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ON THE n-COLOR RADO NUMBER FOR THE EQUATION $x_1 + x_2 + \cdots + x_k + c = x_{k+1}$

S. D. ADHIKARI, L. BOZA, S. ELIAHOU, J. M. MARÍN, M. P. REVUELTA, AND M. I. SANZ

ABSTRACT. For integers k, n, c with $k, n \geq 1$, the n-color Rado number $R_k(n, c)$ is defined to be the least integer N, if it exists or ∞ otherwise, such that for every n-coloring of the set $\{1, 2, \ldots, N\}$, there exists a monochromatic solution in that set to the equation

$$x_1 + x_2 + \cdots + x_k + c = x_{k+1}.$$

In this paper, we mostly restrict to the case $c \geq 0$, and consider two main issues regarding $R_k(n, c)$: is it finite or infinite, and when finite, what is its value? Very few results are known so far on either one.

On the first issue, we formulate a general conjecture, namely that $R_k(n, c)$ should be finite if and only if every divisor $d \leq n$ of $k - 1$ also divides c. The "only if" part of the conjecture is shown to hold, as well as the "if" part in the cases where either $k - 1$ divides c, or $n \geq k - 1$, or $k \leq 7$, except for two instances to be published separately.

On the second issue, we obtain new bounds on $R_k(n, c)$ and determine exact formulae in several new cases, including $R_3(3, c)$ and $R_4(3, c)$. As for the case $R_2(3, c)$, first settled by Schaal in 1995, we provide a new shorter proof.

Finally, the problem is reformulated as a Boolean satisfiability problem, allowing the use of a SAT solver to treat some instances.

1. INTRODUCTION

Throughout the paper, we shall denote by $\mathbb{Z}, \mathbb{N}$ and $\mathbb{N}_+$ the set of integers, nonnegative integers and positive integers, respectively. Let $n \in \mathbb{N}_+$. An n-coloring of a set A is a function

$$\Delta : A \longrightarrow C,$$

where C is some finite set of cardinality $|C| = n$. Here, we shall mostly deal with n-colorings of integer intervals $[1, N]$, where

$$[a, b] = \{a, a + 1, \ldots, b\}$$

for integers $a \leq b$.

Given an n-coloring Δ of $[1, N]$ and a linear equation L in $k + 1$ variables with integer coefficients, a solution $(x_1, \ldots, x_k, x_{k+1})$ to L is said to be *monochromatic* if $\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_{k+1})$.

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1.1. **Some earlier results.** In 1916, Schur [22] proved that for every $n \geq 1$, there exists a least integer $S_2(n) = N$, such that for every n-coloring of $[1, N]$, there exists a monochromatic solution to the equation $x_1 + x_2 = x_3$.

The integers $S_2(n)$ are called the *Schur numbers* and are currently known only for $n \leq 4$, namely: $S_2(1) = 2$, $S_2(2) = 5$, $S_2(3) = 14$ and $S_2(4) = 45$. While these stated values for $n \leq 3$ can easily be settled by hand, the one for $n = 4$ relies on an exhaustive computer search [2]. For $n = 5$, the currently available bounds for $S_2(5)$ are $161 \leq S_2(5) \leq 306$, and it is conjectured in [8] that the lower bound 161, established in [7], is perhaps sharp. For the upper bound 306, see [18]. For general $n \geq 1$, Schur [22] obtained the following bounds:

$$(3^n + 1)/2 + 1 \;\leq\; S_2(n) \;\leq\; \lfloor n!e \rfloor + 1.$$

Slightly improved upper bounds were subsequently provided by Whitehead [23] and Honghui Wan [10], whereas for lower bounds, the inequalities

$$S_2(m + n) \;\geq\; 2S_2(m)S_2(n) - S(m) - S(n) + 1$$

of Abbott and Hanson [1], and $S_2(5) \geq 161$ of Fredricksen and Sweet [8], together yield the sharpening $S_2(n) \geq c\,322^{n/5} \geq c\,3.17^n$ for $n \geq 6$, where c is some absolute positive constant.

In 1933, Rado [13] generalized the work of Schur to arbitrary systems of linear equations. Given $n \geq 1$ and a system of linear equations L, the least integer N (if it exists) such that for every n-coloring of the set $[1, N]$, there is a monochromatic solution to the system L, is called *the n-color Rado number* for L. If no such integer N exists, then this Rado number is defined to be infinite.

Given a linear system L as above, and $n \geq 1$, there are two main issues regarding its corresponding n-color Rado number: is it finite or infinite? When it is finite, what is its exact value?

In particular, for the equation $x_1 + x_2 + \cdots + x_k = x_{k+1}$ where $k \geq 2$, Rado's results imply that for all $n \geq 1$, the corresponding n-color Rado number is actually finite, i.e., there exists a least integer $S_k(n) = N$ such that for every n-coloring of $[1, N]$, there is a monochromatic solution of that equation [13].

In 1982, Beutelspacher and Brestovansky [5] showed that $S_k(2) = k^2 + k - 1$ for $k \geq 2$. More than twenty years later, Sanz [18] established the value $S_3(3) = 43$ with an exhaustive computer search.

Burr and Loo [3] were able to determine the 2-color Rado numbers for the equations $x_1 + x_2 + c = x_3$ and $x_1 + x_2 = kx_3$ for every integer c and for every positive integer k. There are several results due to Schaal and other authors, about 2-color and 3-color Rado numbers for particular equations; see [11, 12, 15, 21].

1.2. **The main conjecture.** Let n, k, c be integers with $n, k \geq 1$ and $c \geq 0$. In this paper, we shall be concerned with the above-mentioned equation

$$x_1 + x_2 + \cdots + x_k + c \;=\; x_{k+1}. \tag{1}$$

Notation 1.1. We shall denote by $R_k(n, c)$ the n-color Rado number corresponding to equation (1), i.e., the smallest positive integer N, if it exists, such that every n-coloring of $[1, N]$ admits a monochromatic solution to it.

For what values of the parameters n, k, c is $R_k(n, c)$ finite? By Rado's result, recalled above, it holds that $R_k(n, 0) = S_k(n)$ is always finite. Now, for $n = 2$, Schaal [19] showed that $R_k(2, c)$ *is finite if and only if* k *or* c *is even*, in which case

$$R_k(2, c) = (k + 1)^2 + (c - 1)(k + 2).$$

Later on, he further showed that the 3-color Rado number $R_2(3, c)$ is always finite [20], and obtained the exact value $R_2(3, c) = 13c + 14$ for all $c \geq 0$.

In this paper, we provide further instances of n, k, c for which $R_k(n, c)$ is shown to be finite or infinite, respectively. These results, as well as those of Schaal on $R_k(2, c)$ and $R_2(3, c)$, fit into one single conjecture.

Conjecture 1.2. *For integers k, n, c with $k \geq 2, n \geq 1$ and $c \geq 0$, the n-color Rado number $R_k(n, c)$ is finite if and only if every divisor $d \leq n$ of $k - 1$ also divides c.*

In the sequel, we settle this conjecture if either $k - 1$ divides c, or $n \geq k - 1$, or $k \leq 7$, except for two special cases which need a completely different approach and will be presented elsewhere.

1.3. **Contents.** In Section 2, we settle the "only if" part of Conjecture 1.2 by giving a sufficient condition on n, k, c ensuring that $R_k(n, c)$ is infinite. In Section 3, we study how $R_k(n, c)$ behaves under changes on n and c, and then exploit the results to settle Conjecture 1.2 in cases $n \geq k-1$ and $k \leq 7$. In Section 4, we introduce new numbers $S_k^*(n)$, smaller and easier to study than $S_k(n)$, and show how they help bounding $R_k(n, c)$ from below. This is then exploited in Section 5, in conjunction with SAT solvers, to get new formulae for some instances of $R_k(n, c)$.

2. An obstacle to finiteness

We start by treating the cases $n = 1$ or $k = 1$. For $n = 1$, the Rado number $R_k(n, c)$ is given by the formula

$$R_k(1, c) = k + c \tag{2}$$

for all $k \geq 1$ and $c \geq 0$, as readily verified. The case $k = 1$ is also easy to determine. Indeed, for $c = 0$ we have

$$R_1(n, 0) = 1$$

for all $n \geq 2$, whereas the following holds for $c \geq 1$.

Proposition 2.1. *For all $n \geq 2$ and $c \geq 1$, we have $R_1(n, c) = +\infty$.*

Proof. The statement follows from the following 2-coloring of $\mathbb{N}_+$:

$$\begin{aligned} \Delta \colon \mathbb{N}_+ &\longrightarrow \{0, 1\}, \\ x &\longmapsto \text{the class of } \lceil x/c \rceil \bmod 2. \end{aligned}$$

Since $\lceil (x + c)/c \rceil = \lceil x/c \rceil + 1$, implying $\Delta(x + c) \equiv \Delta(x) + 1 \bmod 2$, there are no monochromatic solution to the equation $x_1 + c = x_2$, thereby implying $R_1(n, c) = +\infty$ as stated. □

Therefore, from now on, we shall assume $k, n \geq 2$. Here is an obstacle to the finiteness of $R_k(n, c)$, which settles the "only if" part of Conjecture 1.2.

Proposition 2.2. *If there exists a divisor $d \leq n$ of $k - 1$ which does not divide c, then $R_k(n, c) = +\infty$.*

Proof. Color each integer by its class mod d, taken in the set $\{0, 1, \ldots, d-1\}$. This yields a d-coloring of $\mathbb{N}_+$. Let $x_1, \ldots, x_{k+1} \in \mathbb{N}_+$ be monochromatic for this coloring, say all of the same color class r mod d. Then

$$x_1 + \cdots + x_k - x_{k+1} \equiv (k-1)r \equiv 0 \bmod d,$$

where the second congruence follows from the hypothesis that d divides $k-1$. Now since $c \not\equiv 0 \bmod d$ by hypothesis, it follows that $(x_1, \ldots, x_{k+1})$ cannot satisfy the equation

$$x_1 + \cdots + x_k - x_{k+1} = -c.$$

Therefore this equation does not admit any monochromatic solution. It follows that $R_k(d, c) = +\infty$, whence also $R_k(n, c) = +\infty$, as claimed. □

In fact, the above condition is the only general one we are aware of which implies $R_k(n, c) = \infty$. This is what led us to formulate Conjecture 1.2.

3. Varying n and c

In this section, we shall vary the parameters n and c and show how this affects the value of $R_k(n, c)$. This study will ultimately allow us to settle Conjecture 1.2 in the cases where either $k-1$ divides c, or $n \geq k-1$, or $k \leq 7$, except for two key instances which need a different approach and will be published separately.

3.1. Reducing n.

Our first result is a relation between n-color and $(n-1)$-color Rado numbers. Trivially, one has $R_k(n, c) \geq R_k(n-1, c)$, but a sharper inequality holds.

Lemma 3.1. *Let $k, n, c \in \mathbb{N}_+$. Then $R_k(n, c) \geq (k+1)R_k(n-1, c) + c - 1$.*

Proof. To ease notation, set $M = R_k(n-1, c)$ and $N = kM + c$. Thus

$$(k+1)R_k(n-1, c) + c - 1 = N + M - 1,$$

and our aim is to show that $R_k(n, c) \geq N + M - 1$. In order to do that, it suffices to construct an n-coloring of the integer interval $[1, N+M-2]$ for which that interval contains no monochromatic x_i's satisfying the equation

$$x_1 + \cdots + x_k + c = x_{k+1}. \tag{3}$$

The minimality property of $R_k(n, c)$ will then imply the desired inequality.

By definition of M, there exists an $(n-1)$-coloring

$$\Delta \colon [1, M-1] \longrightarrow [1, n-1] \tag{4}$$

such that $[1, M-1]$ contains no Δ-monochromatic x_i's satisfying equation (3). We now extend (4) to an n-coloring

$$\Delta' \colon [1, N+M-2] \longrightarrow [1, n]$$

as follows:

$$\Delta'(x) = \begin{cases} \Delta(x) & \text{if } x \in [1, M-1], \\ n & \text{if } x \in [M, N-1], \\ \Delta(x-(N-1)) & \text{if } x \in [N, N+M-2]. \end{cases}$$

It remains to show that $[1, N+M-2]$ is free of a Δ'-monochromatic solution to (3). Assuming the contrary, let $x_1, \ldots, x_{k+1} \in [1, N+M-2]$ satisfy (3) and be of the same Δ'-color in $[1, n]$.

First, that common color cannot be n, for otherwise all x_i's would belong to $[M, N-1]$, thereby yielding

$$x_{k+1} = x_1 + \cdots + x_k + c \geq kM + c = N,$$

a contradiction.

Therefore, that common color of $x_1, \ldots, x_{k+1}$ belongs to $[1, n-1]$. Hence, some x_i's belong to $[1, M-1]$ and the rest to $[N, N+M-2]$. How do they distribute among these two intervals? First, we may assume that

$$x_1 \leq \cdots \leq x_k.$$

Note further that $x_k < x_{k+1}$, since

$$x_{k+1} = x_1 + \cdots + x_k + c \geq k + c \geq 2$$

by hypothesis on c.

Clearly, the x_i's cannot all belong to $[1, M-1]$ by our hypothesis on (4). It follows that the largest one, namely x_{k+1}, belongs to $[N, N+M-2]$. We claim that

$$x_1, \ldots, x_{k-1} \in [1, M-1] \text{ and } x_k \in [N, N+M-2].$$

Indeed, by (3) and the fact that $x_{k+1} \in [N, N+M-2]$, at most one among $\{x_1, \ldots, x_k\}$ may belong to $[N, N+M-2]$, since

$$2N > N + M - 2$$

as readily verified. Similarly, at least one among $\{x_1, \ldots, x_k\}$ must belong to $[N, N+M-2]$, for otherwise $x_k \leq M-1$, and by (3) we would have

$$x_{k+1} \leq k(M-1) + c < N,$$

a contradiction. Subtracting $N-1$ from x_k and x_{k+1}, it follows that

$$x_1, \ldots, x_{k-1}, x_k - (N-1), x_{k+1} - (N-1)$$

are Δ-monochromatic, belong to $[1, M-1]$, and satisfy (3), a contradiction. Therefore, the interval $[1, N+M-2]$ contains no Δ'-monochromatic solution to (3), and the proof is finished. □

Applying the above result inductively, we obtain the following absolute lower bound.

Theorem 3.2. *Let $k, n, c \in \mathbb{N}_+$. Then $R_k(n, c) \geq \frac{(k+1)^n - 1}{k}(k + c - 1) + 1$.*

Proof. The inequality holds for $n = 1$, since $R_k(1, c) = k + c$ by (2). For general $n \geq 2$, we apply induction and Lemma 3.1. □

3.2. **Reducing c.** We now vary the parameter c and show how $R_k(n, c)$ is affected. Several consequences will then be presented in subsequent sections.

Lemma 3.3. *Let $\alpha, \beta \in \mathbb{Z}$ such that $\alpha \geq 1$ and $\beta \geq 1 - \alpha$. Then for all integers k, n, c with $k, n \geq 2$ and $c \geq 0$, we have*

$$R_k(n, \alpha c - \beta(k-1)) \leq \alpha R_k(n, c) + \beta.$$

Proof. Let I be the integer interval

$$I = [1, \alpha R_k(n,c) + \beta]$$

and let $\Delta\colon I \to [1,n]$ be any n-coloring of I. We must show that there exist $x_1, \dots, x_{k+1} \in I$ which are monochromatic under Δ and which satisfy the equation

$$x_1 + \cdots + x_k + (\alpha c - \beta(k-1)) \;=\; x_{k+1}.$$

By the defining minimality property of $R_k(n, (\alpha c - \beta(k-1)))$, this will suffice to establish the stated inequality.

Let $J = [1, R_k(n,c)]$. We affinely embed J in I as follows:

$$\begin{aligned} h\colon J &\longrightarrow I \\ z &\longmapsto \alpha z + \beta. \end{aligned}$$

Note that our hypotheses on α, β ensure that if $z \geq 1$, then $\alpha z + \beta \geq 1$ and, more generally, that $h(J) \subseteq I$. By composing Δ with h, we obtain an n-coloring $\Delta' = \Delta \circ h$ on J, namely the map $\Delta'\colon J \to [1,n]$ defined by

$$\Delta'(z) \;=\; \Delta(\alpha z + \beta)$$

for all $z \in J$. Now, by definition of the upper bound $R_k(n,c)$ of J, there exists a Δ'-monochromatic solution

$$z_1 + \cdots + z_k + c \;=\; z_{k+1}$$

with $z_i \in J$ for all i. Multiplying by α and adding β's, we get

$$(\alpha z_1 + \beta) + \cdots + (\alpha z_k + \beta) + (\alpha c - \beta(k-1)) \;=\; (\alpha z_{k+1} + \beta).$$

Now $(\alpha z_i + \beta) \in I$ for all i, and since the z_i are Δ'-monochromatic, it follows that the $(\alpha z_i + \beta)$ are Δ-monochromatic. This concludes the proof of the lemma. □

3.3. **The case $n \geq k-1$.** A first consequence of the above lemma is the verification of Conjecture 1.2 in the cases where either $c \equiv 0 \bmod k-1$ (Proposition 3.4) or $n \geq k-1$ (Theorem 3.5).

Proposition 3.4. *If c is a multiple of $k-1$, then $R_k(n,c)$ is finite. More precisely, if $c = q(k-1)$ for some integer $q \geq 1$, then*

$$R_k(n,c) \;\leq\; (q+1)S_k(n) - q.$$

Proof. It follows from Lemma 3.3, with values $\beta = -q$, $\alpha = q+1$ (so that $\alpha + \beta \geq 1$ as required), and $c = 0$ (the c in that lemma, not the present one), that

$$\begin{aligned} R_k(n, q(k-1)) &\leq (q+1)R_k(n,0) - q \\ &= (q+1)S_k(n) - q. \end{aligned}$$

□

We may now settle Conjecture 1.2 in the case $n \geq k-1$.

Theorem 3.5. *Assume $n \geq k-1$ and $c \geq 0$. Then $R_k(n,c)$ is finite if and only if c is a multiple of $k-1$.*

Proof. If c is a multiple of $k-1$, the statement follows from the above proposition. On the other hand, if c is not a multiple of $k-1$, then Proposition 2.2, with divisor $d = k-1$, implies $R_k(n,c) = +\infty$. □

3.4. **Further consequences.** Our next consequence of Lemma 3.3 is a bound on $R_k(n, c)$ in terms of $R_k(n, 1)$.

Proposition 3.6. *For all $c \geq 1$, we have $R_k(n, c) \leq cR_k(n, 1)$. In particular, if $R_k(n, 1)$ is finite, then $R_k(n, c)$ is finite for all $c \geq 0$.*

Proof. Applying Lemma 3.3 with $c = 1$ (the c in that lemma again, not the current one) yields

$$R_k(n, \alpha - \beta(k-1)) \ \leq \ \alpha R_k(n, 1) + \beta$$

for all integer $\alpha \geq 1$ and $\beta \geq 1 - \alpha$. Now, setting $\alpha = c$ (the current one) and $\beta = 0$ in the above relation yields the stated inequality. □

Finally, we find that if $c \geq k - 1$, then $R_k(n, c)$ may be bounded below by $R_k(n, \overline{c}) - q$, where $\overline{c}$ is the class of c mod $k - 1$ and q is the floor of $c/(k-1)$.

Proposition 3.7. *Assume $c \geq k - 1$, and let $c = q(k-1) + \overline{c}$ be the Euclidean division of c by $k - 1$, with $q \geq 0$ and $0 \leq \overline{c} \leq k - 2$. Then*

$$R_k(n, \overline{c}) \ \leq \ R_k(n, c) + q.$$

Proof. It follows from Lemma 3.3, with values $\alpha = 1$ and $\beta = q$, that

$$R_k(n, \overline{c}) \ = \ R_k(n, c - (k-1)q) \ \leq \ R_k(n, c) + q. \qquad \square$$

3.5. **The case $k \leq 7$.** We now verify Conjecture 1.2 in case $k \leq 7$, except for the instances $R_5(3, 2)$ and $R_6(4, 1)$ which require completely different methods and will be presented elsewhere.

Only the "if" part of the conjecture remains open. It states that if every divisor $d \leq n$ of $k - 1$ also divides c, then $R_k(n, c)$ should be finite. This is known to be true in the following cases:

- if $c = 0$, in which case $R_k(n, 0)$ is finite by Rado's results [13].
- if $n = 2$, by Schaal's results recalled in [19, Section 1.2].
- if $c \equiv 0 \bmod k - 1$, in which case $R_k(n, c)$ is finite by Proposition 3.4.
- if $n \geq k - 1$, by Theorem 3.5.

Consequently, for $k \leq 7$, it remains to verify the "if" part of the conjecture only for $n \in [3, k-2]$. In particular, Conjecture 1.2 holds for $k \leq 4$.

The case $k = 5$. It remains to discuss the case $n = 3$ and $c \not\equiv 0 \bmod 4$.

- For c odd, there is a divisor $d \leq n$ of $k - 1 = 4$ not dividing c, namely $d = 2$. Therefore $R_5(3, c) = \infty$ in that case, by the "only if" part of the conjecture, i.e., by Proposition 2.2.
- For $c \equiv 2 \bmod 4$: according to the conjecture, we must show that $R_5(3, c)$ is finite in this case. Now Lemma 3.3 reduces that statement to the sole finiteness of $R_5(3, 2)$. Indeed, it implies $R_5(3, 2\alpha) \leq \alpha R_5(3, 2)$ for all $\alpha \geq 1$, by setting $\beta = 0$ and $c = 2$ there. Finally, it turns out that $R_5(3, 2)$ is indeed finite, as will be proved elsewhere.

We conclude that the conjecture holds for $k = 5$.

The case $k = 6$. We may assume $n \in [3, 4]$ and $c \not\equiv 0 \bmod 5$. The only divisor $d \leq n$ of $k - 1 = 5$ is 1, which divides any c. Therefore, according to the conjecture, we must show that $R_6(n, c)$ is finite for $n \in [3, 4]$ and all $c \geq 1$. Proposition 3.6 and the obvious bound $R_k(3, c) \leq R_k(4, c)$ reduce that statement to the sole finiteness of $R_6(4, 1)$. Again, $R_6(4, 1)$ turns out to be finite, with a proof similar to that for

$R_5(3,2)$ and to appear in the same paper. We conclude that the conjecture also holds for $k=6$.

The case $k=7$. We may assume $n \in [3,5]$ and $c \not\equiv 0 \bmod 6$. The only divisors $d \leq n$ of $k-1=6$ are 1, 2 and 3. If either 2 or 3 does not divide c, then $R_7(n,c)=\infty$ in these cases, by the "only if" part of the conjecture, i.e., by Proposition 2.2. Now, if both 2 and 3 divide c, then $c \equiv 0 \bmod 6$, an already settled case. Therefore, the conjecture holds for $k=7$, with a proof entirely contained in the present paper in contrast to the cases $k=5$ and 6.

The conjecture remains open for $k \geq 8$. However, in order to settle the smallest open case $k=8$, it would suffice, by using the same reduction tools as above, to show that $R_8(6,1)$ is finite.

3.6. **Yet another case of the conjecture.** Here is yet another case where Conjecture 1.2 is shown to hold. Its interest lies in the fact that n is smaller than $k-1$, in contrast with the situation in Section 3.3.

Proposition 3.8. *Let n be an integer such that* $\gcd(n,6)=1$ *and $n \geq 5$. Let $k=6n+1$. The following statements are equivalent.*

(1) *$R_k(n,c)$ is finite.*
(2) *Every divisor $d \leq n$ of $k-1$ also divides c.*
(3) *$k-1$ divides c.*

Proof. That (1) implies (2) directly follows from Proposition 2.2. Now assume (2). Applying this to $d=2$, $d=3$ and $d=n$, it follows that $2 \cdot 3 \cdot n$ divides c, since these numbers are pairwise coprime. That is, $k-1$ divides c, as stated in (3). Finally, that (3) implies (1) directly follows from Proposition 3.4. □

Needless to say, the same proof tools yield the same equivalences under this set of hypotheses: $\gcd(n,30)=1$, $n \geq 5$ and $k=30n+1$; or, for that matter, under the set of hypotheses $\gcd(n,210)=1$, $n \geq 7$ and $k=210n+1$; and so on.

4. Bounds and exact values

In this section, we define new numbers $S_k^*(n)$ which are smaller and easier to determine than $S_k(n)$. We then show how they provide a lower bound to $R_k(n,c)$, and we give an estimate for them. Finally, we use them to obtain a new shorter proof for Schaal's formula on $R_2(3,c)$, exact formulae for $R_3(3,c)$ and $R_4(3,c)$, and sharper bounds in some other cases.

4.1. **The numbers $S_k^*(n)$.** Before defining the $S_k^*(n)$ proper, we introduce and generalize some terminology from additive number theory.

Given nonempty sets A_1, A_2 of integers, their *sumset* A_1+A_2 is defined as

$$A_1+A_2 \;=\; \{x_1+x_2 \mid x_1 \in A_1, x_2 \in A_2\}.$$

In particular, if $A_2=\{c\}$ is a singleton, then $A_1+A_2=\{x+c \mid x \in A_1\}$; it is the translate of A_1 by c, and will be denoted by A_1+c instead of $A_1+\{c\}$. If $A_1=A_2=A$, then we denote $2A=A+A$, and more generally, for any integer $k \geq 1$, we denote by kA the k-fold sumset of A with itself, i.e.,

$$kA \;=\; \underbrace{A+\cdots+A}_{k} \;=\; \{x_1+\cdots+x_k \mid x_i \in A \text{ for all } 1 \leq i \leq k\}.$$

A set A of integers is said to be *sum-free* if $A \cap 2A = \emptyset$. As a generalization well-suited to our purposes here, and with k, c integers with $k \geq 1$, we shall say that A is *$(kX + c)$-free* if

$$(kA + c) \cap A = \emptyset.$$

More generally, given a set C of integers, we shall say that A is *$(kX + C)$-free* if it is $(kX + c)$-free for all $c \in C$, or equivalently, if

$$(kA + C) \cap A = \emptyset.$$

Clearly, for $c \in \mathbb{Z}$, the set A *is $(kX + c)$-free if and only if it contains no solution to the equation*

$$x_1 + \cdots + x_k + c = x_{k+1} \tag{5}$$

with $x_1, \ldots, x_{k+1} \in A$.

Thus, the n-color Rado number $R_k(n, c)$ may equivalently be described as the smallest integer N, if it exists, or ∞ if not, such that for every n-coloring of $[1, N]$, at least one of the color classes A in $[1, N]$ fails to be $(kX + c)$-free.

We are now in a position to define the $S_k^*(n)$.

Definition 4.1. Let n, k be positive integers with $k \geq 2$. Denote by $S_k^*(n)$ the least integer N such that, for every n-coloring of $[1, N]$, there exists a monochromatic solution $(x_1, \ldots, x_{k+1})$ to the equation

$$x_1 + \cdots + x_k + s = x_{k+1} \tag{6}$$

for some $s \in [-k + 2, 1]$.

Equivalently, let $I = [-k+2, 1]$. Then $S_k^*(n)$ *is the least integer N such that, for every n-coloring of* $[1, N]$*, at least one of its color classes fails to be* $(kX + I)$*-free.*

Indeed, a set A is $(kX + I)$-free if and only if it is $(kX + s)$-free for all $s \in I$, or equivalently if, for all $s \in I$, it contains no solution to equation (6).

It directly follows from the definitions that

$$S_k^*(n) \leq \min_{c \in [-k+2,1]} R_k(n, c).$$

In particular, at $c = 0$, we have $S_k^*(n) \leq S_k(n)$.

4.2. **Bounding $R_k(n, c)$ with $S_k^*(n)$.** Our first result constructs a $(kX + c)$-free set A from a $(kX + I)$-free set B, where $I = [-k + 2, 1]$. This will then be used to bound $R_k(n, c)$ in terms of $S_k^*(n)$.

Notation 4.2. For a subset $B \subseteq \mathbb{Z}$ and a positive integer λ, we denote

$$\lambda \cdot B = \{\lambda y \mid y \in B\}.$$

The dot here is important since it helps distinguish $\lambda \cdot B$ from the λ-fold sumset $\lambda B = B + \cdots + B$.

Lemma 4.3. *Let k, c be integers such that $k \geq 2$ and $c \geq 0$. Let $B \subseteq \mathbb{Z}$ be a $(kX + I)$-free subset, where $I = [-k + 2, 1]$. Let*

$$A = (c + k - 1) \cdot B + [-(c + k - 2), 0].$$

Then A is $(kX + c)$-free.

Proof. Assume that A is not $(kX+c)$-free. Then there exist $x_1,\dots,x_{k+1}\in A$ such that

$$x_1+\cdots+x_k+c \;=\; x_{k+1}. \tag{7}$$

By construction, each x_i decomposes as

$$x_i \;=\; (c+k-1)y_i+r_i$$

for some $y_i\in B$ and $r_i\in[-(c+k-2),0]$. Equation (7) then yields

$$(c+k-1)(y_1+\cdots+y_k-y_{k+1})+(r_1+\cdots+r_k-r_{k+1}+c) \;=\; 0.$$

It follows that $(c+k-1)$ divides $(r_1+\cdots+r_k-r_{k+1}+c)$, and that

$$\frac{(r_1+\cdots+r_k-r_{k+1}+c)}{(c+k-1)} \;=\; -(y_1+\cdots+y_k-y_{k+1}). \tag{8}$$

Now, since $r_i\in[-(c+k-2),0]$ for all $i\in[1,k+1]$, we have

$$\begin{array}{rcccl} -(c+k-2)k & \le & r_1+\cdots+r_k & \le & 0,\\ 0 & \le & -r_{k+1} & \le & (c+k-2), \end{array}$$

from which it follows that

$$-(c+k-2)k+c \le (r_1+\cdots+r_k-r_{k+1}+c) \le (c+k-2)+c.$$

Dividing by $(c+k-1)$ and using (8), we get

$$\frac{-(c+k-2)k+c}{(c+k-1)} \;\le\; -(y_1+\cdots+y_k-y_{k+1}) \;\le\; \frac{(c+k-2)+c}{(c+k-1)}. \tag{9}$$

Since the middle term is an integer, inequalities (9) remain valid if we replace the leftmost term by its ceiling $\lceil\;\rceil$ and the rightmost one by its floor $\lfloor\;\rfloor$.

The numerator $-(c+k-2)k+c$ in the leftmost term may be written as $-(c+k-1)k+(c+k-1)+1$, so that

$$\frac{-(c+k-2)k+c}{(c+k-1)} \;=\; -k+1+\frac{1}{(c+k-1)},$$

whose ceiling equals $-k+2$.

In turn, the numerator of the rightmost term of (9) may be written as $(c+k-1)+(c-1)$, so that

$$\frac{(c+k-2)+c}{(c+k-1)} \;=\; 1+\frac{(c-1)}{(c+k-1)},$$

whose floor equals 0 if $c=0$, or 1 if $c\ge 1$. In either case, (9) yields

$$-k+2 \;\le\; -(y_1+\cdots+y_k-y_{k+1}) \;\le\; 1.$$

Thus, setting $s=-(y_1+\cdots+y_k-y_{k+1})$, we have $s\in[-k+2,1]$, and the equality

$$y_1+\cdots+y_k+s \;=\; y_{k+1}$$

implies that B is not $(kX+s)$-free, contrary to the assumption. □

This implies the following lower bound on $R_k(n,c)$ in terms of $S_k^*(n)$.

Proposition 4.4. *Let $k,n\ge 2$. Then $R_k(n,c) \;\ge\; (c+k-1)(S_k^*(n)-1)+1$.*

Proof. Set $M = S_k^*(n)$ and $N = (c+k-1)(M-1)+1$. By the minimality property of $S_k^*(n)$, there exists a special n-coloring Δ of $[1, M-1]$ all of whose color classes B are $(kX + I)$-free, where $I = [-k+2, 1]$. Let

$$\pi\colon [1, N-1] \longrightarrow [1, M-1]$$

be the map defined by

$$\pi(x) = \left\lceil \frac{x}{(c+k-1)} \right\rceil$$

for all $x \in [1, N-1]$, and let Δ' be the n-coloring of $[1, N-1]$ defined by

$$\Delta'(x) = \Delta(\pi(x))$$

for all $x \in [1, N-1]$. Clearly, for all $y \in [1, M-1]$ and all $r \in [-(c+k-2), 0]$, one has

$$\pi((c+k-1)y + r) = y,$$

i.e., the map π is constant on the subinterval $(c+k-1)y + [-(c+k-2), 0]$. It follows that each color class A in $[1, N-1]$ under Δ' is of the form

$$A = (c+k-1) \cdot B + [-(c+k-2), 0] \tag{10}$$

for some color class B in $[1, M-1]$ under Δ. Now, since each such B is $(kX + I)$-free, it follows from (10) and Lemma 4.3 that A is $(kX+c)$-free. By the minimality property of $R_k(n, c)$, it follows that

$$N - 1 \leq R_k(n, c) - 1,$$

as claimed. □

Remark 4.5. Proposition 4.4 provides a lower bound on $R_k(n, c)$ involving $S_k^*(n)$, whereas Proposition 3.4 provides, if c is a nonnegative multiple of $k-1$, an upper bound on $R_k(n, c)$ involving $S_k(n)$. Thus, combining both bounds for $c = q(k-1)$ with $q \in \mathbb{Z}_+$, we get

$$(q+1)(k-1)(S_k^*(n) - 1) + 1 \leq R_k(n, c) \leq (q+1)(S_k(n) - 1) + 1.$$

Remarkably, it turns out that these two bounds sometimes coincide and hence yield the exact value of $R_k(n, c)$, as will be seen later on in some instances. At any rate, we get the following corollary.

Corollary 4.6. *If $S_k(n) - 1 = (k-1)(S_k^*(n) - 1)$, then for all $c = q(k-1)$ with $q \in \mathbb{N}$, we have $R_k(n, c) = (q+1)(S_k(n) - 1) + 1$.*

Proof. This directly follows from the above remark. □

4.3. **A lower bound on $S_k^*(n)$.** We first relate $S_k^*(n)$ to $S_k^*(n-1)$, and then derive an absolute lower bound on it.

Proposition 4.7. *Let $k, n \geq 2$. Then $S_k^*(n) \geq (k+1)S_k^*(n-1) - k + 1$.*

Proof. Set $M = S_k^*(n-1)$ and $N = (k+1)M - k + 1$. By the minimality property of $S_k^*(n-1)$, there exists a special $(n-1)$-coloring Δ of $[1, M-1]$ under which each color class $B \subseteq [1, M-1]$ is $(kX + s)$-free for all $s \in [-k+2, 1]$.

Let us now extend Δ to the n-coloring

$$\Delta'\colon [1, N-1] \longrightarrow [1, n],$$

defined as follows, for $x \in [1, N-1]$:

$$\Delta'(x) = \begin{cases} \Delta(x) & \text{if } x \in [1, M-1], \\ n & \text{if } x \in [M, kM-k+1], \\ \Delta(x-(N-1)) & \text{if } x \in [kM-k+2, N-1]. \end{cases} \tag{11}$$

Note, for later use, that the third interval above is just a translate of the first one, namely:

$$[kM-k+2, N-1] = [1, M-1] + (kM-k+1). \tag{12}$$

It remains to show that each color class in $[1, N-1]$ under Δ' is $(kX+s)$-free. The minimality property of $S_k^*(n)$ will then imply $N-1 \le S_k^*(n)-1$, as desired.

Now, each color class A under Δ' is either equal to $[M, kM-k+1]$, or else it is of the form

$$A = B + \{0, kM-k+1\} = B + (kM-k+1)\cdot[0,1] \tag{13}$$

for some color class $B \subseteq [1, M-1]$ under Δ. This follows from (11) and (12).

So, let $A \subseteq [1, N-1]$ be a color class under Δ', and let $s \in [-k+2, 1]$. We now show that A is $(kX+s)$-free.

Case 1. $A = [M, kM-k+1]$. We then have

$$\min(kA+s) = kM+s \ge kM-k+2 = \max(A)+1.$$

It follows that $(kA+s)\cap A = \emptyset$, as claimed.

Case 2. $A = B+(kM-k+1)\cdot[0,1]$ for some color class $B \subseteq [1, M-1]$, as stated in (13). Assume, for a contradiction, that $(kA+s)\cap A$ is not empty. Since

$$kA+s = kB+s+(kM-k+1)\cdot[0,k],$$

there exist $b_1, \dots, b_{k+1} \in B$, and integers $u \in [0,k], v \in [0,1]$ such that

$$b_1+\cdots+b_k+s+(kM-k+1)u = b_{k+1}+(kM-k+1)v.$$

It follows that $b_1+\cdots+b_k-b_{k+1}+s$ is a multiple of $(kM-k+1)$. Now, since $b_i \in [1, M-1]$ for all i and since $s \in [-k+2, 1]$, we have

$$-M+3 \le b_1+\cdots+b_k-b_{k+1}+s \le kM-k.$$

But the only multiple of $(kM-k+1)$ within this range is 0. Therefore,

$$b_1+\cdots+b_k-b_{k+1}+s = 0,$$

i.e., $b_1+\cdots+b_k+s = b_{k+1}$ and hence belongs to $(kB+s)\cap B$. This contradicts the fact that the color class B is $(kX+s)$-free. □

Corollary 4.8. *Let $k, n \ge 2$. Then $S_k^*(n) \ge \frac{(k+1)^n-1}{k}+1$.*

Proof. Since $S_k^*(1) = 2$, as easily seen, the inequality is satisfied for $n=1$. For general $n \ge 2$, we apply induction and Proposition 4.7. □

4.4. **Revisiting** $R_2(3, c)$**.** An exact formula for $R_2(3, c)$ has been provided by Schaal [20]. We now provide a shorter proof for it, which exploits the above properties of the $S^*_k(n)$ and thereby avoids the case-by-case analysis of [20].

Proposition 4.9. *We have* $S^*_2(3) = 14$.

Proof. Since $S_2(3) = 14$ and $S^*_2(3) \le S_2(3)$, we have $S^*_2(3) \le 14$. The reverse inequality directly follows from Corollary 4.8. Alternatively, it suffices to exhibit a 3-coloring of $[1, 13]$ with all three color classes being $(2X + I)$-free where $I = [0, 1]$ as seen below:

$$\begin{aligned} A_1 &= \{1, 4, 10, 13\}, \\ A_2 &= \{2, 3, 11, 12\}, \\ A_3 &= [5, 9]. \end{aligned}$$

Each A_i satisfies $(2A_i + I) \cap A_i = \emptyset$, as readily checked and as required. □

Corollary 4.10 (Schaal, [20]). *We have* $R_2(3, c) = 13c + 14$ *for all* $c \ge 0$.

Proof. We have $S^*_2(3) = S_2(3) = 14$. The first equality is an instance where the hypothesis

$$S_k(n) = (k - 1)(S^*_k(n) - 1) + 1$$

of Corollary 4.6 is satisfied, here with $k = 2$. That corollary then implies

$$R_2(3, c) = (c + 1)(S_2(3) - 1) + 1,$$

i.e., $R_2(3, c) = 13(c + 1) + 1 = 13c + 14$. □

Using the same method of proof, it is easy to establish the corresponding formula for $R_2(2, c)$, namely:

$$R_2(2, c) = 4c + 5$$

for all $c \ge 0$.

4.5. **A formula for** $S^*_k(2)$**.** We end this section by deriving a formula for $S^*_k(2)$.

Proposition 4.11. *Let* $k \ge 2$*. Then* $S^*_k(2) = k + 3$.

Proof. The bound $S^*_k(2) \ge k + 3$ directly follows from Corollary 4.8. To prove the reverse inequality, it suffices to show that for every 2-coloring of $[1, k + 3]$, one of the two color classes fails to be $(kX + I)$-free, where as usual $I = [-k + 2, 1]$.

Given a 2-coloring of $[1, k+3]$, let A_1, A_2 be its two color classes. We may freely assume that $1 \in A_1$. Since

$$k\{1\} + I = \{k\} + [-k + 2, 1] = [2, k + 1],$$

it follows that if $A_1 \cap [2, k+1]$ failed to be empty, then A_1 would fail to be $(kX+I)$-free and we would be done.

Therefore, we may assume $A_1 \cap [2, k + 1] = \emptyset$, i.e., $[2, k + 1] \subseteq A_2$. Since

$$k[2, k + 1] + I = [2k, k^2 + k] + [-k + 2, 1] = [k + 2, k^2 + k + 1],$$

which contains $\{k + 2, k + 3\}$, we may assume that none of $k + 2, k + 3$ belongs to A_2, for otherwise A_2 would fail to be $(kX + I)$-free and we would again be done.

Therefore, we may assume $\{1, k+2, k+3\} \subseteq A_1$. But then, A_1 fails to be $(kX+I)$-free, since setting

$$x_1 = \cdots = x_{k-1} = 1, \;\; x_k = k+2, \;\; x_{k+1} = k+3, \;\; s = -k+2,$$

we have $x_i \in A_1$ for all i, $s \in I$, and

$$x_1 + \cdots + x_k + s \;=\; x_{k+1}. \qquad \square$$

5. Computer-aided results

The problem of computing $S_k^*(n)$ or $R_k(n,c)$ can be translated as a Boolean satisfiability problem, as detailed in Section 5.3 for the computation of $R_2(4,c)$. The resulting Boolean translation, for given instances, may then be fed to a computer running a suitable SAT solver. All results in this section have been obtained by combining such computations with the theory developed above. The specific SAT solver we have used is March RW [9], the gold medal winner of the 2011 International SAT Competition.

Proposition 5.1. *The following values of $S_k^*(n)$, for $n = 3$ and $2 \leq k \leq 6$, and for $n = 4$ and $2 \leq k \leq 3$, hold:*

(1) $S_2^*(3) = 14,\ S_3^*(3) = 22,\ S_4^*(3) = 32,\ S_5^*(3) = 44,\ S_6^*(3) = 58.$
(2) $S_2^*(4) = 41,\ S_3^*(4) = 86.$

Proof. The formula for $S_2^*(3)$ has been established in Proposition 4.9. All others have been obtained by running a SAT solver on the corresponding Boolean translations. $\square$

The numbers $S_k(n)$ are larger and more difficult to compute than the $S_k^*(n)$'s. However, in a few instances where we have been able to compute them, the hypothesis of Corollary 4.6, namely

$$S_k(n) - 1 \;=\; (k-1)(S_k^*(n) - 1),$$

turned out to be satisfied, thereby allowing an exact formula for the corresponding $R_k(n,c)$'s.

5.1. Exact formulas for $R_3(3,c)$ and $R_4(3,c)$.

We now establish previously unknown formulas for $R_3(3,c)$ and $R_4(3,c)$, as new applications of Corollary 4.6.

5.1.1. *The case $R_3(3,c)$.*

We first need the value of $S_3(3)$, recently obtained in [18].

Theorem 5.2. *We have $S_3(3) \;=\; 43$.*

Proof. The inequality $S_3(3) \leq 43$ is obtained by computer using the SAT solver March [9]. Indeed, the solver established that the Boolean constraints derived from assuming the existence of a 3-coloring of [1,43] with $(3X+0)$-free color classes cannot be satisfied.

The reverse inequality follows from a result of Znam [24] which, for $k = n = 3$, yields the bound

$$S_3(3) \geq \frac{2}{3}(4^3 - 1) + 1 = 43. \qquad \square$$

Theorem 5.3. *For every integer $c \geq 0$, we have*

$$R_3(3,c) \;=\; \begin{cases} \infty & \text{if } c \text{ odd,} \\ 21c + 43 & \text{if } c \text{ even.} \end{cases}$$

Proof. For c odd, we have $R_3(3, c) = +\infty$ by Proposition 2.2. Now assume $c = 2q$ with $q \in \mathbb{N}$. Since $S_3^*(3) = 22$ by Proposition 5.1, and since $S_3(3) = 43$ as stated above, we see that the hypothesis

$$S_k(n) - 1 \;=\; (k-1)(S_k^*(n) - 1)$$

of Corollary 4.6 is again satisfied in this instance. Therefore, that corollary yields the formula

$$R_3(3, 2q) \;=\; (q+1)(S_3(3) - 1) + 1,$$

that is, for $c = 2q$: $R_3(3, c) \;=\; 42(q+1) + 1 \;=\; 21c + 43$, as claimed. □

5.1.2. *The case* $R_4(3, c)$*.* In view of applying Corollary 4.6 again, we now need the value of $S_4(3)$.

Theorem 5.4. *We have* $S_4(3) \;=\; 94$.

Proof. Follows from a three-hour computation with the SAT solver March, using our Boolean translation of the problem along the lines of Section 5.3. □

Theorem 5.5. *For every integer* $c \geq 0$*, we have*

$$R_4(3, c) \;=\; \begin{cases} \infty & \text{if } c \notin 3\mathbb{N}, \\ 31c + 94 & \text{if } c \in 3\mathbb{N}. \end{cases}$$

Proof. For c not divisible by 3, we have $R_4(3, c) = +\infty$ by Proposition 2.2. Assume now $c = 3q$ with $q \in \mathbb{N}$. Since $S_4^*(3) = 32$ by Proposition 5.1, and since $S_4(3) = 94$ as stated above, we see that the hypothesis

$$S_k(n) - 1 \;=\; (k-1)(S_k^*(n) - 1)$$

of Corollary 4.6 is again satisfied in this instance. Therefore, that corollary yields the formula

$$R_4(3, 3q) \;=\; (q+1)(S_4(3) - 1) + 1;$$

that is, for $c = 3q$: $R_4(3, c) \;=\; 93(q+1) + 1 \;=\; 31c + 94$, as claimed. □

5.2. **Some new bounds.** Short of exact formulae, Remark 4.5 also enables us to obtain new bounds on suitable instances of $R_k(n, c)$.

Corollary 5.6. *Let* $c \in \mathbb{N}$*. Then* $40c + 41 \;\leq\; R_2(4, c) \;\leq\; 44c + 45$.

Proof. The lower bound on $R_2(4, c)$ follows from Proposition 4.4 and the value $S_2^*(4) = 41$ stated in Proposition 5.1, whereas the upper bound follows from Proposition 3.4 and the known value $S_2(4) = 45$. □

Corollary 5.7. *Let* $c \in \mathbb{N}$*. Then* $121c + 122 \;\leq\; R_2(5, c) \;\leq\; 305c + 306$.

Proof. It is known that $S_2(5) \leq 316$; see [14, 17]. Moreover, Radziszowski showed in [14] that $r_5(3) \leq 307$, where $r_5(3) = r(3, 3, 3, 3, 3)$ denotes the 5-color Ramsey number for unavoidable monochromatic triangles in any edge-colored complete graph of that order. Applying the relationship between Schur numbers and Ramsey numbers given by Roberston [16], one obtains $S_2(5) \;\leq\; r_5(3) - 1 \;\leq\; 306$.

Propositions 3.4 and 4.4 then yield, for $k = 2$ and $n = 5$, the stated bounds on $R_2(5, c)$. □

5.3. **Seeking $R_2(4,c)$ by computer.** Having fixed $c \geq 0$, we seek successive integers M such that $[1,M]$ admits a 4-coloring all of whose color classes are $(2X+c)$-free; or equivalently, such that no triplet of the form $\{i,j,i+j+c\}$, with $1 \leq i \leq M-c$ and $i \leq j \leq M-i-c$, is monochromatic. When the largest possible such M is found, we are done: $R_2(4,c) = M+1$.

We now reformulate this problem as a Boolean satisfiability problem [6]. We proceed as follows.

First, any 4-coloring of $[1,M]$ may be viewed as a function

$$\Delta\colon [1,M] \longrightarrow \{0,1\}^2.$$

By setting $\Delta(i) = (x_i, x_{i+M})$ for all $i \in [1,M]$, this 4-coloring may be represented by $2M$ binary variables $x_1, \dots, x_{2M}$ with values in $\{0,1\}$.

We now view 0 and 1 as representing the Boolean values False and True, respectively. This allows us to use the logical operators AND, OR and NOT, denoted respectively by $\wedge$, $\vee$ and $\neg$, on the set $\{0,1\}$, with the purpose of translating the *nonequality* of elements by the *validity* of some associated logical formula. Indeed, for any $x, y \in \{0,1\}$, we have

$$x \neq y \iff (x \vee y) \wedge (\neg x \vee \neg y) \text{ is True,} \tag{14}$$

as readily checked. It is easy to similarly translate the non-equality of two given colors in $\{0,1\}^2$, since for $(x_1,y_1),(x_2,y_2) \in \{0,1\}^2$, we have

$$(x_1,y_1) \neq (x_2,y_2) \iff x_1 \neq x_2 \text{ or } y_1 \neq y_2.$$

Let us go back to our generic 4-coloring

$$\Delta\colon [1,M] \longrightarrow \{0,1\}^2$$

represented by the Boolean variables $x_1, \dots, x_{2M}$. Given an arbitrary subset $A \subseteq [1,M]$, we may associate to A, in the way described above, a logical formula $\lambda(A)$ in the variables $x_1, \dots, x_{2M}$, in such a way that *A fails to be monochromatic if and only if $\lambda(A)$ is True.* Therefore, given a family of subsets $A_1, \dots, A_r \subseteq [1,M]$, prohibiting all of them to be monochromatic admits the following Boolean translation:

$$\text{no } A_i \text{ is monochromatic} \iff \bigwedge_{i=1}^{r} \lambda(A_i) \text{ is True.} \tag{15}$$

Now, applying this translation to the above-mentioned set of triplets, namely the subsets $\{i,j,i+j+c\} \subseteq [1,M]$ with $1 \leq i \leq M-c$ and $i \leq j \leq M-i-c$, we obtain by (15) a system of logical formulas that can be simultaneously satisfied if and only if $[1,M]$ admits a 4-coloring all of whose color classes are $(2X+c)$-free. Fixing successive values of M, we feed the associated system to a SAT solver, which will then attempt to say whether that system is satisfiable or not. As long as it is, we increase M. When, for some M_0, we reach nonsatisfiability as an output, we know we are done: $R_2(4,c) = M_0$.

5.4. **Exact values of $R_2(4,c)$ for $c \leq 6$.** Applying the above method, we have obtained, by computer, the following exact value of the Rado numbers $R_2(4,c)$ for $0 \leq c \leq 6$.

Theorem 5.8. *We have*

$$\begin{aligned} R_2(4,0) &= 40 \cdot 0 + 45 = 45, \\ R_2(4,1) &= 40 \cdot 1 + 43 = 83, \text{ and} \\ R_2(4,c) &= 40 \cdot c + 41 \qquad \text{for } 2 \leq c \leq 6. \end{aligned}$$

Proof. The value $R_2(4,0) = 45$ is due to [2]. A 4-coloring of $[1,82]$ with $(2X+1)$-free color classes, implying $R_2(4,1) \geq 83$, is given by the following 4 color classes:

$$\{1,2,15,16,21,22,28,29,35,36,47,54,55,61,62,67,68,81,82\},$$
$$\{3,4,5,6,17,18,19,20,33,34,49,50,63,64,65,66,77,78,79,80\},$$
$$\{23,24,25,26,27,37,38,39,40,41,42,43,44,45,46,56,57,58,59,60\},$$
$$\{7,8,9,10,11,12,13,14,30,31,32,48,51,52,53,69,70,71,72,73,74,75,76\},$$

thereby improving the lower bound $R_2(4,1) \geq 81$ given by Proposition 4.4 and the value $S_2^*(4) = 41$ of Proposition 5.1. For $2 \leq c \leq 6$, the lower bound is achieved by Proposition 4.4, and is revealed to be sharp by computations using the SAT solver March [9]. The running times on a standard desktop computer were as follows:

Values of c	$R_2(4,c)$	Time in seconds
$c = 1$	83	13
$c = 2$	121	50
$c = 3$	161	1260
$c = 4$	201	2810
$c = 5$	241	9270
$c = 6$	281	593000

□

5.5. **Conclusions and open problems.** Combined with a separate forthcoming paper establishing the finiteness of $R_5(3,2)$ and $R_6(4,1)$, Conjecture 1.2 turns out to be true for $k \leq 7$. It remains to settle it in general. The smallest open case is $R_8(6,1)$, conjectured to be finite. In addition, in view of the above determination of $R_2(4,c)$ for $c \leq 6$, it is natural to conjecture that the formula $R_2(4,c) = 40c+41$ also holds for $c \geq 7$. Is it true or not? Along the same line, we are also interested in determining exact values or sharper bounds for $R_2(5,c)$.

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Harish-Chandra Research Institute, Chhathnag Road, Jhunsi, Allahabad-211 019, India

E-mail address: adhikari@hri.res.in

Escuela Técnica Superior de Arquitectura, Departamento de Matemática Aplicada I, Avenida de la Reina Mercedes 2, C.P. 41012 Sevilla, Spain

E-mail address: boza@us.es

ULCO, LMPA J. Liouville, CS 80699 – 62228 Calais Cedex, France — and — CNRS, FR 2956, France

E-mail address: eliahou@lmpa.univ-littoral.fr

Escuela Técnica Superior de Ingeniería de Edificación, Departamento de Matemática Aplicada I, Avenida de la Reina Mercedes 4, C.P. 41012 Sevilla, Spain

E-mail address: jmarin@us.es

Escuela Técnica Superior de Ingeniería de Edificación, Departamento de Matemática Aplicada I, Avenida de la Reina Mercedes 4, C.P. 41012 Sevilla, Spain

E-mail address: pastora@us.es

Escuela Técnica Superior de Ingeniería de Edificación, Departamento de Matemática Aplicada I, Avenida de la Reina Mercedes 4, C.P. 41012 Sevilla, Spain

E-mail address: isanz@us.es