On the Ramsey numbers for stars versus complete graphs

L. Boza, M. Cera, P. García-Vázquez, M.P. Revuelta

Departamento de Matemática Aplicada I, Universidad de Sevilla, Sevilla, Spain

A B S T R A C T

For graphs $G_1, \ldots, G_s$, the multicolor Ramsey number $R(G_1, \ldots, G_s)$ is the smallest integer r such that if we give any edge col-oring of the complete graph on r vertices with s colors then there exists a monochromatic copy of G_i colored with color i, for some $1 \le i \le s$. In this work the multicolor Ramsey number $R(K_{p_1}, \ldots, K_{p_m}, K_{1,q_1}, \ldots, K_{1,q_n})$ is determined for any set of com-plete graphs and stars in terms of $R(K_{p_1}, \ldots, K_{p_m})$.

1. Introduction

All graphs considered are undirected, finite and contain neither loops nor multiple edges. Unless otherwise stated, we follow [2,5] for terminology and definitions.

Let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of the graph G, respectively. $|V(G)|$ is called the order of G, and $|E(G)|$ is called the size of G. For a subset $S \subset V(G)$, the *neighborhood* of S, denoted by $N_G(S)$, is the set of vertices in $V(G) \setminus S$ that are adjacent to some vertex of S. If $S = \{v\}$ we put simply $N_G(v)$. Let $d_G(v)$ be the *degree* of vertex v. The *maximum degree* and *minimum degree* of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. For any subset $S \subseteq V(G)$ (resp. $W \subseteq E(G)$), the induced subgraph of G by S (resp. by W), denoted by $G[S]$ (resp. $G[W]$) is the graph with vertex set S (resp. edge set W) whose edges are the edges of G joining vertices of S (resp. whose vertices are incident to some edge of W). A subset $S \subseteq V(G)$ is called *independent* if $G[S]$ has no edges. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of the largest independent set. Formally, $\alpha(G) = \max\{|S| : S \subset V(G) \text{ is independent}\}$. The *complete graph* on p vertices is denoted by K_p, whereas the *complete bipartite graph* with one vertex in the first class and q vertices in the second class is denoted by $K_{1,q}$ and it is also called a *star* on $q + 1$ vertices.

For graphs $G_1, G_2, \ldots, G_s$, a $(G_1, G_2, \ldots, G_s)$-coloring is a coloring of the edges of a complete graph with s colors, such that it does not contain a subgraph isomorphic to G_i whose all edges are colored with color i, for each $1 \le i \le s$. Similarly, a $(G_1, G_2, \ldots, G_s; r)$-coloring is a $(G_1, G_2, \ldots, G_s)$-coloring

E-mail addresses: boza@us.es (L. Boza), mcera@us.es (M. Cera), pgvazquez@us.es (P. García-Vázquez), pastora@us.es (M.P. Revuelta).

of the complete graph K_r. on r vertices. The multicolor Ramsey number $R(G_1, G_2, \ldots, G_s)$ is defined to be the least positive integer r such that there exist no $(G_1, G_2, \ldots, G_s; r)$-coloring.

In this paper we focus on the multicolor Ramsey number for cliques and stars. Let $p_1, \ldots, p_m, q_1, \ldots, q_n$ be positive integers. Set $P = \{p_1, \ldots, p_m\}$ and $Q = \{q_1, \ldots, q_n\}$. By $R(P, Q) = R(K_{p_1}, \ldots, K_{p_m}, K_{1,q_1}, \ldots, K_{1,q_n})$ we denote the Ramsey number for cliques and stars and by (P, Q)-coloring a $(K_{p_1}, \ldots, K_{p_m}, K_{1,q_1}, \ldots, K_{1,q_n})$-coloring. Following this notation, $R(\emptyset, Q) = R(K_{1,q_1}, \ldots, K_{1,q_n})$ and $R(P, \emptyset) = R(K_{p_1}, \ldots, K_{p_m})$.

Some results concerning the classical multicolor Ramsey number $R(K_{p_1}, \ldots, K_{p_m})$ are known only for a small number of cliques. Exact values for $R(K_3, K_\ell)$, $\ell \in \{3, \ldots, 9\}$, can be found in [6–8,10,12,14]. Moreover, $R(K_4, K_\ell)$, $\ell \in \{4, 5\}$, is determined in [7,11,15] and the only known value of $R(K_{p_1}, \ldots, K_{p_m})$ for $m \geq 3$ up to now is $R(3, 3, 3) = 17$, proved in [7]. Other structures involving multicolor Ramsey number have been studied (see for instance [13,16–18]).

For stars the problem was solved in the following Theorem proved in [4].

Theorem 1.1 (*See [4]*). *Let $q_1, \ldots, q_n$ be positive integers. Then*

$$R(K_{1,q_1}, \ldots, K_{1,q_n}) = \sum_{j=1}^{n} q_j - n + \epsilon_Q,$$

where $\epsilon_Q = 1$ if the number of even integers in the set $\{q_j\}_{j=1}^n$ is even and positive, and $\epsilon_Q = 2$ otherwise.

Regarding to the multicolor Ramsey number $R(K_{p_1}, \ldots, K_{p_m}, K_{1,q_1}, \ldots, K_{1,q_n})$, the exact value is only known for $m = 1$, that is, $R(K_p, K_{1,q_1}, \ldots, K_{1,q_n})$. This result was proved in [9].

Theorem 1.2 (*See [9]*). *Let $p, q_1, \ldots, q_n$ be positive integers with $p \geq 2$. Then*

$$R(K_p, K_{1,q_1}, \ldots, K_{1,q_n}) = (p-1)\left(\sum_{j=1}^{n} q_j - n + \epsilon_Q - 1\right) + 1,$$

where $\epsilon_Q = 1$ if the number of even integers in the set $\{q_j\}_{j=1}^n$ is even and positive, and $\epsilon_Q = 2$ otherwise.

In this work we generalize Theorem 1.2 by determining the multicolor Ramsey number $R(K_{p_1}, \ldots, K_{p_m}, K_{1,q_1}, \ldots, K_{1,q_n})$ for any arbitrary numbers m and n of cliques and stars, respectively. Namely the following result will be proved.

Theorem 1.3. *Let $p_1, \ldots, p_m, q_1, \ldots, q_n$ be positive integers, with $p_i, q_j \geq 2$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Then*

$$R(K_{p_1}, \ldots, K_{p_m}, K_{1,q_1}, \ldots, K_{1,q_n}) - 1 = \left(\sum_{j=1}^{n} q_j - n + \epsilon_Q - 1\right)(R(K_{p_1}, \ldots, K_{p_m}) - 1),$$

where $\epsilon_Q = 1$ if the number of even integers in the set $\{q_j\}_{j=1}^n$ is even and positive, and $\epsilon_Q = 2$ otherwise.

2. Definitions and previous results

In this section we give some definitions and technical results that will be used in order to obtain the main result. We start setting the notation for a suitable partition associated to a given set of vertices.

Notation 2.1. *Given a graph G and a subset $U = \{u_1, \ldots, u_k\} \subseteq V(G)$ of vertices of G, $\mathcal{P}(U) = \{W_1, \ldots, W_k\}$ will denote a partition of the set $N_G(U) \cup U$, chosen in such a way that $W_1 = N_G(u_1) \cup \{u_1\}$ and $W_i = N_G(u_i) \cup \{u_i\} \setminus \bigcup_{j=1}^{i-1} W_j$, for $i = 2, \ldots, k$.*

Observe that if U is a set of independent vertices of G with cardinality $|U| = \alpha(G)$, then $N_G(U) \cup U = V(G)$ and hence, $\mathcal{P}(U)$ is a partition of $V(G)$.

The following result of Brooks [3] proves that the chromatic number of a graph different from an odd cycle and a complete graph is upper bounded by the maximum degree.

Theorem 2.1 (*See [3]*). *If G is a connected graph that is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta(G)$.*

In the next lemma some relationships between the independence number and the maximal degree of a graph are given.

Lemma 2.1. *Let G be a graph and let $G_1, \ldots, G_s$ be its components. Then the next assertions hold:*

(i) *If there exists $i \in \{1, \ldots, s\}$ such that G_i is neither a complete graph nor a cycle of odd length, then $\alpha(G_i) \geq \frac{|V(G_i)|}{\Delta(G_i)}$.*

(ii) *If $G_i \neq K_{\Delta(G)+1}$ for all $i = 1, \ldots, s$, and $\Delta(G) \geq 3$ or G_i is not an odd cycle for all $i = 1, \ldots, s$, then $\alpha(G) \geq \frac{|V(G)|}{\Delta(G)}$.*

(iii) *The inequality $\alpha(G) \geq \frac{|V(G)|}{k}$ holds for any integer $k \geq \Delta(G) + 1$.*

Proof. (i) Suppose that G_i is different from a complete graph and an odd cycle, then by Theorem 2.1 we have $\chi(G_i) \leq \Delta(G_i)$. Let $\gamma_i : V(G_i) \to \{a_1, \ldots, a_{\chi(G_i)}\}$ be a vertex coloring of G with $\chi(G_i)$ colors. By the definition of a vertex coloring, the sets $\gamma_i^{-1}(a_j) \subseteq V(G_i), j = 1, \ldots, \chi(G_i)$ form a partition of $V(G_i)$ and $\gamma_i^{-1}(a_j) \cap \gamma_i^{-1}(a_l) = \emptyset$ for $j \neq l, j, l = 1, \ldots, \chi(G_i)$. Furthermore, $\left|\gamma_i^{-1}(a_j)\right| \leq \alpha(G_i)$ for every $j = 1, \ldots, \chi(G_i)$, which yields

$$|V(G_i)| = \sum_{j=1}^{\chi(G_i)} \left|\gamma_i^{-1}(a_j)\right| \leq \alpha(G_i)\chi(G_i) \leq \alpha(G_i)\Delta(G_i).$$

(ii) Suppose that $G_i \neq K_{\Delta(G)+1}$ for all $i = 1, \ldots, s$. If there exists $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, \Delta(G)\}$ such that $G_i = K_j$, then $\alpha(G_i) = 1 \geq \frac{j}{\Delta(G)} \geq \frac{|V(G_i)|}{\Delta(G)}$. On the other hand, if $\Delta(G) \geq 3$ and there exists $i \in \{1, \ldots, s\}$ such that G_i is an odd cycle C_{2j+1} of length $2j + 1$, with $j \geq 1$, then $\alpha(G_i) = j \geq \frac{2j+1}{3} \geq \frac{|V(G_i)|}{\Delta(G)}$. Finally, if G_i is neither a complete graph nor an odd cycle, then by applying item (i) we have $\alpha(G_i) \geq \frac{|V(G_i)|}{\Delta(G_i)} \geq \frac{|V(G_i)|}{\Delta(G)}$. Therefore,

$$\alpha(G) = \sum_{i=1}^{s} \alpha(G_j) \geq \sum_{i=1}^{s} \frac{|V(G_i)|}{\Delta(G)} = \frac{|V(G)|}{\Delta(G)}.$$

(iii) If $G_i = K_{\Delta(G)+1}$, for some $i \in \{1, \ldots, s\}$, then $\alpha(G_i) = 1 \geq \frac{\Delta(G)+1}{k} = \frac{|V(G_i)|}{k}$. If $G_i \neq K_{\Delta(G)+1}$ for all $i = 1, \ldots, s$, and $\Delta(G) \geq 3$ or G_i is not an odd cycle for all $i = 1, \ldots, s$, then reasoning as in item (ii) we obtain $\alpha(G_i) \geq \frac{|V(G_i)|}{\Delta(G)} \geq \frac{|V(G_i)|}{k}$. Finally, if $\Delta(G) \leq 2$ and G_i is an odd cycle of length $2j + 1$ then $\Delta(G) = 2, k \geq 3$ and $\alpha(G_i) = j \geq \frac{2j+1}{3} \geq \frac{|V(G_i)|}{k}$. Hence,

$$\alpha(G) = \sum_{i=1}^{s} \alpha(G_i) \geq \sum_{i=1}^{s} \frac{|V(G_i)|}{k} = \frac{|V(G)|}{k},$$

and the result follows. □

3. Main results

In [1] a lower bound for the multicolor Ramsey number $R(K_{p_1}, \ldots, K_{p_m}, G_1, \ldots, G_n)$ in terms of the numbers $R(K_{p_1}, \ldots, K_{p_m})$ and $R(G_1, \ldots, G_n)$ was determined.

Theorem 3.1 (*See [1]*). *Let $p_1, \ldots, p_m$ be integers, with $p_i \geq 2$ for $i = 1, \ldots, m$, and let $G_1, \ldots, G_n$ be any arbitrary graphs. Then*

$$R(K_{p_1}, \ldots, K_{p_m}, G_1, \ldots, G_n) \geq \left(R(K_{p_1}, \ldots, K_{p_m}) - 1\right) (R(G_1, \ldots, G_n) - 1) + 1.$$

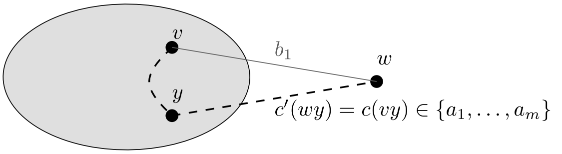

Fig. 1. Definition of c'.

Combining Theorem 3.1 with Theorem 1.1 the following lower bound for the number $R(K_{p_1}, \ldots, K_{p_m}, K_{1,q_1}, \ldots, K_{1,q_n})$ is derived:

$$R(K_{p_1}, \ldots, K_{p_m}, K_{1,q_1}, \ldots, K_{1,q_n}) - 1 \geq \left(\sum_{j=1}^{n} q_j - n + \epsilon_Q - 1 \right) \left(R(K_{p_1}, \ldots, K_{p_m}) - 1 \right) \tag{1}$$

where $\epsilon_Q = 1$ if the number of even integers in the set $\{q_j\}_{j=1}^{n}$ is even and positive, and $\epsilon_Q = 2$ otherwise.

We will prove that inequality (1) is indeed an equality. Before, we need to prove some lemmas.

Lemma 3.1. *Let $p_1, \ldots, p_m, q_1, \ldots, q_n$ be positive integers, with $p_i, q_j \geq 2$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, and denote by K the complete graph on $R(P, Q) - 1$ vertices. Let $c : E(K) \to \{a_1, \ldots, a_m, b_1, \ldots, b_n\}$ be a (P, Q)-coloring of K. Then the graph G with vertex set $V(G) = V(K)$ and edge set $E(G) = c^{-1}(\{b_1, \ldots, b_n\})$ has minimum degree $\delta(G) \geq 1$.*

Proof. Let G be the graph under the hypothesis of the lemma, and we reason by way of contradiction supposing that there exists $v \in V(G)$ such that $d_G(v) = 0$. Let K' be the complete graph obtained from K by adding a new vertex $w \notin V(K)$. We will arrive at a contradiction by proving the existence of a (P, Q)-coloring of K'. Let us consider the application $c' : E(K') \to \{a_1, \ldots, a_m, b_1, \ldots, b_n\}$ defined as follows (see Fig. 1):

$$c'(xy) = \begin{cases} b_1 & \text{if } xy = vw \\ c(vy) & \text{if } x = w \text{ and } y \in V(K) \\ c(xy) & \text{otherwise.} \end{cases}$$

Let us see that c' is a (P, Q)-coloring of the graph K'. First assume that K' contains an a_i-colored copy of K_{p_i} for some $i \in \{1, \ldots, m\}$, then $w \in V(K_{p_i})$, since $c'(E(K)) = c(E(K))$ and c is a (P, Q)-coloring of K. Thus, $v \notin V(K_{p_i})$, because $c'(vw) = b_1 \neq a_i$. Let us denote the set of vertices of K_{p_i} by $V(K_{p_i}) = \{w, v_1, \ldots, v_{p_i - 1}\}$. Since $c'(wv_j) = a_i$ and $c'(wv_j) = c(vv_j)$ for all $j = 1, \ldots, p_i - 1$, and further $a_i = c'(v_j v_\ell) = c(v_j v_\ell)$ for all $j \neq \ell, j, \ell = 1, \ldots, p_i - 1$ then $c\left(E(K[\{v, v_1, \ldots, v_{p_i - 1}\}])\right) = a_i$, contradicting the fact that c is a (P, Q)-coloring of K. Second assume that K' contains a b_j-colored copy of K_{1,q_j} for some $j \in \{1, \ldots, n\}$, then $vw \in E(K_{1,q_1})$, since $c'^{-1}(\{b_1, \ldots, b_n\}) = c^{-1}(\{b_1, \ldots, b_n\}) \cup \{vw\} = E(G) \cup \{vw\}$, due to the fact that $d_G(v) = 0$. Thus $j = 1$ and notice that there is no more incident edges to edge vw with color b_1, because $d_G(v) = 0$ and $c'(wy) = c(vy) \in \{a_1, \ldots, a_m\}$, for every $y \in V(K), y \neq v$. Therefore, we arrive at a contradiction with the hypothesis $q_1 \geq 2$.

Hence, c' is a (P, Q)-coloring of the complete graph K' with $R(P, Q)$ vertices, which is a contradiction. Then G has minimum degree $\delta(G) \geq 1$. □

Given a $(P, Q; R(P, Q) - 1)$-coloring and $Q' \supseteq Q$, the next result leads us to obtain a (P, Q')-coloring of an appropriated complete graph, under certain restrictions.

Lemma 3.2. *Let $p_1, \ldots, p_m, q_1, \ldots, q_n$ be positive integers, with $p_i, q_j \geq 2$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Let $K = K_{R(P,Q)-1}$ be the complete graph on $R(P, Q) - 1$ vertices. Let $c : E(K) \to \{a_1, \ldots, a_m, b_1, \ldots, b_n\}$ be a (P, Q)-coloring of K. Set $B = c^{-1}(\{b_1, \ldots, b_n\})$, $U = \{u_1, \ldots, u_{\alpha(K[B])}\}$ a set of independent vertices of $K[B]$ and $\mathcal{P}(U) = \{W_1, \ldots, W_{\alpha(K[B])}\}$ the partition of $V(K[B])$ following Notation 2.1. Let k be a non negative integer and set $Q' = \{q_1, \ldots, q_n, \ldots, q_{n+k}\}$ such that $|W_i| \leq R(\emptyset, Q') - 1$ for all $i = 1, \ldots, \alpha(K[B])$. We consider the complete graph K_i^* whose set of vertices* *is obtained from W_i by adding new vertices not belonging to $V(K)$ such that $|V(K_i^*)| = R(\emptyset, Q') - 1$, and $V(K_i^*) \cap V(K_j^*) = \emptyset$ for all $1 \le i \ne j \le \alpha(K[B])$. Let K^* be the complete graph with set of vertices $\bigcup_{i=1}^{\alpha(K[B])} V(K_i^*)$ and let $c_i^* : E(K_i^*) \to \{b_1, \dots, b_{n+k}\}$ be a $(\emptyset, Q')$-coloring of K_i^*. Let $c^* : E(K^*) \to \{a_1, \dots, a_m, b_1, \dots, b_{n+k}\}$ be an application define as follows:*

$$c^*(vw) = \begin{cases} c_i^*(vw) & \text{if } vw \in E(K_i^*) \\ c(u_iu_j) & \text{if } v \in V(K_i^*) \text{ and } w \in V(K_j^*) \text{ with } i \ne j. \end{cases}$$

Then the next assertions hold:

(i) *c^* is a (P, Q')-coloring of K^*.*
(ii) $|V(K^*)| \le (R(P, \emptyset) - 1)(R(\emptyset, Q') - 1)$.

Proof. (i) First, suppose that there exists $j \in \{1, \dots, n+k\}$ such that $(c^*)^{-1}(\{b_j\}) \supseteq E(K_{1,q_j})$. Since $c^*(u_iu_j) = c(u_iu_j) \in \{a_1, \dots, a_m\}$ for all $1 \le i \ne j \le \alpha(K[B])$, then there exists $h \in \{1, \dots, \alpha(K[B])\}$ such that $E(K_{1,q_j}) \subseteq E(K_h^*)$. Moreover, the definition of c^* implies that $c^*(E(K_{1,q_j})) = c_h^*(E(K_{1,q_j})) = \{b_j\}$ and this is not possible because c_h^* is a $(\emptyset, Q')$-coloring of K_h^*.

Second, suppose that there exists $i \in \{1, \dots, m\}$ such that $(c^*)^{-1}(\{a_i\}) \supseteq E(K_{p_i})$, and set $V(K_{p_i}) = \{v_1, \dots, v_{p_i}\}$. Let us see that $v_\ell \in V(K_{j_\ell}^*)$ for every $\ell = 1, \dots, p_i$ and $j_\ell \ne j_{\ell'}$ if $1 \le \ell \ne \ell' \le p_i$. Otherwise, there exist $h \in \{1, \dots, \alpha(K[B])\}$ and $\ell, \ell' \in \{1, \dots, p_i\}$ such that $v_\ell, v_{\ell'} \in V(K_h^*)$. Then $c^*(v_\ell v_{\ell'}) = c_h^*(v_\ell v_{\ell'}) \in \{b_1, \dots, b_{n+k}\}$ and this not possible since $c^*(E(K_{p_i})) = \{a_i\}$.

We consider the set of vertices $\{u_{j_1}, \dots, u_{j_{p_i}}\} \subseteq U$. From the definition of c^*, it follows that $c^*(u_{j_\ell}u_{j_{\ell'}}) = c(u_{j_\ell}u_{j_{\ell'}}) = c^*(v_{j_\ell}v_{j_{\ell'}}) = a_i$ for all $1 \le \ell \ne \ell' \le p_i$. Hence, $c(E(K[\{u_{j_1}, \dots, u_{j_{p_i}}\}])) = \{a_i\}$ and therefore $c^{-1}(\{a_i\}) \supseteq E(K_{p_i})$. This is an contradiction since c is an (P, Q)-coloring of K, and the result follows.

(ii) From the definition of K^*, we know that $V(K^*) = \bigcup_{i=1}^{\alpha(K[B])} V(K_i^*)$ and $K_i^* \cap K_j^* = \emptyset$ for $1 \le i \ne j \le \alpha(K[B])$. Thus, $|V(K^*)| = \sum_{j=1}^{\alpha(K[B])} |V(K_j^*)| = \alpha(K[B])\big(R(\emptyset, Q') - 1\big)$. Notice that the restriction of c^* to $K^*[\{u_1, \dots, u_{\alpha(K[B])}\}]$ is a $(P, \emptyset)$-coloring of $K^*[\{u_1, \dots, u_{\alpha(K[B])}\}]$, because $c^*(u_iu_j) = c(u_iu_j) \in \{a_1, \dots, a_m\}$. Thus,

$$\alpha(K[B]) = |V(K^*[\{u_1, \dots, u_{\alpha(K[B])}\}])| \le R(P, \emptyset) - 1$$

and therefore, $|V(K^*)| \le (R(P, \emptyset) - 1)\big(R(\emptyset, Q') - 1\big)$. □

The next result provides an upper bound on the Ramsey number $R(P, Q)$ when the number of integers q_j that are even is 0 or odd.

Proposition 3.1. *Let $p_1, \dots, p_m, q_1, \dots, q_n$ be positive integers, with $p_i, q_j \ge 2$, for $i = 1, \dots, m$ and $j = 1, \dots, n$. If the number of integers q_j that are even is 0 or odd, then*

$$R(K_{p_1}, \dots, K_{p_m}, K_{1,q_1}, \dots, K_{1,q_n}) - 1 \le \left(\sum_{j=1}^{n} q_j - n + 1\right)\big(R(K_{p_1}, \dots, K_{p_m}) - 1\big).$$

Proof. Set $K = K_{R(P,Q)-1}$ and let $c : E(K) \to \{a_1, \dots, a_m, b_1, \dots, b_n\}$ be a (P, Q)-coloring of K. Set $Q' = Q$ and let $U = \{u_1, \dots, u_{\alpha(K[B])}\}$ be a set of independent vertices of $K[B]$, where $B = c^{-1}(\{b_1, \dots, b_n\})$. Let $W_1, \dots, W_{\alpha(K[B])}$ be the partition $\mathcal{P}(U)$ of $K[B]$ according to Notation 2.1. By the construction of W_i, we have $|W_i| \le \Delta(K[B]) + 1 \le \sum_{j=1}^{n}(q_j - 1) + 1 = \sum_{j=1}^{n} q_j - n + 1$, for $i = 1, \dots, \alpha(K[B])$. From Theorem 1.1, it follows that $R(\emptyset, Q') - 1 = \sum_{j=1}^{n} q_j - n + 1$, which implies $|W_i| \le R(\emptyset, Q') - 1$ for all $i = 1, \dots, \alpha(K[B])$. Then, by applying Lemma 3.2, we may construct a (P, Q')-coloring of a complete graph $K^* \supseteq K$ such that $|V(K^*)| \le (R(P, \emptyset) - 1)\big(R(\emptyset, Q') - 1\big)$. Hence,

$$R(P, Q) - 1 = |V(K)| \le |V(K^*)| \le (R(P, \emptyset) - 1)\big(R(\emptyset, Q') - 1\big)$$

and the result follows. □

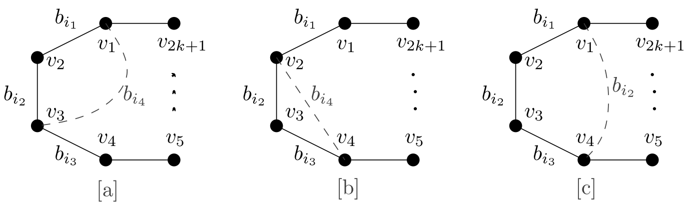

Fig. 2. Color assignment in *Case* 1.

To determine an upper bound for the multicolor Ramsey number $R(P, Q)$ when the number of integers q_j that are even is even and positive, we need to show this lemma.

Lemma 3.3. *Let $p_1, \dots, p_m, q_1, \dots, q_n$ be positive integers, with $p_i, q_j \geq 2$, for $i = 1, \dots, m$ and $j = 1, \dots, n$, and set $K = K_{R(P,Q)-1}$. If the number of integers q_j that are even is even and positive, then there exists a (P, Q)-coloring $c : E(K) \to \{a_1, \dots, a_m, b_1, \dots, b_n\}$ such that $\Delta(K[B]) \geq 3$ or $K[B]$ contains no an odd cycle, where $B = c^{-1}(\{b_1, \dots, b_n\})$.*

Proof. Let $c : E(K) \to \{a_1, \dots, a_m, b_1, \dots, b_n\}$ be a (P, Q)-coloring such that $|B| = |c^{-1}(\{b_1, \dots, b_n\})|$ is maximum.

Without lost of generality we may assume that $|V(K)| \geq 4$. Otherwise, we are done unless $K = K[B] = C_3$. In this case, it is enough to replace the color b_j of any edge of $K[B]$ with any color a_i, and the result holds.

By way of contradiction, suppose that $\Delta(K[B]) \leq 2$ and there exists an odd cycle C_{2k+1} in $K[B]$. First, let us see that $q_j = 2$ for all $j = 1, \dots, n$. Otherwise, there would exist $j \in \{1, \dots, n\}$ such that $q_j \geq 3$. Since $|V(K)| \geq 4$ and $\Delta(K[B]) \leq 2$, we can find an edge $uv \in E(K)$ such that $c(uv) \in \{a_1, \dots, a_m\}$. Since the number of integers q_j that are even is even and positive, then $n \geq 2$. Given $b_i \neq b_j$, let $c' : E(K) \to \{a_1, \dots, a_m, b_1, \dots, b_n\}$ be the application defined as follows:

$$c'(xy) = \begin{cases} b_j & \text{if } xy \in B \\ b_i & \text{if } xy = uv \\ c(xy) & \text{otherwise.} \end{cases}$$

It is clear that c' is a (P, Q)-coloring of K and $\left|(c')^{-1}(\{b_1, \dots, b_n\})\right| > |B|$, which is not possible since B has maximum cardinality. Hence, $q_j = 2$ for all $j = 1, \dots, n$.

Two cases need to be distinguished according to the length of the cycle C_{2k+1} contained in $K[B]$.

Case 1. Assume $k \geq 2$ and denote by $\{v_1, \dots, v_{2k+1}\}$ the set of vertices of C_{2k+1}.

Assume $k \geq 2$. Since C_{2k+1} has odd length and $q_j = 2$ for all $j = 1, \dots, n$, then $n \geq 3$, that is, the cycle $C_{2k+1} \subseteq K[B]$ must be colored with at least three different colors in order to avoid the existence of two incident edges with the same color. Indeed, there must exist a path of length three whose edges are colored with three different colors $b_{i_1}, b_{i_2}, b_{i_3}$. Denote by $\{v_1, \dots, v_{2k+1}\}$ the set of vertices of C_{2k+1} so that $c(v_1v_2) = b_{i_1}, c(v_2v_3) = b_{i_2}$ and $c(v_3v_4) = b_{i_3}$ (see Fig. 2). Since the number of integers q_j that are even is even, then $n \geq 4$, and therefore, there exists a color $b_{i_4} \notin \{b_{i_1}, b_{i_2}, b_{i_3}\}$.

If $c(v_1v_{2k+1}) \in \{b_{i_2}, b_{i_3}\}$ (see Fig. 2 [a]), we can define

$$c'(xy) = \begin{cases} b_{i_4} & \text{if } xy = v_1v_3 \\ c(xy) & \text{otherwise.} \end{cases}$$

If $c(v_4v_5) \in \{b_{i_1}, b_{i_2}\}$ (see Fig. 2 [b]), we can define

$$c'(xy) = \begin{cases} b_{i_4} & \text{if } xy = v_2v_4 \\ c(xy) & \text{otherwise.} \end{cases}$$

If $c(v_1v_{2k+1}) = c(v_4v_5) = b_{i_4}$ (see Fig. 2 [c]), we can define

$$c'(xy) = \begin{cases} b_{i_2} & \text{if } xy = v_1v_4 \\ c(xy) & \text{otherwise.} \end{cases}$$

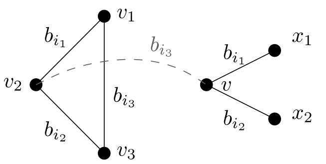

Fig. 3. Color assignment for *Case* 2.

In any case, c' is a (P, Q)-coloring of K such that $\left|(c')^{-1}(\{b_1, \ldots, b_n\})\right| > |B|$, a contradiction.

Case 2. Assume $k = 1$. Let denote by $\{v_1, v_2, v_3\}$ the set of vertices of $C_3 \subseteq K[B]$. Since c is a (P, Q)-coloring of K and $q_j = 2$ for all $j = 1, \ldots, n$, we may assume that $c(v_1v_2) = b_{i_1}$, $c(v_2v_3) = b_{i_2}$ and $c(v_1v_3) = b_{i_3}$ with $b_{i_1} \neq b_{i_2} \neq b_{i_3}$ (see Fig. 3). As $|V(K)| \geq 4$ there exists $v \in V(K)$ such that $v \notin V(C_3)$. Observe that $d_{K[B]}(v) \leq \Delta(K[B]) \leq 2$, and so without lost of generality, we may suppose that $c(\{vy \mid y \in V(K[B])\}) \subseteq \{b_{i_1}, b_{i_2}\}$. Let c' be the following application:

$$c'(xy) = \begin{cases} b_{i_3} & \text{if } xy = vv_2 \\ c(xy) & \text{otherwise.} \end{cases}$$

It is easy to check that c' is a (P, Q)-coloring of K verifying $\left|(c')^{-1}(\{b_1, \ldots, b_n\})\right| > |B|$, which is again a contradiction. Then, the result follows. □

Proposition 3.2. *Let $p_1, \ldots, p_m, q_1, \ldots, q_n$ be positive integers, with $p_i, q_j \geq 2$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. If the number of integers q_j that are even is even and positive, then*

$$R(K_{p_1}, \ldots, K_{p_m}, K_{1,q_1}, \ldots, K_{1,q_n}) - 1 \leq \left(\sum_{j=1}^{n} q_j - n\right)\left(R(K_{p_1}, \ldots, K_{p_m}) - 1\right).$$

Proof. Set $K = K_{R(P,Q)-1}$. By Lemma 3.3, there exists a (P, Q)-coloring $c : E(K) \to \{a_1, \ldots, a_m, b_1, \ldots, b_n\}$ of K such that the set $B = c^{-1}(\{b_1, \ldots, b_n\})$ verifies $\Delta(K[B]) \geq 3$ or $K[B]$ does not contain an odd cycle. By Lemma 3.1, we know that $|V(K[B])| = R(P, Q) - 1$ and $V(K[B]) = V(K)$. From the definition of $K[B]$, it follows that $d_{K[B]}(v) = \sum_{i=1}^{n} |\{x \mid vx \in B,\ c(vx) = b_i\}| \leq \sum_{i=1}^{n}(q_i - 1)$, for any vertex $v \in V(K)$. Thus, $\Delta(K[B]) \leq \sum_{i=1}^{n}(q_i - 1)$ and by Theorem 1.1,

$$R(\emptyset, Q) = \sum_{i=1}^{n} q_i - n + 1 = \sum_{i=1}^{n}(q_i - 1) + 1 \geq \Delta(K[B]) + 1.$$

Two cases need to be distinguished:

Case 1. Assume $\sum_{i=1}^{n}(q_i - 1) + 1 = \Delta(K[B]) + 1$. That is, $\Delta(K[B]) = \sum_{i=1}^{n} q_i - n = R(\emptyset, Q) - 1$.

If $K_{\Delta(K[B])+1} \subseteq K[B]$ then the restriction of the (P, Q)-coloring c on $E(K_{\Delta(K[B])+1})$ must be a $(\emptyset, Q)$-coloring of $K_{\Delta(K[B])+1}$, which is not possible because $|V(K_{\Delta(K[B])+1})| = \Delta(K[B]) + 1 = R(\emptyset, Q)$. Hence, by applying Lemma 2.1 (item (ii)) we have

$$\alpha(K[B]) \geq \frac{|V(K[B])|}{\Delta(K[B])} = \frac{R(P, Q) - 1}{\sum\limits_{i=1}^{n} q_i - n}.$$

Case 2. Assume $\sum_{i=1}^{n}(q_i - 1) + 1 > \Delta(K[B]) + 1$. That is, $\Delta(K[B]) < \sum_{i=1}^{n} q_i - n$. Denoting by $k = \sum_{i=1}^{n} q_i - n$ and applying Lemma 2.1 (item (iii)) we have

$$\alpha(K[B]) \geq \frac{|V(K[B])|}{k} = \frac{R(P, Q) - 1}{\sum\limits_{i=1}^{n} q_i - n}.$$

Hence, in any case it follows that

$$\alpha(K[B]) \geq \frac{R(P,Q)-1}{\sum_{i=1}^{n} q_i - n}. \tag{2}$$

Let $Q' = Q \cup \{2\} = \{q_1, \ldots, q_n, q_{n+1} = 2\}$. By Theorem 1.1, we deduce that

$$R(\emptyset, Q') = \sum_{i=1}^{n+1} q_i - (n+1) + 2 = \sum_{i=1}^{n} q_i - n + 3. \tag{3}$$

Let $U = \{u_1, \ldots, u_{\alpha(K[B])}\}$ be a set of independent vertices of $K[B]$ and let us consider the partition $\mathcal{P}(U) = \{W_1, \ldots, W_{\alpha(K[B])}\}$ of $K[B]$ constructed following Notation 2.1. For every $i = 1, \ldots, \alpha(K[B])$ we know by (3) that

$$|W_i| \leq \left|N_{K[B]}(u_i) \cup \{u_i\}\right| \leq \Delta(K[B]) + 1 \leq \sum_{i=1}^{n} q_i - n + 1 \leq R(\emptyset, Q') - 1.$$

Then the hypothesis of Lemma 3.2 are satisfied, hence we consider the graphs K_i^* and K^* constructed following Lemma 3.2. Thus,

$$R(P,Q') - 1 \geq |V(K^*)| = \sum_{i=1}^{\alpha(K[B])} \left|V(K_i^*)\right| = \sum_{i=1}^{\alpha(K[B])} \left(R(\emptyset, Q') - 1\right) = \alpha(K[B]) \left(R(\emptyset, Q') - 1\right).$$

Combining (2) and (3), we have

$$R(P,Q') - 1 \geq \frac{R(P,Q)-1}{\sum_{i=1}^{n} q_i - n} \left(\sum_{i=1}^{n} q_i - n + 2\right),$$

yielding to

$$R(P,Q) - 1 \leq \frac{\left(R(P,Q') - 1\right)\left(\sum_{i=1}^{n} q_i - n\right)}{\sum_{i=1}^{n} q_i - n + 2}. \tag{4}$$

Since the number of integers q_j of Q' that are even is odd, by applying Theorem 3.1 and by assuming equality (3), it follows that

$$R(P,Q') - 1 \leq (R(P,\emptyset) - 1)\left(R(\emptyset, Q') - 1\right) = (R(P,\emptyset) - 1)\left(\sum_{i=1}^{n} q_i - n + 2\right). \tag{5}$$

Finally, from inequalities (4) and (5), we have

$$\begin{aligned} R(P,Q) - 1 &\leq \frac{\left(\sum_{i=1}^{n} q_i - n\right)}{\sum_{i=1}^{n} q_i - n + 2} (R(P,\emptyset) - 1)\left(\sum_{i=1}^{n} q_i - n + 2\right) \\ &= \left(\sum_{i=1}^{n} q_i - n\right)(R(P,\emptyset) - 1), \end{aligned}$$

which proves the result. □

As a consequence of (1), Propositions 3.1 and 3.2, the following theorem determines the multicolor Ramsey number $R(K_{p_1}, \ldots, K_{p_m}, K_{1,q_1}, \ldots, K_{1,q_n})$ for any set of complete graphs and stars in terms of $R(K_{p_1}, \ldots, K_{p_m})$. This result generalizes Theorem 1.2 proved in [9].

Theorem 3.2. *Let m, n be positive integers. Then*

$$R(K_{p_1}, \ldots, K_{p_m}, K_{1,q_1}, \ldots, K_{1,q_n}) - 1 = \left(\sum_{j=1}^{n} q_j - n + \epsilon_Q - 1\right) \left(R(K_{p_1}, \ldots, K_{p_m}) - 1\right),$$

where $\epsilon_Q = 1$ if the number of even integers in the set $\{q_j\}_1^n$ is even and positive, and $\epsilon_Q = 2$ otherwise.

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