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Extended eigenvalues of composition operators

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A R T I C L E I N F O

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A B S T R A C T

A complex scalar λ is said to be an *extended eigenvalue* of a bounded linear operator A on a complex Hilbert space if there is a nonzero operator X such that $AX = \lambda XA$. The results in this paper provide a full solution to the problem of computing the extended eigenvalues for those composition operators C_φ induced on the Hardy space $H^2(\mathbb{D})$ by linear fractional transformations φ of the unit disk.

1. Introduction

The goal of this paper is the calculation of the extended eigenvalues for composition operators C_φ induced on the Hardy space $H^2(\mathbb{D})$ by linear fractional transformations φ of the unit disk. This problem leads to intriguing questions on the interface between complex analysis and operator theory.

This is the first part of a twofold project. The second part will appear in a forthcoming paper and it concerns the structure of the linear manifold of the extended eigenoperators (to be defined below) corresponding to a fixed extended eigenvalue, when that manifold is regarded as a bilateral module over the operator's commutant.

The main results of this paper are summarized in Table 1. The first column shows the acronyms that are used for the symbols φ. The second column exhibits the fixed points of the symbols φ when they are expressed in a standard form. The third column collects the explicit expressions of such standard forms.

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Table 1
Extended eigenvalues of linear fractional composition operators.

Symbol	Fixed points	Standard form of the symbol φ	Ext (C_φ)	Thm.
EA	$0, \infty$	$\varphi(z) = \omega z, \quad \lvert\omega\rvert = 1$	$\{\omega^n : n \in \mathbb{Z}\}$	3.1
LOX/HNA III	$c \in \mathbb{D}, \infty$	$\varphi(z) = a(z-c) + c, \quad \lvert a\rvert + \lvert 1-a\rvert \cdot \lvert c\rvert \le 1$	$\{\varphi'(c)^n : n \in \mathbb{Z}\}$	3.2
HA	$1, -1$	$\varphi(z) = \dfrac{z+r}{1+rz}, \quad 0 < r < 1$	$\partial\mathbb{D}$	4.4
HNA I	$1, \infty$	$\varphi(z) = rz + 1 - r, \quad 0 < r < 1$	$\overline{\mathbb{D}} \setminus \{0\}$	5.3
HNA II	$1, 0$	$\varphi(z) = \dfrac{rz}{1-(1-r)z}, \quad 0 < r < 1$	$\mathbb{C} \setminus \mathbb{D}$	5.7
PA	$1, 1$	$\varphi(z) = \dfrac{(2-a)z + a}{-az + 2 + a}, \quad \mathrm{Re}\, a = 0$	$\partial\mathbb{D}$	6.2
PNA	$1, 1$	$\varphi(z) = \dfrac{(2-a)z + a}{-az + 2 + a}, \quad \mathrm{Re}\, a > 0$	$\{e^{-at} : t \in \mathbb{R}\}$	6.5

The fourth column presents the results for the extended eigenvalues of the composition operators, and the fifth column points at the places in the paper where those results can be found.

1.1. Extended eigenvalues and extended eigenoperators

Our setting is an infinite dimensional, complex separable Hilbert space H. A complex scalar λ is called an *extended eigenvalue* of a bounded linear operator A provided that there exists a nonzero operator X such that

$$AX = \lambda XA. \tag{1.1}$$

Such an operator X is called an *extended eigenoperator* of the operator A. The family of all the extended eigenvalues of an operator A is called the *extended spectrum* of A, and it is denoted by Ext (A). Further, the notation $\mathcal{E}xt(\lambda, A)$ stands for the collection of all the extended eigenoperators corresponding to a given extended eigenvalue $\lambda \in$ Ext (A). If we accept the zero operator as an extended eigenoperator, then that collection becomes a weakly closed, linear manifold.

These notions have their roots in the simultaneous and independent works of Scott Brown [6], and Kim, Moore and Pearcy [11], about a generalization of the celebrated theorem of Victor Lomonosov [20]. They proved that if an operator A has a nonzero, compact extended eigenoperator, then the operator A has a non trivial, closed hyperinvariant subspace.

A systematic study of the extended eigenvalues and the extended eigenoperators of classical operators started with the work of Biswas, Lambert and the third author [2] for the Volterra operator, and continued soon after with the work of Lambert [17], who also gave sufficient conditions for an operator to have non trivial, closed hyperinvariant subspaces, and who calculated the extended spectra for scalar perturbations of the Volterra operator.

This line of research has become a very active field of investigation with many contributions, both in the search for invariant subspaces [1,12,16–18], and in the computation of the extended spectral picture for some special classes of operators [2–4,8,9,13,14,19,24,28].

1.2. Composition operators on the Hardy space

We shall be dealing with composition operators defined on the Hardy space $H^2(\mathbb{D})$. An analytic self map φ of the unit disk $\mathbb{D}$ induces a linear operator C_φ defined on the space of analytic functions on the unit disk by the expression

$$C_\varphi f = f \circ \varphi, \qquad f : \mathbb{D} \to \mathbb{C} \text{ analytic.}$$

Littlewood's subordination principle [27] ensures that C_φ can be restricted to a bounded linear operator on the Hardy space $H^2(\mathbb{D})$. Perhaps this is the most natural Hilbert space of analytic functions. Recall that the map that takes each function f to its radial limit f^*, defined almost everywhere by the expression $f^*(e^{i\theta}) := \lim f(re^{i\theta})$ as $r \to 1^-$, is an isometry from $H^2(\mathbb{D})$ into $L^2(\partial\mathbb{D})$, that is,

$$\|f\|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |f^*(e^{i\theta})|^2 \, d\theta \right)^{1/2}, \quad f \in H^2(\mathbb{D}).$$

A whole theory has been developed around composition operators on the Hardy space, that relates the function theoretic properties of φ with the operator theoretic properties of C_φ. A very nice treatise on this subject is the book of Joel Shapiro [27].

1.3. Classification of linear fractional transformations

Recall that a *linear fractional map* defined on the Riemann sphere $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is any map of the form

$$\varphi(z) = \frac{az + b}{cz + d}. \tag{1.2}$$

The condition $ad - dc \neq 0$ ensures that φ is not constant. Applying the usual conventions to the point at infinity, it is well known that φ becomes an automorphism of the Riemann sphere, and from a metric point of view, φ is also an isometry of the Riemann sphere provided with the chordal distance.

We are interested in *linear fractional transformations,* that is, the class LFT of linear fractional maps φ with the property that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. A necessary and sufficient condition for this property is that

$$|b\overline{d} - a\overline{c}| + |ad - bc| \leq |b|^2 - |d|^2.$$

This simple condition appears very seldom in the literature. It can be found, for instance, in the work of María J. Martín [21], or in the work of Contreras, Díaz-Madrigal, Martín, and Vukotić [7]. The latter reference provides another condition, that is somehow more involved, but has an easier proof.

There is a classification of LFTs according to their fixed point configuration. We refer the reader to the paper of Joel Shapiro [26] for that classification. What is important for our purposes is that there are eight classes: elliptic automorphic (EA), parabolic automorphic (PA), parabolic non automorphic (PNA), hyperbolic automorphic (HA), hyperbolic non automorphic of the first, second or third kind (HNA I), (HNA II), (HNA III), and loxodromic (LOX). After conjugation by suitable linear fractional maps, every LFT can be expressed in a standard form that is collected in Table 1.

The class HNA III of hyperbolic, non automorphic LFTs without a fixed point on $\partial\mathbb{D}$ is missing in Shapiro's classification, but there is an easy fix for this issue, because those maps have the same standard form as the loxodromic ones.

2. Preliminary results

In this section we will establish several lemmas that will be used throughout the paper. We start with a folk result that applies to any Hilbert space operator. We leave its proof to the reader as a simple but important exercise.

Lemma 2.1. *An operator $A \in \mathcal{B}(H)$ is injective if and only if $0 \notin \operatorname{Ext}(A)$.*

The following result is well known. It is a consequence of the open mapping theorem and the principle of analytic continuation.

Lemma 2.2. *Let us suppose that φ is a nonconstant, analytic self map of the open unit disk. Then C_φ is injective.*

If we combine Lemma 2.1 with Lemma 2.2, we conclude that 0 is never an extended eigenvalue of a composition operator.

Proposition 2.3. *If φ is an non constant analytic self map of the open unit disk, then $0 \notin \mathrm{Ext}(C_\varphi)$.*

Next, we present two basic general results about the behavior of the extended spectrum under taking the adjoint and a scalar multiplication. They can be easily derived from equation (1.1).

Lemma 2.4. *If $A \in \mathcal{B}(H)$ then $\mathrm{Ext}(A^*) \setminus \{0\} = \{1/\overline{\lambda} \colon \lambda \in \mathrm{Ext}(A) \setminus \{0\}\}$.*

Lemma 2.5. *If $A \in \mathcal{B}(H)$ and $\alpha \in \mathbb{C} \setminus \{0\}$ then $\mathrm{Ext}(\alpha A) = \mathrm{Ext}(A)$.*

The following result is a necessary condition for a complex number to be an extended eigenvalue of an injective operator with a total set of eigenvectors. We will use the notation $\sigma_p(A)$ for the *point spectrum* of an operator $A \in \mathcal{B}(H)$, that is, the set of all eigenvalues of A.

Lemma 2.6. *Let $A \in \mathcal{B}(H)$ be an injective operator with nonempty point spectrum. If A admits a total family of eigenvectors, then $\mathrm{Ext}\,(A) \subseteq \{\alpha/\beta \colon \alpha, \beta \in \sigma_p(A)\}$.*

Proof. Let $\mathcal{F}$ be a total subset of H consisting of eigenvectors for A. Then, let $\lambda \in \mathrm{Ext}\,(A)$ and let $X \in \mathcal{E}xt(\lambda, A)$. Next, there exists $f \in \mathcal{F}$ such that $Xf \neq 0$ and there exists $\beta \in \sigma_p(A)$ such that $Af = \beta f$. It follows that $AXf = \lambda XAf = \lambda\beta Xf$. Since $Xf \neq 0$ we conclude that $\alpha := \lambda\beta \in \sigma_p(A)$. Since A is injective, we have $\beta \neq 0$ so that $\lambda = \alpha/\beta$, as we wanted. □

Multiplication operators are a nice source of extended eigenoperators for composition operators. Recall that every bounded analytic function $b \in H^\infty(\mathbb{D})$ induces a *multiplication operator* $M_b \in \mathcal{B}(H^2(\mathbb{D}))$ defined by $M_b f = b \cdot f$. It is easy to check that M_b is indeed a bounded linear operator with $\|M_b\| = \|b\|_\infty$.

Lemma 2.7. *Let C_φ be the composition operator induced by $\varphi \colon \mathbb{D} \to \mathbb{D}$. Let $\lambda \in \sigma_p(C_\varphi)$ and let b be a function in $H^\infty(\mathbb{D})$ such that $C_\varphi b = \lambda b$. Then, $\lambda \in \mathrm{Ext}(C_\varphi)$ and $C_\varphi M_b = \lambda M_b C_\varphi$.*

Proof. Notice that for every $f \in H^2(\mathbb{D})$ we have

$$C_\varphi M_b f = C_\varphi(b \cdot f) = (b \circ \varphi) \cdot (f \circ \varphi) = (C_\varphi b) \cdot (C_\varphi f) = \lambda b \cdot C_\varphi f = \lambda M_b C_\varphi f,$$

so that $C_\varphi M_b = \lambda M_b C_\varphi$, as we wanted. □

The last two results of this section about the extended spectrum of a direct sum of two operators will be useful to reduce the problem of a hyperbolic, non automorphic composition operator of the second kind (HNA II), to the one of the first kind (HNA I).

Lemma 2.8. *Consider a direct sum decomposition $H = H_1 \oplus H_2$, let $A_1 \in \mathcal{B}(H_1)$ and let $A_2 \in \mathcal{B}(H_2)$. Then $\mathrm{Ext}(A_1) \cup \mathrm{Ext}(A_2) \subseteq \mathrm{Ext}(A_1 \oplus A_2)$.*

Proof. Clearly, it suffices to prove that $\operatorname{Ext}(A_1) \subseteq \operatorname{Ext}(A_1 \oplus A_2)$. To that end, let $\lambda \in \operatorname{Ext}(A_1)$ and let $X \in \mathcal{B}(H_1)$ be a nonzero operator such that $A_1 X = \lambda X A_1$. It is not hard to verify that $X \oplus 0$ is a nonzero operator in $\mathcal{B}(H)$ satisfying $(A_1 \oplus A_2)(X \oplus 0) = \lambda(X \oplus 0)(A_1 \oplus A_2)$. Therefore, $\lambda \in \operatorname{Ext}(A_1 \oplus A_2)$. □

Lemma 2.9. *Consider a direct sum decomposition $H = H_1 \oplus H_2$, let I_1 be the identity operator on H_1, and let A be an injective operator in $\mathcal{B}(H_2)$. Then*

$$\operatorname{Ext}(I_1 \oplus A^*) = \sigma_p(A^*) \cup \left[\overline{\sigma_p(A)}\right]^{-1} \cup \operatorname{Ext}(A^*).$$

Proof. Let $\lambda \in \operatorname{Ext}(I_1 \oplus A^*)$ and let $X \in \mathcal{E}xt(I_1 \oplus A^*, \lambda)$. Since $X \in \mathcal{B}(H)$, there is a representation of X as a block matrix with respect to the decomposition $H = H_1 \oplus H_2$, say

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

A simple computation shows that the equation $(I_1 \oplus A^*)X = \lambda X(I_1 \oplus A^*)$ is equivalent to the following system of equations:

$$(1-\lambda)X_{11} = 0, \quad X_{12}(I_1 - \lambda A^*) = 0,$$
$$(A^* - \lambda I_1)X_{21} = 0, \quad A^* X_{22} - \lambda X_{22} A^* = 0.$$

Since $X \neq 0$, one of the four operators X_{jk} must be nonzero. If $X_{11} \neq 0$, then $\lambda = 1$, which is always an extended eigenvalue of any operator. If $X_{22} \neq 0$ then $\lambda \in \operatorname{Ext}(A^*)$, and if $X_{21} \neq 0$, then $\lambda \in \sigma_p(A^*)$. Finally, if $X_{12} \neq 0$ then $\lambda \neq 0$ and $(I_1 - \overline{\lambda} A)X_{12}^* = 0$, so that $1/\overline{\lambda} \in \sigma_p(A)$, and the proof of the first inclusion is complete. Conversely, if one operator $X_{j_0 k_0}$ is nonzero, then we get an extended eigenoperator for $I_1 \oplus A^*$ by taking the other blocks X_{jk} equal to zero, and this fact completes the proof of the second inclusion, which yields the desired result. □

3. Elliptic and loxodromic maps

Recall that if φ is an *elliptic automorphism*, then φ has two fixed points, one in $\mathbb{D}$ and another in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Given this fixed point configuration, upon conjugation by disk automorphisms, we can suppose that the fixed points are $0, \infty$, which leads to the standard form $\varphi(z) = \omega z$. Let us observe that the conjugation by disk automorphisms induces a similarity at the operator level, and the extended spectra are invariant under similarities. Notice that ω^n is an eigenvalue of C_φ, and a corresponding eigenfunction is the monomial $e_n(z) = z^n$, for every $n \in \mathbb{N}_0$.

Theorem 3.1. *Let $\varphi(z) = \omega z$ be an elliptic automorphism of the unit disk. Then*

$$\operatorname{Ext}(C_\varphi) = \{\omega^n : n \in \mathbb{Z}\}.$$

Proof. The inclusion $\operatorname{Ext}(C_\varphi) \subseteq \{\omega^n : n \in \mathbb{Z}\}$ is a consequence of Lemma 2.6. Indeed, we know from Lemma 2.2 that C_φ is injective. On the other hand, the set $\{e_n : n \in \mathbb{N}_0\}$ of eigenfunctions for C_φ is a total subset of $H^2(\mathbb{D})$.

Next, each function e_n is bounded on $\mathbb{D}$, and it follows from Lemma 2.7 that $\{\omega^n : n \in \mathbb{N}_0\} \subseteq \operatorname{Ext}(C_\varphi)$. Thus, it is sufficient to show the inclusion $\{\omega^{-n} : n \in \mathbb{N}\} \subseteq \operatorname{Ext}(C_\varphi)$. Indeed, fix $k \in \mathbb{N}$ and let $X = M_{e_k}^*$. Notice that $X e_n = e_{n-k}$ for $n \geq k$, so that

$$C_\varphi X e_n = C_\varphi e_{n-k} = \omega^{n-k} e_{n-k} = \omega^{n-k} X e_n = \omega^{-k} X C_\varphi e_n. \tag{3.1}$$

Since $Xe_n = 0$ for all $n < k$, the equation (3.1) is valid for all $n \in \mathbb{N}_0$. Finally, the family $\{e_n : n \in \mathbb{N}_0\}$ spans a dense linear manifold in $H^2(\mathbb{D})$, so that $C_\varphi X = \omega^{-k} X C_\varphi$, and the proof is complete. □

Recall that if φ is a loxodromic map or a hyperbolic nonautomorphism of the third kind φ can be assumed to have one fixed point at infinity, while the other one, say c, belongs to $\mathbb{D}$. In this situation φ has the standard form

$$\varphi(z) = a(z - c) + c, \tag{3.2}$$

with $|a| + |1 - a| \cdot |c| \le 1$, if $c \ne 0$, and $|a| < 1$, if $c = 0$. The hyperbolic nonautomorphism of the third kind corresponds to the case $a > 0$. In this situation, the composition operator C_φ has a countable point spectrum, namely $\sigma_p(C_\varphi) = \{\varphi'(c)^n : n \in \mathbb{N}_0\}$. Further, each eigenvalue is of multiplicity one and a corresponding eigenfunction is given by

$$\sigma(z) = z - c. \tag{3.3}$$

We also know that $C_\varphi \sigma^n = \varphi'(c)^n \sigma^n$, and that the span of the eigenfunctions $\{\sigma^n : n \in \mathbb{N}_0\}$ is a dense linear manifold in $H^2(\mathbb{D})$.

Notice that $\sigma^{-1} \in L^\infty(\mathbb{T})$, hence the multiplication operator $M_{\sigma^{-1}}$ is bounded on $L^2(\mathbb{T})$. One knows that $H^2(\mathbb{D})$ can be identified with a subspace of $L^2(\mathbb{T})$ consisting of functions whose negative Fourier coefficients vanish. Therefore, let us regard $M_{\sigma^{-1}}$ as a bounded operator from $H^2(\mathbb{D})$ to $L^2(\mathbb{T})$. Let P be the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$, and consider the Toeplitz operator T defined by

$$Tf = PM_{\sigma^{-1}} f, \qquad \text{for all } f \in H^2(\mathbb{D}). \tag{3.4}$$

Clearly, $T\sigma^m = \sigma^{m-1}$, for $m \ge 1$. On the other hand,

$$\frac{1}{\sigma(e^{i\theta})} = \frac{1}{e^{i\theta} - c} = e^{-i\theta} \frac{1}{1 - ce^{-i\theta}} = e^{-i\theta} \sum_{n=0}^{\infty} c^n e^{-in\theta},$$

which shows that $T1 = P\sigma^{-1} = 0$.

Theorem 3.2. *Let φ be a map that is either loxodromic or hyperbolic non automorphic of the third kind. Then, we have*

$$\operatorname{Ext}(C_\varphi) = \{\varphi'(c)^n : n \in \mathbb{Z}\}.$$

Proof. Since $\sigma_p(C_\varphi) = \{\varphi'(c)^n : n \in \mathbb{N}_0\}$, it follows from Lemma 2.6 and Proposition 2.3 that $\operatorname{Ext}(C_\varphi) \subseteq \{\varphi'(c)^n : n \in \mathbb{Z}\}$. By Lemma 2.7, $\{\varphi'(c)^n : n \in \mathbb{N}_0\} \subseteq \operatorname{Ext}(C_\varphi)$. Therefore, it remains to establish that $\{\varphi'(c)^{-n} : n \in \mathbb{N}\} \subseteq \operatorname{Ext}(C_\varphi)$. We will show for the operator T defined by (3.4), that T^n is an extended eigenoperator of C_φ corresponding to the extended eigenvalue $\varphi'(c)^{-n}$. Since the set $\{\sigma^m : m \in \mathbb{N}_0\}$ spans a dense linear manifold in $H^2(\mathbb{D})$, it suffices to check that

$$C_\varphi T^n \sigma^m = \varphi'(c)^{-n} T^n C_\varphi \sigma^m,$$

for all $m \in \mathbb{N}_0$ and all $n \in \mathbb{N}$. The last identity follows from a straightforward computation. □

4. Hyperbolic automorphic maps

If φ is a hyperbolic automorphism of the unit disk, its fixed points lie on the boundary of the unit circle, and upon conjugation by disk automorphisms, we can suppose that the fixed point of φ are the points 1 and -1. Moreover, we can suppose that φ has the following standard form

$$\varphi(z) = \frac{z+r}{1+rz}, \qquad \text{for some } 0 < r < 1. \tag{4.1}$$

In what follows we will take advantage of some spectral information about C_φ that can be found in [26]. Let $R = (1+r)/(1-r)$. Every point in the open annulus

$$G := \{\alpha \in \mathbb{C} \colon R^{-1/2} < |\alpha| < R^{1/2}\}, \tag{4.2}$$

belongs to the point spectrum of C_φ. Moreover, every eigenvalue of C_φ can be written as

$$\gamma(w) := R^w, \tag{4.3}$$

where w belongs to the open strip

$$\Omega := \{w \in \mathbb{C} : -1/2 < \mathrm{Re}\,(w) < 1/2\}. \tag{4.4}$$

A corresponding eigenfunction is given by

$$e_w(z) := \left(\frac{1+z}{1-z}\right)^w. \tag{4.5}$$

The notion of an operator with rich point spectrum has been introduced recently as a way to determine the extended eigenvalues for Cesàro operators [14] and bilateral weighted shifts [13,14].

An operator $A \in \mathcal{B}(H)$ is said to have a *rich point spectrum* provided that $\operatorname{int}\sigma_p(A) \neq \emptyset$ and for every open disk $D \subseteq \sigma_p(A)$, the family of the corresponding eigenvectors

$$\bigcup_{z\in D} \ker(A-z) \tag{4.6}$$

is a total subset of H. The following result [14, Lemma 7.1] is a sufficient condition for an operator A to have rich point spectrum.

Lemma 4.1. *Let $A \in \mathcal{B}(H)$ and assume there are an open connected set $\Omega \subseteq \mathbb{C}$, an analytic mapping $h \colon \Omega \to H$ and a nonconstant analytic function $\gamma \colon \Omega \to \mathbb{C}$ such that*

(i) *$h(w) \in \ker[A - \gamma(w)] \setminus \{0\}$ for all $w \in \Omega$,*
(ii) *$\{h(w) \colon w \in \Omega\}$ is a total subset of H, and*
(iii) *$\sigma_p(A) \subseteq \operatorname{clos}\gamma(\Omega)$.*

Then A has rich point spectrum.

It turns out that Lemma 4.1 applies to the composition operators under consideration.

Theorem 4.2. *If φ is a hyperbolic automorphism of $\mathbb{D}$, then C_φ has rich point spectrum.*

Proof. The property of having rich point spectrum is invariant under similarity, so we may assume that φ is in the standard form given by the equation (4.1). It is fairly easy to check that the hypotheses of Lemma 4.1 are satisfied with $A = C_\varphi$, and with Ω, e_w and γ as in the equations (4.3), (4.4) and (4.5), and the mapping $h\colon \Omega \to H^2(\mathbb{D})$ defined by $h(w) := e_w$. It is clear that Ω is an open, connected set and that γ is an analytic function such that $\gamma(\Omega) = G = \sigma_p(C_\varphi)$, so that the condition (iii) is satisfied. Finally, the condition (i) means that $h(w)$ is an eigenfunction of C_φ, and we refer the reader to [23] for the proof of the condition (ii). Thus, it remains to show that h is an analytic mapping, that is, for every $g \in H^2(\mathbb{D})$, the function $f\colon \Omega \to \mathbb{C}$ defined by $f(w) = \langle h(w), g\rangle$ is analytic. This follows easily from a suitable application of [15, Lemma 6.6]. □

The next result [14, Theorem 3.3] establishes a useful fact about operators with rich point spectrum whose point spectrum is between the interior and the closure of an annulus.

Lemma 4.3. *If an operator $A \in \mathcal{B}(H)$ has rich point spectrum and there are constants $C, c > 0$ so that $\{z \in \mathbb{C}\colon c < |z| < C\} \subseteq \sigma_p(A) \subseteq \{z \in \mathbb{C}\colon c \le |z| \le C\}$, then* $\operatorname{Ext}(A) \subseteq \partial\mathbb{D}$.

Now we are ready to describe the family of all the extended eigenvalues for a hyperbolic automorphic composition operator.

Theorem 4.4. *If C_φ is the composition operator induced by a hyperbolic automorphism of the unit disk then the set of all extended eigenvalues for C_φ is the unit circle, that is,* $\operatorname{Ext}(C_\varphi) = \partial\mathbb{D}$.

Proof. First, we show that $\partial\mathbb{D} \subseteq \operatorname{Ext}(C_\varphi)$. Notice that $\partial\mathbb{D} = \{\gamma(it)\colon t \in \mathbb{R}\}$, where γ is given by (4.3). Further, $\gamma(it)$ is an eigenvalue of C_φ and a corresponding eigenfunction e_{it} is bounded on the unit disk. It follows from Lemma 2.7 that $\gamma(it) \in \operatorname{Ext}(C_\varphi)$.

In the other direction, we know from Theorem 4.2 that C_φ has rich point spectrum, and moreover, its point spectrum $\sigma_p(C_\varphi)$ is the open annulus given by equation (4.2). Hence, the inclusion $\operatorname{Ext}(C_\varphi) \subseteq \partial\mathbb{D}$ follows at once from Lemma 4.3. □

5. Hyperbolic non automorphic maps

In this section, φ is a hyperbolic, non automorphic linear fractional transformation of the unit disk. First, we consider the class HNA I, which corresponds to the following fixed point configuration: the first fixed point on $\partial\mathbb{D}$ and the second in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. In this case, we may assume that the fixed points are $z = 1$ and $z = \infty$. This leads to the standard form

$$\varphi(z) = rz + (1 - r), \quad \text{for some} \quad 0 < r < 1. \tag{5.1}$$

Deddens [10, Theorem 3 (iv)] proved that the point spectrum of C_φ is the punctured disk

$$\sigma_p(C_\varphi) = \{\lambda \in \mathbb{C}\colon 0 < |\lambda| < r^{-1/2}\}. \tag{5.2}$$

Moreover, he proved that for every $w \in \mathbb{C}$ with $\operatorname{Re}(w) > -1/2$, the function $e_w(z) = (1 - z)^w$ is an eigenfunction of C_φ corresponding to the eigenvalue $\lambda = r^w$, that is,

$$(C_\varphi e_w)(z) = r^w e_w(z). \tag{5.3}$$

In order to compute the extended eigenvalues of C_φ we make use of the following result.

Theorem 5.1. *Let φ be a hyperbolic, non automorphic linear fractional selfmap of the unit disk, with a fixed point in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Then C_φ has rich point spectrum.*

Proof. The strategy of the proof is exactly the same as in the proof Theorem 4.2. Namely, we will apply Lemma 4.1 to the operator $A = C_\varphi$, the open half plane $\Omega := \{w \in \mathbb{C} : \operatorname{Re}(w) > -1/2\}$, the mapping $h: \Omega \to H^2(\mathbb{D})$ defined by $h(w)(z) = (1-z)^w$, and the function $\gamma: \Omega \to \mathbb{C}$ defined by the expression $\gamma(w) = r^w$. Then, conditions (i) and (iii) of Lemma 4.1 follow from (5.3) and (5.2). Also, regarding condition (ii), it suffices to show that the linear span of $\{e_n : n \in \mathbb{N}_0\}$ is dense in $H^2(\mathbb{D})$. This is obvious because, for every $n \in \mathbb{N}_0$, the polynomials $\{(1-z)^k : 0 \le k \le n\}$ generate the same linear manifold as the monomials $\{z^k : 0 \le k \le n\}$. Finally, the fact that h is an analytic mapping can be derived in the same way as in the proof of Theorem 4.2. □

It was shown in [14, Theorem 3.1] that the shape of the point spectrum determines the extended spectral picture in the following sense.

Lemma 5.2. *If $A \in \mathcal{B}(H)$ has rich point spectrum and $\lambda \in \operatorname{Ext}(A)$ then $\lambda \cdot \operatorname{int} \sigma_p(A) \subseteq \operatorname{clos} \sigma_p(A)$.*

This result leads to the description of the set of all extended eigenvalues of C_φ.

Theorem 5.3. *Let φ be a hyperbolic, nonautomorphic linear fractional selfmap of the unit disk with a fixed point in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Then, we have $\operatorname{Ext}(C_\varphi) = \overline{\mathbb{D}} \setminus \{0\}$.*

Proof. Once again, we may assume that φ is in the standard form given by equation (5.1). Let $\lambda \in \overline{\mathbb{D}} \setminus \{0\}$. Then, there exists $w \in \mathbb{C}$ such that $\operatorname{Re}(w) \ge 0$ and $\lambda = r^w$. Thus, λ is an eigenvalue of C_φ and a corresponding eigenfunction is the bounded analytic function $e_w(z) = (1-z)^w$. It follows from Lemma 2.7 that $\overline{\mathbb{D}} \setminus \{0\} \subseteq \operatorname{Ext}(C_\varphi)$.

Conversely, let $\lambda \in \operatorname{Ext}(C_\varphi)$. We know from Proposition 2.3 that $\lambda \ne 0$. Since Theorem 5.1 states that φ has rich point spectrum, an application of Lemma 5.2 gives $\lambda \cdot [D(0, r^{-1/2}) \setminus \{0\}] \subseteq \overline{D}(0, r^{-1/2})$, and this easily yields $|\lambda| \le 1$, as we wanted. □

Now, let us consider the maps in the class HNA II, that is, the LFTs with one fixed point on $\partial\mathbb{D}$ and the other one in $\mathbb{D}$. We can assume without loss of generality that a fixed point is $z = 0$ and the other one is $z = 1$. This yields the standard form

$$\varphi(z) = \frac{rz}{1-(1-r)z}, \quad \text{for some } 0 < r < 1. \tag{5.4}$$

The case HNA II can be reduced to the case HNA I by using Lemma 2.8 and Lemma 2.9, in combination with the following result, due to Shapiro [26, p. 864] using ideas from his joint paper with Bourdon [5].

Theorem 5.4. *If φ is given by formula (5.4) and $\psi(z) = rz + (1-r)$, then C_φ is unitarily equivalent to the operator $I_1 \oplus rC_\psi^*$, where I_1 is the identity operator on the subspace of constant functions.*

Now we immediately get a result about the extended spectrum of C_φ.

Theorem 5.5. *If φ is a hyperbolic, non automorphic linear fractional transformation of the unit disk with a fixed point in $\mathbb{D}$, then $\operatorname{Ext}(C_\varphi) \supseteq \mathbb{C} \setminus \mathbb{D}$.*

Proof. First, we may assume without loss of generality that φ is given by the formula (5.4). We know from Theorem 5.4 that C_φ is unitarily equivalent to $I_1 \oplus rC_\psi^*$, so that $\operatorname{Ext}(C_\varphi) = \operatorname{Ext}(I_1 \oplus rC_\psi^*)$. Next, it follows from Lemma 2.5 and Lemma 2.8 that

$$\mathrm{Ext}\,(I_1 \oplus rC_\psi^*) \supseteq \mathrm{Ext}\,(rC_\psi^*) = \mathrm{Ext}\,(C_\psi^*).$$

Recall from the proof of Theorem 5.1 that the functions $e_n(z) = (1-z)^n$, $n \in \mathbb{N}_0$, span a dense linear manifold in $H^2(\mathbb{D})$, and since $C_\psi e_n = r^n e_n$ for all $n \in \mathbb{N}_0$, it follows that C_ψ has dense range, and therefore C_ψ^* is injective. Hence, we get from Lemma 2.1 that $0 \notin \mathrm{Ext}(C_\psi^*)$. Finally, using Theorem 5.3 and Lemma 2.4, we obtain $\mathrm{Ext}\,(C_\psi^*) = \mathbb{C} \setminus \mathbb{D}$. This shows that $\mathrm{Ext}(C_\varphi) \supseteq \mathbb{C} \setminus \mathbb{D}$, as we wanted. □

Our strategy to prove the inclusion $\mathrm{Ext}\,(C_\varphi) \subseteq \mathbb{C} \setminus \mathbb{D}$ is to apply Lemma 2.9 to the operator $A = C_\psi^*$. We know that $\sigma_p(C_\psi)$ is given by the formula (5.2). Also, we showed in the proof of Theorem 5.5 that $\mathrm{Ext}\,(C_\psi^*) = \mathbb{C} \setminus \mathbb{D}$. Therefore, we only need to compute $\sigma_p(C_\psi^*)$.

Recall that the Euler operator E_r is defined for $0 < r < 1$ on the Hilbert space ℓ^2 by the expression

$$(E_r f)(n) := \sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k} f(k). \tag{5.5}$$

Deddens [10, p. 862] proved that C_ψ^* is unitarily equivalent to E_r. On the other hand, we know from [25, Theorem 2] that the Euler operator E_r has empty point spectrum. Summarizing, we have

Corollary 5.6. *If $0 < r < 1$ and $\psi(z) = rz + (1-r)$, then the point spectrum of C_ψ^* is empty.*

Now we are all set to describe the extended spectrum of C_φ.

Theorem 5.7. *If φ is a hyperbolic, non automorphic linear fractional transformation of the unit disk with a fixed point in $\mathbb{D}$, then* $\mathrm{Ext}\,(C_\varphi) = \mathbb{C} \setminus \mathbb{D}$.

Proof. We already know from Theorem 5.5 that $\mathrm{Ext}(C_\varphi) \supseteq \mathbb{C} \setminus \mathbb{D}$. Conversely, it follows from Theorem 5.4 and Lemma 2.9 that

$$\mathrm{Ext}(C_\varphi) = \mathrm{Ext}(I_1 \oplus rC_\psi^*) \subseteq \sigma_p(rC_\psi^*) \cup \left[\overline{\sigma_p(rC_\psi)}\right]^{-1} \cup \mathrm{Ext}(rC_\psi^*).$$

We also know from Corollary 5.6 that $\sigma_p(rC_\psi^*) = \emptyset$. Using equation (5.2) we obtain that

$$\left[\overline{\sigma_p(rC_\psi)}\right]^{-1} = \{\lambda \in \mathbb{C} : |\lambda| > r^{-1/2}\} \subseteq \mathbb{C} \setminus \mathbb{D}.$$

Finally, it follows from Lemma 2.5 and the fact that $\mathrm{Ext}\,(C_\psi^*) = \mathbb{C} \setminus \mathbb{D}$, (see the proof of Theorem 5.5), that $\mathrm{Ext}\,(rC_\psi^*) = \mathrm{Ext}(C_\psi^*) = \mathbb{C} \setminus \mathbb{D}$, and the theorem is proved. □

6. Parabolic maps

In this section we consider parabolic LFTs, that is, linear fractional transformations of the unit disk with exactly one fixed point, that necessarily lies on the boundary of the unit disk. After conjugation with a suitable linear fractional map, such an LFT adopts the standard form

$$\varphi(z) = \frac{(2-a)z + a}{-az + 2 + a}, \tag{6.1}$$

where $a \in \mathbb{C}$ is a constant such that $\mathrm{Re}\,(a) \geq 0$. The family of functions $\{e_t(z) : t \geq 0\}$ defined by

$$e_t(z) = \exp\left(-t\,\frac{1+z}{1-z}\right), \tag{6.2}$$

consists of eigenfunctions for the operator C_φ corresponding to the eigenvalues e^{-at}. The point spectrum of the parabolic case is also known to be the set $\sigma_p(C_\varphi) = \{e^{-at} : t \geq 0\}$.

First of all, we suppose that φ is a disk automorphism, that is, φ belongs to the class PA. Then, $\mathrm{Re}\,(a) = 0$, hence $\sigma_p(C_\varphi) = \partial\mathbb{D}$. The following result is due to Nordgren, Rosenthal and Wintrobe [23].

Lemma 6.1. *The family of eigenfunctions* $\{e_t : t \geq 0\}$ *spans a dense linear manifold in* $H^2(\mathbb{D})$.

We now proceed to compute the extended spectrum of C_φ.

Theorem 6.2. *If* φ *is a parabolic automorphism of the unit disk, then* $\mathrm{Ext}(C_\varphi) = \partial\mathbb{D}$.

Proof. We know from Lemma 6.1 that C_φ has a total family of eigenfunctions. Since C_φ is invertible, Lemma 2.6 applies and the inclusion $\mathrm{Ext}(C_\varphi) \subseteq \partial\mathbb{D}$ follows from the fact that $\sigma_p(C_\varphi) = \partial\mathbb{D}$. The reverse inclusion $\mathrm{Ext}(C_\varphi) \supseteq \partial\mathbb{D}$ follows from Lemma 2.7. $\square$

Next, we turn our attention to the class PNA of parabolic non automorphic LFTs. The function φ is still given by (6.1) but the side condition now is that $\mathrm{Re}\,(a) > 0$.

Recall that the Sobolev space $W^{1,2}[0, +\infty)$ is the linear space of all functions $f \in L^2[0, +\infty)$ that are absolutely continuous on each bounded subinterval of the interval $[0, +\infty)$ and such that $f' \in L^2[0, +\infty)$.

It is a well known fact that when $W^{1,2}[0, +\infty)$ is provided with the norm

$$\|f\|_{1,2} = \left(\int_0^\infty (|f(t)|^2 + |f'(t)|^2)\, dt \right)^{1/2},$$

it becomes separable, complex Hilbert space.

We refer to the appendix of our paper [15] for a proof of the following well known fact.

Theorem 6.3. *If* $f, g \in W^{1,2}[0, +\infty)$ *then* $fg \in W^{1,2}[0, +\infty)$, *and moreover,*

$$\|fg\|_{1,2} \leq \sqrt{2}\, \|f\|_{1,2} \|g\|_{1,2}.$$

The inequality above means that, after rescaling the norm, $W^{1,2}[0, +\infty)$ becomes a Banach algebra.

Montes-Rodríguez, Ponce-Escudero and Shkarin [22] proved that C_φ is similar to a multiplication operator defined on the Sobolev algebra. More precisely, they proved the following result.

Theorem 6.4. *Let* S *be the linear map defined by* $(Sf)(t) := \langle f, e_t \rangle$. *Then,* S *is a linear isomorphism from* $H^2(\mathbb{D})$ *onto* $W^{1,2}[0, +\infty)$. *Moreover, if* φ *is given by the formula (6.1), then* $C_\varphi^* = S^{-1} M_\psi S$, *where* M_ψ *denotes the operator of multiplication by the function* $\psi(t) = e^{-\overline{a}t}$, $\mathrm{Re}\,(a) > 0$, *on the Sobolev algebra* $W^{1,2}[0, +\infty)$. *Further,* ψ *is a cyclic function for the operator* M_ψ, *that is, the span of* $\{\psi^n : n \in \mathbb{N}\}$ *is a dense linear manifold in* $W^{1,2}[0, +\infty)$.

Now we have at hand all the results that we need to tackle the parabolic non automorphic case. Notice that, even if φ is not given in a standard form, there is a unique $a \in \mathbb{C}$ such that $\mathrm{Re}\, a > 0$ and $\sigma_p(C_\varphi) = \{e^{-at} : t \geq 0\}$.

Theorem 6.5. *If* φ *is a parabolic, non automorphic linear fractional transformation of the unit disk, then* $\mathrm{Ext}\,(C_\varphi) = \{e^{-at} : t \in \mathbb{R}\}$.

Proof. There is no loss of generality in assuming that the symbol φ is of the form (6.1). We know that $\sigma_p(C_\varphi) = \{e^{-at}: t \geq 0\}$. Also, Lemma 6.1 shows that C_φ has a total set of eigenfunctions. Thus, it follows from Lemma 2.6 that $\mathrm{Ext}\,(C_\varphi) \subseteq \{e^{-at}: t \in \mathbb{R}\}$. Also, the inclusion $\{e^{-at}: t \geq 0\} \subseteq \mathrm{Ext}\,(C_\varphi)$ is a consequence of Lemma 2.7 and the fact that every eigenfunction e_t given by the formula (6.2) is a bounded analytic function.

Now, we need to show that $\{e^{at}: t > 0\} \subseteq \mathrm{Ext}\,(C_\varphi)$. It follows from Proposition 2.3 and Lemma 2.4 applied to $A = C_\varphi^*$ that $\mathrm{Ext}\,(C_\varphi) = \{1/\overline{\lambda}: \lambda \in \mathrm{Ext}\,(C_\varphi^*) \setminus \{0\}$. We know from Theorem 6.4 that C_φ^* is similar to M_ψ, where $\psi(t) = e^{-\overline{a}t}$. Thus, we get $\mathrm{Ext}\,(C_\varphi) = \{1/\overline{\lambda}: \lambda \in \mathrm{Ext}\,(M_\psi) \setminus \{0\}\}$. Consequently, we will be done if we can show that $e^{-\overline{a}t} \in \mathrm{Ext}(M_\psi)$.

Let $t_0 > 0$ and consider the translation operator T defined by the expression $(Tf)(t) = f(t + t_0)$. It is easy to check that T is a bounded linear operator on $W^{1,2}[0, +\infty)$ with $\|T\| \leq 1$. Let $h \in W^{1.2}[0, +\infty)$ be a nonzero function such that $h(t_0) = 0$, and consider the operator X defined by

$$(Xf)(t) \colon = \begin{cases} h(t)f(t - t_0), & \text{if } t \geq t_0, \\ 0, & \text{if } 0 \leq t < t_0. \end{cases}$$

First we will show that X is a bounded linear operator on $W^{1,2}[0, +\infty)$. Notice that

$$\begin{aligned} \|Xf\|_{1,2}^2 &= \int_{t_0}^{+\infty} |h(t)f(t - t_0)|^2\,dt + \int_{t_0}^{\infty} |[h(t)f(t - t_0)]'|^2\,dt \\ &= \int_0^{+\infty} |h(u + t_0)f(u)|^2\,du + \int_0^{\infty} |[h(u + t_0)f(u)]'|^2\,du \\ &= \|(Th)f\|_{1,2}^2 \leq 2\|Th\|_{1,2}^2 \|f\|_{1,2}^2 \leq 2\|f\|_{1,2}^2, \end{aligned}$$

where we have used Theorem 6.3 in the inequality next to the last step.

Finally, we show that $M_\psi X = e^{-\overline{a}t_0} X M_\psi$. It suffices to test this identity on the total set $\{\psi^n: n \in \mathbb{N}\}$. If $t \in [0, t_0)$, both sides of the equality $(M_\psi X \psi^n)(t) = e^{-\overline{a}t_0}(X M_\psi \psi^n)(t)$ vanish, and if $t \geq t_0$, we have

$$\begin{aligned} (M_\psi X \psi^n)(t) &= e^{-\overline{a}t} h(t) e^{-n\overline{a}(t - t_0)} \\ &= e^{-\overline{a}t_0} h(t) e^{-(n+1)\overline{a}(t - t_0)} \\ &= e^{-\overline{a}t_0} (X \psi^{n+1})(t) \\ &= e^{-\overline{a}t_0} (X M_\psi \psi^n)(t), \end{aligned}$$

and the theorem is proved. □

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