Exact value of 3 color weak Rado number

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Abstract

For integers k, n, c with k, $n \geq 1$ and $c \geq 0$, the n color weak Rado number $WR_k(n, c)$ is defined as the least integer N, if it exists, such that for every n-coloring of the set $\{1, 2, ..., N\}$, there exists a monochromatic solution in that set to the equation $x_1 + x_2 + ... + x_k + c = x_{k+1}$, such that $x_i \neq x_j$ when $i \neq j$. If no such N exists, then $WR_k(n, c)$ is defined as infinite.

In this work, we consider the main issue regarding the 3 color weak Rado number for the equation $x_1 + x_2 + c = x_3$ and the exact value of the $WR_2(3, c) = 13c + 22$ is established.

Keywords:
Schur numbers, sum-free sets, weak Schur numbers, weakly sum-free sets, Rado numbers, weak Rado numbers.

1 Introduction

In terms of coloring, the Schur number $S_2(n)$ [14] is the least positive integer N such that for every n-coloring of $\{1, 2, ..., N\}$,

$\Delta : \{1, 2, ..., N\} \longrightarrow \{1, 2, ..., n\}$, there exists a monochromatic solution to the equation $x_1 + x_2 = x_3$, such that $\Delta(x_1) = \Delta(x_2) = \Delta(x_3)$ where x_1 and x_2 need not be distinct.

In 1933, Rado [9], [10] generalized the work of Schur to arbitrary systems of linear equations. Given a system of linear equations L and a natural number n, the least integer N (if it exists) such that for every coloring of the set $\{1, 2, ..., N\}$ with n colors there is a monochromatic solution to L, which is called the n color Rado number for L. If no such integer N exists, then the n color Rado number for the system L is taken to be infinite.

Eighty-three years after the first Rado results, very little progress has been obtained for some systems of linear equations. Bur and Loo [2] were able to determine the 2 color Rado number for the equations $x_1 + x_2 + c = x_3$ and $x_1 + x_2 = kx_3$ for every integer c and for every positive integer k [3].

In 1993, Schaal [12] determined the 2 color Rado number $R_k(2, c)$ for the equation $x_1 + x_2 + \ldots + x_k + c = x_{k+1}$. He also obtained [13] the 3 color Rado number $R_2(3, c)$. There are several results due to Schaal and other authors concerning 2 color and 3 color Rado numbers for particular equations, see [7], [8], [11] and other authors [6]. In addition, recently we have studied when $R_k(n, c)$ is finite or infinite and we have obtained new exacts values [1]. In this work, we consider a generalization of the Rado numbers. For every integer $c \geq 0$, $n \geq 1$, let $WR_2(n, c)$ be the least integer N (if it exists) such that, for every coloring of the set $\{1, 2, ..., N\}$ with n colors, there exists a monochromatic solution to the equation $x_1 + x_2 + c = x_3$, where $x_1 \neq x_2$. The numbers $WR_2(n, c)$ are called *weak Rado numbers*.

$WR_2(n, c)$ can be defined equivalently as the greatest N, such that the set $\{1, 2, ..., N - 1\}$ can be partitioned into n sets $\{A_1, A_2, ..., A_n\}$, such that for any $x_1, x_2 \in A_i$ then $x_1 + x_2 + c \notin A_i$, $\forall i$ where $x_1 \neq x_2$. The sets $\{A_1, A_2, ..., A_n\}$ are *weakly sum free for the equation* $x_1 + x_2 + c = x_3$.

In 1952, Walker [15] claimed the value $WR_2(5, 0) = 196$, without proof. Sixty years later, we have shown $WR_2(5, 0) \geq 196$ [4] and Schaal et al.[5] have obtained the number $WR_2(2, c)$ for every integer c.

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2 Main Result

Theorem 2.1 $WR_2(3,c) = 13c + 22$ *for any* $c > 0$.

2.1 Lower bound

Lemma 2.2 $WR_3(2,c) \geq 13c + 22$ *for any* $c > 0$.

Proof.

Consider the following partition $\{A_1, A_2, A_3\}$ of $\{1, ..., 13c + 21\}$:

$$\begin{cases} A_1 = \{1, 2, \ldots, c+2\} \cup \{3c+7, \ldots, 4c+7\} \cup \{9c+17, \ldots, \\ \qquad 10c+17\} \cup \{12c+21, \ldots, 13c+21\} \\ A_2 = \{c+3, c+4, \ldots, 3c+6\} \cup \{10c+18, \ldots, 12c+20\} \\ A_3 = \{4c+8, 4c+9, \ldots, 9c+16\} \end{cases}$$

$\{A_1, A_2, A_3\}$ is a partition of $\{1, \ldots, 13c + 21\}$.

We prove that this partition is weakly sum free, i.e. if $x_1, x_2 \in A_i$, with $x_1 \neq x_2$ then $x_1 + x_2 + c \notin A_i$.

We assume, without any loss of generality, that $x_1 < x_2$.

Case 1: $x_1, x_2 \in A_1$

- If $x_2 \leq c + 2$, then $c + 3 \leq x_1 + x_2 + c \leq 3c + 3$, therefore $x_1 + x_2 + c \notin A_1$.
- If $3c + 7 \leq x_2 \leq 4c + 7$ then $4c + 8 \leq x_1 + x_2 + c \leq 9c + 13$, therefore $x_1 + x_2 + c \notin A_1$.
- If $9c + 17 \leq x_2 \leq 10c + 17$, we have:
  · If $x_1 \leq c+2$ then $10c+18 \leq x_1+x_2+c \leq 12c+19$, therefore $x_1+x_2+c \notin A_1$.
  · If $3c + 7 \leq x_1$ then $13c + 24 \leq x_1 + x_2 + c$, therefore $x_1 + x_2 + c \notin A_1$.
- If $x_2 \geq 12c + 21$ then $x_1 + x_2 + c \geq 13c + 22$, therefore $x_1 + x_2 + c \notin A_1$.

Case 2: $x_1, x_2 \in A_2$ and $x_1 \geq c + 3$

- If $x_2 \leq 3c+6$, then $3c+7 \leq x_1+x_2+c \leq 7c+11$, therefore $x_1+x_2+c \notin A_2$.
- If $x_2 \geq 10c + 18$ then $12c + 21 \leq x_1 + x_2 + c$, therefore $x_1 + x_2 + c \notin A_2$.

Case 3: $x_1, x_2 \in A_3$

Since $9c + 17 \leq x_1 + x_2 + c$, then $x_1 + x_2 + c \notin A_3$.

□

2.2 Upper bound

Lemma 2.3 *$WR_3(2, c) \leq 13c + 22$ for any $c > 0$.*

Proof.

The upper bound is obtained considering all 3-colorings of the positive integers 1, 2 and 3. To the elements of the sets A_1, A_2 and A_3, we assign the following colors $\Delta(\{A_1\}) = i_1$, $\Delta(\{A_2\}) = i_2$, $\Delta(\{A_3\}) = i_3$, where i_1, i_2, i_3 are three different colors.

Five main cases are considered:

Case 1 $A_1 \supseteq \{1,\ 2,\ 3\}$.

Case 2 $A_1 \supseteq \{1,\ 2\}$ and $A_3 \supseteq \{3\}$.

Case 3 $A_1 \supseteq \{1,\ 3\}$ and $A_2 \supseteq \{2\}$.

Case 4 $A_1 \supseteq \{1\}$ and $A_2 \supseteq \{2,\ 3\}$.

Case 5 $A_1 \supseteq \{1\}$, $A_2 \supseteq \{2\}$ and $A_3 \supseteq \{3\}$.

We have to obtain weakly sum free subsets for the equation $x_1+x_2+c = x_3$.

Let $f(\{A_i\})$ be subsets containing the monochromatic solutions of the elements of the sets A_i, $i = 1, 2, 3$.

The key of the proof is the following:

- If $a \in f(\{A_i\}) \cap f(\{A_j\})$, with $i \neq j$ then $a \in A_k$ with $k \neq i, j$.
- If $a \in f(\{A_1\}) \cap f(\{A_2\}) \cap f(\{A_3\})$, then $a \notin A_i$, $i = 1, 2, 3$.

□

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