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A CASE STUDY ON HOW PRIMARY-SCHOOL IN-SERVICE TEACHERS CONJECTURE AND PROVE: AN APPROACH FROM THE MATHEMATICAL COMMUNITY

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Abstract

This paper studies how four primary-school in-service teachers develop the mathematical practices of conjecturing and proving. From the consideration of professional development as the legitimate peripheral participation in communities of practice, these teachers' mathematical practices have been characterised by using a theoretical framework (consisting of categories of activities) that describes and explains how a research mathematician develops these two mathematical practices. This research has adopted a qualitative methodology and, in particular, a case study methodological approach. Data was collected in a working session on professional development while the four participants discussed two questions that invoked the development of the mathematical practices of conjecturing and proving. The results of this study show the significant presence of informal activities when the four participants conjecture, while few informal activities have been observed when they strive to prove a result. In addition, the use of examples (an informal activity) differs in the two practices, since examples support the conjecturing process but constitute obstacles for the proving process. Finally, the findings are contrasted with other related studies and several suggestions are presented that may be derived from this work to enhance professional development.

Keywords: conjecturing, proving, primary-school in-service teachers, professional development, research mathematicians

Abstrak

Artikel ini mempelajari tentang bagaimana empat guru sekolah dasar mengembangkan latihan mengajar matematika dari dugaan dan pembuktian. Berdasarkan pertimbangan pengembangan profesional sebagai partisipasi periferal yang sah dalam komunitas pembelajaran, latihan mengajar matematika guru ini telah dicirikan dengan menggunakan kerangka kerja teoretis (terdiri dari kategori kegiatan) yang menggambarkan dan menjelaskan bagaimana seorang peneliti matematika mengembangkan dua latihan mengajar matematika ini. Penelitian ini menggunakan metodologi kualitatif dan khususnya pendekatan metodologi studi kasus. Data dikumpulkan dalam sesi pelatihan pengembangan profesional sementara empat peserta membahas dua pertanyaan yang meminta pengembangan pembelajaran matematika dari dugaan dan pembuktian. Hasil penelitian ini menunjukkan adanya aktivitas informal yang signifikan ketika keempat partisipan berspekulasi, sementara beberapa aktivitas informal telah diamati ketika mereka berusaha untuk membuktikan hasil. Selain itu, penggunaan contoh (kegiatan informal) berbeda dalam dua latihan mengajar, karena contoh mendukung proses dugaan tetapi merupakan hambatan untuk proses pembuktian. Akhirnya, temuan ini dibandingkan dengan studi terkait lainnya dan beberapa saran disajikan yang mungkin berasal dari pekerjaan ini untuk meningkatkan pengembangan profesional.

Kata kunci: melakukan konjektur (dugaan), membuktikan, guru SD, pengembangan profesional, meneliti matematikawan

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In mathematics education, research into mathematical practices is receiving increasing attention. This growth in interest is, on the one hand, motivated by studies from the philosophy of mathematics that focus on the processes of construction of mathematical knowledge (Lakatos, 1976; Tymoczko, 1998) and, on the other hand, by suggestions of a curricular nature that explicitly indicate the inclusion of mathematical practices as academic mathematical content (National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; RAND Mathematics Study Panel, 2003; U.K. Department of Education, 2014). Throughout this paper, *mathematical practices* (disciplinary practices in Rasmussen, Wawro, & Zandieh, 2015) are considered as being those mathematical activities developed by research mathematicians when building mathematical knowledge during their research. The present study focuses on the mathematical practices of conjecturing and proving, for their distinguished role in the teaching and learning of problem-solving (Chong, Shahrill, & Li, 2019) and in view of their relevance for the development of mathematical knowledge (Turrisi, 1997). Indeed, the existing literature brings to light the significance of these practices as a means for in-depth learning in all the mathematical areas, whereby students' engagement in these activities can significantly influence their mathematical training, thereby giving them a wide background for conceptual understanding (Stylianides & Ball, 2008; Watson, 1980).

Research into mathematical practices developed specifically by in-service teachers, which may yield valuable information for the improvement of the teaching and learning of such practices, is a field that could be further explored (Ball, Thames, & Phelps, 2008; Carrillo-Yañez et al., 2018). This research field may contribute, among others, towards the design of professional development activities, towards the professional noticing of mathematical practices, and consequently towards the promotion of students' understanding of these practices and towards their engagement therein. In this respect, several interesting contributions focus on conjecturing and proving, and include: the research by Knuth (2002), which analyses teachers' conceptions of the role and nature of proof; the study by Ko (2010), which reviews mathematics teachers' conceptions of proof and discusses possible implications for educational research; the paper by Melhuish, Thanheiser, and Guyot (2018) on professional noticing of the mathematical practices of generalising (conjecturing) and justifying (argumentation and proof); and the research by Lesseig (2016), which studies teachers' conjecturing, generalising and justifying behaviour when becoming involved in a classic number theory task that invokes these three mathematical practices. Other recent contributions on this topic by Astawa, Budayasa, and Juniati (2018) and by Oktaviyanthi, Herman, and Dahlan (2018) have studied the process of future mathematics teacher cognition in constructing mathematical conjecture and the way in which future mathematics teacher can prove the limit of a function by formal definition, respectively. In summary, mathematics in-service teachers must know how to make conjectures and provide proofs to foster these mathematical practices in their students, which motivates this research field and, in particular, the present study. Therefore, this paper studies the mathematical practices of primary-school in-service teachers and, specifically, focuses on the way in which these teachers develop the mathematical practices of conjecturing and proving in a professional training context.

Conjecturing and Proving Activities

Conjecturing and proving constitute two mathematical practices that have been extensively studied by the research community over the years. Consequently, many authors from various fields have referred to these practices and have assigned them different connotations. For this reason, this section begins by specifying certain terminology to frame this research. As mentioned earlier, the term *mathematical practice* is employed to refer to the mathematical activities developed by research mathematicians when they build mathematical knowledge during their research. In particular, the term *research mathematician* refers to those researchers who have a Ph.D. and have published research papers in mathematics. Furthermore, *conjecturing* and *proving* are the mathematical practices that generate conjectures and proofs respectively. Specifically, a conjecture is assumed to be a statement that can be true or false, appears reasonable, "has not been convincingly justified and yet it is not known to be contradicted by any examples, nor is it known to have any consequences which are false" (Mason, Burton, & Stacey, 1982, p. 58). Moreover, the definition of proof by Weber and Mejia-Ramos (2011) is adopted: "the socially sanctioned written product that results from mathematicians' attempts to justify why a conjecture is true" (p. 331).

In reviewing the literature, many studies can be encountered that advocate conjecturing and proving as two mathematical and very closely related practices which constitute two sides of the same coin. Several significant authors who support this consideration include: Peirce (Turrisi, 1997) and Lakatos (1976), from the philosophy of mathematics; and Polya (1954), Alibert and Thomas (2002), Boero and collaborators (Boero, 2007; Boero, Garuti, & Lemut, 2007), and Rasmussen et al. (2015), from mathematics education. All these authors highlight, from different points of view, that these two mathematical practices are interrelated and are essential for the construction of mathematical knowledge. For these reasons, the joint study of these two mathematical practices is considered in this paper.

In this study, the theoretical framework that is employed to analyse the data consists of several categories of activities that were identified to characterise how research mathematicians' conjecture and prove (Fernández-León, Gavilán-Izquierdo, & Toscano, 2020). The use of this framework for this research is justified below. In this work, professional development is considered as the legitimate peripheral participation in communities of practice (Lave & Wenger, 1991). This theoretical perspective maintains that a teacher learns mathematics when he/she engages in research mathematicians' standard practices by approaching the way in which research mathematicians themselves participate. For this reason, knowing how research mathematicians develop mathematical practices may be conducive to describing and explaining how in-service teachers develop such practices. In particular, a characterisation of a research mathematician's mathematical practices of conjecturing and proving (Fernández-León et al., 2020) is employed in this paper to study how several primary-school in-service teachers conjecture and prove. Indeed, research mathematicians' mathematical practices are also a source of information to improve the teaching and learning of these practices at any academic level, since they report on "what it is that we want students to learn and how instruction should be designed" (Weber & Dawkins, 2018, p. 70). In recent decades, many other studies that try to gain an accurate understanding of how research mathematicians develop mathematical practices have been conducted. For instance, Burton (1998, 2004) proposes a model regarding how mathematicians learn mathematical practices through their own research, while Ouvrier-Buffet (2015) and Martín-Molina, González-Regaña, and Gavilán-Izquierdo (2018) study the mathematicians' mathematical practice of defining, and Weber (2008) researches into the mathematicians' practice of proving.

The characterisation of the mathematical practices of conjecturing and proving given by Fernández-León et al. (2020) is now described. This characterisation arises from a case study with a single research mathematician. The characterisation encompasses categories of activities that describe and explain how that particular research mathematician conjectures and proves. To be precise, Fernández-León et al. (2020) found five categories of conjecturing activities (C.H.a, C.H.b, C.H.c, C.V.a, C.V.b) and six categories of proving activities (P.H.a, P.H.b, P.V.a, P.V.b, P.V.c, P.V.d). The letter C in the code of certain categories refers to the act of conjecturing and the letter P refers to proving. In their paper, Fernández-León et al. (2020) consider the constructs of *horizontal mathematising* and *vertical mathematising* of Rasmussen, Zandieh, King, and Teppo (2005) to organise the mathematical activities (the categories) that are identified when conjecturing and proving. Rasmussen et al. (2005) slightly adapt these two constructs introduced by Treffers (1987), who uses them to describe the so-called progressive mathematisation. Thus, the codes of the aforementioned categories also include either the letter H, if the category of activities is of a horizontal nature, or V, if the category is of a vertical nature. Rasmussen et al. (2005) use these two constructs (horizontal and vertical mathematising) to describe how certain students develop the mathematical practices of defining, symbolising, and algorithmatising and thus characterise their so-called *advancing mathematical activity*. On the one hand, horizontal mathematising "is mainly related to initial or informal ways of reasoning" (Fernández-León et al., 2020, p. 3) and, on the other hand, "vertical mathematising refers to those activities built on horizontal activities with the aim of creating new mathematical ideas or realities" (Fernández-León et al., 2020, p. 3). A brief description is now offered of each of the categories of the theoretical framework (see Tables 1 and 2). Firstly, the categories of activities related to the mathematical practice of conjecturing are described.

Table 1. Categories of conjecturing activities (Fernández-León et al., 2020)

	Horizontal mathematising	**Vertical mathematising**
Conjecturing	C.H.a) *Detecting patterns*: this category of activities refers to the experimentations with mathematical objects (such as a square, a function, and a vector space) in relation to a certain characteristic or observable property. In order to be precise, this category refers to such logical reasoning and informal activities with mathematical objects that give rise to the detection of a certain pattern in a specific mathematical context.	C.V.a) *Formalising patterns*: this category refers to the generalisation and formalisation of a pattern observed when experimenting with mathematical objects in relation to a certain observable property. Specifically, a previously observed pattern is generalised to formulate a conjecture.

C.H.b) *Testing conjectures*: this category includes those experimentations carried out with specific mathematical objects that satisfy the hypotheses of a conjecture to check whether this conjecture does hold. C.H.c) *Modifying statements*: this category refers to the experimentations with the components of an already existing conditional proposition (regardless of whether it has been proved or not, that is, either a proved proposition or a conjecture) that involve the modification of its hypotheses or conclusion. Specifically, this experimentation involves proposing possible changes in the hypotheses or thesis of a certain statement. The reasons behind a proposal of changes may be of a different nature. For instance, the finding of a counterexample for a conjecture may motivate the consideration of changes in parts of a statement and the study of which of those changes are possible.	C.V.b) *Formalising modifications of statements*: this category refers to the formalisation of the modifications of the hypotheses or the conclusion of an already existing conditional proposition (regardless of whether it has been proved or not). Notice that this formalisation gives rise to a conjecture.

The categories of activities related to the mathematical practice of proving are now described in Table 2.

Table 2. Categories of proving activities (Fernández-León et al., 2020)

	Horizontal mathematising	Vertical mathematising
Proving	P.H.a) *Detecting techniques or tools within proofs*: this category involves a careful study and examination of the characteristics and steps of other proofs related to the proof to be constructed. In order to be precise, this category refers to the search for proof techniques or tools used in other proofs that may fit in well with the new proof. P.H.b) *Detecting patterns in examples*: this category includes experimentations with mathematical objects, which satisfy the hypotheses of a certain conjecture, with the	P.V.a) *Selecting and applying proving methods*: this category involves the selection and application of proving methods (by contraposition, by contradiction, by induction, etc.). P.V.b) *Using proof techniques or tools found within other proofs*: this category involves the application and use of proof techniques or tools found in other proofs. P.V.c) *Applying known results*: this category of activities appears when known

aim of detecting patterns that may be extended to the general setting where the conjecture to be proved is formulated.	results are applied in order to build chains of logical implications. P.V.d) *Formalising findings with examples*: this category refers to the extension and formalisation of the calculations and experimentations with certain mathematical objects that satisfy the hypotheses of a given conjecture. Those patterns that were previously detected in examples are formalised, thereby giving rise to part of the mathematical proof that is being constructed.

This paper focuses on the study of the way primary-school in-service teachers conjecture and prove and tackles the following research question: how can the mathematical practices of conjecturing and proving of primary-school in-service teachers be characterised by using the categories of activities from the theoretical framework?

METHOD

Participants and Context

The participants of this study were four primary-school in-service teachers (whose students were between 6 and 12 years old), named Julia, Poppy, Ivy, and Rose (pseudonyms). Poppy and Ivy were teachers who had more than 20 years of experience, while Rose and Julia were newly qualified teachers. Each of the four had attained a three-year Bachelor's degree in Primary Education and had completed a complementary course on the teaching and learning of mathematical problem-solving. It should be borne in mind that the Bachelor's degree in Primary Education attained by these participants included an approach to mathematical proofs (both numerical and geometrical) in its curriculum, although of a low-level.

These teachers voluntarily participated in a professional training context where they worked together with educator-researchers from a Spanish University with the aim of improving their teaching of mathematics. These four teachers authorised that the data collected during this professional training context may be used for research purposes.

Data

The data of this study has been extracted from the annexes of the Ph.D. thesis by Muñoz-Catalán (2009). While reading this dissertation for other purposes, one of the authors of the present paper informed the others that said dissertation included the transcripts of certain conversations between four primary-school in-service teachers that could be interesting and useful for their research, since these conversations included these teachers' reasoning and reflections when constructing conjectures and proofs.

For this reason, we studied in detail the data included in that dissertation and, finally, considered for research, as data of this study, the transcript of a working session where the participants (the four in-service teachers described above) and the educator-researchers discussed two questions of a questionnaire (also designed by Muñoz-Catalán, 2009) on professional development. These two questions, laid out below (see Table 3), invoke the mathematical practices of conjecturing and proving.

Table 3. Questions 8 and 11 from a questionnaire on professional development (Muñoz-Catalán, 2009, vol. I, p. 232)

QUESTION 8: Is the following statement true: "The sum of a multiple of 2 and a multiple of 10 is a multiple of 10"?
☐ Yes, because 20+40 is a multiple of 10.
☐ Yes, because it holds true for the following examples: 10+10, 20+10, 50+20.
☐ No, because the result of the sum 2+10 is not a multiple of 10.
☐ I would need a mathematical proof because there are cases where it holds true and cases where it does not.
QUESTION 11: Prove whether the following statement is true: "The sum of a multiple of 2 and a multiple of 10 is an even number".

Notice here that, although most of the questions in the questionnaire (which included 30 questions) studied teachers' professional identities and their conceptions on the teaching and learning of mathematics, Muñoz-Catalán (2009) also took the opportunity to ask about professional knowledge and, specifically, about content knowledge (in Questions 8 and 11).

Data Analysis

A description of the process of analysis is now given. Firstly, each researcher individually distinguished events in the data, which are excerpts of the transcript which report on at least one mathematical activity in which the participants are engaged when conjecturing or proving. Subsequently, each researcher analysed the identified events by using the characteristics that describe each category of activities of the theoretical framework (see Tables 1 and 2). Specifically, when an identified event was totally in line with the description of a category, that event was characterised by that category. In those cases where the identified event was related to (but not totally in line with) the characteristics of a certain category of activities, the slight differences between the event and the category were highlighted. It should be borne in mind that no event in the data related to the mathematical practices of conjecturing and proving has been detected that is not related (at least partially) to any category of the theoretical framework.

Finally, a contrast analysis was carried out between the assignments of events to categories conducted by each researcher. Those assignments that were common to the three researchers were accepted, and the non-common ones were discussed, in order to reach a consensus.

RESULTS AND DISCUSSION

This section is devoted to answering the research question of this inquiry and to discussing the findings. The classification of activities given in Fernández-León et al. (2020) is employed to describe and explain how four primary-school in-service teachers develop the mathematical practices of conjecturing and proving. Specifically, several relevant events identified in the transcript that are related to these two mathematical practices are shown. This section is divided into four subsections. In the first two subsections, each of those events has been characterised by using the categories of activities of the theoretical framework. In each event shown, those words that connect such an event with the category that characterises it have been underlined. In the third subsection, the slight differences found between the characteristics of some events and the descriptions of certain categories have been highlighted and analysed. In this case, the words that connect each event with the category to which it is partially related have been underlined, although these events are not actually characterised by any category. In order to facilitate the reading of the results, the labels Q.8 or Q.11 have been assigned to most of the events that are shown, depending on whether the event is related to Question 8 or Question 11 (see Table 3). Moreover, the statement "The sum of a multiple of 2 and a multiple of 10 is a multiple of 10" shown in Q.8 will be referred to as C.8, while the statement "The sum of a multiple of 2 and a multiple of 10 is an even number" given in Q.11 will be referred to as C.11. In this paper, C.8 and C.11 are assumed to be conjectures since, for the participants, these statements fulfil the conditions given in the definition of conjecture (see Conjecturing and Proving Activities subsection). Finally, in the fourth subsection, the findings of this study have been discussed.

In this paper, the findings are reported through the separation of the activities of the participants that are linked to the mathematical practice of conjecturing from those that are related to the mathematical practice of proving and, specifically, by following the order of the categories presented in Tables 1 and 2.

How Primary-School In-Service Teachers Develop the Mathematical Practice of Conjecturing

Evidence is first provided of horizontal mathematising activities of the participants related to the practice of conjecturing.

C.H.a) *Detecting patterns.* This category of activities has been identified when the participants experimented with specific numbers while carrying out arithmetic operations with them and, as a consequence of this experimentation, the participants detected a pattern.

A representative protocol of this category identified in the data is given below. In this protocol, Ivy was striving to answer Q.11. Since she did not know how to prove C.11, she started off by testing this conjecture with numerical examples. In those examples, she observed a property that was different from the statement of C.11.

71. Ivy: No, I said I don't know how to prove it; the only thing I can say is that I have checked, <u>I have observed it, in the sense that I have seen</u>, I have sounded it out, <u>I have seen what happens</u> and later I have obtained a deduction from what I have observed: that an even number plus another even number is always even. But of course, it is what I have checked, <u>I have not made a complete list with all the numbers and I have seen that it always happens in this way</u>: even number plus even number results in an even number. (Q.11)

In this protocol, Ivy was repeating a reasoning previously given in line 16 of the transcript, where she explicitly said that "[16. Ivy:] I have written that I don't know how to prove it, I just know how to check them, and after observation and deduction from my observation, I say that an even number plus another even number is another even number". Although both ways of reasoning (see lines 16 and 71 of the transcript above) were given in the process of proving of C.11, it may be seen that Ivy started off by testing C.11 ("I have checked") and that later she detected a pattern from a list of numbers ("I have observed it, in the sense that I have seen […], I have seen what happens, […] I have not made a complete list with all the numbers and I have seen that it always happens in this way"). Specifically, from the expressions "I have seen what happens, […] I have not made a complete list with all the numbers and I have seen that it always happens in this way", it has been inferred that Ivy had realised, after observing a list of numbers, that something special or worthy of note might be taken into account (the pattern). Moreover, it has been noted that Ivy finished her contribution by formalising the detected pattern in a conjecture (see the C.V.a category below). In this excerpt of the transcript, the detected pattern indicates a property that is different from the statement she was trying to prove (C.11) and the formalisation of such a property (when stating the conjecture) has been essential for us to link this horizontal mathematical activity to the category *Detecting patterns*.

There now follows certain protocols that show the appearance of the category *Testing conjectures* in the data.

C.H.b) *Testing conjectures.* This category of activities has been identified when the participants checked a previously given conjecture (C.8 or C.11) by using calculations with different numbers that satisfied the hypotheses of that conjecture.

Four representative protocols of this category identified in the data are now shown. In this case, more examples are given to illustrate different consequences of the testing process. In the first three protocols, the teachers were trying to check whether C.8 was true. In the last protocol, Ivy was testing conjecture C.11.

1. Poppy: I first wrote down that I would need a mathematical proof because there are cases where it holds true and cases where it doesn't, but because I was sounding out, […], then I said no, because the result of the sum 2+10 is not a multiple of 10 and then I said that the moment it does not hold true for a case, a mathematical proof cannot be given. […]. (Q.8)

35. Julia: Let's see, a multiple of 2 plus a multiple of 10, and now the 11th [she refers to Question 11] says that it has to be even and I find the fact of being even very logical but that it is multiple of 10 implies fulfilling many conditions, I will have to take 2 out as common factor. This is obvious because even numbers, by definition, are multiples of 2 but a multiple of 10 is more complicated because to be a multiple of 10, this written here in parentheses [x + 5y] will have to be 5 or a multiple of 5 and this is more complicated. This fact holds true in some cases but not in all [in the transparency this is shown as follows (see Figure 1)]. (Q.8)

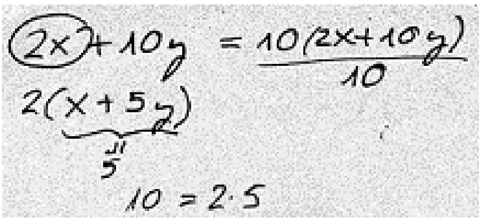

Figure 1. Hand-written note included in line 35 in the transcript

98. Julia: Anyway, this case is too specific because it holds true in fewer cases than where it does not hold. Normally, it holds true for number 5 and number 10, but it does not hold for 6, 7, 8 nor for 9, that is, there are many cases where it does not hold true, double, compared with those where it holds. Then, I don't see either much need of more proof, this was seen as very clear, that there are proofs where something must be adapted because it is much more questionable, maybe in this case when one proves two million times, then you realise but since this case was so clear. (Q.8)

71. Ivy: […]; the only thing I can say is that I have checked, […] I have sounded it out, I have seen what happens and later I have obtained a deduction from what I have observed: […]. But of course, it is what I have checked, I have not made a complete list with all the numbers […]. (Q.11)

In the first protocol, Poppy tried to answer Q.8. With this aim, she tested the conjecture C.8 with the numerical example offered as third possible answer to Q.8 (see the expression 2+10 in Table 3). Consequently, she found a counterexample and rejected C.8.

In the second and third protocols, Julia studied under which additional conditions on the hypotheses of C.8, that conjecture could be true. First, it should be highlighted that she had previously noticed, at the beginning of the working session, that C.8 did not hold since she had found a counterexample by looking at the third possible answer to Q.8. In the first of these two protocols, Julia stated that the fact of being a multiple of 10 (see the thesis of C.8) “implies fulfilling many conditions”. For this reason, she tried to find which additional conditions on the hypotheses of C.8 would be necessary to ensure that the sum of a multiple of 2 and a multiple of 10 was a multiple of 10. In particular, she based on the symbolic expressions used in the proof of C.11 to formulate a new conjecture, “[x + 5y] will have to be 5 or a multiple of 5” (for every natural number y), which provided details about the new additional conditions on the hypotheses of C.8. Moreover, she said that this new conjecture was true “in some cases but not in all”, which has allowed us to infer that she had tested the new conjecture. In the third protocol, Julia explicitly mentioned the numbers she had used to test the conjecture C.8 and specified in which cases that conjecture held (5 and 10) and in which cases it did not hold (6, 7, 8, 9). Here, it is significant that she highlighted the need for more than one numerical example to reject the examined conjecture. Notice that such a type of behaviour has been also found in many other protocols of the transcript.

Finally, in the last example, it may be seen that Ivy tested the conjecture C.11 when she stated that “I have checked, […] I have sounded it out”.

A representative protocol that shows the appearance of the category *Modifying statements* in the data is now given.

C.H.c) *Modifying statements.* This category has been identified when the participants experimented with the components of an existing conditional proposition (C.8, C.11 or other propositions that appear in the data), by modifying its hypotheses or conclusion.

In the following protocol, Julia suggested additional conditions that could be assumed on the hypotheses of C.8.

> 35. Julia: […] implies fulfilling many conditions, […] multiple of 10 is more complicated because to be a multiple of 10, this written here in parentheses [x + 5y] will have to be 5 or a multiple of 5 and this is more complicated. This fact holds true in some cases but not in all [in the transparency this is shown as follows (see Figure 1)]. (Q.8)

In this protocol, already shown and described above, Julia proposed an extra condition on the hypotheses of C.8 that would guarantee that a new, although weaker, result would be true (if x + 5y is multiple of 5 then 2x + 10y is multiple of 10). To be precise, she suggested that the sum of a multiple of 2 ($2x$) and a multiple of 10 ($2 \cdot 5y$) would be a multiple of 10 in the case that $x + 5y$ were multiple of 5. This last condition, which appeared in the proof process of C.8, was an extra condition on the hypotheses of C.8. We feel that Julia did not formalise that possible modification on the hypotheses of C.8 since she might not have realised that she was carrying out that modification. That is, while she was striving to prove C.8, she realised that the conclusion of that conjecture was too strong in the light of the written algebraic expressions she was working with. However, she might not be aware that the new condition considered in the proof could imply that she had proved a new result. More data would be needed to establish stronger conclusions on this matter.

The evidence of vertical mathematising activities of the participants linked to the practice of conjecturing is laid out below.

C.V.a) *Formalising patterns.* This category has been identified when the participants of this study generalised and formalised a certain previously detected pattern.

In this protocol, Ivy formalised a pattern detected from a list of numbers.

71. Ivy: […] I have obtained a deduction from what I have observed: that an even number plus another even number is always even. […] I have not made a complete list with all the numbers and I have seen that it always happens in this way: even number plus even number results in an even number. (Q.11)

In this case, it can be observed that Ivy had formalised the pattern detected while testing conjecture C.11 by giving a new conjecture (see an exhaustive description of this protocol at the beginning of this subsection). Figure 2 shows the new conjecture.

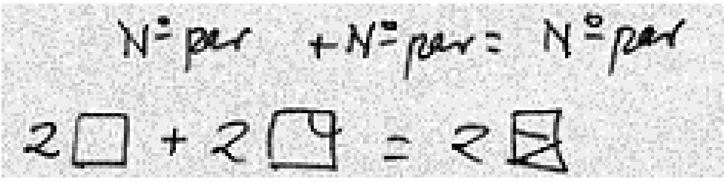

Figure 2. Hand-written note found in the data that shows the formalised pattern. Notice that the term "par" in the note means "even"

Notice that this last new conjecture has a very similar structure to conjecture C.11. This may be due to the fact that Ivy detected the referred pattern, which was later formalised, when she was testing conjecture C.11. However, there is insufficient data to assert that the construction of the new conjecture (see Figure 2) was based directly on the modification of the statement of C.11. For this reason, this protocol has not been included as empirical evidence of the category *Modifying statements.*

C.V.b) *Formalising modifications of statements.* This category has been identified when the participants formalised, in a new conjecture, the modifications they had previously considered on the hypotheses or conclusion of an existing statement.

The following protocol shows how Ivy included a new condition on the thesis of a statement to gain confidence that such a statement was true.

122. Ivy: Sometimes yes, and sometimes no; I have not stopped to look at it, what happens is that an odd number plus another odd one is even, but it may be any even number, it does not have to be just the even-number multiples of 8.

This protocol is part of the discussion among the participants about the veracity of a conjecture posed by an educator-researcher during the analysed session, which states *the sum of a multiple of 5 and* *a multiple of 3 is a multiple of 8*. In this protocol, Ivy gave a new conjecture, which has been called C', that claims that *the sum of two odd numbers is an even number that is not necessarily a multiple of 8*. It is important to observe what happened in lines 116 and 117 of the transcript to understand how the conjecture C' arose.

116. Educator-researcher: Your argument is: "odd plus odd is not even" [he/she writes down in the transparency: odd + odd ≠ even].

117. Ivy: No, no, odd plus odd is even, but it does not have to be a multiple of 8.

Specifically, the new conjecture C' arose from the following process: in line 116, the educator-researcher deduced that Ivy, in her previous appearances, was asserting that "odd plus odd is not even". Later, in line 117, Ivy denied that assertion and gave the new conjecture C' by modifying the previous assertion deduced by the educator-researcher. It can be observed that Ivy created C' in two almost simultaneous steps: the first part of the conjecture (*the sum of two odd numbers is an even number*) appeared when Ivy modified the educator-researcher's assertion ("odd plus odd is not even") by referring to a mathematical property ("odd plus odd is even") that she already knew. Furthermore, the second part of the conjecture (*that is not necessarily a multiple of 8*) was closely related to the conclusion of the conjecture posed by the educator-researcher that she was originally trying to prove (*the sum of a multiple of 5 and a multiple of 3 is a multiple of 8*).

How Primary-School In-Service Teachers Develop the Mathematical Practice of Proving

Evidence of horizontal mathematising activities of the participants of this study related to the practice of proving is first provided.

P.H.a) *Detecting techniques or tools within proofs.* This category has been identified when one of the participants reflected on the steps and characteristics of one existing proof with the aim of finding techniques or tools that could fit in well with the construction of a new proof.

In this protocol, Julia analysed the steps of a proof of C.11 with the aim of proving C.8.

44. Julia: Look, <u>both statements are the same</u>, the sum of a number multiple of 2 and another number multiple of 10; a multiple of 2 is 2x and a multiple of 10 is 10y; the sum of both numbers [writing down the sign + between both algebraic expressions (see Figure 3)]. Then<u> what I have done is to take out 2 as common factor, so yes it is an even number since 2 times anything is always even</u>, because an even number, by definition, is always going to be multiple of 2. And <u>the other has the same statement</u>, it says: <u>the sum of a number multiple of 2 and another number multiple of 10, that is similar to this one, has to result in a multiple of 10.</u> Therefore, for this [pointing out the expression $2(x + 5y)$ in the transparency (see Figure 3)] to be a multiple of 10 what I have thought is that if 10 is equal to $2 \cdot 5$ then $x + 5y$ has to be equal to 5 or a multiple of 5. (Q.8)

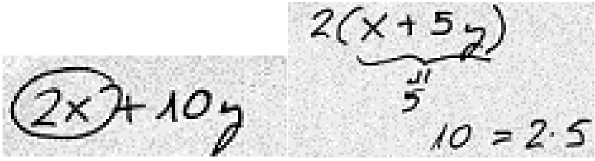

Figure 3. Julia's hand-written note that shows her steps to prove C.8

At the beginning of this protocol, Julia described the main steps that she had followed to prove C.11. She subsequently carefully reflected on these steps with the aim of finding techniques that could fit in well with the proof of C.8. To be precise, it can be observed at the end of the protocol that she tried to apply the scheme of the proof of C.11 to prove C.8 but did not succeed. This behaviour is highlighted again in P.V.b.

The evidence of vertical mathematising activities of the participants of this study linked to the practice of proving is laid out below.

P.V.b) *Using proof techniques found within other proofs.* This category has been identified when one of the participants used techniques of one existing proof in order to prove a new result.

In this protocol, Julia applied techniques found in a proof of C.11 to prove C.8.

44. Julia: Look, both statements are the same, […]. And the other has the same statement, it says: the sum of a number multiple of 2 and another number multiple of 10, that is similar to this one, has to result in a multiple of 10. Therefore, for this [pointing out the expression $2(x + 5y)$ in the transparency (see Figure 3)] to be a multiple of 10 what I have thought is that if 10 is equal to $2 \cdot 5$ then $x + 5y$ has to be equal to 5 or a multiple of 5. (Q.8)

In particular, in this protocol she decomposed numbers into prime factors and takes out common factors. Furthermore, it may be observed that she was aware of what should happen to ensure that this more complicated case (the proof of C.8) would work. However, we feel that the complexity of formalising such ideas (the fact that $x + 5y$ is a multiple of 5) meant that, as mentioned earlier in this paper, she did not raise any new conjecture related to them.

Certain protocols can now be presented that show the appearance of the category *Applying known results* in the data.

P.V.c) *Applying known results.* This category has been identified when the participants applied certain mathematical properties or results to build chains of logical implications.

In the following protocols, the teachers applied mathematical properties or results to complete the proof of C.11.

19. Julia: I have considered that an even number by definition is a multiple of 2 and, therefore $2x + 2 \cdot 5y = 2(x + 5y)$; then it is multiple of 2. (Q.11)

24. Rose: And the sum of two even numbers is always an even number. (Q.11)

In the first example, Julia applied the Fundamental Theorem of Arithmetic that states *every natural number can be written as a product of prime numbers*, together with the property of taking out common factors. In the second example, Rose applied a known result to finish a chain of logical implications that proved the conjecture C.11. To be precise, Rose firstly applied, in line 20, the Fundamental Theorem of Arithmetic to decompose 2 and 10 into the product of prime factors. She explicitly stated there that "[20. Rose:] I have written down 2x +, I have decomposed number 10 as $5 \cdot 2$ and then, I have written down that $2x + 5 \cdot 2y$ is equal to another number, equal to z, […]". Subsequently, instead of taking out common factors in the same way as Julia had carried out (see the preceding protocol), Rose directly applied a known result, "the sum of two even numbers is always an even number", to the algebraic expression $2x + 5 \cdot 2y$ to conclude that C.11 was true. It is important to notice here that several of the techniques or tools that the participants applied in this context of divisibility to answer the questions (Q.8 and Q.11) coincided with the application of classic theorems (such as the Fundamental Theorem of Arithmetic) or properties that are often used in this branch of mathematics.

Finally, it should be emphasised that certain categories of activities of the theoretical framework were not identified in the data: P.H.b, P.V.a and P.V.d. However, regarding P.H.b and P.V.d, it was observed that at least one participant of this study was aware that the educator-researcher behaved in accordance with the description of such categories when carrying out proving activities. The following example illustrates this.

133. Julia: This proof is in reverse. It starts from a specific case, you have not done it in the same way as the other proof.

In this protocol, Julia realised that the educator-researcher drew from particular cases (specific examples) to construct a proof (see Figure 4), which was a different method compared to that which the participants had carried out before.

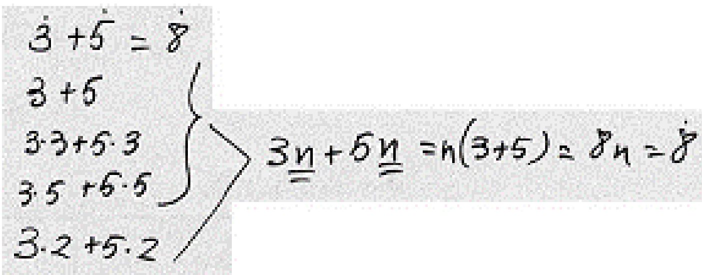

Figure 4. An educator-researcher's hand-written note found in line 132

More on Proving

There is a group of events identified in the transcript which, while providing information on how the participants prove, has not been characterised by any category, since none of these events fulfil all the characteristics of a category. However, it is interesting to note that each of these events is partially related to some category of the theoretical framework, since there may always be found certain similarities and also some differences between the characteristic of an event and the description of the related category. For this reason, these events cannot be directly characterised by the related category. For instance, the following protocol is closely related with the P.H.a category, although its characteristics are slightly different to the description of this category.

17. Poppy: I was also testing [as Ivy did in line 16] but, maybe with an equation I can do something, and I also did it […]. (Q.11)

In this protocol, Poppy was considering the technique of using "equations" (a word that she seems to use as a synonym of *algebraic expression*) to prove statements related to divisibility in natural numbers: a topic studied in her Bachelor's degree. In particular, this protocol shows how primary-school in-service teachers may sometimes consider proof techniques that they already know while trying to prove certain new mathematical statements. It should be borne in mind that Ivy had not been observed as finding the referred proof technique in another proof, as the P.H.a category considers. For this reason, this event has not been characterised by this category.

Other examples are shown in the two protocols below. In this case, these protocols are closely related with the P.V.b category, although their characteristics are slightly different to those in the description of this category.

19. Julia: I have considered that an even number by definition is a multiple of 2 and, therefore $2x + 2 \cdot 5y = 2(x + 5y)$; then it is multiple of 2. (Q.11)

66. Poppy: Yes, I have said: $2n + 10n$, but later I said: well, number 10 can be decomposed; then I wrote 5 to take out the common factor. (Q.11)

In these two protocols, the participants used the techniques of decomposing numbers into prime factors or of taking out common factors when proving certain results. Notice that both techniques are widely used in divisibility. It should also be emphasised that, in these two protocols, not only had the two techniques described above been observed but also the technique of translating a statement into symbolic language (mathematical symbols). This technique encompasses the translation of the hypotheses or conclusion of a statement into symbolic language in the proof process of such a statement. For instance, in line 19, Julia considered the definition of an even number and wrote down its translation (the algebraic expression $2x$) with the aim of proving a statement. Specifically, this technique has been found several times in the data when the participants tried to prove conjectures (C.8, C.11, or other conjectures that the educator-researchers posed), and translated the statements of such conjectures into mathematical symbols.

In addition, these two protocols show that primary-school in-service teachers sometimes employ proof techniques that they already know while trying to prove certain new mathematical statements. As before, it should be borne in mind that Julia and Poppy had not been observed to have found the referred proof techniques in another existing proof, as the P.V.b category demands.

In line 19 of the transcript, another interesting event may be highlighted. Specifically, the event "I have considered that an even number by definition is a multiple of 2 [...] $2(x + 5y)$; then it is multiple of 2." is closely related with the P.V.c category, although the characteristics of this event are slightly different to the description of this category of activities. In this event, Julia firstly took into account the definition of an even number to be aware of what she had to obtain in order to prove C.11. Subsequently, she wrote down algebraic expressions related to the hypotheses of C.11 "$2x + 2 \cdot 5y = 2(x + 5y)$", and finally she applied the definition of even number in order to conclude that C.11 was true. This event shows that primary-school in-service teachers may sometimes apply definitions or similar statements (axioms) when building chains of logical implications. It should be noted that, in this event, Julia did not apply a known result but a known definition, which allowed her to construct a chain of logical implications in a similar way to that in which the P.V.c category describes.

Some Reflections on the Results

In this paper, the mathematical practices of conjecturing and proving of four primary-school in-service teachers have been characterised. The various categories of activities of the theoretical framework identified in the data have helped us to describe and explain how these teachers develop these two mathematical practices. Moreover, we have highlighted that certain events identified in the data are related to specific categories of activities, although those events present certain characteristics that differ from the characteristics of the categories to which they are related. Thus, the findings reported in this work report that not all the mathematical activities developed by primary-school in-service teachers when conjecturing and proving may be exhaustively explained by the categories of activities defined in the study by Fernández-León et al. (2020). Notice also that three categories of activities of the theoretical framework have not been identified in the data (P.H.b, P.V.a, and P.V.d) and that these three categories are linked to the mathematical practice of proving.

Furthermore, it is also highlighted that the majority of the events identified in the data and subsequently categorised with the theoretical framework are of a horizontal nature, that is, they mainly include informal ways of reasoning. In fact, the main behaviour that guides the participants' practice of conjecturing is that of the use of examples. In particular, they use mathematical objects that satisfy the hypotheses of certain statements to test said statements, which help these teachers to reject such statements or convince themselves of their truth. As a consequence of the testing process, these teachers sometimes observe regularities that motivate the appearance of new conjectures. These findings show the potential of exploration with examples when conjecturing (Huang, 2016; Lesseig, 2016; Morselli, 2006). In previous studies conducted with pre- and in-service teachers (see, for instance, Knuth, 2002; Martin & Harel, 1989), the role of empirical evidence in obtaining conviction regarding the truth of a statement is also highlighted. In particular, Knuth (2002) indicates that teachers "reach a stronger level of conviction" (p. 401) by testing with specific mathematical objects. In the present work, it can even be observed that the participants sometimes show confusion between the process of testing with specific mathematical objects and the mathematical proof, a fact that must not be overlooked, since “teachers’ use of examples to verify statements contributes to students’ belief that testing a number of examples is sufficient proof” (Lesseig, 2016, p. 23). Nevertheless, what it can be concluded is that the participants of this study consider the experimentation with mathematical objects an important role in the development of the mathematical practices of conjecturing and proving (Lesseig, 2016; Lynch & Lockwood, 2019; Morselli, 2006).

The literature on students’ difficulties with proofs has documented a variety of persistent misconceptions held by the students (Stylianides & Stylianides, 2018). One such misconception is that *a single example is insufficient to reject an assertion or mathematical property*, and another is the fact that *the verification of a property in some particular cases is enough to guarantee its general validity*. The results of this study with primary-school in-service teachers show that these two misconceptions persist with certain teachers. This fact reveals a *pernicious circle* where students maintain these two misconceptions from school or high school even when they become in-service teachers themselves, and they then continue transmitting the same misconceptions to their own students. We suggest that giving opportunities for professional development on these mathematical practices may help to break this vicious circle. For instance, the results of this study underline the need for the promotion of aspects of a more formal nature of the practice of conjecturing (vertical mathematising activities) and of aspects of the practice of proving such as the use of examples that guide the proving process but do not constitute a proof. From our learning approach, it is stated that learning in the processes of professional development aims, through the peripheral participation, to alter the way in-service teachers develop mathematical practices to the way the mathematicians’ community of practice develops such practices (Blömeke, Kaiser, König, & Jentsch, 2020; Lave & Wenger, 1991; Podkhodova, Snegurova, Stefanova, Triapitsyna, & Pisareva, 2020; Yilmaz, 2020), always taking into consideration that there are differences between both contexts (Weber, Inglis, & Mejia-Ramos, 2014).

The literature has also documented that primary-school in-service teachers often face similar difficulties that students and pre-service teachers face, not only with proofs but also with many other mathematical contents (see, for instance, Ubuz & Yayan, 2010). For this reason, we agree with Ubuz and Yayan (2010) when stating that “an important step to improving subject matter knowledge should be better subject matter preparation for primary teachers” (p. 799). We also believe that addressing difficulties in the content knowledge of primary-school teachers (see, for instance, Oflaz, Polat, Altaylı Özgül, Alcaide, & Carrillo, 2019) would shed light on where to focus so that teachers to acquire essential content knowledge to teach in the various mathematical domains (regarding proving as mathematical content, see, for instance, Siswono, Hartono, & Kohar, 2020; van Dormolen, 1977). In this way, studying teachers’ activities when they strive either to prove mathematical assertions or to formulate conjectures constitutes the first step and exploring the reasons behind the difficulties found when conjecturing and proving would be an interesting topic for further research.

Moreover, it is also interesting to make a comment regarding line 71 of the transcript (see C.H.a category above). In this protocol, where Ivy explains how she poses the conjecture: *the sum of two even numbers is an even number*, it may be identified a chain of categories of activities (*chain of progressive mathematisations* in Rasmussen et al., 2005) that explicitly shows how the theoretical framework is able to explain the way primary-school in-service teachers develop the mathematical practice of conjecturing. This chain begins when Ivy tests C.11 (C.H.b) and this chain continues when she later observes a pattern in her calculations (C.H.a) that it is not immediately deducible from C.11. Moreover, it can be seen that this observed pattern is formalised in a new conjecture: *the sum of two even numbers is an even number* (C.V.a). As mentioned in a previous subsection, the fact that the new statement has a similar structure to that of C.11 could inform us that Ivy has based the construction of the new conjecture on the statement of C.11. However, more data would be required to conclude that the C.H.c or C.V.b categories appear in line 71. The chain of categories found in this protocol reveals possible interconnections between the categories of activities of the theoretical framework and shows how vertical activities build on horizontal activities. Furthermore, this chain, which shows conjecturing activities while a participant is proving a statement (C.11), is a clear example of what Lesseig (2016) calls a "cycle of empirical exploration, conjecturing, generalizing and justifying" (p. 18). In line 71, Ivy reasons "through examples to generate additional conjectures and generalizations" (Lesseig, 2016, p. 18).

CONCLUSION

This work characterises how four primary-school in-service teachers develop the mathematical practices of conjecturing and proving. Melhuish et al. (2018) have recently affirmed that a "teacher must be able to notice mathematical reasoning forms such as *justifying* and *generalizing*" (p. 2) (herein named proving and conjecturing). In this regard, the results of this study highlight aspects of these practices (such as the use of examples or the need for more formal mathematical activities) which might report on professional development so that in-service teachers would be able to notice different forms of mathematical reasoning (Hidayah, Sa'dijah, Subanji, & Sudirman, 2020; Lesseig, 2016) in order to foster mathematical practices in the classroom.

Furthermore, other mathematical tasks that invoke the same mathematical practices (conjecturing and proving), although from different mathematical fields (such as analysis and geometry), should be considered in order to complement this exploratory study. We also maintain that the mathematical practices of conjecturing and proving of more primary-school in-service teachers should be studied to achieve findings of a more representative nature. We consider that the limited data of this study may have influenced how useful the theoretical framework has been in describing how these primary-school in-service teachers conjecture and prove. Outside the scope of the theoretical framework, we have only observed that the participants sometimes consider and employ proof techniques that they already know and that they sometimes apply definitions or similar statements (axioms) when building chains of logical implications. Nevertheless, we believe that these latter findings constitute a starting point for the broadening of the theoretical framework defined by Fernández-León et al. (2020), although for this purpose more research mathematicians should be studied, since the categories of activities were generated by that population. Indeed, we hypothesise that more research on this topic could reveal new mathematical activities when conjecturing and proving that motivate the appearance of new categories thereof.

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