end{array} \right. , \tag{5}$$

for every $r \in \mathcal{R}$. Note that this optimization problem is parameterized in r and that, except for some very special cases [14], its closed form parametric solution might be hard to compute and/or to handle.

In this paper we propose a systematic method to compute a good polynomial approximation of $\Gamma^*(r)$, denoted by $\hat{\Gamma}(r) \in \mathbb{R}[r]$ such that $\hat{\Gamma}(r) \leq \Gamma^*(r),\; \forall r \in \mathcal{R}$. Note that for any lower bound $\hat{\Gamma}(r) \leq \Gamma^*(r)$, the set $\mathcal{S}_{\hat{\Gamma}}(r)$ is a safe reference dependent invariant set.

For the sake of simplicity, in the sequel we will compute one $\hat{\Gamma}(r)$ at a time. This is without loss of generality, since the multi-constraint case can be expressed as the composition of single constraint cases as follows:

$$\Gamma^*(r) = \min_i \{\Gamma_i^*(r)\},\; i = 1, \ldots, n_c,$$

where $\Gamma_i^*(r)$ is the value of $\Gamma^*(r)$ if only the i-th constraint $c_i(x,r)$ is considered.

Remark 3 *In the case of local stability, it is possible to impose the additional constraint* $\delta V(x,r) < 0$ *in* (5), *which will yield the largest safe level set of the local RDLF.*

3 Approximating the Largest positively Invariant Set via SOS Techniques

In this paper we propose a method to compute a parameterized solution for (5) using SOS programming. This framework is able to tackle convex relaxations of nonconvex optimization problems through polynomial optimization [24]. The following theorem presents the main contribution of this paper. The theorem states that it is possible to find a good solution to Problem 1 by solving a Semi-Definite Programming (SDP) optimization problem.

Theorem 4 *Consider system* (2) *subject to* $c_j(x,r) \geq 0$, $j = 1, \ldots, n_c$ *and assume that a polynomial RDLF* $V(x,r) \in \mathbb{R}[x,r]$ *is known. Let* $\mathcal{D} = \{(x,r) : c_i(x,r) \geq 0\}$, $\mathcal{R} = \{r : \overline{c}_j(r) \geq 0,\, j = 1, \ldots, n_c\}$, $\hat{\Gamma}_i(r) \in \mathbb{R}[r]$ *and* $\mathcal{S}_{\hat{\Gamma}_i}(r) = \{x \in \mathbb{R}^n : V(x,r) \leq \hat{\Gamma}(r)\}$. *Then* $\mathcal{S}_{\hat{\Gamma}_i}(r) \subseteq \mathcal{D}$, $\forall r \in \mathcal{R}$, *if there exist* $q \in \mathbb{R}[x,r]$, $\{s_j\}_{j=1}^{n_c} \in \Sigma[x,r]$ *such that*

$$V - \hat{\Gamma}_i + q \cdot c_i - \sum_{j=1}^{n_c} s_j \cdot \overline{c}_j \in \Sigma[x,r]. \tag{6}$$

PROOF. The condition an admissible $\hat{\Gamma}_i(r)$ must fulfill is

$$\left\{(x,r) : V(x,r) \leq \hat{\Gamma}_i(r)\right\} \subseteq \mathcal{D},\; \forall r \in \mathcal{R}. \tag{7}$$

Reformulating (7) as a set emptiness condition yields

$$\left\{(x,r) : r \in \mathcal{R},\, c_i(x,r) = 0,\, V(x,r) > \hat{\Gamma}_i(r)\right\} = \emptyset,$$

which in turn is equivalent to [2]

$$\left\{(x,r) : V - \hat{\Gamma}_i \geq 0,\, V - \hat{\Gamma}_i \neq 0,\, c_i = 0,\, \{\bar{c}_j\}_{j=1}^{n_c} \geq 0\right\} = \emptyset. \tag{8}$$

At this point, the Krivine–Stengle Positivstellensatz (P-satz) [21,25] states that (8) is empty if and only if there exist polynomials g, h, l such that

$$g + h^2 + l = 0,$$

where $g \in \mathbf{K}(\{V - \hat{\Gamma}_i,\, \{\bar{c}_j\}_{j=1}^{n_c}\})$, $h \in \mathbf{M}(V - \hat{\Gamma}_i)$, and $l \in \mathbf{I}(c_i)$. For the definitions of the algebraic structures $\mathbf{M}(\cdot)$, $\mathbf{I}(\cdot)$ and $\mathbf{K}(\cdot)$ please refer to Appendix A. Directly applying the definitions of these algebraic structures leads to

$$\begin{aligned} g = {} & \sigma_0 + \sigma_1 \left(V - \hat{\Gamma}_i\right) + \sigma_2 \bar{c}_1 + \ldots + \sigma_{n_c+1} \bar{c}_{n_c} + \\ & \sigma_{n_c+2} \left(V - \hat{\Gamma}_i\right) \bar{c}_1 + \sigma_{n_c+3} \left(V - \hat{\Gamma}_i\right) \bar{c}_2 + \ldots + \\ & \sigma_{2n_c+1} \left(V - \hat{\Gamma}_i\right) \bar{c}_{n_c} + \ldots \\ h = {} & \left(V - \hat{\Gamma}_i\right)^k \\ l = {} & \bar{q} c_i, \end{aligned}$$

where $\sigma_j \in \Sigma[x,r]$, $k \in \mathbb{Z}_{\geq 0}$, and $\bar{q} \in \mathbb{R}[x,r]$. Setting to zero every σ_j that is either not factored in $V - \hat{\Gamma}_i$ or has a power higher than 1 yields the following sufficient condition

$$\begin{aligned} - s_0 \cdot \left(V - \hat{\Gamma}_i\right) + \left(V - \hat{\Gamma}_i\right)^{2k} + \bar{q} \cdot c_i & \\ - \left(V - \hat{\Gamma}_i\right) \sum_{j=1}^{n_c} s_j \cdot \bar{c}_j = 0, & \end{aligned} \tag{9}$$

where the SOS multipliers σ_j have been renamed as $\{s_j\}_{j=0}^{n_c}$ for simplicity. Imposing $\bar{q} = q\left(V - \hat{\Gamma}_i\right)$ with $q \in \mathbb{R}[x,r]$ and $k = 1$ it follows that (8) is empty if

$$\left(V - \hat{\Gamma}_i\right)\left(-s_0 + \left(V - \hat{\Gamma}_i\right) + q \cdot c_i - \sum_{j=1}^{n_c} s_j \cdot \bar{c}_j\right) = 0,$$

which allows us to formulate the following condition

$$s_0 = V - \hat{\Gamma}_i + q \cdot c_i - \sum_{j=1}^{n_c} s_j \cdot \bar{c}_j. \tag{10}$$

Since s_0 is an SOS polynomial, (10) implies that $\hat{\Gamma}_i$ is a lower bound of Γ_i^* if there exist SOS polynomials $s_1, \ldots, s_{n_c}$ and a polynomial q such that (6) holds, which concludes the proof. □

Since in this paper we are interested in the largest safe level set of V, a possible way to do so is to maximize the integral of $\hat{\Gamma}_i$ over $\mathcal{R}_d \subseteq \mathcal{R}$, a compact domain chosen so as to avoid improper integrals in the case where $\mathcal{R}$ is unbounded. Additionally, $\mathcal{R}_d$ can be chosen so as to prioritize the accuracy of $\hat{\Gamma}_i$ in a certain region of interest of $\mathcal{R}$, *e.g.*, around likely operation points of the system.

Once the structure of the polynomials $\hat{\Gamma}_i$, q, and $\{s_j\}_{j=1}^{n_c}$ is set, it is possible to use Theorem 4 to formulate the following SOSP optimization problem

$$\begin{cases} \quad \max \int_{r \in \mathcal{R}_d} \hat{\Gamma}_i(r)\, \mathrm{d}r \\ \text{s.t.} \\ \quad V - \hat{\Gamma}_i + q \cdot c_i - \sum_{j=1}^{n_c} s_j \cdot \bar{c}_j \in \Sigma[x,r] \\ \quad \{s_j\}_{j=1}^{n_c} \in \Sigma[x,r] \\ \quad q \in \mathbb{R}[x,r] \end{cases}, \tag{11}$$

which was proven to be equivalent to an SDP optimization problem in [17].

Remark 5 *Note that whenever $\mathcal{R}_d$ is a normal domain [26] described by polynomials, $\int_{r \in \mathcal{R}_d} \hat{\Gamma}_i(r)\, \mathrm{d}r$ can easily be computed in closed form and is polynomial [27], which implies that the objective function of* (11) *remains linear in the coefficients of $\hat{\Gamma}_i$. When this is not the case, a practical approach is to randomly select a (possibly large) number of points in $\mathcal{R}_d$, $p_1, \ldots, p_{n_w}$, and use the following objective function*

$$\frac{1}{n_w} \sum_{w=1}^{n_w} \hat{\Gamma}_i(p_w). \tag{12}$$

Note that for a sufficiently large n_w, optimizing over (12) *is equivalent to optimizing over $\int_{\mathcal{R}_d} \hat{\Gamma}_i(r)\, \mathrm{d}r$.*

4 Example and practical considerations

It is widely known that the LMI problems that arise from SOS theory can become exceedingly sparse [28] and numerically ill-conditioned. This may limit the applicability of the method presented in Section 3. In this section

[2] For brevity, we will omit the arguments of functions whenever there is no risk of confusion.

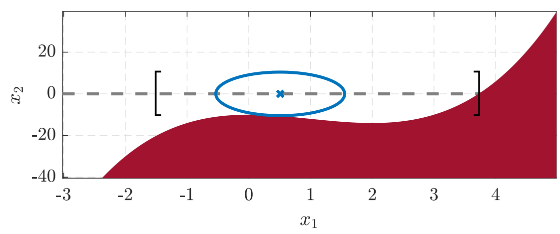

Fig. 2. Visual representation of the case study. The set of possible points of equilibrium of system (13) is depicted as a grey dashed line, the set of points where $c(x,r) < 0$ is represented by a solid dark red shaded area, and the set of points within $\mathcal{R}_d$ are encapsulated by black square brackets. For a given $r = \tilde{r}$, the set of points such that $V(x,\tilde{r}) = \Gamma^*$ is represented as a solid blue ellipsoid, and the steady state associated to $\tilde{r}$, $\overline{x}_{\tilde{r}}$, is represented by a blue cross-shaped marker.

we propose two techniques that might aid in mitigating numerical issues in the proposed methodology. The first technique is based on exploiting the structure of the problem at hand to utilize some *a priori* knowledge in the structure of $\hat{\Gamma}(r)$. The second technique is based on a division of the domain $\mathcal{R}_d$.

In order to show the practical improvements due to the application of these two techniques, we introduce a numerical case study consisting of a continuous time double integrator controlled with a PD control law

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix} x + \begin{bmatrix} 0 \\ \omega^2 \end{bmatrix} r, \tag{13}$$

where $\omega = 10$ is the angular frequency and $\zeta = 0.2$ is the damping ratio. This system is subject to the constraint

$$c(x,r) = x_2 - x_1^3 + 3x_1^2 + 10 \geq 0. \tag{14}$$

Since $\overline{x}_r = [r\ 0]^{\mathrm{T}}$, constraint (14) defines $\mathcal{R}$ as $\mathcal{R} = \{r : r \leq 3.721\}$. The domain $\mathcal{R}_d$ is chosen as $\mathcal{R}_d = \{r : -1.5 \leq r \leq 3.721\}$. For every equilibrium point, stability can be proved using the following quadratic RDLF

$$V(x,r) = (x - \overline{x}_r)^{\mathrm{T}} \begin{bmatrix} 12.6450 & -0.005 \\ -0.005 & 0.1263 \end{bmatrix} (x - \overline{x}_r)\,.$$

The problem is depicted in Fig. 2. In Fig. 3 we show the results obtained by solving (11) with constraint (14) for several values of $\partial\hat{\Gamma}$. The obtained results are compared with $\Gamma^*(r)$ which was computed using a 100-point grid with a resolution of 0.0527, which was made possible due to the low dimensionality of the system. As expected, the accuracy of the approximation increases with the degree of $\hat{\Gamma}$. However, as it can be seen in Fig. 3, a relatively high value of $\partial\hat{\Gamma}$ is required to get a fairly good

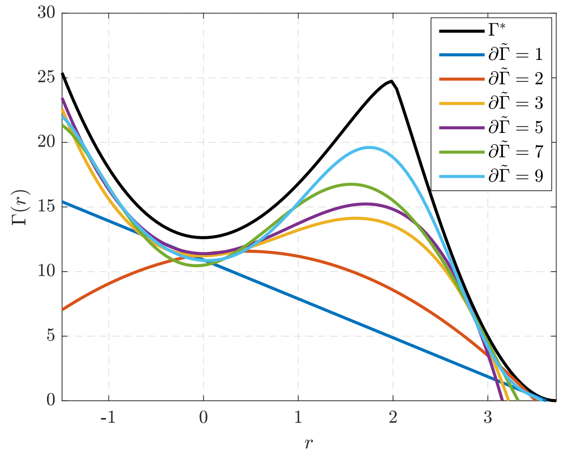

Fig. 3. Approximation of Γ^* for different values of $\partial\hat{\Gamma}$.

approximation. Furthermore, in this particular case, the optimizer is not able to find a solution for $\partial\hat{\Gamma} > 15$. In the following subsection we will introduce a method to increase the accuracy of $\hat{\Gamma}$ without increasing $\partial\hat{\Gamma}$.

4.1 Exploiting the structure of $\Gamma^*(r)$

A first way to reduce the required degree of $\hat{\Gamma}_i(r)$ consists in exploiting the fact that the set $\{r : \Gamma_i^*(r) = 0\}$ coincides with the set $\{r : \overline{c}_i(r) = 0\}$. This is proved by the following lemma.

Lemma 6 *Let system* (2) *subject to the constraint* $c_i(x,r) \geq 0$, *and whose global stability can be proved through an RDLF* $V(x,r)$. *Then for any* $r \in \mathcal{R}$, $\overline{c}_i(r) > 0$ *implies that* $\Gamma_i^*(r) > 0$, *and* $\overline{c}_i(r) = 0$ *implies that* $\Gamma_i^*(r) = 0$.

PROOF. Let $r' \in \mathcal{R}$ be such that $\overline{c}_i(r') > 0$ (*i.e.*, r' is in the interior of $\mathcal{R}$) and $\overline{x}_{r'}$ its associated steady-state. The definition of $\Gamma_i^*(r)$ in (5) implies that

$$\exists x' \in \{x : V(x,r') = \Gamma_i^*(r')\}, \text{ such that } c_i(x',r') = 0. \tag{15}$$

Since from the definition of RDLF $V(x,r') > 0$ if $x \neq \overline{x}_{r'}$, it follows from (15) that $\Gamma_i^*(r') > 0$, and finally that $\overline{c}_i(r') > 0 \Leftrightarrow \Gamma_i^*(r') > 0$. Using the same logic, it follows that $\Gamma_i^*(r'') = 0 \Leftrightarrow \overline{c}_i(r'') = 0$ (*i.e.*, r'' is on the border of $\mathcal{R}$). □

We can use the results of Lemma 6 to our advantage by using the following structure for $\hat{\Gamma}_i$

$$\hat{\Gamma}_i = \overline{c}_i^k \cdot \tilde{\Gamma}_i, \tag{16}$$

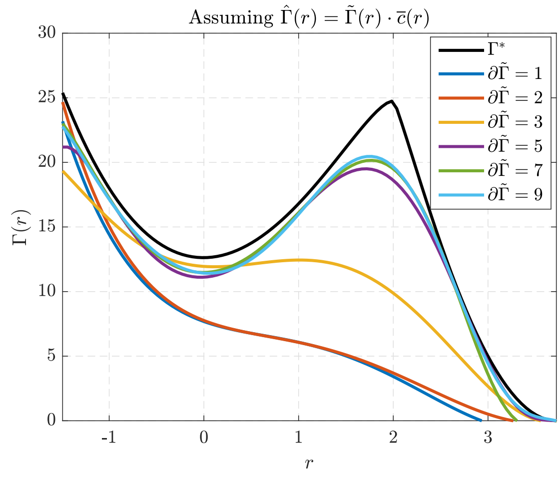

Fig. 4. Approximation of Γ^* for different values of $\partial\tilde{\Gamma}$, assuming $\hat{\Gamma} = \tilde{\Gamma} \cdot \overline{c}$.

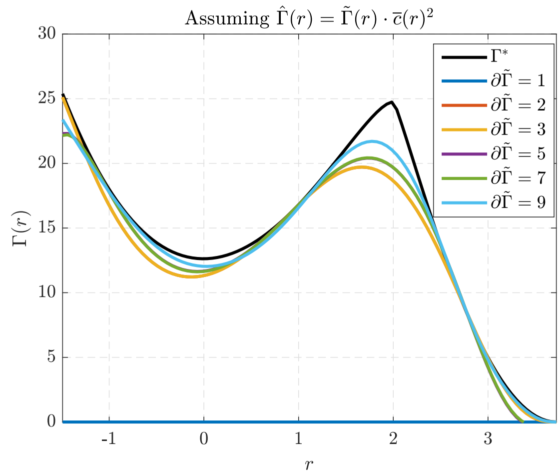

Fig. 5. Approximation of Γ^* for different values of $\partial\tilde{\Gamma}$, assuming $\hat{\Gamma} = \tilde{\Gamma} \cdot \overline{c}^2$. Note that the curve corresponding to $\partial\tilde{\Gamma} = 2$ overlaps with $\partial\tilde{\Gamma} = 3$, and $\partial\tilde{\Gamma} = 5$ overlaps with $\partial\tilde{\Gamma} = 7$.

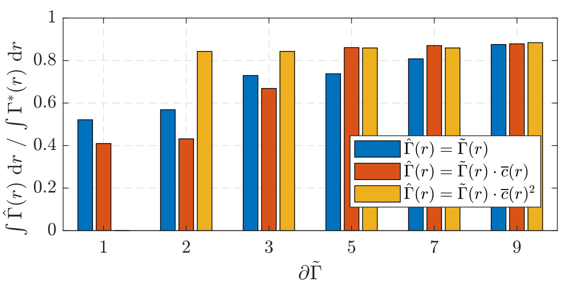

Fig. 6. Accuracy of $\hat{\Gamma} = \overline{c}^k \cdot \tilde{\Gamma}$ with respect to the degree of $\tilde{\Gamma}$ for $k \in \{0, 1, 2\}$.

where $k \in \mathbb{Z}_{\geq 0}$ and $\tilde{\Gamma}_i \in \mathbb{R}[r]$. Intuitively, the use of this term may require a lower $\partial\tilde{\Gamma}_i$ to achieve a higher accuracy, since assuming that $\hat{\Gamma}_i$ is factorized in $\overline{c}_i$ automatically sets $\hat{\Gamma}_i$ to 0 wherever $\overline{c}_i$ vanishes. In Fig. 4 and 5 we show the results of applying this notion to our case study, assuming that $\hat{\Gamma}$ is factorized in $\overline{c}$ and $\overline{c}^2$, respectively. As it can be seen in Fig. 6, when assuming $\hat{\Gamma} = \tilde{\Gamma} \cdot \overline{c}$ the required degree to obtain an accuracy of more than 80% is $\partial\tilde{\Gamma} = 5$, compared to the required degree of $\partial\hat{\Gamma} = 7$ if this assumption is not made. The required degree is further decreased to $\partial\tilde{\Gamma} = 2$ if we assume $\hat{\Gamma} = \tilde{\Gamma} \cdot \overline{c}^2$. In these cases, the optimizer fails to find a sensible solution for $\partial\tilde{\Gamma} > 10$ for the first case, and for $\partial\tilde{\Gamma} > 11$ in the second one.

4.2 Piece-wise polynomial approach

In some cases the SDP solver may not reach a sensible approximation of Γ_i^* due to the impossibility of $\hat{\Gamma}$ to fulfill the required characteristics in a large domain. To tackle this, a possible approach is to divide $\mathcal{R}_d$ in several subsets $\mathcal{R}_\ell,\ \ell = 1, \ldots, n_r$ such that $\mathcal{R}_d = \bigcup_{\ell=1}^{n_r} \mathcal{R}_\ell$.

These subsets are described by the additional constraints $\overline{c}_j^\ell(r)$, with $j = 1, \ldots, n_\ell$. Subsequently the polynomial approximation of Γ^* associated with the ℓ-th subset $\mathcal{R}_\ell$, denoted by $\hat{\Gamma}_{i,\ell}$, can be computed through the following SOS condition

$$V - \hat{\Gamma}_{i,\ell} + q \cdot c_i - \sum_{j=1}^{n_\ell} s_j \cdot \overline{c}_j \in \Sigma[x, r],$$

Once the $\hat{\Gamma}_{i,\ell}$ have been computed, $\hat{\Gamma}_i(r)$ can be recovered as follows:

$$\hat{\Gamma}_i(r) = \max_\ell \{\hat{\Gamma}_{i,\ell}(r)\},$$

where $\hat{\Gamma}_{i,\ell} = 0\ \forall r \notin \mathcal{R}_\ell$. By dividing $\mathcal{R}_d$, not only we reduce the size of the domain over which we compute each $\hat{\Gamma}_{i,\ell}$, but also the term $\sum_{j=1}^{n_\ell} s_j \cdot \overline{c}_j$ becomes simpler since typically $n_c > n_\ell$.

A possible approach for the division of $\mathcal{R}_d$ is the Delaunay tessellation [29,30], which divides the space into p-simplices. It is well known that a p-simplex requires $p+1$ inequalities to be described, therefore, this tessellation achieves the minimal amount of inequalities describing a given inner $\mathcal{R}_\ell$ for a polyhedral division of $\mathcal{R}_d$.

We show the results of applying this approach to the case study in Fig. 7, for $n_r = 9$, $\partial\hat{\Gamma}_{1,\ell} = \partial s_j = 4$. As it can be seen, $\hat{\Gamma}$ and Γ^* overlap, thus yielding a very good approximation for this case study.

Remark 7 *Note that the two techniques presented in this section are not mutually exclusive and can be used at the same time.*

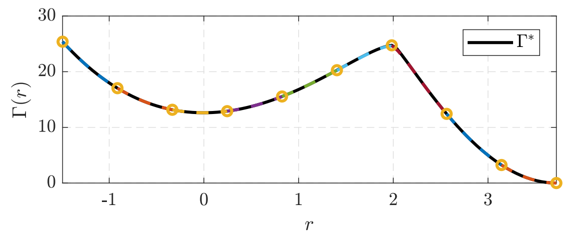

Fig. 7. Piece-wise polynomial approach for the case study, using $\partial\hat{\Gamma}_\ell = 4$, $\ell \in \{1,\ldots,9\}$. Γ^* is depicted as a solid black line, each of the divisions is marked as a yellow circular marker, and every $\hat{\Gamma}_\ell$ is depicted as a dashed colored line.

5 Applications in constrained control

In this section we will discuss two different applications of the proposed reference dependent positively invariant sets in constrained control problems. The first application is in discrete time and allows us to compute the terminal set in the MPC for Tracking framework for discrete time systems. The second case shows that continuous time positively invariant sets can be used to calculate the dynamic safety margin in the ERG framework.

5.1 Safe Invariant Sets As Terminal Conditions In the MPC for Tracking Framework

In the last two decades, MPC schemes have been widely used to address constrained control problems [2,31]. These schemes have the remarkable feature of being able to drive the system state to a fixed set-point while optimizing the control performances and satisfying the constraints to which the system is subject. However, classical MPC schemes might lose feasibility under sudden set-point changes [32]. To tackle this, the MPC for Tracking scheme was presented in [33]. This control scheme minimizes at every tome step t the following cost function

$$J_{N_p,N_c}(x,r;\mathbf{u},v) = \sum_{j=0}^{N_c-1} J_s(x(j)-\overline{x}_v, u(j)-\overline{u}_v) + \sum_{j=N_c}^{N_p-1} J_s(x(j)-\overline{x}_v, \kappa(x(j),v)-\overline{u}_v) + J_f(x(N_p)-\overline{x}_v) + J_o(v-r) \tag{17}$$

subject to

$$\min_{\mathbf{u},v} \; J_{N_p,N_c}(x,r;\mathbf{u},v), \tag{18a}$$
s.t.
$$x(0) = x(t), \tag{18b}$$
$$x(j+1) = \phi(x(j),u(j)), \quad j = 0,\ldots,N_c-1, \tag{18c}$$
$$(x(j),u(j)) \in \mathcal{Z}, \quad j = 0,\ldots,N_c-1, \tag{18d}$$
$$x(j+1) = \phi(x(j),\kappa(x(j),v)),\ j = N_c,\ldots,N_p-1, \tag{18e}$$
$$(x(j),\kappa(x(j),v)) \in \mathcal{Z}, \quad j = N_c,\ldots,N_p-1, \tag{18f}$$
$$\overline{x}_v = g_x(v), \tag{18g}$$
$$\overline{u}_v = g_u(v), \tag{18h}$$
$$(x(N_p),v) \in \Omega, \tag{18i}$$

where N_p and N_c are the prediction and control horizons, respectively, $\kappa : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ is a stabilizing control law as in Section 2, $\mathcal{Z} = \mathcal{X} \times \mathcal{U}$, $v \in \mathcal{R}$ is the auxiliary reference, $g_x : \mathcal{R} \to \mathbb{R}^n$ and $g_u : \mathcal{R} \to \mathbb{R}^m$ are two locally Lipschitz functions that map the auxiliary reference to its corresponding steady state and input $(\overline{x}_v, \overline{u}_v)$ (*i.e.*, $\overline{x}_v = \phi(g_x(v), g_u(v))$, $J_s : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $J_o : \mathbb{R}^p \to \mathbb{R}$, and $J_f : \mathbb{R}^n \to \mathbb{R}$ are convex positive definite functions that represent the stage, offset, and terminal costs, respectively (see [34] for more details). Ω is an invariant set for tracking which can be defined as follows:

Definition 8 *For a given set of constraints $\mathcal{Z} = \mathcal{X}\times\mathcal{U}$, a set of admissible references $\mathcal{R}$, and a local control law $u = \kappa(x,v)$, a set $\Omega \subset \mathbb{R}^n\times\mathbb{R}^p$ is an (admissible) invariant set for tracking for system* (2) *if for all $(x,v) \in \Omega$, we have $(x,\kappa(x,v)) \in \mathcal{Z}$, $v \in \mathcal{R} \ominus \mathcal{B}_p(\varepsilon)$, and $(\phi(x,\kappa(x,v)),v) \in \Omega$.*

This set represents the MPC for Tracking counterpart of the classical terminal set used in MPC schemes to guarantee stability and recursive feasibility. The computation of such a reference dependent set is one of the most challenging problems when designing an MPC for tracking.

To approximate these sets, in [34] the authors resort to a partition of the set of points of equilibrium paired with linearization of the system for each partition based on a Linear Time Varying characterization [35,36]. However, this *ad-hoc* technique is conservative and has a solution only for limited classes of constraints.

Note that once a control law $\kappa(x,v)$ acting as a terminal control law has been fixed [22,23], terminal sets for this scheme can be characterized as safe level sets of RDLFs, meaning that the proposed method in Section 3 can be employed to compute these sets in the design phase. In the rest of this subsection we will show an example where an invariant set for tracking is computed using the methodology proposed in this paper.

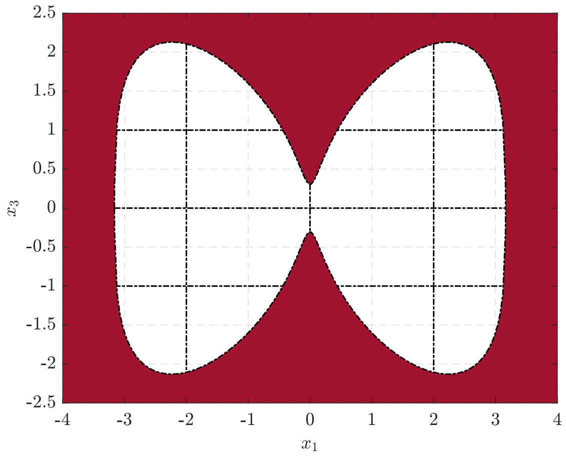

Fig. 8. Representation of $\mathcal{R}$ and its division used in the example at hand.

Example: Consider the following 4-dimensional model of a ball-and-plate system in discrete time

$$x(t+1) = I_2 \otimes \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} x(t) + I_2 \otimes \begin{bmatrix} 0.125 \\ 0.5 \end{bmatrix} u(t) \quad (19)$$

The positions of this system are constrained as follows

$$c(x, v) = x_1^4 + x_3^4 - 10x_1^2 + x_3^2 - 0.1 \leq 0, \qquad (20)$$

which describes the *bow tie* set depicted in Fig. 8. In order to build the MPC for Tracking, we chose $\kappa(x, v) = I_2 \otimes [-4 \;\; -2.73]x + 4I_2 v$ as the stabilizing terminal control law. When system (19) is controlled with this law and v is kept constant, the stability of every point of equilibrium $\overline{x}_v = [v_1 \; 0 \; v_2 \; 0]^{\mathrm{T}}$ can be proven using the RDLF

$$V(x, v) = (x - \overline{x}_v)^{\mathrm{T}} \left(I_2 \otimes \begin{bmatrix} 5.3933 & 0.8668 \\ 0.8668 & 1.1946 \end{bmatrix} \right) (x - \overline{x}_v).$$

To compute $\hat{\Gamma}$, we used the approach presented in Section 4.2: we divided $\mathcal{R}$ until the optimizer arrived at a sensible solution for each $\hat{\Gamma}_\ell$ (see Fig.8), and we did not assume any dependence on $\overline{c}$ (*i.e.*, $k = 0$ in (16)). For what concerns the maximal degrees of the decision polynomials, we set $\partial q_\ell = 6$, $\partial s_{j,\ell} = 4$ and $\partial \hat{\Gamma}_\ell = 8$, where $\ell \in \{1, \ldots, 16\}$.

Since the set described by (20) is non-convex, we used the results in [37] to ensure convergence. For what concerns the MPC for Tracking design parameters, the control and prediction horizons were set to $N_c = 1$ and $N_p = 2$, respectively. The stage, offset, and terminal cost functions in the objective function of the MPC optimization

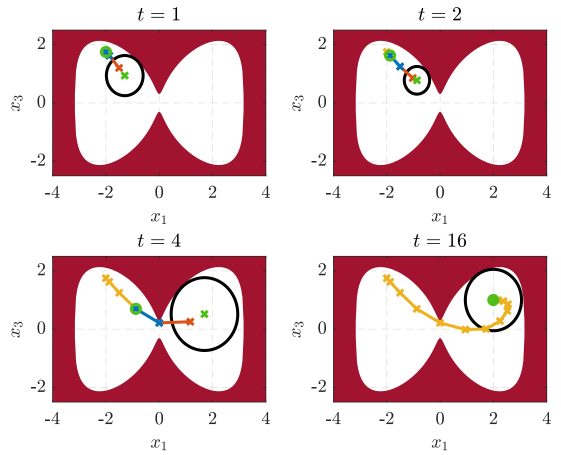

Fig. 9. Simulation results; previous state trajectory, its predicted evolution and terminal set at different time steps. The previous state trajectory is depicted in a solid yellow line, the predicted trajectory for the first N_c steps is depicted in a solid blue line, the last $N_p - N_c$ steps is depicted in a solid orange line, and the terminal invariant set is outlined in a solid black line. The initial point is marked with a green circle, and the steady state associated to the auxiliary reference is represented with a green cross.

problem are $J_s(x - \overline{x}_v, u - \overline{u}_v) = \|x - \overline{x}_v\|_Q^2 + \|u - \overline{u}_v\|_R^2$, $J_o(v - r) = \|v - r\|_T^2$, and $J_f(x - \overline{x}_v) = \|x(N_p) - \overline{x}_v\|_P^2$, respectively with weights $Q = I_4$, $R = 0.1I_2$, and $T = 10I_2$. In Fig. 9 we report the simulation results of the resulting MPC scheme assuming the initial state $x_0 = [-2 \; 0 \; 1.75 \; 0]^{\mathrm{T}}$, and setting the reference to $r = [2 \; 1]^{\mathrm{T}}$. As it can be seen in Fig. 9 (which shows only for some time steps) the MPC is able to drive the system in N_p steps to the terminal invariant set, whose size allows for a large domain of attraction of the MPC for Tracking.

5.2 *Reference Dependent Positively Invariant Sets in the ERG Framework*

The ERG [4] is an add-on unit that is able to provide constraint satisfying capabilities to prestabilized continuous time systems (see Fig. 10). The main idea behind the ERG is to feed the precompensated system with a filtered version of the desired reference r, $v \in \mathbb{R}^p$, computed such that if v remains constant, the system will not violate any constraints. In particular, the ERG [15] manipulates the time derivative of the auxiliary reference v as

$$\dot{v} = \Delta(x, v) \cdot \rho(r, v),$$

where $\rho(r, v)$ and $\Delta(x, v)$ are the two fundamental components of the ERG, called the Navigation Field (NF) and the Dynamic Safety Margin (DSM), respectively.

The NF is a vector field such that for any two steady-state admissible references v and r, the trajectory of sys-

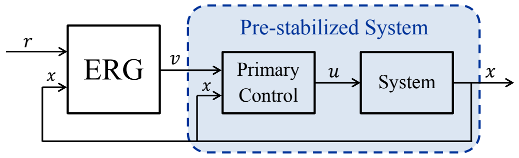

Fig. 10. The general structure of the ERG scheme.

tem $\dot{v} = \rho(r, v)$ goes from v to r through a path of strictly steady-state admissible references. This problem can be addressed using standard path planning algorithms *e.g.*, [38].

The DSM is a measure of the distance between the constraints and the system trajectory that would emanate from the state x for a constant reference v. A possible way to construct a DSM [4,15] is by using an RDLF $V(x, v)$ and a bound $\hat{\Gamma}$ such that $\mathcal{S}_{\hat{\Gamma}}$ is a safe invariant set as follows:

$$\Delta(x, v) = \lambda \cdot (\hat{\Gamma}(v) - V(x, v)),$$

where $\lambda > 0$ is a tuning parameter. Note that this implies that $\Delta(x, v) > 0$ whenever x is in the interior of $\mathcal{S}_{\hat{\Gamma}}(v)$, and $\Delta(x, v) = 0$ when x is on the border of $\mathcal{S}_{\hat{\Gamma}}(v)$.

Accordingly, the methodology presented in this paper can be applied directly to compute the bound $\hat{\Gamma}(v)$ in the DSM within the ERG framework.

Example: Consider a ball-and-plate system in continuous time stabilized with a PD control law

$$\dot{x} = I_2 \otimes \begin{bmatrix} 0 & 1 \\ -100 & -4 \end{bmatrix} x + I_2 \otimes \begin{bmatrix} 0 \\ 100 \end{bmatrix} v,$$

whose positions are constrained to lie within the *bow tie* set (20). For every equilibrium point $\overline{x}_v = [v_1\ 0\ v_2\ 0]^{\mathrm{T}}$, stability can be proved using the following quadratic RDLF

$$V(x, v) = (x - \overline{x}_v)^{\mathrm{T}} \left(I_2 \otimes \begin{bmatrix} 12.645 & 0.005 \\ 0.005 & 0.1263 \end{bmatrix} \right) (x - \overline{x}_v).$$

We computed $\hat{\Gamma}$ by dividing $\mathcal{R}$ and by setting the degrees of the polynomials in the same manner as in the example presented in Section 5.1.

For what concerns the NF, since $\mathcal{R}$ is non-convex [4], it is enough to choose it as

$$\rho(r, v) = \frac{\nabla\Phi(v)^{-1}(\Phi(r) - \Phi(v))}{\max\{\nabla\Phi(v)^{-1}(\Phi(r) - \Phi(v)), \theta\}},$$

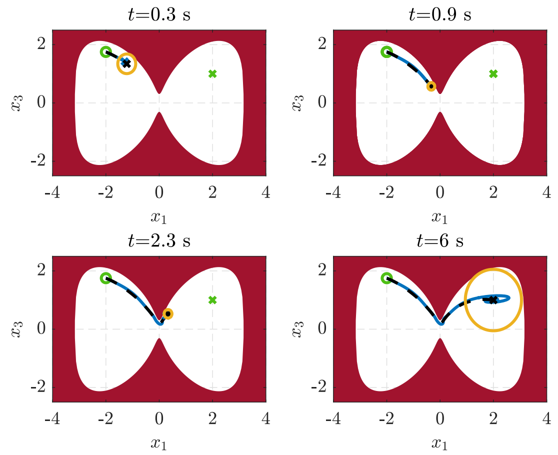

Fig. 11. Simulation results; state trajectory, trajectory of v and invariant set associated to v at different time steps. The previous state trajectory is depicted in a solid blue line, the trajectory of v is depicted in a dashed black line, and the invariant set is outlined in a solid yellow line. The initial point is marked with a green circle, the steady state associated to the current v is represented with a black cross, the current state is depicted with a blue cross, and the reference is depicted with a green cross.

where $\Phi : \mathcal{R}_d \to \mathcal{R}_c$ is a diffeomorphism that maps the interior of $\mathcal{R}$ to a convex set $\mathcal{R}_c$ and $\theta = 0.01$ is a smoothing factor. A possible choice for Φ is

$$\Phi(r) = \begin{bmatrix} r_1 \\ \dfrac{r_2}{\sqrt{\dfrac{1}{2}\sqrt{-4r_1^4 + 40r_1^2 + \dfrac{7}{5}} - \dfrac{1}{2}}} \end{bmatrix}.$$

Simulation results are shown in Fig. 11 for an initial state $x_0 = [-2\ 0\ 1.75\ 0]^{\mathrm{T}}$, and a desired reference $r = [2\ 1]^{\mathrm{T}}$. As it can be seen, the proposed ERG is able to steer the system state to the desired reference point while fulfilling the constraints at all times. The reader is referred to [39] for a video containing extra material.

6 Conclusions

In this paper we proposed a systematic method to approximate safe reference dependent positively invariant sets parameterized for systems that are subject to general polynomial constraints. To do so, first, we demonstrated that such sets can be determined through an optimization problem. It was then shown that we can approximate a parameterized solution of this optimization problem by making use of SOS techniques. We later showed that it is possible to alleviate some of the numerical issues that the SOS framework presents if the underlying structure of the largest safe level set is exploited, and if the overall optimization problem is broken down into several, smaller better numerically conditioned problems.

The proposed method has relevant applications in constrained control schemes. In particular, the proposed method can be used in the MPC for Tracking framework to determine the terminal set, and in the ERG framework to determine the Dynamic Safety Margin. Corresponding formulations in the context of the mentioned applications were discussed, and a numerical simulation for each of the mentioned applications were presented in order to evaluate the effectiveness of the proposed method.

We believe that the proposed methodology can be applied to systems subject to polynomial constraints such as robotic manipulators or the aging-aware charge of Li-ion batteries [40]. Possible future research lines include simultaneously computing the RDLF and its largest safe level set and generalizing the methodology beyond polynomials, *e.g.* for rational functions.

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A Krivine–Stengle Positivstellensatz

Before presenting the Krivine–Stengle Positivstellensatz, a few definitions need to be introduced. For the sake of simplicity, the following concepts will not be explained in depth, and will rather be mathematically characterized. For further information on the matter, the reader is referred to [17,41,42] and the references therein.

Definition 9 *(Multiplicative Monoid) The multiplicative monoid generated by a set of polynomials $P = \{p_i\}_{i=1}^m$, $\mathbf{M}(P)$ where $p_i \in \mathbb{R}[x_1, \dots, x_n],\ i = 1, \dots, m$ is defined as*

$$\mathbf{M}(P) = \left\{ \prod_{i=1}^m p_i^{k_i},\ k_i \in \mathbb{Z}_{\geq 0},\ i = 1, \dots, m \right\},$$

e.g., for a set of polynomials $P' = \{p_1, p_2\}$ the multiplicative monoid generated by P' is the following:

$$\mathbf{M}(P') = \left\{ p_1^{k_1} p_2^{k_2},\ k_1, k_2 \in \mathbb{Z}_{\geq 0} \right\} = \left\{ 1,\ p_1,\ p_2,\ p_1^2,\ p_2^2,\ p_1p_2,\ p_1^3,\ p_1^2p_2,\ p_1p_2^2,\ p_2^3,\ \dots \right\}.$$

Definition 10 *(Cone) The cone generated by a set of polynomials $P = \{p_i\}_{i=1}^m$, $\mathbf{K}(P)$, where $p_i \in \mathbb{R}[x_1, \dots, x_n],\ i = 1, \dots, m$ is defined as*

$$\mathbf{K}(P) = \left\{ s_0 + \sum_i s_i g_i : s_i \in \Sigma[x_1, \dots, x_n], g_i \in \mathbf{M}(P) \right\},$$

e.g., for a set of polynomials $P' = \{p_1, p_2\}$ with $p_1, p_2 \in \mathbb{R}[x_1, \dots, x_n]$ the cone generated by P' is defined as

$$\mathbf{K}(P') = \{s_0 + s_1p_1 + s_2p_2 + s_3p_1p_2 + s_4p_1^2 + \dots : s_i \in \Sigma[x_1, \dots, x_n]\}.$$

Definition 11 *(Ideal) The ideal generated by a set of polynomials $P = \{p_i\}_{i=1}^m$, $p_i \in \mathbb{R}[x_1, \dots, x_n],\ i = 1, \dots, m$ is defined as*

$$\mathbf{I}(P) = \left\{ \sum_{i=1}^m t_i p_i : t_i \in \mathbb{R}[x_1, \dots, x_n],\ i = 1, \dots, m \right\},$$

e.g., for a set of polynomials $P' = \{p_1, p_2\}$ with $p_1, p_2 \in \mathbb{R}[x_1, \dots, x_n]$ the ideal generated by P' is the following set of polynomials:

$$\mathbf{I}(P') = \{t_1p_1 + t_2p_2 : t_1, t_2 \in \mathbb{R}[x_1, \dots, x_n]\}.$$

At this point, the Krivine–Stengle Positivstellensatz can be expressed as follows:

Theorem 12 *(Krivine–Stengle Positivstellensatz) Let $f_i(x),\ i \in \mathcal{I},\ g_j(x),\ j \in \mathcal{J},\ h_k(x),\ k \in \mathcal{K}$ be finite sets of polynomials in $\mathbb{R}[x]$, $K = \mathbf{K}(f_i)$, $M = \mathbf{M}(g_j)$, and $I = \mathbf{I}(h_k)$, then the set*

$$\{x : f_i(x) \geq 0, i \in \mathcal{I}, g_j(x) \neq 0, j \in \mathcal{J}, h_k(x) = 0, k \in \mathcal{K}\}$$

is empty if and only if

$$\exists f \in K,\ g \in M,\ h \in I\ :\ f + g^2 + h = 0.$$