

Homomorphisms and polynomial invariants of graphs

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A B S T R A C T

This paper initiates a general study of the connection between graph homomorphisms and the Tutte polynomial. This connection can be extended to other polynomial invariants of graphs related to the Tutte polynomial such as the transition, the circuit partition, the boundary, and the coboundary polynomials. As an application, we describe in terms of homomorphism counting some fundamental evaluations of the Tutte polynomial in abelian groups and statistical physics. We conclude the paper by providing a homomorphism view of the uniqueness conjectures formulated by Bollobás, Pebody and Riordan.

1. Introduction

Counting homomorphisms between graphs arises in many different areas including extremal graph theory, partition functions in statistical physics and property testing of large graphs [6]. Given two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, a *homomorphism of G to H* , written as $f : G \rightarrow H$, is a mapping $f : V(G) \rightarrow V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. When G and H are multigraphs, that is, they might have parallel edges and loops, a *homomorphism of G to H* is a function $f_V : V(G) \rightarrow V(H)$ together with an associated function $f_E : E(G) \rightarrow E(H)$ consistent with f_V in that $f_E(uv) = f_V(u)f_V(v)$, and such that f_E maps parallel edges (resp., loops) in G to parallel edges (resp., loops) in H .

The number of homomorphisms of G to H is denoted by $\text{hom}(G, H)$. This number, considered as a function of G with H fixed is a *graph parameter*, that is, a function of graphs invariant under isomorphisms. A broader class of parameters related to homomorphisms has been recently intensively studied in the context of statistical physics in [6].

The motivation of this paper is to show the usefulness of the homomorphism perspective in the study of polynomial invariants of graphs. Thus, our main contribution is to prove that there exists a

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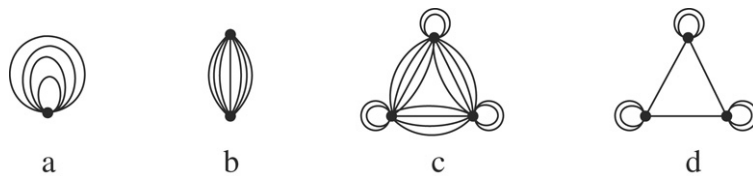


Fig. 1. (a) $K_1^{4,0}$, (b) $K_2^{0,7}$, (c) $K_3^{2,4}$, (d) $K_3^{2,1}$.

strong connection between counting graph homomorphisms and evaluating polynomials associated with graphs. The importance of this approach lies in its applicability. For instance, it can put in a new context some well-known problems such as the uniqueness questions formulated by Bollobás, Pebody and Riordan in [4].

One of the most studied polynomial invariants in combinatorics is the *Tutte polynomial*, or *dichromate* of [32]. This is an isomorphism-invariant function from the set of finite multigraphs with loops allowed to $\mathbb{Z}[x, y]$ which can be defined in several ways, see for instance [5,7,32]. Throughout this paper, we shall consider its contraction–deletion formulae. Given a finite graph $G = (V(G), E(G))$ and $e \in E(G)$, we denote by $G - e$ and G/e , respectively, the result of deleting and contracting the edge e in G . Thus, the Tutte polynomial of G , denoted by $T(G; x, y)$, can be defined by the following recurrence relations:

1. If G has no edges then $T(G; x, y) = 1$.
2. $T(G; x, y) = T(G - e; x, y) + T(G/e; x, y)$ provided that $e \in E(G)$ is neither a loop nor a bridge.
3. $T(G; x, y) = yT(G - e; x, y)$ whenever $e \in E(G)$ is a loop.
4. $T(G; x, y) = xT(G/e; x, y)$ whenever $e \in E(G)$ is a bridge.

It is well known that homomorphisms of a graph G to the complete graph K_n are just n -colorings of G (see [16]). In [18], Joyce showed that the number of homomorphisms of any graph G to a complete graph with non-multiple edges and p loops at each vertex, is an evaluation of the coboundary polynomial of G . This polynomial was first defined in [8] as a generalization of the chromatic polynomial. Since the Tutte polynomial can be regarded as an extension of the chromatic and the coboundary polynomials, a natural question arises: can we find other graphs H such that the number of homomorphisms of any graph G to H is given (up to a determined term) by an evaluation of the Tutte polynomial of G ?

This paper contains two main results. We first prove that every complete graph with p loops at each vertex and constant multiplicity q at the non-loop edges can play the role of H , whenever p is different than q . Just as the Tutte polynomial is an extension of the chromatic and the coboundary polynomials, this complete graph which we denote by $K_n^{p,q}$ is a natural extension of both the complete graph K_n and the Joyce graph $K_n^{p,1}$. Fig. 1 shows four instances of this family of graphs.

Our second main result is the characterization, by assuming a local condition, of those graphs H such that the parameter $\text{hom}(_, H)$ can be recovered from the Tutte polynomial. We prove that such graphs are necessarily isomorphic to graphs of the family $K_n^{p,q}$. The local condition is not too restrictive since it is satisfied by all the multiplicative invariants of graphs that can be deduced from the Tutte polynomial.

The Tutte polynomial extends not only the chromatic and the coboundary polynomials but also, among others, the *flow*, the *boundary*, the *transition* and the *circuit partition* polynomials. Thus, our characterization leads to important connections between homomorphism counting and these polynomials, which have a special role in the field of graph theory. Indeed, the boundary polynomial was introduced in [33] as a generalization of the flow polynomial, and it is the dual coboundary polynomial [7]. Both polynomials have been recently used to obtain new evaluations of the Tutte polynomial at some points on the hyperbolae $H_\alpha = \{(x, y) | (x - 1)(y - 1) = \alpha\}$ for $\alpha \in \mathbb{N}$ (see [13]). The boundary polynomial is also the coboundary polynomial of the dual of the cycle matroid of the graph. We refer the reader to [33] for its particular interpretation for graphs.

The transition polynomial arose in [17] as a tool for summarizing and generalizing a number of results obtained by Martin [26,27] and Las Vergnas [22–24]. It has many interesting applications in

knot theory, see for example [19]. The circuit partition polynomial was first defined in [9], and was so named in [3]. This polynomial is a simple transform of the original *Martin polynomial*, which was developed by Martin in [26] to study families of cycles in 4-regular Eulerian graphs. Furthermore, the circuit partition polynomial has surprising applications to many areas including infrastructure networks and reconstruction of DNA sequences, see for instance [1].

As an application, we use our characterization to describe in terms of homomorphism counting some important evaluations of the Tutte polynomial in abelian groups and statistical physics. Specifically, we sketch applications to difference sets in abelian groups, the Potts model, and the random cluster model in statistical mechanics.

We shall conclude the paper by introducing a new type of uniqueness of graphs related to homomorphism counting, which we call *coloring uniqueness*, and by showing its relation with *Tutte uniqueness* and *chromatic uniqueness*.

2. Graph homomorphisms and the Tutte polynomial

In this section we establish a connection between counting graph homomorphisms and evaluating the Tutte polynomial.

We first introduce some notation that will be used throughout this paper. The graphs considered are finite and not necessarily simple. Thus, let us denote by Ω the set of finite multigraphs with loops allowed. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$. An edge $e \in E(G)$ can be either a loop uu or a non-loop edge uv with $u \neq v$. The multiplicity of an edge $e = uv \in E(G)$, written as $m(e)$, is the number of edges joining u and v when $u \neq v$, and the number of loops attached at u when $u = v$. The set of homomorphisms of a graph G to a graph H is denoted by $\text{Hom}(G, H)$, and its order is $\text{hom}(G, H)$, which is given by the following expression:

$$\text{hom}(G, H) = \sum_{f:V(G) \rightarrow V(H)} \prod_{u,v \in V(G)} m(f(u)f(v))^{m(uv)}$$

where the multiplicities are zero when the vertices are not adjacent, and considering $0^0 = 1$.

Given $f \in \text{Hom}(G, H)$ and $e = uv \in E(G)$, we write $f(e) = f(u)f(v) \in E(H)$. For a fixed $H \in \Omega$, a constant x that depends on H is written as x_H . The graphs $K_n^{p,q}$ defined in the previous section satisfy the following conditions: $p, q, n \in \mathbb{N}$, $n \geq 1$ and $p, q \geq 0$. When $n = 1$, we consider $q = 0$ and $p > 0$.

One of the most important properties of the Tutte polynomial is the existence of a contraction–deletion formula. In fact, this polynomial is an example of a *Tutte–Grothendieck invariant* [7,33], that is, a function f from the set of graphs to a fixed commutative ring satisfying the following:

- Contraction–Deletion Formula: $f(G) = f(G - e) + f(G/e)$ when G is connected and e is neither a loop nor a bridge.
- Multiplicativity: the invariant of a graph is the product of the invariants of its connected components.
- Isomorphism Invariance: the invariants of two isomorphic graphs are the same.

The following result states that every Tutte–Grothendieck invariant is essentially an evaluation of the Tutte polynomial.

Theorem 2.1 ([33]). *Let f be any function from the set of graphs to a fixed commutative ring $\mathbb{Z}[x_0, y_0, n, a, b]$ which is multiplicative and isomorphism invariant. Further, let f satisfy the following recurrence relations:*

- $f(G) = n^\lambda$ if G has no edges and λ vertices.
- $f(G) = af(G - e) + bf(G/e)$ provided that $e \in E(G)$ is neither a loop nor a bridge.
- $f(G) = x_0f(G/e)$ whenever $e \in E(G)$ is a bridge.
- $f(G) = y_0f(G - e)$ whenever $e \in E(G)$ is a loop.

Then $f(G) = n^c a^{m-\lambda+c} b^{\lambda-c} T(G; \frac{x_0}{b}, \frac{y_0}{a})$ where G is a graph with λ vertices, m edges and c connected components.

Observe that the parameter $\text{hom}(_, K_n^{p,q})$ is multiplicative, and $\text{hom}(G, K_n^{p,q}) = n^\lambda$ if G is the graph with λ vertices and no edges. Thus, the following result is the key tool to relate this parameter to the Tutte polynomial. It defines $\text{hom}(_, K_n^{p,q})$ in terms of a contraction–deletion formulae.

Lemma 2.2. *The number of homomorphisms of any graph G to $K_n^{p,q}$ satisfies the following recurrence relations:*

- (1) $\text{hom}(G, K_n^{p,q}) = q \text{hom}(G - e, K_n^{p,q}) + (p - q) \text{hom}(G/e, K_n^{p,q})$ provided that $e \in E(G)$ is neither a loop nor a bridge.
- (2) $\text{hom}(G, K_n^{p,q}) = p \text{hom}(G - e, K_n^{p,q})$ whenever $e \in E(G)$ is a loop.
- (3) $\text{hom}(G, K_n^{p,q}) = (p + q(n - 1)) \text{hom}(G/e, K_n^{p,q})$ whenever $e \in E(G)$ is a bridge.

Proof. Assume first that $K_n^{p,q}$ is a simple graph, that is $p = 0$ and $q = 1$. Statements (1), (2) and (3) are the contraction–deletion formulae of the chromatic polynomial:

- $\text{hom}(G - e, K_n^{0,1}) = \text{hom}(G, K_n^{0,1}) + \text{hom}(G/e, K_n^{0,1})$ whenever $e \in E(G)$ is neither a loop nor a bridge.
- If G has loops then $\text{hom}(G, K_n^{0,1}) = 0$.
- $\text{hom}(G, K_n^{0,1}) = (n - 1) \text{hom}(G/e, K_n^{0,1})$ whenever $e \in E(G)$ is a bridge.

Suppose now that $K_n^{p,q}$ is not a simple graph. For $n > 1$, we consider three cases:

- (1) First suppose that $e = uv$ is an edge of a graph G which is neither a loop nor a bridge. Let $f \in \text{Hom}(G, K_n^{p,q})$. Obviously, $f(e) = f(u)f(v)$ might be either a loop or a non-loop edge with multiplicity q . In the first case, there are clearly $p \cdot \text{hom}(G/e, K_n^{p,q})$ homomorphisms of G to $K_n^{p,q}$. In the second case, we have to count the number of homomorphisms of G to $K_n^{p,q}$ satisfying that $f(u) \neq f(v)$. Thus, we have to exclude from the set $\text{Hom}(G - e, K_n^{p,q})$ those homomorphisms such that $f(e)$ is a loop, that is, the set $\text{Hom}(G/e, K_n^{p,q})$. Therefore, there are $q \text{hom}(G - e, K_n^{p,q}) - q \text{hom}(G/e, K_n^{p,q})$ homomorphisms so that the image of e is a non-loop edge with multiplicity q . Hence, the total number is $\text{hom}(G, K_n^{p,q}) = q \text{hom}(G - e, K_n^{p,q}) + (p - q) \text{hom}(G/e, K_n^{p,q})$. This recurrence relation includes the possibilities of $p = 0$ and $p = q > 0$.
- (2) When $e \in E(G)$ is a loop, $f(e)$ is a loop in $K_n^{p,q}$. Since there are p loops attached at each vertex of $K_n^{p,q}$, we have $\text{hom}(G, K_n^{p,q}) = p \text{hom}(G - e, K_n^{p,q})$.
- (3) Let $e \in E(G)$ be a bridge. Then $f(e)$ is either a loop xx or a non-loop edge xy with $m(xy) = q$. The graph $K_n^{p,q}$ has p loops attached at x and $(n - 1)q$ non-loop edges incident with x (or y). Hence, $\text{hom}(G, K_n^{p,q}) = (p + q(n - 1)) \text{hom}(G/e, K_n^{p,q})$.

The result also holds for $n = 1$. In this case $K_1^{p,0}$ is a graph with one vertex and p loops. It is clear that $\text{hom}(G, K_1^{p,0}) = p^{|E(G)|}$ for every graph G . \square

Theorem 2.1 and **Lemma 2.2** imply the following relationship for all $n \geq 1$ (the case $n = 1$ is straightforward since $T(G; 2, 2) = 2^m$).

Theorem 2.3. *For every graph G with λ vertices, m edges and c connected components, the following holds:*

1. $\text{hom}(G, K_n^{p,q}) = n^c (p - q)^{\lambda - c} q^{m - \lambda + c} T\left(G; \frac{p+q(n-1)}{p-q}, \frac{p}{q}\right)$ with $q \geq 1$ and $p \neq q$.
2. $\text{hom}(G, K_1^{p,0}) = (p/2)^m T(G; 2, 2)$ with $p > 0$.

Remark. When $p = q$, $\text{hom}(G, K_n^{p,p}) = n^\lambda p^m$ is considered an evaluation of the Tutte polynomial in [2], hence it could be included as case 3 in the previous theorem. However we have not considered this case since the characterization provided by **Theorem 2.3** and the following results hold for $p \neq q$.

Our next aim is to characterize the graphs H such that the parameter $\text{hom}(_, H)$ satisfies a contraction–deletion formula. We first state the result for complete graphs, that is, graphs with the property that between any two distinct vertices there is at least one edge, and at a vertex there may be any number of loops.

Definition 2.4. A function $h : \Omega \rightarrow \mathbb{Q} - \{0\}$ is called *local* if for every graph $G \in \Omega$ the quotients $h(G)/h(G - e)$ and $h(G)/h(G/e)$ depend only on whether the edge e is a loop, a bridge or none of them, but they do not depend on the choice of G and e (up to its status as a loop, bridge or neither).

Remark. A local function h cannot take the value zero for any graph G since in that case the quotients $h(G)/h(G - e)$ and $h(G)/h(G/e)$ are *undefined* for every graph G . Indeed, since these quotients do not depend on the graph, if there exists a graph G_0 such that $h(G_0) = 0$ then $h(G)/h(G - e)$ and

$h(G)/h(G/e)$ are either zero for every graph G or *undefined* for every graph G . To show that the first case is impossible it suffices to note that $h \equiv 0$ and hence the quotients are *undefined*.

Lemma 2.5 ([29]). Let q_1, \dots, q_t be different non-zero real numbers and let k be positive integer. If there exist real numbers a_1, \dots, a_t such that for all $m \geq k$ it holds that,

$$a_1 q_1^m + a_2 q_2^m + \dots + a_t q_t^m = 0$$

then $a_1 = a_2 = \dots = a_t = 0$.

Proposition 2.6. For every complete graph $H \in \Omega$, the two following statements are equivalent:

- (1) There exist two rational numbers x_H and y_H , and a local function h_H such that for every graph G , $\text{hom}(G, H) = h_H(G)T(G; x_H, y_H)$.
- (2) There exist $p, q, n \in \mathbb{N}$ with $p \neq q$ such that $H \cong K_n^{p,q}$.

Proof. (\Leftarrow) When $H \cong K_n^{p,q}$, **Theorem 2.3** provides the numbers x_H and y_H , and the local function h_H satisfying statement (1).

(\Rightarrow) Suppose that there exist such two rational numbers x_H and y_H , and a local function h_H . We have to show that H is isomorphic to some $K_n^{p,q}$. If so, then the values x_H, y_H and $h_H(G)$ are necessarily those stated in **Theorem 2.3**.

Since H is a complete graph, we can assume that H is not simple (if not, $H \cong K_n$ and the result holds). Thus, it suffices to prove the following: (1) every non-loop edge of H has multiplicity q , (2) there are p loops attached at each vertex of H , (3) $p \neq q$. Let us consider two cases according to the number of loops in H .

Case 1. The graph H has n vertices and no loops.

For the graph $K_1^{p,0}$ we have $0 = \text{hom}(K_1^{p,0}, H) = h_H(K_1^{p,0})T(K_1^{p,0}, x_H, y_H) = h_H(K_1^{p,0})y_H^p$ which implies that $y_H = 0$ since $h_H(K_1^{p,0}) \neq 0$.

Suppose now on the contrary that there are t different non-zero edge multiplicities that appear in H , denoted by q_1, \dots, q_t . Since H is not simple and has no loops, there is at least one multiplicity bigger than one. Let e_i be the number of pairs of vertices joined by an edge of multiplicity q_i (thus, the total number of edges is $\sum e_i q_i$). We have,

$$\text{hom}(K_2^{0,m}, H) = \sum 2e_i q_i^m = h_H(K_2^{0,m})T(K_2^{0,m}, x_H, 0) = h_H(K_2^{0,m})x_H.$$

Since h_H is a local function, there exist some value c depending only on H and not on m such that,

$$c = \frac{h_H(K_2^{0,m})}{h_H(K_2^{0,m} - e)} = \frac{h_H(K_2^{0,m})}{h_H(K_2^{0,m-1})} = \frac{\sum e_i q_i^m}{\sum e_i q_i^{m-1}}.$$

Hence,

$$0 = \sum e_i q_i^m - c \sum e_i q_i^{m-1} = \sum q_i^{m-1} e_i (q_i - c).$$

By **Lemma 2.5** it follows that $q_i = c$ for $i = 1, \dots, t$ and therefore all the edge multiplicities are the same, as needed.

Case 2. The graph H has n vertices and at least one loop.

Let q_1, \dots, q_t and e_1, \dots, e_t be defined as in the previous case. Assume also on the contrary that there are s different loop multiplicities, denoted by p_1, \dots, p_s , and let f_j be the number of vertices with p_j loops attached. Then,

$$\begin{aligned} \text{hom}(K_2^{0,m}, H) &= \sum 2e_i q_i^m + \sum f_j p_j^m. \\ \text{hom}(K_2^{0,m} - e, H) &= \sum 2e_i q_i^{m-1} + \sum f_j p_j^{m-1}. \\ \text{hom}(K_2^{0,m}/e, H) &= \sum f_j p_j^{m-1}. \end{aligned}$$

Since H satisfies statement (1) and has loops we have,

$$0 \neq \text{hom}(K_2^{0,m}/e, H) = h_H(K_2^{0,m}/e)T(K_2^{0,m}/e; x_H, y_H) = h_H(K_2^{0,m}/e)y_H^{m-1} \Rightarrow y_H \neq 0.$$

Furthermore,

$$\begin{aligned}
\text{hom}(K_2^{0,m}, H) &= h_H(K_2^{0,m})T(K_2^{0,m}; x_H, 0) \\
&= h_H(K_2^{0,m}) \left[T(K_2^{0,m} - e; x_H, 0) + T(K_2^{0,m}/e; x_H, 0) \right] \\
&= \frac{h_H(K_2^{0,m})}{h_H(K_2^{0,m} - e)} \text{hom}(K_2^{0,m} - e, H) + \frac{h_H(K_2^{0,m})}{h_H(K_2^{0,m}/e)} \text{hom}(K_2^{0,m}/e, H).
\end{aligned}$$

Thus,

$$\sum 2e_i q_i^m + \sum f_j p_j^m = \frac{h_H(K_2^{0,m})}{h_H(K_2^{0,m} - e)} \left[\sum 2e_i q_i^{m-1} + \sum f_j p_j^{m-1} \right] + \frac{h_H(K_2^{0,m})}{h_H(K_2^{0,m}/e)} \left[\sum f_j p_j^{m-1} \right].$$

Denoting c and \tilde{c} the quotients $\frac{h_H(K_2^{0,m})}{h_H(K_2^{0,m} - e)}$ and $\frac{h_H(K_2^{0,m})}{h_H(K_2^{0,m}/e)}$ respectively, one obtains the following equation,

$$\sum 2e_i q_i^m + \sum f_j p_j^m - c \left(\sum 2e_i q_i^{m-1} + \sum f_j p_j^{m-1} \right) - \tilde{c} \sum f_j p_j^{m-1} = 0.$$

By Lemma 2.5, we conclude that $q_i = c$ for $i = 1, \dots, t$ and $p_j = c + \tilde{c}$ for $j = 1, \dots, s$. Hence, all non-loop edge multiplicities are the same, say equal to q , and all loop multiplicities are also equal, say equal to p . To show that $p \neq q$ it suffices to observe that \tilde{c} is not zero. \square

We next prove the non-complete version of the previous proposition.

Theorem 2.7. *For every connected graph $H \in \Omega$, the following statements are equivalent:*

- (1) *There exist two rational numbers x_H and y_H , and a local function h_H such that for every graph G , $\text{hom}(G, H) = h_H(G)T(G; x_H, y_H)$.*
- (2) *There exist $p, q, n \in \mathbb{N}$ with $p \neq q$ such that $H \cong K_n^{p,q}$.*

Proof. By Proposition 2.6, it suffices to prove that if H satisfies statement (1) then H is a complete graph. We consider two cases according to the number of loops in H .

Case 1. The graph H has n vertices and no loops.

We can assume that the non-zero multiplicities that appear in H are all equal, say equal to q since in proving Proposition 2.6, we only use that H is complete to exclude the case in which H is simple and so $H \cong K_n$. Thus, the same argument proves that all the non-zero multiplicities in H are the same.

Since H is loopless then $0 = \text{hom}(K_1^{p,0}, H) = h_H(K_1^{p,0})y_H^p$ which leads to $y_H = 0$. Moreover, since H is simple we have,

$$\frac{h_H(K_2^{0,m})}{h_H(K_2^{0,m-1})} = \frac{\text{hom}(K_2^{0,m}, H)T(K_2^{0,m-1}, x_H, 0)}{\text{hom}(K_2^{0,m-1}, H)T(K_2^{0,m}, x_H, 0)} = \frac{2|E(H)|q^m x_H}{2|E(H)|q^{m-1} x_H} = q.$$

Therefore for every graph G and $e \in E(G)$ that is neither a loop nor a bridge we have $h_H(G)/h_H(G-e) = q$, since the value of this quotient does not depend on the graph G .

Let $A(H)$ denote the adjacency matrix of H . A m -walk in a graph is an alternating sequence of vertices and edges $v_0, e_1, v_1, \dots, v_{m-1}, e_m, v_m$ where $e_{i+1} = v_i v_{i+1}$ for $0 \leq i \leq m-1$. An m -walk is closed if $v_0 = v_m$.

The matrix $A(H)^m$ has (i, j) entry the sum of m -walks from i to j . Moreover, a closed m -walk corresponds to a homomorphic image of C_m . Thus, for a cycle of length m , denoted by C_m , we have that $\text{hom}(C_m, H) = \text{trace}(A(H)^m)$ [14]. On the other hand, if P_m denotes the path on m vertices, then $\text{hom}(P_m, H)$ counts the m -walks in H not necessarily closed, and so $\text{hom}(P_m, H) = \text{trace}(J \cdot A(H)^{m-1})$ where J is the all-one-matrix [14]. Hence, when m is even it follows that,

$$\begin{aligned}
0 &\neq \text{hom}(C_m, H) = h_H(C_m)T(C_m, x_H, 0), \\
0 &\neq \text{hom}(P_{m-1}, H) = h_H(P_{m-1})T(P_{m-1}, x_H, 0).
\end{aligned}$$

Thus, $\text{hom}(C_m, H) = q \cdot \text{hom}(P_{m-1}, H) + \frac{h_H(C_m)}{h_H(C_{m-1})} \text{hom}(C_{m-1}, H)$. Denoting by \tilde{c} the quotient $\frac{h_H(C_m)}{h_H(C_{m-1})}$ it follows that,

$$\text{trace}(A(H)^m) = q \cdot \text{trace}(J \cdot A(H)^{m-1}) + \tilde{c} \cdot \text{trace}(A(H)^{m-1}).$$

Therefore, $\text{trace}((A(H) - qJ - \tilde{c}I)A(H)^{m-1}) = 0$ for every m which implies that all the elements of the matrix $A(H) - qJ - \tilde{c}I$ are equal to zero. Since H is loopless then $\tilde{c} = -q$ and $A(H)$ is the adjacency matrix of a complete graph with no loops and multiplicity q in all the non-loop edges.

Case 2. The graph H has n vertices and at least one loop.

Suppose on the contrary that H is not a complete graph. By [Proposition 2.6](#) we can assume that all the non-zero multiplicities of the non-loop edges that appear in H are equal, say equal to q . Denote by α the number of non-loop edges with multiplicity zero in H . We can also assume that there are $n - \beta$ vertices with the same number of loops attached, say equal to p different than q , and β vertices with no loops attached.

The graph H has loops which implies that $0 \neq \text{hom}(K_1^{p,0}, H) = h_H(K_1^{p,0})y_H^p$ and so $y_H \neq 0$. Since h is a local function and H satisfies statement (1) we have,

$$c = \frac{h_H(K_2^{0,m})}{h_H(K_2^{0,m-1})} = \frac{\left[\binom{n(n-1)}{2} - \alpha \right] q^m + (n - \beta)p^m}{\left[\binom{n(n-1)}{2} - \alpha \right] q^{m-1} + (n - \beta)p^{m-1}} (x_H + y_H + \dots + y_H^{m-2})$$

where c is a constant that does not depend on m . Thus if $y_H = 1$ then,

$$\left[\left(\frac{n(n-1)}{2} - \alpha \right) [c(x_H + m - 1) - q(x_H + m - 2)] \right] q^{m-1} + (n - \beta)[c(x_H + m - 1) - p(x_H + m - 2)]p^{m-1} = 0$$

and if $y_H \neq 1$ it follows that,

$$\begin{aligned} & \left[\left(\frac{n(n-1)}{2} - \alpha \right) \left(x_H - \frac{y_H}{y_H - 1} \right) (q \cdot c - q^2) \right] q^{m-2} \\ & + \left[(n - \beta) \left(x_H - \frac{y_H}{y_H - 1} \right) (p \cdot c - p^2) \right] p^{m-2} \\ & + \left[\left(\frac{n(n-1)}{2} - \alpha \right) \left(\frac{y_H}{y_H - 1} \right) (c \cdot q \cdot y_H - q^2) \right] (q \cdot y_H)^{m-2} \\ & + \left[(n - \beta) \left(\frac{y_H}{y_H - 1} \right) (c \cdot p \cdot y_H - p^2) \right] (p \cdot y_H)^{m-2} = 0. \end{aligned}$$

In both cases since $p \neq q$, by [Lemma 2.5](#) we conclude that $n(n-1)/2 = \alpha$ and $n = \beta$ which leads to the desired contradiction. Hence $H \cong K_n^{p,q}$. \square

[Theorems 2.1](#) and [2.7](#) lead to the characterization of the graphs H such that the parameter $\text{hom}(_, H)$ satisfies a Tutte–Grothendieck contraction–deletion formula.

Corollary 2.8. *For every connected graph $H \in \Omega$ with n vertices, the following two statements are equivalent:*

1. *There exist rational numbers a_H, b_H, c_H and d_H such that the parameter $\text{hom}(_, H)$ satisfies the following recurrence relations:*
 - (i) $\text{hom}(G, H) = n^\lambda$ if G has no edges and λ vertices.
 - (ii) $\text{hom}(G, H) = a_H \text{hom}(G - e, H) + b_H \text{hom}(G/e, H)$ provided that $e \in E(G)$ is neither a loop nor a bridge.
 - (iii) $\text{hom}(G, H) = c_H \text{hom}(G/e, H)$ whenever $e \in E(G)$ is a bridge.
 - (iv) $\text{hom}(G, H) = d_H \text{hom}(G - e, H)$ whenever $e \in E(G)$ is a loop.
2. *There exist $p, q \in \mathbb{N}$ with $p \neq q$ such that $H \cong K_n^{p,q}$.*

Another consequence of [Theorem 2.7](#) is the characterization of the finite simple graphs H such that the parameter $\text{hom}(_, H)$ can be recovered from the Tutte polynomial, up to a local function. It turns out that these graphs are just those that can be recovered from the chromatic polynomial.

Corollary 2.9. *Let H be a finite connected simple graph with n vertices. The following statements are equivalent:*

1. *There exist two rational numbers x_H and y_H , and a local function h_H such that for every graph G , $\text{hom}(G, H) = h_H(G)T(G; x_H, y_H)$.*
2. *$H \cong K_n$.*

We conclude this section by establishing a relationship for homomorphisms of dual graphs.

Proposition 2.10. *Let G be a planar graph with λ vertices, m edges and c connected components, and G^* be its dual graph. Then the following holds:*

1. $\text{hom}(G, K_n^{p,q}) = \left(\frac{p-q}{q}\right)^m n^{\lambda-m-1} \text{hom}\left(G^*, K_n^{q+\frac{q^2n}{p-q}, q}\right)$ with $q + \frac{q^2n}{p-q} \in \mathbb{N}$, $n > 1$, $q \geq 1$ and $p \geq 0$.
2. $\text{hom}(G, K_1^{p,0}) = \text{hom}(G^*, K_1^{p,0})$ with $p > 0$.

Proof. If G is a planar graph with λ vertices, m edges and c connected components, then the dual graph, G^* , is a connected graph with $1 - \lambda + m + c$ vertices and m edges. Given p, q and $n > 1$ satisfying that $q + \frac{q^2n}{p-q} \in \mathbb{N}$, let us choose $x = \frac{p+q(n-1)}{p-q}$ and $y = p/q$. Thus, $x \neq 1$ and $xq \neq q$. Moreover, $n = (x-1)(y-1)$. By [Theorem 2.3](#) we have,

$$\begin{aligned} T(G; x, y) &= [(x-1)(y-1)]^{-c} (yq-q)^{c-\lambda} q^{-m+\lambda-c} \text{hom}(G, K_n^{yq,q}) \\ &= (x-1)^{-c} (y-1)^{-\lambda} q^{-m} \text{hom}(G, K_n^{yq,q}). \\ T(G^*; y, x) &= [(x-1)(y-1)]^{-1} (yq-q)^{\lambda-m-c} q^{-\lambda+c} \text{hom}(G^*, K_n^{xq,q}) \\ &= (x-1)^{-1+\lambda-m-c} (y-1)^{-1} q^{-m} \text{hom}(G^*, K_n^{xq,q}). \end{aligned}$$

In [32] it is proved that for every planar graph G , $T(G; x, y) = T(G^*; y, x)$. Hence,

$$\begin{aligned} \text{hom}(G, K_n^{p,q}) &= (x-1)^{-1+\lambda-m} (y-1)^{\lambda-1} \text{hom}(G^*, K_n^{xq,q}) \\ &= \left(\frac{p-q}{q}\right)^m n^{\lambda-m-1} \text{hom}\left(G^*, K_n^{q+\frac{q^2n}{p-q}, q}\right) \end{aligned}$$

where $x-1 = qn/(p-q)$ and $y-1 = (p-q)/q$.

When $q = 0$, it is straightforward to show that the result also holds. \square

We want to stress that the conditions $xq = q + \frac{q^2n}{p-q} \in \mathbb{N}$ and $n > 1$ imply $q < p$ and $(x-1)(y-1) > 1$. Therefore the previous result provides a connection between the number of homomorphisms of G and G^* to $K_n^{yq,q}$ and $K_n^{xq,q}$ respectively, for an infinite number of points over the hyperbolae $(x-1)(y-1) = n$.

3. Homomorphisms and other polynomial invariants of graphs

There are many polynomial invariants that can be recovered from the Tutte polynomial. Among those, there are some (but not all) that can be related to homomorphism counting. In this section, we establish connections between the parameter $\text{hom}(_, H)$ and the *boundary*, the *coboundary*, the *transition*, and the *circuit partition* polynomials. We start by considering the boundary and the coboundary polynomials which have a special role in the theory of the Tutte polynomial, see for example [13,33].

Let G be a graph and ω a fixed orientation of its edges. For every $v \in V(G)$, we can divide the edges incident with v according to the orientation ω into two sets, $\omega^+(v)$ and $\omega^-(v)$, that is the edges directed *into* the vertex and the edges directed *out of* the vertex.

Given an abelian group A of order r , a function $f : E(G) \rightarrow A$ is called an A -flow of G with orientation ω if for each vertex $v \in V(G)$,

$$\sum_{e \in \omega^+(v)} f(e) = \sum_{e \in \omega^-(v)} f(e).$$

In particular, a *nowhere-zero A -flow* is a $A \setminus \{0\}$ -flow. The number of A -flows of a graph G depends only on the order of A and not on its particular structure.

Let us denote by $\Theta_A(G)$ the set of A -flows of G . The *boundary polynomial*, or *bad flow polynomial* of [33], is defined as follows.

$$F(G; r, x) = \sum_{f \in \Theta_A(G)} x^{|f^{-1}(0)|}$$

where $|f^{-1}(0)|$ is the number of zero-edges in the A -flow f . Clearly this polynomial is an extension of the flow polynomial since it considers not only nowhere-zero A -flows of a graph G , but also A -flows in which there are i zero-edges with $1 \leq i \leq |E(G)|$. Thus, $F(G; r, 0)$ is the flow polynomial of G .

Similarly, the *coboundary polynomial*, or *monochrome polynomial* of [33], is defined for any abelian group A of order r by

$$P(G; r, y) = \sum_{g \in \mathcal{C}_r(G)} y^{|\Gamma_g(G)|}$$

where $\mathcal{C}_r(G)$ is the set of vertex r -colorings of G , and $\Gamma_g(G)$ is the set of *monochrome edges* in a given $g \in \mathcal{C}_r(G)$, that is, the edges which have endpoints of the same color. Since the chromatic polynomial only considers proper vertex r -colorings of a graph, it is clear that $P(G; r, 0)$ is the chromatic polynomial of G .

The following relationships define the boundary and the coboundary polynomials as evaluations of the Tutte polynomial, up to local functions.

Theorem 3.1 ([33]). *For any graph G with λ vertices, m edges and c connected components the following holds:*

- (1) $F(G; r, x) = (x - 1)^{m-\lambda+c} T\left(G; x, \frac{x-1+r}{x-1}\right)$.
- (2) $P(G; r, y) = r^c (y - 1)^{\lambda-c} T\left(G; \frac{y-1+r}{y-1}, y\right)$.

Remark. We refer the reader to [2] for how to interpret $F(G; r, 1) = r^{m-\lambda+c}$ and $P(G; r, 1) = r^\lambda$ as sort of evaluations of the Tutte polynomial.

Now, we relate these polynomials to the parameter $\text{hom}(_, H)$.

Proposition 3.2. *For every graph G with λ vertices, m edges and c connected components, the following holds:*

- (1) $\text{hom}(G, K_n^{p,q}) = n^{\lambda-m} (p - q)^m F\left(G; n, \frac{p+q(n-1)}{p-q}\right)$ with $n > 1, q \geq 1, p \geq 0$ and $p \neq q$.
- (2) $\text{hom}(G, K_n^{p,q}) = q^m P(G; n, p/q)$ with $n > 1, q \geq 1, p \geq 0$ and $p \neq q$.

Proof. To prove statement (1), let us choose $x = \frac{p+q(n-1)}{p-q}$ and $r = n$. Then,

$$\frac{x - 1 + r}{x - 1} = \frac{p}{q}$$

and **Theorem 3.1** provides the following relationship,

$$T\left(G; \frac{p + q(n - 1)}{p - q}, \frac{p}{q}\right) = \left(\frac{qn}{p - q}\right)^{\lambda - m - c} F\left(G; n, \frac{p + q(n - 1)}{p - q}\right).$$

Hence by [Theorem 2.3](#), we have

$$\begin{aligned} \text{hom}(G, K_n^{p,q}) &= n^c (p-q)^{\lambda-c} q^{m-\lambda+c} T\left(G; \frac{p+q(n-1)}{p-q}, \frac{p}{q}\right) \\ &= n^c (p-q)^{\lambda-c} q^{m-\lambda+c} \left(\frac{qn}{p-q}\right)^{\lambda-m-c} F\left(G; n, \frac{p+q(n-1)}{p-q}\right) \\ &= n^{\lambda-m} (p-q)^m F\left(G; n, \frac{p+q(n-1)}{p-q}\right). \end{aligned}$$

Statement (2) is proved analogously. It suffices to choose $y = p/q$ and $r = n$. Then,

$$\frac{y-1+r}{y-1} = \frac{p+q(n-1)}{p-q}$$

and by [Theorems 2.3](#) and [3.1](#) it follows that,

$$\begin{aligned} \text{hom}(G, K_n^{p,q}) &= n^c (p-q)^{\lambda-c} q^{m-\lambda+c} T\left(G; \frac{p+q(n-1)}{p-q}, \frac{p}{q}\right) \\ &= n^c (p-q)^{\lambda-c} q^{m-\lambda+c} n^{-c} \left(\frac{p-q}{q}\right)^{c-\lambda} P\left(G; n, \frac{p}{q}\right) \\ &= q^m P\left(G; n, \frac{p}{q}\right). \quad \square \end{aligned}$$

Theorem 3.3. For every connected graph $H \in \Omega$, the following two statements are equivalent:

- (1) There exist a rational number x_H , a positive integer number $r_H > 1$, and a local function h_H satisfying that for every graph G , $\text{hom}(G, H) = h_H(G)F(G; r_H, x_H)$.
- (2) There exist $p, q, n \in \mathbb{N}$ with $p \neq q$ such that $H \cong K_n^{p,q}$.

Proof. (\Leftarrow) When H is isomorphic to some $K_n^{p,q}$, [Proposition 3.2](#) provides the numbers x_H and r_H , and the function h_H satisfying statement (1).

(\Rightarrow) Suppose now that we are given a graph H satisfying statement (1). Let $G \in \Omega$ be a graph with λ vertices, m edges and c connected components. [Theorem 3.1](#) implies that,

$$\begin{aligned} \text{hom}(G, H) &= h_H(G)F(G; r_H, x_H) \\ &= h_H(G)(x_H - 1)^{m-\lambda+c} T\left(G; x_H, \frac{x_H - 1 + r_H}{x_H - 1}\right). \end{aligned}$$

Since the function $h_H(G)(x_H - 1)^{m-\lambda+c}$ is local, [Theorem 2.7](#) leads to $H \cong K_n^{p,q}$. \square

Observe that [Theorem 3.1](#) and the connection between the parameter $\text{hom}(_, H)$ and the Tutte polynomial, enable us to state and prove a similar characterization for the coboundary polynomial. We omit it for the sake of brevity.

The transition polynomial was first defined in [17] on 4-regular planar graphs in terms of a weight function Λ . One can find many interesting applications of this polynomial mostly in knot theory, see for instance [19]. The Tutte polynomial of a connected planar graph G with set of faces $R(G)$ is related to the transition polynomial $Q(M(G); \Lambda, \tau)$ of its medial graph $M(G)$. The following relationship is proved in [17] for special values of μ and δ , and a special weight function Λ . For the sake of brevity, we do not describe the roles of these parameters but refer the reader to [17] for a complete description of them.

$$Q(M(G); \Lambda, \tau) = \delta^{1-|V(G)|} \mu^{1-|R(G)|} T\left(G; 1 + \frac{\delta\tau}{\mu}, 1 + \frac{\mu\tau}{\delta}\right).$$

The *medial graph* $M(G)$ is the planar connected 4-regular graph obtained from G as follows: The vertices of $M(G)$ correspond to the edges of G and two vertices of $M(G)$ are joined by an edge if the

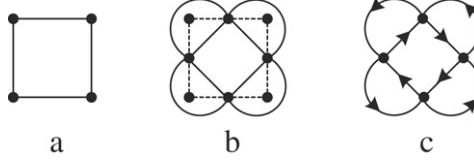


Fig. 2. (a) The cycle C_4 , (b) the medial graph of C_4 , (c) the directed medial graph of C_4 .

corresponding edges of G are neighbors in the cyclic order around the vertex (see Fig. 2). We do not go into more details and just state the connection between the transition polynomial and the parameter $\text{hom}(_, H)$. This connection is given by the following two results.

Proposition 3.4. *Let G be a connected planar graph with λ vertices and m edges. The number of homomorphisms of G to $K_n^{p,q}$ with $p \neq q$ and $n > 1$ is given by*

$$\text{hom}(G, K_n^{p,q}) = n^{m-\lambda+1} (p-q)^m \delta^m Q(M(G), \Lambda, \sqrt{n}) \quad \text{if } p-q \neq q\sqrt{n}.$$

$$\text{hom}(G, K_n^{p,q}) = (\sqrt{n})^{\lambda+1} q^m \delta^m Q(M(G), \Lambda, \sqrt{n}) \quad \text{if } p-q = q\sqrt{n}.$$

Proof. Assume first that $p-q \neq q\sqrt{n}$. By Theorem 2.3 we have,

$$\text{hom}(G, K_n^{p,q}) = n(p-q)^{\lambda-1} q^{m-\lambda+1} T\left(G; \frac{p+q(n-1)}{p-q}, \frac{p}{q}\right).$$

Let us choose $\delta \neq 0$, $\tau = \sqrt{n}$ and $\mu = \delta \frac{p-q}{q\sqrt{n}}$. Then,

$$\frac{p+q(n-1)}{p-q} = 1 + \frac{\delta\sqrt{n}}{\mu} \quad \text{and} \quad \frac{p}{q} = 1 + \frac{\mu\sqrt{n}}{\delta}.$$

We prove this result by using Euler's formula, $|R(G)| = 2 - \lambda + m$, and by considering the connection between the Tutte polynomial and the transition polynomial. We proceed as follows.

$$\begin{aligned} \text{hom}(G, K_n^{p,q}) &= n(p-q)^{\lambda-1} q^{m-\lambda+1} T\left(G; \frac{p+q(n-1)}{p-q}, \frac{p}{q}\right) \\ &= n(p-q)^{\lambda-1} q^{m-\lambda+1} T\left(G; 1 + \frac{\delta\sqrt{n}}{\mu}, 1 + \frac{\mu\sqrt{n}}{\delta}\right) \\ &= n(p-q)^{\lambda-1} q^{m-\lambda+1} \delta^{\lambda-1} \mu^{1+m-\lambda} Q(M(G), \Lambda, \sqrt{n}) \\ &= n(p-q)^{\lambda-1} q^{m-\lambda+1} \delta^m \left(\frac{p-q}{q\sqrt{n}}\right)^{1+m-\lambda} Q(M(G), \Lambda, \sqrt{n}) \\ &= n^{1+m-\lambda} (p-q)^m \delta^m Q(M(G), \Lambda, \sqrt{n}). \end{aligned}$$

Suppose now that $p-q = q\sqrt{n}$. Then, $\mu = \delta$ and

$$\frac{p}{q} = 1 + \sqrt{n} = \frac{p+q(n-1)}{p-q}.$$

Thus, the following expressions are obtained.

$$\begin{aligned} \text{hom}(G, K_n^{p,q}) &= n(p-q)^{\lambda-1} q^{m-\lambda+1} T(G; 1 + \sqrt{n}, 1 + \sqrt{n}) \\ &= n((1 + \sqrt{n})q - q)^{\lambda-1} q^{m-\lambda+1} \delta^m Q(M(G), \Lambda, \sqrt{n}) \\ &= (\sqrt{n})^{1+\lambda} q^m \delta^m Q(M(G), \Lambda, \sqrt{n}). \quad \square \end{aligned}$$

Theorem 3.5. *For every connected graph $H \in \Omega$, the following statements are equivalent:*

1. *There exist a constant τ_H , and a local function h_H such that for every connected planar graph G , $\text{hom}(G, H) = h_H(G)Q(M(G); \Lambda, \tau_H)$.*
2. *There exist $p, q, n \in \mathbb{N}$ with $p \neq q$ and $n > 1$ such that $H \cong K_n^{p,q}$.*

Proof. (\Leftarrow) When H is isomorphic to some $K_n^{p,q}$, [Proposition 3.4](#) proves that statement (1) holds. (\Rightarrow) Consider now the constant τ_H , and the local function h_H satisfying statement (1). Let G be a connected planar graph with λ vertices and m edges. We have,

$$Q(M(G), \Lambda, \tau_H) = \delta_H^{1-\lambda} \mu_H^{\lambda-m-1} T \left(G; 1 + \frac{\delta_H \tau_H}{\mu_H}, 1 + \frac{\mu_H \tau_H}{\delta_H} \right).$$

Then,

$$\text{hom}(G, H) = h_H(G) \delta_H^{1-\lambda} \mu_H^{\lambda-m-1} T \left(G; 1 + \frac{\delta_H \tau_H}{\mu_H}, 1 + \frac{\mu_H \tau_H}{\delta_H} \right).$$

Clearly, the function $h_H(G) \delta_H^{1-\lambda} \mu_H^{\lambda-m-1}$ is local. Hence, [Theorem 2.7](#) implies that H is isomorphic to some $K_n^{p,q}$. \square

We obtain similar results for the circuit partition polynomial which was first defined in [9] as a generating function for the number of Eulerian partitions of an Eulerian graph or digraph into s components. This polynomial is a generalization, for a specific weight function, of one of Jaeger's transition polynomials (see [9]). It has many applications to several areas, including non-mathematical fields (see for instance [1,3]).

The Tutte polynomial of a planar graph G with c connected components is related to the circuit partition polynomial of its directed medial graph $\overrightarrow{M(G)}$ (see [26,27]). This relationship is given by the following expression,

$$j(\overrightarrow{M(G)}; x) = x^c T(G; x + 1, y + 1)$$

The *directed medial graph* $\overrightarrow{M(G)}$ results from directing the edges of $M(G)$ as follows. We first color the faces of $M(G)$ black or white, depending on whether they contain or do not contain, respectively, a vertex of the original graph G . The edges of $M(G)$ are directed so that the black face is on the left of an incident edge (see [Fig. 2](#)). We just state the connection between the circuit partition polynomial and the parameter $\text{hom}(_, H)$, without going into more detail.

Proposition 3.6. *Let G be a planar graph with λ vertices, m edges and c connected components. For every $q, n \in \mathbb{N}$ with $q \geq 1$ and $n > 1$ such that $\sqrt{n} \in \mathbb{N}$, the following holds:*

$$\text{hom} \left(G, K_n^{(1+\sqrt{n})q,q} \right) = (\sqrt{n})^\lambda q^m j(\overrightarrow{M(G)}; \sqrt{n}).$$

Proof. Given q and n , let us choose $p = (1 + \sqrt{n})q$. Then,

$$\frac{p + q(n-1)}{p-q} = \frac{p}{q} = 1 + \sqrt{n} \Rightarrow T \left(G; \frac{p + q(n-1)}{p-q}, \frac{p}{q} \right) = T(G; 1 + \sqrt{n}, 1 + \sqrt{n}).$$

Hence by [Theorem 2.3](#) we have,

$$\begin{aligned} \text{hom}(G, K_n^{(1+\sqrt{n})q,q}) &= n^c (p-q)^{\lambda-c} q^{m-\lambda+c} T \left(G; \frac{p + q(n-1)}{p-q}, \frac{p}{q} \right) \\ &= n^c ((1 + \sqrt{n})q - q)^{\lambda-c} q^{m-\lambda+c} (\sqrt{n})^{-c} j(\overrightarrow{M(G)}; \sqrt{n}) \\ &= (\sqrt{n})^\lambda q^m j(\overrightarrow{M(G)}; \sqrt{n}). \quad \square \end{aligned}$$

[Theorem 2.7](#) and the connection between the Tutte polynomial and the circuit partition polynomial allow us to prove the following result.

Theorem 3.7. *Let $H \in \Omega$ be a connected graph. Suppose that there exist a constant x_H , and a local function h_H such that for every planar graph G , $\text{hom}(G, H) = h_H(G) j(\overrightarrow{M(G)}; x_H)$. Then there exist $p, q, n \in \mathbb{N}$ with $p \neq q$ and $n > 1$ such that $H \cong K_n^{p,q}$.*

Remark. The Penrose polynomial is an instance of graph polynomial that cannot be related to the homomorphism counting by using our technique. Indeed, this polynomial can be described in terms

of a transition polynomial for specific values of δ , μ and τ (see [17] for more details). The Penrose polynomial is defined for $\tau = -2$ and Proposition 3.4 considers $\tau = \sqrt{n} > 0$.

4. Applications to abelian groups and statistical physics

The aim of this section is to sketch connections between homomorphism counting and fundamental evaluations of the Tutte polynomial in abelian groups and statistical physics. Concretely, we focus on difference sets in abelian groups, the Potts model and the random cluster model in statistical mechanics.

4.1. Difference sets in abelian groups

Let G be a graph with a fixed orientation ω of its edges, and consider uniform probability in the space of pairs (f_1, f_2) of functions from $E(G)$ into a subset B of an abelian group A of order r . A (r, k, l) -difference set in A is a subset $B \subseteq A$ of k elements with $2 \leq k \leq r$ such that, for all $0 \neq a \in A$ there exist l pairs $(b_1, b_2) \in B \times B$ with $b_1 - b_2 = a$.

Let $A^{V(G)}$ denote the set of all functions $g : V(G) \rightarrow A$, and $A^{E(G)}$ the set of all functions $f : E(G) \rightarrow A$. The boundary operator $d^* : A^{E(G)} \rightarrow A^{V(G)}$ is defined for a given function $f : E(G) \rightarrow A$ and each vertex $v \in V(G)$ by,

$$d^*f(v) = \sum_{e \in \omega^+(v)} f(e) - \sum_{e \in \omega^-(v)} f(e).$$

Observe that the kernel of d^* is the space of the A -flows of G .

In this subsection, we prove that when B is a difference set in A with $k < r$, the event that f_1 and f_2 have the same boundary has probability equal, up to a factor, to $\text{hom}(G, K_n^{p,q})$ for some values of n , p and q . We first recall the relationship between such probability and the boundary polynomial from [13].

Lemma 4.1 ([13]). *If B is a (r, k, l) -difference set then,*

$$\Pr(d^*f_1 = d^*f_2) = k^{-2m} l^m F\left(G; r, \frac{k}{l}\right)$$

where G is a graph with m edges.

This lemma and Proposition 3.2 imply the following result.

Proposition 4.2. *Let A be an abelian group on r elements, $B \subseteq A$ a (r, k, l) -difference set in A , and q any positive integer number such that $(\frac{rl}{k-l} + 1)q \in \mathbb{N}$. If two functions $f_1, f_2 : E(G) \rightarrow B$ are chosen uniformly at random then for every graph G with λ vertices and m edges it holds that,*

$$\Pr(d^*f_1 = d^*f_2) = k^{-2m} (k-l)^m r^{-\lambda} q^{-m} \text{hom}\left(G, K_r^{\left(\frac{rl}{k-l} + 1\right)q, q}\right).$$

Proof. Let us choose $p = (\frac{rl}{k-l} + 1)q$ and $n = r$. Then,

$$\frac{p + q(n-1)}{p-q} = \frac{k}{l} \quad \text{and} \quad F\left(G; r, \frac{k}{l}\right) = F\left(G; n, \frac{p + q(n-1)}{p-q}\right).$$

Proposition 3.2 and Lemma 4.1 lead to the following expressions,

$$\begin{aligned} \Pr(d^*f_1 = d^*f_2) &= k^{-2m} l^m F\left(G; r, \frac{k}{l}\right) \\ &= k^{-2m} l^m r^{m-\lambda} (p-q)^{-m} \text{hom}\left(G, K_r^{\left(\frac{rl}{k-l} + 1\right)q, q}\right) \\ &= k^{-2m} l^m r^{m-\lambda} \left(\frac{qrl}{k-l}\right)^{-m} \text{hom}\left(G, K_r^{\left(\frac{rl}{k-l} + 1\right)q, q}\right) \\ &= k^{-2m} (k-l)^m r^{-\lambda} q^{-m} \text{hom}\left(G, K_r^{\left(\frac{rl}{k-l} + 1\right)q, q}\right). \quad \square \end{aligned}$$

Observe that the condition $(\frac{rl}{k-l} + 1)q \in \mathbb{N}$ implies $k > l$. Hence, we can consider the value $q = k - l$ and the following result is a particular case of the above-stated proposition.

Corollary 4.3. *Let B be a proper (r, k, l) -difference set in A . If two functions $f_1, f_2 : E(G) \rightarrow B$ are chosen uniformly at random then for every graph G with λ vertices and m edges we have,*

$$\Pr(d^*f_1 = d^*f_2) = k^{-2m} r^{-\lambda} \text{hom}(G, K_r^{(r-1)l+k, k-l}).$$

Note also that $B = A \setminus \{0\}$ is an $(r, r - 1, r - 2)$ -difference set. Thus, we can state the following corollary.

Corollary 4.4. *Let q be any positive integer number. If two functions $f_1, f_2 : E(G) \rightarrow A \setminus \{0\}$ are chosen uniformly at random then for every graph G with λ vertices and m edges it holds that,*

$$\Pr(d^*f_1 = d^*f_2) = (r - 1)^{-2m} r^{-\lambda} q^{-m} \text{hom}(G, K_r^{(r-1)^2q, q}).$$

4.2. The Potts model and the Gibbs probability

For the combinatorial analysis of the Potts model on a finite graph G , it is assumed that the interaction energy, which measures the strength of the interaction between neighboring pairs of vertices, is constant and equal to J . Consider that each vertex can be in S different states, and let $K = 2\beta J$ where β is a parameter of the model determined by the temperature. The following relationship between the partition function Z of the Potts model and the coboundary polynomial of G is proved for example in [33],

$$Z(G) = e^{-K|E(G)|} P(G; S, e^K).$$

This relationship and Proposition 3.2 lead to the connection between counting graph homomorphisms and the partition function Z of the Potts model.

Proposition 4.5. *Let q be any positive integer such that $e^K q \in \mathbb{N}$. Then for every graph G with m edges it holds that,*

$$Z(G) = e^{-Km} q^{-m} \text{hom}(G, K_S^{e^K q, q}).$$

The random cluster model on a finite graph G can be regarded as the analytic continuation of the Potts model to non-integer S (see [31]). This model is a correlated bond percolation model in statistical mechanics, introduced by Fortuin and Kasteleyn in [10] (see also [31,33]) and defined by a probability distribution, called the *Gibbs probability*, as follows. For every subset $A \subseteq E(G)$,

$$\mu(A) = N^{-1} \left(\prod_{e \in A} t_e \right) \left(\prod_{e \notin A} (1 - t_e) \right) S^{k(A)}$$

where $k(A)$ is the number of connected components of the graph $(V(G), A)$, the value t_e is a probability assigned to every edge $e \in E(G)$, $S \geq 0$ is a parameter of the model, and N is the normalizing constant so that $\sum_{A \subseteq E(G)} \mu(A) = 1$.

When each of the t_e are made equal, the Gibbs probability appears as a two-parameter family of probability measure $\mu = \mu(t, S)$ where $0 \leq t \leq 1$ and $S > 0$. In this case, this probability is essentially an evaluation of the Tutte polynomial of G (see [33]). Indeed,

$$\mu(A) = \frac{\left(\frac{t}{1-t}\right)^{|A|} S^{-r(A)}}{\left(\frac{t}{S(1-t)}\right)^{r(E(G))} T\left(G; 1 + \frac{S(1-t)}{t}, \frac{1}{1-t}\right)}$$

where $r(A) = |V| - k(A)$ is the *rank* of A .

We now reformulate this relationship in terms of homomorphism counting.

Proposition 4.6. Let G be a finite graph, and $A \subseteq E(G)$. For every $\ell \in \mathbb{N}$ such that $(1-t)\ell$ is a positive integer, the Gibbs probability is given by

$$\mu(A) = \frac{\left(\frac{t}{1-t}\right)^{|A|} S^{-r(A)+|V(G)|} (1-t)^{|E(G)|} \ell^{|E(G)|}}{\text{hom}\left(G, K_S^{\ell, (1-t)\ell}\right)}.$$

Proof. Let us write $n = S$ and $q = (1-t)\ell$. Then,

$$T\left(G; \frac{\ell + q(n-1)}{\ell - q}, \frac{\ell}{q}\right) = T\left(G; 1 + \frac{S(1-t)}{t}, \frac{1}{1-t}\right).$$

By [Theorem 2.3](#) we have,

$$\begin{aligned} T\left(G; \frac{\ell + q(n-1)}{\ell - q}, \frac{\ell}{q}\right) &= [n^{-k(G)} (\ell - q)^{k(G)-|V(G)|} q^{-|E(G)|+|V(G)|-k(G)}] \text{hom}\left(G, K_n^{\ell, q}\right) \\ &= [S^{-k(G)} t^{-r(E(G))} (1-t)^{r(E(G))-|E(G)|} \ell^{-|E(G)|}] \text{hom}\left(G, K_n^{\ell, (1-t)\ell}\right). \end{aligned}$$

Hence,

$$\begin{aligned} \mu(A) &= \frac{\left(\frac{t}{1-t}\right)^{|A|} S^{-r(A)}}{\left(\frac{t}{S(1-t)}\right)^{r(E(G))} S^{-k(G)} t^{-r(E(G))} (1-t)^{r(E(G))-|E(G)|} \ell^{-|E(G)|} \text{hom}\left(G, K_n^{\ell, (1-t)\ell}\right)} \\ &= \frac{\left(\frac{t}{1-t}\right)^{|A|} S^{-r(A)+|V(G)|} (1-t)^{|E(G)|} \ell^{|E(G)|}}{\text{hom}\left(G, K_S^{\ell, (1-t)\ell}\right)}. \quad \square \end{aligned}$$

5. Coloring uniqueness

Since the Tutte polynomial contains a great deal of information about the graph to which it is associated, a question that arises naturally is whether a graph can be recovered up to isomorphism from its Tutte polynomial. A graph G is said to be *Tutte-unique* if $T(G; x, y) = T(H; x, y)$ implies $H \cong G$, for every other graph H . Tutte uniqueness has been studied for several families of graphs, such as complete multipartite graphs, wheels, hypercubes (see [\[28\]](#)), locally grid graphs [\[11,25\]](#), and hexagonal tilings [\[12\]](#). In 2000, Bollobás, Pebody and Riordan [\[4\]](#) conjectured that almost all graphs are Tutte-unique. Since then there has been little progress on this conjecture.

The problem of finding graphs determined by polynomial invariants has been studied also for other polynomials [\[30\]](#), such as the chromatic polynomial [\[20,21\]](#). Since the chromatic polynomial of a 2-connected graph can be recovered from its Tutte polynomial, we obtain that 2-connected chromatically-unique graphs are Tutte-unique. It is also conjectured that almost all graphs are chromatically-unique [\[4\]](#). Following this line of research, we introduce the concept of *coloring-uniqueness*.

Definition 5.1. A finite graph G is coloring-unique if $\text{hom}(G, K_n^{p,q}) = \text{hom}(H, K_n^{p,q})$ for all $n \geq 1$, $p, q \geq 0$ and $p \neq q$ implies $H \cong G$, for every other graph H .

Observe that chromatically-unique graphs are coloring-unique.

Theorem 5.2. Let G be a simple, 2-connected graph. If G is coloring-unique then G is Tutte-unique.

Proof. Let H be a simple, 2-connected graph non-isomorphic to G . It suffices to prove that there exist two values x_0 and y_0 such that $T(G; x_0, y_0) \neq T(H; x_0, y_0)$. We can assume that $m = |E(G)| = |E(H)|$ (if not $T(G; 2, 2) \neq T(H; 2, 2)$). Thus, since G and H are 2-connected simple graphs, we have

Table 1

Homomorphisms–Tutte polynomial	
$\text{hom}(G, K_n^{p,q}) = n^c (p - q)^{\lambda - c} q^{m - \lambda + c} T\left(G; \frac{p+q(n-1)}{p-q}, \frac{p}{q}\right), p \geq 0, q \geq 1, p \neq q$	G any graph with λ vertices, m edges, c connected components
$\text{hom}(G, K_1^{p,0}) = (p/2)^m T(G, 2, 2)$ with $p > 0$	
Homomorphisms–Transition polynomial	
$\text{hom}(G, K_n^{p,q}) = n^{m-\lambda+1} (p - q)^m \delta^m Q(M(G), A, \sqrt{n})$ if $p - q \neq q\sqrt{n}$	G connected planar graph $M(G)$ medial graph
$\text{hom}(G, K_n^{p,q}) = (\sqrt{n})^{\lambda+1} q^m \delta^m Q(M(G), A, \sqrt{n})$ if $p - q = q\sqrt{n}$	
Homomorphisms–Circuit partition polynomial	
$\text{hom}(G, K_n^{(1+\sqrt{n})q,q}) = (\sqrt{n})^\lambda q^{mj}(\overrightarrow{M(G)}; \sqrt{n}), q \geq 1, n > 1, \sqrt{n} \in \mathbb{N}$	G planar graph. $\overrightarrow{M(G)}$ directed medial graph
Homomorphisms–Boundary polynomial	
$\text{hom}(G, K_n^{p,q}) = n^{\lambda-m} (p - q)^m F\left(G; n, \frac{p+q(n-1)}{p-q}\right), p \geq 0, q \geq 1, p \neq q$	G any graph with λ vertices, m edges, c connected components
Homomorphisms–Coboundary polynomial	
$\text{hom}(G, K_n^{p,q}) = q^m P(G; n, p/q), p \geq 0, q \geq 1, p \neq q$	G any graph with λ vertices, m edges, c connected components

$\lambda = |V(H)| = |V(G)|$. The graph G is coloring-unique and so there exist $n_0, p_0, q_0 \in \mathbb{N}$ such that $\text{hom}(G, K_{n_0}^{p_0, q_0}) \neq \text{hom}(H, K_{n_0}^{p_0, q_0})$. By [Theorem 2.3](#) it follows that,

$$\begin{aligned} & n_0(p_0 - q_0)^{\lambda-1} q_0^{m-\lambda+1} T\left(G; 1 + \frac{q_0 n_0}{p_0 - q_0}, \frac{p_0}{q_0}\right) \\ & \neq n_0(p_0 - q_0)^{\lambda-1} q_0^{m-\lambda+1} T\left(H; 1 + \frac{q_0 n_0}{p_0 - q_0}, \frac{p_0}{q_0}\right) \end{aligned}$$

which implies that,

$$T\left(G; 1 + \frac{q_0 n_0}{p_0 - q_0}, \frac{p_0}{q_0}\right) \neq T\left(H; 1 + \frac{q_0 n_0}{p_0 - q_0}, \frac{p_0}{q_0}\right). \quad \square$$

Observe that the above result enables us to state the following result which relates to the uniqueness conjectures of Bollobás, Pebody and Riordan [\[4\]](#).

Theorem 5.3. *If almost all graphs are coloring-unique then almost all graphs are Tutte-unique.*

6. Concluding remarks

1. Some of the connections provided in this paper are summarized in [Table 1](#). In particular, those between counting graph homomorphisms and evaluating polynomials associated with graphs.

2. The connection between graph homomorphisms and graph invariants is well known and useful. One can here quote not only classical (and not so classical) results such as those covered in [\[33,16\]](#), but also recent works for reaching an algebraical approach of this connection, see for example [\[6\]](#). It is perhaps surprising how tight (in certain very concrete instances) this connection is. This paper shows both the connection in the case of polynomial invariants, and also its limitations. But perhaps this approach can put in a new context some well-known problems such as uniqueness questions. Andrew Goodall [\[15\]](#) very recently showed the converse of [Theorem 5.2](#): Every Tutte unique graph is coloring unique. A bit surprisingly, the concepts of Tutte- and coloring-uniqueness coincide.

3. Finally, note that in proving [Theorem 2.7](#) we used the properties of the homomorphism function $\text{hom}(_, H)$ for very special graphs only: multiple edges and cycles (and their minors). It is sufficient to assume the locality of the function h_H for this small minor closed family. We shall postpone the details of these questions to a sequel of this paper.

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