The dimension of a graph

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Abstract

For each graph G *the dimension of* G is defined as the smallest dimension in the Euclidean Space where there is an embedding in which all the edges of G are segments of a straight line of length one. The exact value is calculated for some important families of graphs and this value is compared with other invariants. An infinite quantity of forbidden graphs for dimension 2 is also shown.

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1 Introduction

Let G be a graph. The *dimension* of graph G, denoted by $\dim(G)$, is defined as the smallest natural number $n \in N$ such that G has an embedding in R^n

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where all the edges of G are segments of a straight line of length one.

In this work the exact values of the dimension of graph G are calculated for some important families of graphs: the complete graphs, the bipartite complete and the tripartite complete graphs. This value is also compared with other invariants: the chromatic number and the genus of a graph. In addition, bounds of the dimension are given by planar and outerplanar graphs.

It is easy to check that the graphs with dimension of maximum n, with $n > 1$, can neither be characterized by minors nor even by forbidden topological minors. The following question arises: Can the graphs with dimension of maximum n be characterized by a finite quantity of forbidden subgraphs? A negative answer is obtained by showing an infinite family of forbidden graphs for dimension 2.

2 Dimension of some families of graphs

In this section the exact values of the dimension of graph G are calculated for some important families of graphs.

2.1 Dimension of the complete graphs

In the following result the dimension of a complete graph is obtained.

Theorem 2.1 $\dim(K_n) = n - 1$

Proof.

The following n points are considered in the Euclidean Space R^{n-1}, $y_0 = (0, 0, 0, ..., 0), \ldots, y_k = (z_1, z_2, ..., z_{k-1}, (k+1)z_k, 0, ..., 0)$, where $k = 1, 2, ..., n-1$ and $z_k = \frac{1}{\sqrt{2k(k+1)}}$. One can easily check that these n points in R^{n-1} are at distance one. Therefore $\dim(K_n) \leq n - 1$.

We can prove, by induction in n, that the embedding of these points is unique up to movements. Given n points in R^{n-1}, let us take the first $n-1$ points. They are contained in a lineal variety of dimension $n-2$. By a movement, this variety can be taken to the hyperplane $X_{n-1} = 0$. As this variety is isomorphic to R^{n-2}, by hypothesis of induction it can be supposed that these first $n-1$ points are $y_0, \ldots, y_{n-2}$, by adding the value 0 as the $(n-1)$-th coordinate.

It is not difficult to show, by induction in the component, that the two unique points of R^{n-1} that are at distance one from the $n-1$ previous points are $y_{n-1} = (z_1, \ldots, z_{n-1}, nz_{n-1})$ and $(z_1, \ldots, z_{n-1}, -nz_{n-1})$. By making a symmetry with respect to the hyperplane $X_{n-1} = 0$, the $n-2$ previous vertices become invariant, while the two last points transform into each other.

As a consequence of the uniqueness of the embedding and since the points $y_0, \ldots, y_{n-1}$ are not contained in a lineal variety of dimension $n-2$, $\dim(K_n) > n-2$ and therefore the result is obtained.

□

The following consequences are deduced from the previous result:

Corollary 2.2 *The embedding of K_n in R^{n-1} is unique up to movements.*

Corollary 2.3 *Given n vertices in R^n such that the distance between any pair of vertices is equal to 1, there are only two vertices whose distance is 1 from all the vertices.*

Corollary 2.4 *Let x be an edge of K_n,* $\dim(K_n - x) = n - 2$.

Furthermore, the following result is also obtained:

Corollary 2.5 $\dim(P_7 + K_2) = 4$.

Proof.

Let us suppose the opposite case where the graph P_7+K_2 has an embedding in R^3. This graph has 9 vertices and the edge u_1u_2, v_iv_{i+1}, with $i = 1, \ldots, 6$, and u_iv_j, with $i = 1, 2$ and $j = 1, \ldots, 7$. The vertices u_1, u_2, v_1 and v_2 induce K_4. It can be supposed that their coordinates are $(0, 0, 0)$, $(1, 0, 0)$, $(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3})$.

From Corollary 2.3, given 3 points which form an equilateral triangle of unitary sides, only two points exist that are at distance 1 from all the points in R^3. Hence, it can be deduced that given the points u_1, u_2 and v_i, with $i < 1$, two points will exist at distance 1 from all of the points. These are v_{i-1} and v_{i+1}.

The coordinates of the vertices v_i, with $i = 3, \ldots, 7$, are respectively, $(\frac{1}{2}, -\frac{7\sqrt{3}}{18}, \frac{2\sqrt{6}}{9})$, $(\frac{1}{2}, -\frac{23\sqrt{3}}{54}, -\frac{5\sqrt{6}}{27})$, $(\frac{1}{2}, \frac{17\sqrt{3}}{162}, -\frac{28\sqrt{6}}{81})$, $(\frac{1}{2}, \frac{241\sqrt{3}}{486}, -\frac{11\sqrt{6}}{243})$ and $(\frac{1}{2}, -\frac{329\sqrt{3}}{1458}, \frac{230\sqrt{6}}{729})$.

In this case, segments v_1v_2 and v_6v_7 intersect at $(\frac{1}{2}, \frac{23\sqrt{3}}{66}, \frac{5\sqrt{6}}{33})$, therefore a contradiction is obtained.

An embedding of $P_7 + K_2$ in R^4, can be given by assigning the coordinates $(\frac{1}{2}, \frac{2433\sqrt{3}}{658}, 0, \frac{\sqrt{36894}}{329})$ to v_7 and by conserving the coordinates of the other vertices.

2.2 Dimension of the multipartite complete graphs

Let us now enunciate the result that indicates the dimension of the bipartite complete graphs:

Theorem 2.6 (i) $\dim(K_{m,n}) = 4$, *con* $3 \leq m \leq n$.

(ii) $\dim(K_{2,n}) = 3$, *con* $n \geq 3$.

(iii) $\dim(K_{1,n}) = 2$, *si* $n \geq 2$.

(iv) $\dim(K_{1,1}) = 1$.

Items ii, iii and iv are straightforward. Item i will be proved in the next section.

The following result calculates the dimension of the tripartite complete graphs:

Theorem 2.7 (i) $\dim(K_{m,n,p}) = 6$, *con* $3 \leq m \leq n \leq p$.

(ii) $\dim(K_{m,n,p}) = 5$, *con* $m \leq 2$ *y* $3 \leq n \leq p$.

(iii) $\dim(K_{m,2,p}) = 4$, *con* $m \leq 2$ *y* $p \geq 3$.

(iv) $\dim(K_{m,2,2}) = 3$, *con* $m \leq 2$.

(v) $\dim(K_{1,1,p}) = 3$, *con* $p \geq 3$.

(vi) $\dim(K_{1,1,p}) = 2$, *con* $p \leq 2$.

3 Relationship between the dimension and other invariants

In this section we study if a graph has a given invariant then the dimension of a graph is bounded or if a graph has a given dimension then the invariant of a graph is bounded.

First, we relate the dimension of a graph with the chromatic number of a graph:

Lemma 3.1 $\dim(G) \leq 2\chi(G)$.

Corollary 3.2 *If there exists a coloring of the vertices of* G *with* $\chi(G)$ *colors so that there exist* k *colors that have been assigned to a maximum of 2 vertices, then* $\dim(G) \leq 2\chi(G) - k$.

Corollary 3.3 $\dim(K_{p_1,\ldots,p_k,p_{k+1},\ldots,p_{k+h}}) \leq k + 2h$, *when* $p_1 \leq \ldots \leq p_k < 3 \leq p_{k+1} \leq \ldots \leq p_{k+h}$.

As was seen in Theorems 2.6 and 2.7, in some cases the inequality is strict and in other cases it is an equality.

From this result we can approach the proof of item i of Theorem 2.6.

Proof. By Corollary 3.3 $\dim(K_{m,n}) \leq 4$ is obtained. On the other hand, it is clear that $\dim(K_{m,n}) \geq \dim(K_{3,3})$, therefore it is enough to prove that $\dim(K_{3,3}) \geq 4$.

Let us suppose the opposite. An embedding of $K_{3,3}$ in R^3 exists with all the edges of length 1. Three disjoint vertices are considered taken two by two, and the three spheres whose radii are equal to 1, centered in these vertices. Two of these spheres intersect, a maximum of a circle and this circle intersects with the other sphere at a maximum of two points. However, an adjacent vertex with the centers of the 3 spheres has to be in their intersection, therefore there would be a maximum of two of these vertices and a contradiction would be obtained. Hence the result holds.

By Four Colors's Theorem (see [1]), we know that if a graph is plane then its chromatic number is at most 4, therefore the following result is obtained:

Corollary 3.4 *If G is a plane graph then* $\dim(G) \leq 8$.

One open problem is to calculate the maximum value that the dimension of a plane graph can have. By Corollary 2.5 and 3.4, we know that this number ranges from 4 to 8.

As the chromatic number of the outerplanar graphs is at most 3, the dimension of the outerplanar graphs is at most 6. However this result can be improved in the following way:

Theorem 3.5 *If G is an outerplanar graph, then* $\dim(G) \leq 3$.

Proof.

Let us suppose that the result is not true and that G is the smallest outerplanar graph with $\dim(G) > 3$. G has a vertex v of degree 2. Let G' be a graph which is obtained when the vertex v of G is removed and the edge between u and w is added(u and w are the neighbors of v). G' is an outerplanar graph and it has a vertex less than G, hence $\dim(G') \leq 3$. An embedding of G' in R^3 is considered with all the unitary edges and such that u and w are at distance 1. Since there are infinite points in R^3 which are at distance 1 from u and w, vertex v can be placed at one of them, where the edges uv and wv don't intersect with the rest of the graph. Therefore we would have an embedding of G in R^3 with all the unitary edges which is a contradiction.

This result cannot be improved, since it is easy to check that $\dim(K_1 + P_6) = 3$.

Let us relate the dimension of a graph with the genus of a graph (remember that the genus of a graph is the smallest genus on the compact surfaces where the graph admits an embedding). The following result is required:

Proposition 3.6 (i) *Let G be a graph, there exists a subdivision of G, denoted by $S(G)$, such that* $\dim(S(G)) = 3$.

(ii) *Let G be a planar graph, there exists a subdivision of G, denoted by $S(G)$, such that* $\dim(S(G)) = 2$.

Proof.

It is clear that any finite graph can be represented in R^3 so that their edges are segments of a straight line. In addition, the complete graph whose cardinal is the continuous $2^{\aleph_0}$, can also be represented in this way (see [3]). On the other hand, all finite planar graphs can be represented in R^2 such that all the edges are segments of a straight line (see [4]).

Let us distinguish three cases:

Case 1: The lengths of the edges are natural numbers.

In this case the edges are subdivided such that all edges have length 1.

Case 2: The lengths of the edges are rational numbers.

If the lengths of the edges are rational numbers then a homothecy of dilation factor the minimum common multiple of the denominators of rational numbers is considered and the previous case is applied.

Case 3: The lengths of the edges are irrational numbers.

If some of the lengths of the edges are irrational numbers, then an edge x with irrational length l_x and a rational number r sufficiently near to $\frac{l_x}{2}$ are taken, such that x can be replaced by two new adjacent edges of length r. Therefore, these edges are sufficiently near to the edge x such that these edges don't intersect with the rest of the edges of the graph.

By repeating the process with all the edges of irrational length, a subdivision of the graph is obtained with all the edges of rational length and the previous case is applied.

Considering an appropriate subdivision of any graph with genus $n \geq 1$, graphs with genus n and with dimension 3 can be found. However, there exists an upper bound of the dimension in function of the genus, as the following result shows:

Proposition 3.7 $\dim(G) \leq 2\lfloor \frac{7+\sqrt{1+48\gamma(G)}}{2} \rfloor$.

Proof.

Ringel and Youngs in [2] proved that $\chi(G) \leq \lfloor \frac{7+\sqrt{1+48\gamma(G)}}{2} \rfloor$. By Lemma 3.1, the result is obtained.

4 Finite or infinite set of the forbidden graph of bounded dimension

A forbidden graph in a partial order for a specific property, is a graph that doesn't have the property, however any smaller graph, in this partial order, has this property. It is clear that the graph with the property "to have a maximum dimension of 1" can be characterized by forbidden minors. The unique forbidden finite graphs are $K_{1,3}$ and K_3. There is one forbidden numerable graph:

- The graph composed of two connected components, each one of which is an infinite path and an infinite numerable quantity, $\aleph_0$, of the disjoint edges.

There are also two forbidden non numerable graphs:

(i) The graph composed of $\aleph_1$ (the cardinal $\aleph_0$), disjoint edges.

(ii) The graph composed of $2^{\aleph_0}$ (the cardinal following the cardinal of the continuum) disjoint vertices (in the generalized continuum hypothesis this cardinal is $\aleph_2$).

However, even in the finite case, the graphs with the property "to have a maximum dimension of 2" are not been able to characterize by forbidden minors, not even by topological forbidden minors. Logically, these graphs are been able to characterize by forbidden subgraphs. The question arises in a natural way: Is the forbidden subgraph set for the property "to have a maximum dimension of 2" finite or not?.

The answer is negative, as we can deduce from the following result:

Theorem 4.1 *The graphs in the Figure 1 are forbidden graphs with the property "to have a maximum dimension of 2" with* $n \geq 2$.

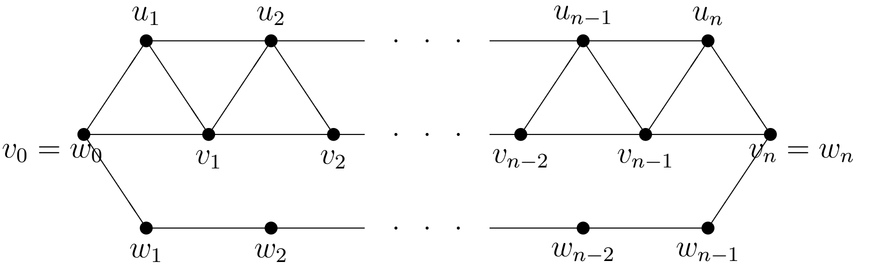

Fig. 1. Forbidden graphs with the property "to have a maximum dimension of 2"

Proof.

Since K_3 has an unique embedding in R^2 up to movements, it is easy to see that, extending this embedding, the graph minus the edges $w_k w_{k+1}$, with $k = 0, \ldots, n-1$, has an unique embedding in R^2. In this embedding the vertices $v_0$and v_n are at distance $n-1$, therefore extending this embedding to the graph, the edges $w_k w_{k+1}$ are overlapped on $v_k v_{k+1}$, hence the dimension of the graph has to be bigger than 2.

On the other hand, it is not difficult to find an embedding in R^2 when an edge is eliminated. It is enough to consider four cases depending on the type of eliminated edge: $u_k u_{k+1}$, $v_k v_{k+1}$, $w_k w_{k+1}$ and $u_k v_k$ or $u_{k+1} v_k$.

As a consequence the following result is obtained:

Corollary 4.2 *The forbidden graph set with the property "to have a maximum dimension of 2" is infinite.*

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