A Fuzzy DEA Slacks-Based Approach

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Abstract

This paper deals with the problem of efficiency assessment using Data Envelopment Analysis (DEA) when the input and output data are given as fuzzy sets. In particular, a fuzzy extension of the measure of inefficiency proportions, a well-known slacks-based additive inefficiency measure, is considered. The proposed approach also provides fuzzy input and output targets. Computational experiences and comparison with other fuzzy DEA approaches are reported.

Keywords: `Relative efficiency; fuzzy data; DEA; measure of inefficiency proportions; fuzzy targets`
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1. Introduction

In order to assess the relative efficiency of a set of homogeneous Decision Making Units (DMUs) a non-parametric methodology, namely Data envelopment analysis (DEA), can be used. DEA only requires data about the inputs consumed and the outputs produced by the DMUs and, from that, using some optimization models, an efficiency score and an efficient target can be computed for each DMU ([49], [12]). There are different ways of carrying out the

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projection onto the efficient frontier and computing the corresponding efficiency scores: radial approaches (e.g. [29]), hyperbolic approaches (e.g. [18]), directional distance function approaches (e.g. [32]), slacks-based approaches (e.g. [43]), multidirectional efficiency approaches (e.g. [33]), efficiency potential approaches (e.g. [40]), etc.

One of the challenges in DEA applications is the difficulty in quantifying the exact value of some of the input and output data in real-world problems, where the observed values are often undetermined or incomplete. One way to handle this uncertainty in DEA data is using fuzzy sets. Thus, imprecise or vague data in DEA can be represented by linguistic terms characterized by fuzzy numbers.[23].

There are different fuzzy DEA (FDEA) approaches that can be used when the data are fuzzy. The best reference on the subject is [17], which extended a previous review by [22] and proposed a taxonomy that classifies the FDEA methods into α-level set approaches (e.g. [27], [38]), fuzzy ranking approaches (e.g. [21], [20]), possibility approaches (e.g. [44]), fuzzy arithmetic approaches (e.g. [45]) and fuzzy random/type-2 fuzzy sets (e.g. [42]). [48] presents a more recent survey of the FDEA literature. A summary of existing FDEA approaches is shown in Table 1. This list does not pretend to be exhaustive but it includes a number of representative approaches. For comparison, the proposed approach has also been included.

It can be seen in the Table 1 that the α-level set, the fuzzy ranking and the possibility approaches are the most common. Most FDEA approaches involve radial, input-oriented, multiplier formulations. There are some non-radial approaches, mostly based on the Enhanced Russell Graph Measure (ERGM) and Slacks-Based Measure (SBM), as well as a few additive approaches, using both multiplier and envelopment formulations. Some approaches compute fuzzy efficiency scores while others compute a crisp efficiency or an efficiency score for each possibility level. Also some FDEA approaches allow ranking the DMUs. What most approaches neglect (or, more exactly, are unable to compute) are efficient targets. Only a few FDEA approaches can compute targets, crisp

Table 1: Summary of existing fuzzy DEA approaches. Notes: CRS = Constant Returns to Scale; VRS = Variable Returns to Scale; SBM = Slacks-Based Measure; ERGM = Enhanced Russell Graph Measure

Method	Reference	Main features
$\alpha-$level	Kao and Liu (2000)	Ratio, multiplier formulation; CRS; fractional parametric programming; fuzzy efficiency scores
	Saati et al. (2002)	Radial, input-oriented, multiplier formulation; CRS; interval linear programming; $\alpha-$level efficiency scores
	Saati and Memariani (2009)	Non-oriented SBM, multiplier formulation; CRS; interval linear programming; $\alpha-$level efficiency scores
	Hsiao et al. (2011)	Non-oriented SBM, envelopment formulation; VRS; fuzzy efficiency and super-efficiency scores
	Hatami-Marbini et al. (2012)	Additive, non-oriented, envelopment formulation; VRS; interval linear programming; $\alpha-$level efficiency scores
	Puri and Yadav (2013)	Radial and non-radial, input-oriented, envelopment formulations; CRS; fuzzy efficiency scores
	Wu et al. (2015)	Non-radial, non-oriented ERGM envelopment formulation; CRS; undesirable outputs; fuzzy efficiency scores
Fuzzy ranking	Guo and Tanaka (2001)	Radial, input-oriented, multiplier formulation; CRS; fuzzy linear programming; fuzzy efficiency scores
	León et al. (2003)	Radial, input-oriented, envelopment formulation; VRS; fuzzy linear programming; possibility-level efficiency scores
	Soleimani-damaneh et al. (2006)	Radial, input-oriented, envelopment formulation; VRS; crisp efficiency scores; crisp targets
	Ghasemi et al. (2015)	Multiplier formulation; generalized DEA; VRS; Fuzzy expected value; DMU ranking
	Khaleghi et al. (2015)	Radial, input-oriented, multiplier formulation; CRS; fully fuzzy; level-sum method; fuzzy efficiency scores, DMU ranking
	Kordrostami et al. (2016)	Radial, input-oriented, envelopment formulation; CRS; integer data; crisp efficiency scores; crisp targets
	Arana-Jiménez et al. (2020)	Radial, input-oriented, envelopment formulation; CRS; fully fuzzy; fuzzy partial order; lexicographic weighted Tchebycheff method; fuzzy efficiency scores, fuzzy targets
	Proposed approach	Normalized additive, non-oriented, envelopment formulation; VRS; fuzzy partial order; crisp efficiency scores; fuzzy targets
Possibility approach	Lertworasirikul et al. (2003)	Radial, input-oriented, multiplier and envelopment formulations; VRS; possibility-level efficiency scores
	Wang and Chin (2011)	Radial, input-oriented, multiplier formulation; CRS; Fuzzy expected value; double frontier analysis; crisp efficiency scores; DMU ranking
	Ruiz and Sirvent (2017)	Radial, input-oriented, multiplier formulation; CRS; benevolent and aggressive possibility-level cross-efficiency scores
	Izadikhah and Khoshroo (2018)	Non-radial, non-oriented modified ERGM envelopment formulation; CRS; undesirable outputs; possibility-level super-efficiency scores
	Perykani et al. (2019)	Radial, input-oriented, multiplier formulation; CRS; adjustable pessimistic-optimistic parameter; possibility-level efficiency scores
Fuzzy arithmetic	Wang et al. (2009)	Radial, input-oriented, multiplier formulation; CRS; fuzzy efficiency scores; DMU ranking
	Azar et al. (2016)	Additive, non-oriented, multiplier formulation; CRS; common set of weights; fuzzy efficiency scores; DMU ranking

in some cases and fuzzy in others.

As shown in Table 1, the proposed fuzzy ranking approach is based on the normalized additive envelopment formulation. Unlike the conventional additive DEA model considered in other FDEA approaches and which does not normalize the input and output slacks, the proposed approach is units invariant. Moreover, it makes use of a fuzzy partial order, which has proved to be a flexible FDEA modelling tool (e.g. [4]). This combination of features leads to a simple and elegant linear programming (LP) formulation that provides crisp efficiency scores as well as fuzzy targets. This target computing feature and its focus on quantifying the potential input and output improvements are the main advantages of the proposed approach with respect to other more complex approaches that do not provide efficient targets. We believe that such information is essential for managers to take action and orientate their improvement efforts.

The structure of the paper is as follows. In Section 2, some notions related to the Fuzzy sets and DEA methodologies are introduced. In Section 3, uncertainty on inputs and outputs is addressed with trapezoidal fuzzy numbers. The corresponding fuzzy DEA technology is defined, and a fuzzy DEA model is proposed to compute an inefficiency measure as well as a fuzzy target for each DMU. Section 4 illustrates the proposed approach on a dataset from the literature, comparing it with some existing fuzzy DEA approaches. Finally, Section 5 summarizes and concludes.

2. Preliminaries

2.1. Fuzzy sets Theory

A fuzzy set on $\mathbb{R}^n$ is a mapping $u : \mathbb{R}^n \to [0, 1]$. For each fuzzy set u and for any $\alpha \in (0, 1]$, let us define the α-level set as $[u]^\alpha = \{x \in \mathbb{R}^n \mid u(x) \geq \alpha\}$. Let us denote the support of u by $supp(u)$ where $supp(u) = \{x \in \mathbb{R}^n \mid u(x) > 0\}$. The closure of $supp(u)$ defines the 0-level set of u, .i.e. $[u]^0 = cl(supp(u))$ where $cl(M)$

means the closure of the subset $M \subset \mathbb{R}^n$. A fuzzy number is a type of fuzzy set (see [14, 15]) defined as follows.

Definition 1. *A fuzzy set u on $\mathbb{R}$ is said to be a fuzzy number if:*

1. *u is normal, i.e. there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;*
2. *u is an upper semi-continuous function;*
3. *$u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\}$, $x, y \in \mathbb{R}$, $\lambda \in [0,1]$;*
4. *$[u]^0$ is compact.*

Definition 2. *A fuzzy number $\tilde{u} = (u^1, u^2, u^3, u^4)$ is said to be a trapezoidal fuzzy number if its membership function is given by*

$$\tilde{u}(x) = \begin{cases} \frac{x-u^1}{u^2-u^1}, & \text{if } u^1 \leq x < u^2, \\ 1, & \text{if } u^2 \leq x \leq u^3, \\ \frac{u^4-x}{u^4-u^3}, & \text{if } u^3 < x \leq u^4, \\ 0, & \text{otherwise}. \end{cases} \tag{1}$$

Its corresponding α-levels are determined by $[\tilde{u}]^\alpha = [u^1 + \alpha(u^2 - u^1), u^4 - \alpha(u^4 - u^3)]$. We denote the set of all trapezoidal fuzzy numbers as $TrFN$. The subset of non-negative TrFNs is denoted by $TrFN_+$. A TrFN $\tilde{u}$ is a triangular fuzzy number if and only if $u^2 = u^3$. Figure 1 shows illustrative examples of a trapezoidal and a triangular fuzzy number.

Definition 3. *Given two trapezoidal fuzzy numbers $\tilde{a} = (a^1, a^2, a^3, a^4) \in TrFN$ and $\tilde{b} = (b^1, b^2, b^3, b^4) \in TrFN$, the following arithmetical operations are defined:*

(i) Addition,

$$\tilde{a} + \tilde{b} = (a^1 + b^1, a^2 + b^2, a^3 + b^3, a^4 + b^4) \tag{2}$$

(ii) Multiplication by a scalar $\lambda \in \mathbb{R}$,

$$\lambda\tilde{a} = \begin{cases} (\lambda a^1, \lambda a^2, \lambda a^3, \lambda a^4) & \text{if } \lambda \geq 0; \\ (\lambda a^4, \lambda a^3, \lambda a^2, \lambda a^1) & \text{if } \lambda < 0. \end{cases} \tag{3}$$

(iii) Multiplication of two TrFN, $\tilde{a}\tilde{b} = \tilde{c} = (c^1, c^2, c^3, c^4)$, where

$$\begin{aligned} c^1 &= \min\{a^1b^1, a^1b^4, a^4b^1, a^4b^4\} \qquad & c^4 &= \max\{a^1b^1, a^1b^4, a^4b^1, a^4b^4\} \\ c^2 &= \min\{a^2b^2, a^2b^3, a^3b^2, a^3b^3\} \qquad & c^3 &= \max\{a^2b^2, a^2b^3, a^3b^2, a^3b^3\} \end{aligned} \tag{4}$$

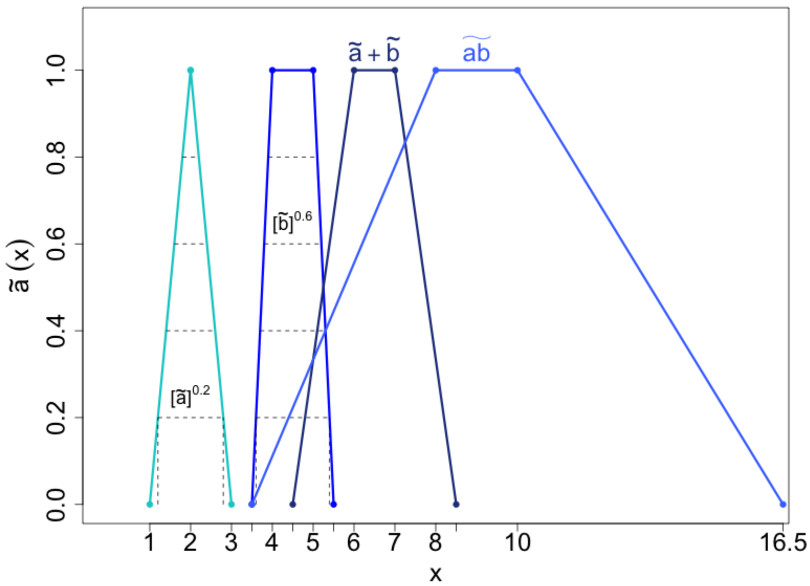

Figure 1: Two trapezoidal fuzzy numbers, their sum and their product, as defined in 3. In cyan, the particular case of a triangular fuzzy number $\tilde{a} = (1, 2, 3) = (1, 2, 2, 3)$. In blue, a trapezoidal fuzzy number $(3.5, 4, 5, 5.5)$. The $[\tilde{a}]^\alpha$ and $[\tilde{b}]^\alpha$ levels, for $\alpha = 0.2, 0.4, 0.6, 0.8$, are represented with dashed lines.

In the particular case of two non-negative TrFN $\tilde{a}$ and $\tilde{b}$, their multiplication is just $\tilde{a}\tilde{b} = (a^1b^1, a^2b^2, a^3b^3, a^4b^4)$. Examples of the addition and of the multiplication of a trapezoidal and a triangular fuzzy number are shown in Figure 1.

In the Fuzzy DEA problem addressed in this research, we will consider that the input and output variables (and some model variables) are $TrFN_+$. The arithmetic operations between them are those established in Definition 3. Besides, it is necessary to provide a partial order relationship between two trapezoidal fuzzy numbers. To this aim, we will use LU-fuzzy partial orders, which are well known in the literature (see, e.g., [46, 41]):

$$\mu \leqq (\geqq)\nu \text{ if and only if } \underline{\mu}_\alpha \leq (\geq)\underline{\nu}_\alpha \text{ and } \overline{\mu}_\alpha \leq (\geq)\overline{\nu}_\alpha, \text{ for all } \alpha \in [0, 1].$$

In the particular case of TrFNs, and following the characterization result given in [3] for triangular fuzzy numbers, we can say that given two trapezoidal fuzzy numbers $\tilde{u} = (u^1, u^2, u^3, u^4)$ and $\tilde{v} = (v^1, v^2, v^3, v^4)$,

$$\tilde{u} \leqq (\geqq) \tilde{v} \text{ if and only if } u^i \leq (\geq) v^i, \text{ for all } i = 1, 2, 3, 4. \tag{5}$$

2.2. DEA methodology

Consider a set of N DMUs. For $j \in \{1, \ldots, N\}$, each DMU_j has M inputs $X_j = (x_{1j}, \ldots, x_{Mj}) \in \mathbb{R}^M$, and produces S outputs $Y_j = (y_{1j}, \ldots, y_{Sj}) \in \mathbb{R}^S$. DEA aims to measure the relative efficiency of these DMUs. A DMU is inefficient if it can reduce its inputs without reducing its outputs or it can increase its outputs without increasing its inputs. The first step in the DEA methodology is establishing the production possibility set (PPS) (a.k.a. DEA technology), which contains all the operating points that are deemed feasible. The usual axioms for a basic DEA technology, called T, are the following:

(A1) Envelopment: $(X_j, Y_j) \in T$, for all $j \in \{1, \ldots, N\}$.

(A2) Free disposability: $(x, y) \in T$, $(x', y') \in \mathbb{R}^{M+S}$, $x' \geqq x$, $y' \leqq y \Rightarrow (x', y') \in T$.

(A3) Constant Returns to Scale (CRS): $(x, y) \in T \Rightarrow (\lambda x, \lambda y) \in T$, for all $\lambda \in \mathbb{R}_+$.

(A4) Convexity: $(x, y), (x', y') \in T$, then $\lambda(x, y) + (1 - \lambda)(x', y') \in T$, for all $\lambda \in [0, 1]$.

Let us recall that the technology which verifies (A1)-(A4) is known as Constant Returns to Scale (VRS), and noted as T_{CRS}. Following the minimum extrapolation principle, the DEA PPS is the intersection of all the sets that satisfy the above axioms and can be mathematically formulated as ([9])

$$T_{CRS} = \left\{ (x, y) \in \mathcal{R}_+^{M+S} : \;\; x \geqq \sum_{j=1}^{N} \lambda_j X_j, \;\; y \leqq \sum_{j=1}^{N} \lambda_j Y_j, \;\; \lambda_j \geq 0 \right\}$$

There is another common DEA technology labelled Variable Returns to Scale (VRS) which corresponds to dropping Axiom (A3) ([7])

$$T_{VRS} = \left\{ (x, y) \in \mathcal{R}_+^{M+S} : \ x \geqq \sum_{j=1}^{N} \lambda_j X_j, \ \ y \leqq \sum_{j=1}^{N} \lambda_j Y_j, \ \ \sum_{j=1}^{N} \lambda_j = 1, \lambda_j \geq 0 \right\}$$

Note that the CRS and VRS DEA technologies differ basically in that the latter imposes the convexity constraint $\sum_{j=1}^{N} \lambda_j = 1$. Recall that, given a technology T (T_{CRS}, T_{VRS}, or any other under consideration), a DMU p is said to be efficient if and only if for any $(x, y) \in T$ such that $x \leqq X_p$ and $y \geqq Y_p$, then $(x, y) = (X_p, Y_p)$.

Radial efficiency measures can be computed by:

a) reducing all the inputs equi-proportionally without decreasing the outputs (input-oriented model):

$$E(x, y) = \min\{\theta | (\theta x, y) \in T\}$$

b) expanding all the outputs equi-proportionally without increasing the inputs (output-oriented model):

$$E^{-1}(x, y) = \max\{\gamma | (x, \gamma y) \in T\}$$

Instead of a radial measure of efficiency, an additive DEA model can be used considering the input and output slacks. i.e. the amount that each input can be reduced, and each output can be increased. This leads to the following additive measure of inefficiency ([10])

$$\begin{aligned}
\text{Max} \quad & \sum_{i=1}^{M} s_i^- + \sum_{r=1}^{S} s_r^+ && (6)\\
\text{s.t.} \quad & \sum_{j=1}^{N} \lambda_j x_{ij} + s_i^- = x_{ip}, \quad i = 1, \ldots, M \\
& \sum_{j=1}^{N} \lambda_j y_{rj} = y_{rp} + s_r^+, \quad r = 1, \ldots, S, \\
& s_i^- \geq 0, \ \ i = 1, \ldots, M \\
& s_r^+ \geq 0, \ \ r = 1, \ldots, S \\
& \lambda_j \geq 0, \quad j = 1, \ldots, N,
\end{aligned}$$

Note that a certain *DMUp* is efficient if the optimal solution of the above model has an objective function equal to zero. This means that all input slacks s_i^-, and all output slacks s_r^+ must be zero at the optimum ([13]).

A drawback of the above model is that its objective function sums input and output slacks that may be measured in different units. One way of solving this is to normalize the slacks, i.e. the input and output improvements, and express them in relative terms. Then, for a given *DMUp*, the corresponding Measure of Inefficiency Proportions (MIP) ([11]) results from changing the objective function of the above model to

$$\text{Max} \quad \textstyle\sum_{i=1}^{M} s_i^- / x_{ip} + \sum_{r=1}^{S} s_r^+ / y_{rp} \tag{7}$$

Another relevant inefficiency measure is the directional distance function (DDF), which uses a directional vector $g = (g^x, g^y) = (g_1^x, \ldots, g_M^x, g_1^y, \ldots, g_S^y) \neq 0$, and leads to the following DEA model ([8])

$$\begin{aligned} \delta_p = \quad & \text{Max} \quad \beta \qquad\qquad (8) \\ & \text{s.t.} \quad \sum_{j=1}^{N} \lambda_j x_{ij} \leq x_{ip} - \beta g_i^x, \quad i = 1, \ldots, M \\ & \qquad\; \sum_{j=1}^{N} \lambda_j y_{rj} \geq y_{rp} + \beta g_r^y, \quad r = 1, \ldots, S, \\ & \qquad\; \sum_{j=1}^{N} \lambda_j = 1 \\ & \qquad\; \lambda_j, \beta \geq 0, \quad j = 1, \ldots, N, \end{aligned}$$

Finally, another slacks-based measure of inefficiency was proposed by [19] as a generalization of the above optimization problem considering the sum of directional distance functions based on the unit vectors e_i^x, $i = 1, \ldots, M$ and e_r^y, $r = 1, \ldots, S$ (instead of a single directional vector $g = (g^x,\ g^y)$) and different variables β_i and γ_r, instead of a single variable β for all inputs and outputs. These β_i and γ_r dimensionless variables can be interpreted as the number of units that each input (respectively, output) can decrease (respectively, increase) for DMU_p.

$$\begin{aligned}
\sigma_p = \quad & \text{Max} \quad \sum_{i=1}^{M} \beta_i + \sum_{r=1}^{S} \gamma_r \qquad (9) \\
& \text{s.t.} \quad \sum_{j=1}^{N} \lambda_j x_{ij} \leq x_{ip} - \beta_i e_i^x, \quad i = 1, \ldots, M \\
& \sum_{j=1}^{N} \lambda_j y_{rj} \geq y_{rp} + \gamma_r e_r^y, \quad r = 1, \ldots, S, \\
& \sum_{j=1}^{N} \lambda_j = 1 \\
& \lambda_j \geq 0, \quad j = 1, \ldots, N \\
& \beta_i \geq 0, \quad i = 1, \ldots, M \\
& \gamma_r \geq 0, \quad r = 1, \ldots, S
\end{aligned}$$

3. Fuzzy DEA Slacks-Based model

In this section we present a slack-based measure of inefficiency in DEA based on the sum of multiple directional distance functions in a similar way as [19], but applied to the fuzzy data context and using the observed inputs and outputs to define the corresponding directional vectors along each input and output dimension.

Let us assume that there are $j = 1, \ldots, N$ DMUs, each of them with M fuzzy inputs $\tilde{X}_j = (\tilde{x}_{1j}, \ldots, \tilde{x}_{Mj}) \in (TrFN)_+^M$, and S fuzzy outputs $\tilde{Y}_j = (\tilde{y}_{1j}, \ldots, \tilde{y}_{Sj}) \in (TrFN)_+^S$. Let us define the following fuzzy DEA technology as a natural extension of the conventional T_{VRS} DEA technology

$$T_{FDEA} = \left\{ (\tilde{x}, \tilde{y}) \in (TrFN)_+^{M+S} : \tilde{x} \geqq \sum_{j=1}^{N} \lambda_j \tilde{x}_{ij} \ \ \forall i, \ \ \tilde{y} \leqq \sum_{j=1}^{N} \lambda_j \tilde{y}_{rj} \ \ \forall r, \sum_{j=1}^{N} \lambda_j = 1, \ \ \lambda \in \mathbb{R}_+^N \right\}$$

For T_{FDEA}, and as a natural extension of efficiency in a crisp technology, a DMU p is said to be efficient if and only if for any $(\tilde{x}, \tilde{y}) \in T_{FDEA}$ such that

$\tilde{x} \leqq \tilde{X}_p$ and $\tilde{y} \geqq \tilde{Y}_p$, then $(\tilde{x}, \tilde{y}) = (\tilde{X}_p, \tilde{Y}_p)$. Under the given technology T_{FDEA}, for each DMU_p, an additive, slacks-based measure of inefficiency $\tilde{I}(\tilde{X}_p, \tilde{Y}_p)$, that is similar to MIP model (7) and also similar to (9) (if x_{ip} and y_{rp} were used as directional vectors instead of e_i^x and e_r^y) can be computed as:

$$\text{(FDEA)} \;\; \tilde{I}(\tilde{X}_p, \tilde{Y}_p) = \text{Max} \;\; \sum_{i=1}^{M} \tilde{\beta}_i + \sum_{r=1}^{S} \tilde{\gamma}_r \tag{10}$$

$$\text{s.t.} \qquad \sum_{j=1}^{N} \lambda_j \tilde{x}_{ij} + \tilde{\beta}_m \tilde{x}_{ip} \leqq \tilde{x}_{ip}, \quad i = 1, \ldots, M, \tag{11}$$

$$\sum_{j=1}^{N} \lambda_j \tilde{y}_{rj} \geqq \tilde{y}_{rp} + \tilde{\gamma}_r \tilde{y}_{rp}, \quad r = 1, \ldots, S, \tag{12}$$

$$\sum_{j=1}^{N} \lambda_j = 1 \tag{13}$$

$$\tilde{\beta}_i, \tilde{\gamma}_r \in (TrFN)_+ \quad i = 1, \ldots, M, r = 1, \ldots, S, \tag{14}$$

$$\lambda_j \geq 0, \quad j = 1, \ldots, N. \tag{15}$$

Note that $\tilde{\beta}_i \tilde{x}_{ip}$ and $\tilde{\gamma}_r \tilde{y}_{rp}$ play the same role as the input and output slacks in conventional DEA. Here, however, the inputs $\tilde{x}_{ij}$ and outputs $\tilde{y}_{rj}$, as well as the variables $\tilde{\beta}_i$ and $\tilde{\gamma}_r$ are assumed to be trapezoidal fuzzy numbers $(TrFN)_+$, i.e.

$$\begin{aligned}
\tilde{x}_{ij} &= (x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4), \quad i = 1, \ldots, M, \;\; j = 1, \ldots, N \\
\tilde{y}_{rj} &= (y_{rj}^1, y_{rj}^2, y_{rj}^3, y_{rj}^4), \quad r = 1, \ldots, S, \;\; j = 1, \ldots, N \\
\tilde{\beta}_i &= (\beta_i^1, \beta_i^2, \beta_i^3, \beta_i^4), \quad i = 1, \ldots, M \\
\tilde{\gamma}_r &= (\gamma_r^1, \gamma_r^2, \gamma_r^3, \gamma_r^4), \quad r = 1, \ldots, S
\end{aligned}$$

The objective function is also a trapezoidal fuzzy number, and only the intensity variables λ_j are non-negative real numbers, $j = 1, \ldots, N$. Let $\tilde{\boldsymbol{\beta}} = (\tilde{\beta}_1, \ldots, \tilde{\beta}_M)$, $\tilde{\boldsymbol{\gamma}} = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_S)$, and let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ be a feasible solution of (FDEA) and let us define the following fuzzy Pareto solution for the above fuzzy DEA problem.

Definition 4. *A feasible solution for (FDEA) $(\tilde{\boldsymbol{\beta}}^*, \tilde{\boldsymbol{\gamma}}^*, \boldsymbol{\lambda}^*)$ is a fuzzy Pareto solution of (FDEA) if there does not exist a feasible solution $(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, \boldsymbol{\lambda})$ for (FDEA) such that $\sum_{i=1}^{M} \tilde{\beta}_i + \sum_{r=1}^{S} \tilde{\gamma}_r \geqq \sum_{i=1}^{M} \tilde{\beta}_i^* + \sum_{r=1}^{S} \tilde{\gamma}_r^*$, and $\sum_{i=1}^{M} \tilde{\beta}_i + \sum_{r=1}^{S} \tilde{\gamma}_r \neq \sum_{i=1}^{M} \tilde{\beta}_i^* + \sum_{r=1}^{S} \tilde{\gamma}^*{}_r$.*

Moreover, using the above fuzzy DEA model, efficient DMUs have a null inefficiency measure, i.e.

Proposition 1. *If DMU p is efficient, then $\tilde{I}(\tilde{X}_p, \tilde{Y}_p) = \tilde{0}$, i.e. there is no fuzzy Pareto solution of (FDEA) with $\sum_{i=1}^{M} \tilde{\beta}_i^* + \sum_{r=1}^{S} \tilde{\gamma}^*{}_r \neq \tilde{0}$.*

Proof. Suppose that $\tilde{I}(\tilde{X}_p, \tilde{Y}_p) \neq \tilde{0}$, with $(\tilde{\boldsymbol{\beta}}^*, \tilde{\boldsymbol{\gamma}}^*, \boldsymbol{\lambda}^*)$ a fuzzy Pareto solution for (FDEA). As $\tilde{I}(\tilde{X}_p, \tilde{Y}_p) \geqq \tilde{0}$, then $(\tilde{\boldsymbol{\beta}}^*, \tilde{\boldsymbol{\gamma}}^*) \neq \tilde{0}$, and we have two possible cases: $\tilde{\boldsymbol{\beta}}_{i_0}^* \geqq \tilde{0}$, $\tilde{\boldsymbol{\beta}}_{i_0}^* \neq \tilde{0}$ for some i_0, or $\tilde{\boldsymbol{\gamma}}_{r_0}^* \geqq \tilde{0}$, $\tilde{\boldsymbol{\gamma}}_{r_0}^* \neq \tilde{0}$ for some r_0. Define $(\tilde{x}^*, \tilde{y}^*) = (\sum_{j=1}^{N} \lambda_j^* \tilde{x}_{ij}, \sum_{j=1}^{N} \lambda_j^* \tilde{y}_{rj})$. In the first case, and by (11), $\tilde{x}_{i_0}^* \leqq \tilde{x}_{i_0 p}$, $\tilde{x}_{i_0}^* \neq \tilde{x}_{i_0 p}$, and then $\tilde{x}^* \leqq \tilde{x}_p$, $\tilde{x}^* \neq \tilde{x}_p$. Furthermore, by (12), it follows that $\tilde{y}^* \geqq \tilde{y}_p$, which implies that DMU p is not efficient, reaching a contradiction. In the second case, reasoning similarly we reach again a contradiction. □

Note that, from the above definition and proposition, $\tilde{I}(\tilde{X}_p, \tilde{Y}_p)$ is the set associated to the fuzzy Pareto solutions of (FDEA) and this set does not have to be necessarily a singleton. To address this fact, we propose an approach for solving the previous Fuzzy DEA problem computing a crisp inefficiency measure $I(\tilde{X}_p, \tilde{Y}_p)$ for the DMU p solving

$$\text{(FDEA2)} \quad I(\tilde{X}_p, \tilde{Y}_p) = \text{Max} \quad \sum_{i=1}^{M} \sum_{k=1}^{4} \beta_i^k + \sum_{r=1}^{S} \sum_{k=1}^{4} \gamma_r^k \qquad (16)$$
$$\text{s.t.} \qquad (11) - (15)$$

Proposition 2. *If a DMU p is efficient, then $I(\tilde{X}_p, \tilde{Y}_p) = 0$.*

Proof. If DMU p is efficient, then, by Proposition 1 it follows that $\tilde{I}(\tilde{X}_p, \tilde{Y}_p) = \sum_{i=1}^{M} \tilde{\beta}_i^* + \sum_{r=1}^{S} \tilde{\gamma}_r^* = 0$ for any $(\tilde{\boldsymbol{\beta}}^*, \tilde{\boldsymbol{\gamma}}^*, \boldsymbol{\lambda}^*)$ fuzzy Pareto solution for (FDEA). Now, suppose that the optimal solution of (16) is $I(\tilde{X}_p, \tilde{Y}_p) > 0$. Then, there exists $(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, \boldsymbol{\lambda})$ feasible for (FDEA2), such that $\sum_{i=1}^{M} \sum_{k=1}^{4} \beta_i^k + \sum_{r=1}^{S} \sum_{k=1}^{4} \gamma_r^k > 0$. Hence

$(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}) \neq 0$, and therefore, $\sum_{i=1}^{M} \tilde{\beta}_i + \sum_{r=1}^{S} \tilde{\gamma}_r \geqq 0$, $\sum_{i=1}^{M} \tilde{\beta}_i + \sum_{r=1}^{S} \tilde{\gamma}_r \neq 0$. Since $(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, \boldsymbol{\lambda})$ is feasible for (FDEA), it implies that $\tilde{I}(\tilde{X}_p, \tilde{Y}_p) \neq 0$, which contradicts Proposition 1. □

The optimization problem (FDEA2), with fuzzy data and constraints, can be reformulated with crisp values and inequalities by means of the corresponding fuzzy numbers parametrization. To this end, let us consider the 4-tuple representation of $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ as $\tilde{x}_{ij} = (x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4) \in \mathbb{R}^4$ and $\tilde{y}_{rj} = (y_{rj}^1, y_{rj}^2, y_{rj}^3, y_{rj}^4) \in \mathbb{R}^4$, respectively. Using these parametrizations, and the order relationships between fuzzy numbers and their parameters given in (5), we can reformulate the inefficiency measure $I(\tilde{X}_p, \tilde{Y}_p)$ (16) as

$$\text{(FDEA3)} \quad I(X_p, Y_p) = \text{Max} \sum_{i=1}^{M} \sum_{k=1}^{4} \beta_i^k + \sum_{r=1}^{S} \sum_{k=1}^{4} \gamma_r^k \tag{17}$$

$$\text{s.t.} \quad \sum_{j=1}^{N} \lambda_j x_{ij}^k + \beta_i^k x_{ip}^k \leq x_{ip}^k, \quad k = 1, \ldots, 4, \; i = 1, \ldots, M \tag{18}$$

$$\sum_{j=1}^{N} \lambda_j y_{rj}^k \geq y_{rp}^k + \gamma_r^k y_{rp}^k, \quad k = 1, \ldots, 4, \; r = 1, \ldots, S, \tag{19}$$

$$\sum_{j=1}^{N} \lambda_j = 1 \tag{20}$$

$$\beta_i^k \leq \beta_i^{k+1}, \quad k = 1, 2, 3 \;\; i = 1, \ldots, M, \tag{21}$$

$$\gamma_r^k \leq \gamma_r^{k+1}, \quad k = 1, 2, 3 \;\; r = 1, \ldots, S, \tag{22}$$

$$\beta_i^k, \gamma_r^k, \lambda_j \geq 0 \;\; i = 1, \ldots, M, r = 1, \ldots, S, j = 1, \ldots, N \tag{23}$$

There exists a strong relationship between (FDEA2) and (FDEA3) as the following theorem establishes.

Proposition 3. *$(\tilde{\beta}, \tilde{\gamma}, \lambda)$ is an optimal solution of (FDEA2), with $\lambda \in \mathbb{R}_+^N$, $\tilde{\beta} \in (TrFN)_+^M$ and $\tilde{\gamma} \in (TrFN)_+^S$, if and only if its corresponding representation is an optimal solution of (FDEA3), with $\lambda \in \mathbb{R}_+^N$, $\tilde{\beta} = (\beta_1^1, \beta_1^2, \beta_1^3, \beta_1^4, \ldots, \beta_M^1, \beta_M^2, \beta_M^3, \beta_M^4) \in \mathbb{R}_+^{4M}$ and $\tilde{\gamma} = (\gamma_1^1, \gamma_1^2, \gamma_1^3, \gamma_1^4, \ldots, \gamma_M^1, \gamma_M^2, \gamma_M^3, \gamma_M^4) \in \mathbb{R}_+^{4S}$*

Proof. It is not difficult to see that the constraint conditions (11) - (15) in (FDEA2) are equivalent to the constraint conditions (18) - (23) in (FDEA3). The rest of the proof is straightforward. □

Corollary 1. *The optimal objective function value of models (FDEA2) and (FDEA3) coincide and, therefore, given* $(\tilde{X}, \tilde{Y}) \in T_{FDEA}$ *with corresponding representation given by* $(X, Y) \in \mathbb{R}^{4M} \times \mathbb{R}^{4S}$*, it follows that* $I(\tilde{X}, \tilde{Y}) = I(X, Y)$.

Corollary 2. *If a DMU p is efficient, then* $I(X_p, Y_p) = 0$.

In addition to providing the inefficiency measure for each a DMU_p, the proposed (FDEA3) model provides fuzzy input and output targets $(\tilde{X}_p^{target}, \tilde{Y}_p^{target})$ given as

$$\tilde{X}_p^{target} = \sum_{j=1}^{N} \lambda_j^* \tilde{X}_j \tag{24}$$

$$\tilde{Y}_p^{target} = \sum_{j=1}^{N} \lambda_j^* \tilde{Y}_j \tag{25}$$

Note that the targets computed by model (FDEA3) for a given a DMU p are efficient, i.e. $\tilde{I}(\tilde{X}_p^{target}, \tilde{Y}_p^{target}) = \tilde{0}$.

Before presenting a numerical example to illustrate the proposed approach it may be interesting, as one of the reviewers inquired, to clarify the meaning of the fuzzy input and output slacks computed by the proposed approach. In this regard, we have to distinguish the interpretation in mathematical terms from its interpretation in economic/managerial terms. As regards the latter, the interpretation is the same as in conventional DEA, i.e. input slacks represent excess input consumption and output slacks represent output shortfalls. The former imply an inefficient use of the resources and the latter correspond to an underperformance in the production of the outputs. It is interesting to note that both effects occur simultaneously, i.e. the input reduction and the output increase can (and should) both be achieved at the same time.

The mathematical interpretation of the slacks is simple in the crisp case but more subtle in the fuzzy case. That is probably the reason why some FDEA

Table 2: Observed DMUs and results of proposed FDEA approach

DMU	Input	Output	$\tilde{I}(\tilde{x}, \tilde{y})$	Input Target	Output Target
	$\tilde{x}$	$\tilde{y}$		$\tilde{x}^{target}$	$\tilde{y}^{target}$
A	$(1, 3, 4)$	$(2, 3, 4)$	$(0, 0, 0)$	$(1, 3, 4)$	$(2, 3, 4)$
B	$(3.5, 4, 4.5)$	$(1.5, 2.5, 3.5)$	$(0.36, 0.36, 0.36)$	$(1.50, 3.375, 4.50)$	$(2.75, 3.75, 4.75)$
C	$(3, 4.5, 6)$	$(5, 6, 7)$	$(0, 0, 0)$	$(3, 4.5, 6)$	$(5, 6, 7)$
D	$(6, 6.5, 7)$	$(2.75, 4, 5.25)$	$(0.47, 0.47, 0.47)$	$(3, 4.5, 6)$	$(5, 6, 7)$
E	$(5, 7, 9)$	$(4.5, 5, 5.5)$	$(0.44, 0.53, 0.6)$	$(3, 4.5, 6)$	$(5, 6, 7)$
F	$(7.5, 8, 8.5)$	$(3, 3.5, 4)$	$(0.96, 1, 1.04)$	$(3, 4.5, 6)$	$(5, 6, 7)$
G	$(9, 10, 11)$	$(5.5, 6, 6.5)$	$(0, 0, 0)$	$(9, 10, 11)$	$(5.5, 6, 6.5)$
H	$(5.5, 6, 6.5)$	$(0.5, 2, 3.5)$	$(1.08, 1.08, 1.08)$	$(3, 4.5, 6)$	$(5, 6, 7)$

approaches consider crisp slacks. However, we believe that, since the observed inputs and outputs are fuzzy, their corresponding slacks should also be fuzzy. This makes a lot of sense. Thus, since there is uncertainty in the observed data, it is logical that there be uncertainty on the amount that they can be improved. Similarly, this uncertainty also reaches the input and output targets to be achieved to be efficient.

4. Numerical example

Let us consider the dataset from León et al. [30], with only one fuzzy input and one output fuzzy, to illustrate the proposed approach. Table 2 shows the observed data, given as triangular fuzzy numbers, in the first two columns. The third column reports the fuzzy inefficiency measure $\tilde{I}(\tilde{X}, \tilde{Y})$, which in this single-input, single-output case simplifies to $\tilde{I}(\tilde{x}, \tilde{y}) = \tilde{\beta} + \tilde{\gamma}$. The variables $\tilde{\beta}$ and $\tilde{\gamma}$ have been computed solving (FDEA3) problem and re-writing the solution in its equivalent fuzzy parametrization as described at the previous section. Finally, the two last columns provide the input and output targets computed using (24) and (25), respectively. Note that, among these 8 DMUs, we have that five of them are clearly inefficient, since their $\tilde{I}(\tilde{x}_p, \tilde{y}_p) \neq \tilde{0}$, whereas we have only three possible efficient DMUs, namely A, C, and G. Recall that a null

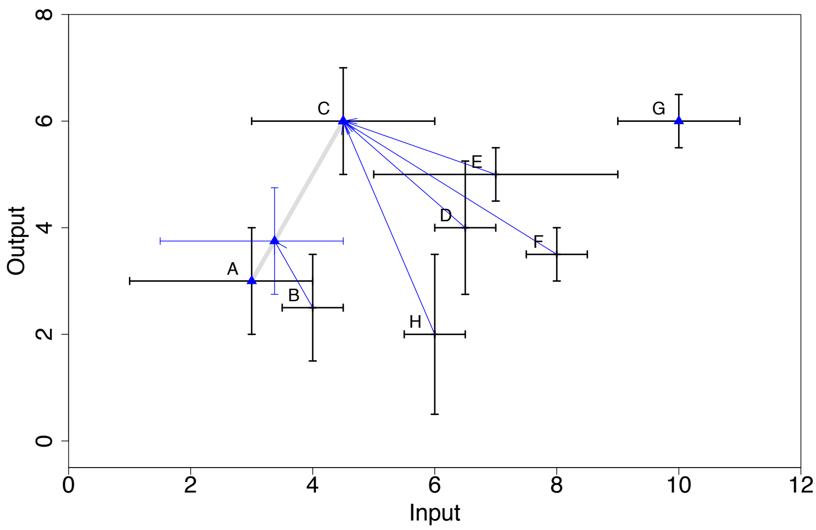

Figure 2: Observed DMUs and their corresponding targets marked with blue triangles. The thick, gray line joining DMUs A and C is a crisp approximation of the true fuzzy efficient frontier.

inefficiency measure is a necessary but not a sufficient condition for establishing the efficiency of a DMU.

This simple example also allows us to illustrate the proposed FDEA approach graphically as regards the efficiency status classification. Thus, Figure 2 shows the observed DMUs, labeled A to H. The plotted horizontal and vertical error bars represent the $0-$level (i.e., the closure of the support) of the corresponding trapezoidal fuzzy number, for both the input and output dimensions. The targets computed by the proposed approach are marked with blue triangles and its corresponding horizontal and vertical error bars. Note that DMUs A and C are efficient, and hence, they are projected onto themselves. The efficiency status of DMU G is undetermined, its inefficiency score $\tilde{I}(\tilde{x}_G, \tilde{y}_G)$ is zero although it is not projected onto itself but onto DMU C. Inefficient DMUs D, E, F, and H are projected onto DMU C while the remaining inefficient DMU (namely DMU B) is projected onto a target that is a linear convex combination

Table 3: Comparison with other FDEA approaches

Method	α-level	A	B	C	D	E	F	G	H
León et al. (2003) [30]	0	1	1	1	0.7500	0.6429	0.6050	1	0.6923
(Radial efficiency)	0.5	1	0.9412	1	0.6623	0.6172	0.5227	1	0.6400
	0.75	1	0.8675	1	0.6144	0.6010	0.4776	1	0.5854
	1	1	0.7500	1	0.5385	0.5714	0.4062	0.45	0.5000
Saati et al. (2002) [38]	0	1	1	1	1	1	0.640	0.867	0.764
(Radial efficiency)	0.5	1	0.764	1	0.706	0.835	0.462	0.628	0.457
	0.75	1	0.602	1	0.574	0.668	0.390	0.533	0.343
	1	0.750	0.469	1	0.462	0.536	0.328	0.450	0.250
Hatami-Marbini	0	6	7	2.333	8.25	8.5	9.5	9.5	10
et al. (2012) [24]	0.5	0	5	0	6.125	6	7.75	7.5	7.75
(Sum of slacks ineff.)	0.75	0	4	0	5.063	4.75	6.875	6.5	6.625
	1	0	2.5	0	4	3.5	6	5.5	5.5
$I(x_p, y_p)$	–	0	1.071	0	1.429	1.584	3.013	0	3.231

of DMUs A and C. The thick grey line joining these DMUs is an approximate crisp representation of the true fuzzy efficient frontier.

Figures 3 and 4 show the solutions, for each of the DMUs, computed by the proposed approach (FDEA). For each DMU, the left panel shows, in blue color, the fuzzy inefficiency measure $\tilde{I}(\tilde{x}, \tilde{y}) = \tilde{\beta} + \tilde{\gamma}$. The middle panel shows the input data (in black color) and the input target (in magenta color and dash line). The right panel, shows the output data (in black color) and the output target (in magenta color and dash line). When a DMU p is inefficient not only $\tilde{I}(\tilde{x}_p, \tilde{x}_e) \neq \tilde{0}$ but also its input target is shifted to the left (i.e. the original input is reduced) whereas its output target is shifted to the right (i.e. the original output is increased).

Finally, in Table 3, the results of the proposed approach are compared with those of other fuzzy DEA approaches, namely [30], [38] and [24]. The last row of the table considers the crisp inefficiency measure $I(x_p, y_p)$ computed by model (FDEA3). For the inefficient DMUs, with $I(x_p, y_p) > 0$, their efficiency measurements from [30] and [38] are less than one for nearly all the α levels. Equivalently in the case of [24], the corresponding sum of slacks is greater than zero for all α levels. On the other hand, for the three DMUs with $I(x_p, y_p) = 0$, and therefore possibly efficient, their efficiency scores from [30] and [38] are equal to one for all α levels in the case of DMUs A and C and for some alpha levels in the case of DMU G. The sum of slacks computed by [24] are zero for DMUs A and C for most $\alpha-$levels and greater than zero for DMU C for all $\alpha-$levels.

5. Conclusions

In this paper, a new fuzzy DEA slacks-based measure of inefficiency has been proposed. It requires solving a crisp linear optimization model and allows computing corresponding input and output targets. The normalized character of additive metric used in the objective function implies that it is units invariant. Computational experiments have been presented to validate the proposed approach and compare it with existing fuzzy DEA methods.

One of the advantages of the proposed approach with respect to other more complex approaches is that, apart from computing efficiency scores, it focuses on the input and output improvements that can be achieved, called slacks in DEA parlance. These improvements are also expressed in the form of efficient targets and provide managers with useful information on the variables on which the improvement effort should be concentrated and the extent of those efforts. Most existing FDEA approaches, particularly but not only those that use multiplier formulations, do not provide efficient targets and thus their managerial usefulness is limited.

A limitation of the proposed approach is that a null value of the computed inefficiency score is a necessary nut not sufficient condition for efficiency, i.e. it cannot discriminate between efficient and weakly efficient DMUs. This issue does not happen in the crisp DEA case, in which maximizing the sum of input and output slacks guarantees efficiency. This highlights the differences between the crisp and fuzzy data scenarios. Devising an appropriate phase II, as in radial DEA approaches, is a topic for further research. Extending the

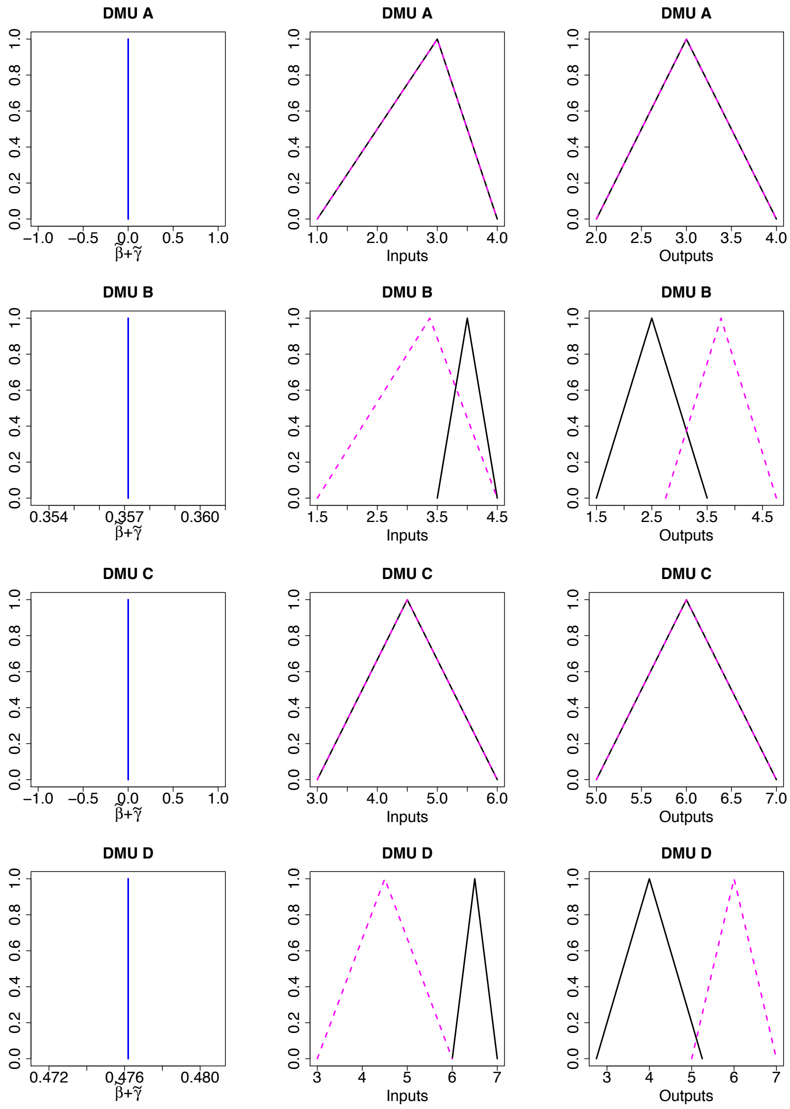

Figure 3: Solutions of proposed FDEA approach for DMUs A to D.

proposed to include undesirable outputs is also a worthy endeavour.

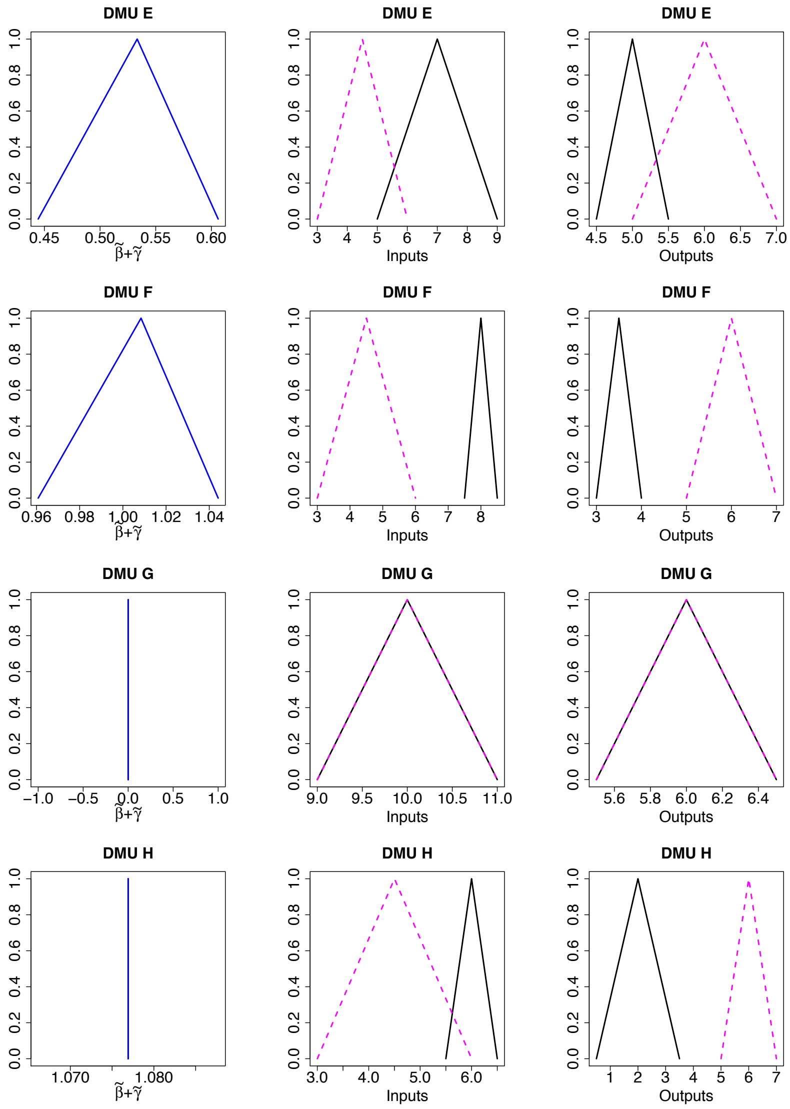

Figure 4: Solutions of proposed FDEA approach for DMUs E to H.

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