On the finiteness of some n-color Rado numbers

S.D. Adhikari [a], L. Boza [b], S. Eliahou [c,d], J.M. Marín [e], M.P. Revuelta [e,*], M.I. Sanz [e]

[a] *Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad—211 019, India*

[b] *Escuela Técnica Superior de Arquitectura, Departamento de Matemática Aplicada I, Universidad de Sevilla, Avenida de la Reina Mercedes 2, C.P. 41012 Sevilla, Spain*

[c] *Univ. Littoral Côte d'Opale, EA 2597 - LMPA - Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville, F-62228 Calais, France*

[d] *CNRS, FR 2956, France*

[e] *Escuela Técnica Superior de Ingeniería de Edificación, Departamento de Matemática Aplicada I, Universidad de Sevilla, Avenida de la Reina Mercedes 4, C.P. 41012 Sevilla, Spain*

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For integers k, n, c with $k, n \geq 1$, the n-color Rado number $R_k(n, c)$ is defined to be the least integer N if any, or infinity otherwise, such that for every n-coloring of the set $\{1, 2, \dots, N\}$, there exists a monochromatic solution in that set to the linear equation $x_1 + x_2 + \cdots + x_k + c = x_{k+1}$.

A recent conjecture of ours states that $R_k(n, c)$ should be finite if and only if every divisor $d \leq n$ of $k-1$ also divides c. In this paper, we complete the verification of this conjecture for all $k \leq 7$. As a key tool, we first prove a general result concerning the degree of regularity over subsets of $\mathbb{Z}$ of some linear Diophantine equations.

1. Introduction

Denote $\mathbb{N}_+ = \{1, 2, 3, \dots\}$. Let k, n, c be integers with $k, n \in \mathbb{N}_+$. In this paper, we shall be concerned with the linear Diophantine equation

$$x_1 + x_2 + \cdots + x_k + c = x_{k+1} \tag{1}$$

and, more precisely, with the existence of monochromatic solutions to it under n-colorings of $\mathbb{N}_+$.

Definition 1.1. The *n-color Rado number*, denoted $R_k(n, c)$, is the smallest positive integer N, if any, such that for every n-coloring of the integer interval $[1, N] = \{1, \dots, N\}$, there exists a monochromatic solution to Eq. (1). If no such N exists, then $R_k(n, c)$ is defined to be infinite.

In this paper, we are interested in the question of determining for what values of the parameters k, n, c the number $R_k(n, c)$ is finite. The general answer is not known. For $c = 0$, a result of Rado states that $R_k(n, 0)$ is always finite [9]. See [2–4,6,8,10–15] for other selected partial results. To add to this list, we present the following conjecture [1].

* Corresponding author.

E-mail addresses: adhikari@hri.res.in (S.D. Adhikari), boza@us.es (L. Boza), eliahou@lmpa.univ-littoral.fr (S. Eliahou), jmarin@us.es (J.M. Marín), pastora@us.es (M.P. Revuelta), isanz@us.es (M.I. Sanz).

Conjecture 1.2. *Let k, n, c be integers with $k \geq 2$, $n \geq 1$ and $c \geq 0$. Then $R_k(n, c)$ is finite if and only if every divisor $d \leq n$ of $k-1$ also divides c.*

The "only if" part of the conjecture is settled in [1]. As for the "if" part, we showed in that same paper that it holds in the following cases:

- if $k-1$ divides c,
- or if $n \geq k-1$,
- or if $k \leq 7$, provided $R_5(3, 2)$ and $R_6(4, 1)$ are shown to be finite.

Our present purpose is to complete the proof of Conjecture 1.2 for $k \leq 7$, precisely by showing that both $R_5(3, 2)$ and $R_6(4, 1)$ are finite. This is done in the last Section 4, using two tools set up in the earlier sections. The main one is Theorem 2.5 of Section 2. It allows one to compare, in quite general circumstances, the *degree of regularity*, over $\mathbb{N}_+$ and over $\mathbb{Z}$, of some linear Diophantine equations. The second tool is provided in Section 3, where the degree of regularity is expressed in terms of the chromatic number of a suitable hypergraph. As noted in the concluding Section 4.3, the smallest open case of Conjecture 1.2 is now $R_8(6, 1)$.

2. Regularity

The Rado number $R_k(n, c)$ can be expressed in terms of a variant of the degree of regularity of Eq. (1), as done in Proposition 2.4 of Section 2.3. This variant is obtained by restricting the subset of $\mathbb{Z}$ over which solutions to the given equation are sought. Our main result, Theorem 2.5, is established in Section 2.4.

2.1. The equations (L) and (L_0)

We shall consider here a somewhat more general version of Eq. (1). For integer vectors $\alpha, x \in \mathbb{Z}^{k+1}$, let us denote by $\alpha \cdot x$ their standard dot product. That is, if $\alpha = (\alpha_1, \ldots, \alpha_{k+1})$ and $x = (x_1, \ldots, x_{k+1})$, then $\alpha \cdot x = \sum_{i=1}^{k+1} \alpha_i x_i$. Now, given $c \in \mathbb{Z}$, consider the linear Diophantine equation $(L) : \alpha \cdot x = -c$ and its associated homogeneous version, $(L_0) : \alpha \cdot x = 0$. Note that Eq. (1) is a special case of equation (L), namely where $\alpha = (1, 1, \ldots, 1, -1) \in \mathbb{Z}^{k+1}$.

2.2. Degree of regularity

The following notion was introduced by Rado [9]. Given $n \in \mathbb{N}_+$, the equation (L) is said to be *n-regular* if, for every n-coloring of $\mathbb{N}_+$, there exists a *monochromatic* solution $x \in \mathbb{N}_+^{k+1}$ to (L).

Clearly, for $n \geq 2$, n-regularity implies $(n-1)$-regularity. This motivates the following definition. See e.g. [5].

Definition 2.1. The *degree of regularity* of (L) is the largest integer $n \geq 0$, if any, such that (L) is n-regular. This (possibly infinite) number is denoted by $dor(L)$. If $dor(L) = \infty$, then (L) is said to be *regular*.

We shall need the following particular case of a general theorem of Rado on systems of linear Diophantine equations.

Theorem 2.2 ([9]). *The homogeneous equation $(L_0) : \alpha \cdot x = 0$ is regular if and only if some nonempty subsequence of the coordinates of α sums up to 0.*

2.3. Over a subset of $\mathbb{Z}$

We now introduce a refined version of regularity, by focusing on solutions x to (L) all of whose coordinates belong to some given subset A of $\mathbb{Z}$.

Definition 2.3. Let $A \subseteq \mathbb{Z}$. Let $n \in \mathbb{N}_+$. We say that (L) is *n-regular over A* if, for every n-coloring of A, there exists a *monochromatic* solution $x \in A^{k+1}$ to (L). The *degree of regularity of (L) over A* is the largest integer $n \geq 0$, if any, such that (L) is n-regular over A. We shall denote this (possibly infinite) number by $dor_A(L)$.

Note that if $A = \mathbb{N}_+$, then $dor_{\mathbb{N}_+}(L)$ coincides with $dor(L)$ as defined in the previous section.

Clearly, for any $A \subseteq B \subseteq \mathbb{Z}$, we have $0 \leq dor_A(L) \leq dor_B(L)$. Thus

$$dor_{\mathbb{N}_+}(L) \leq dor_{\mathbb{Z}}(L). \tag{2}$$

Whether the reverse inequality also holds is addressed in the next section. But first, let us connect this notion with the Rado numbers $R_k(n, c)$.

Proposition 2.4. *Let k, n, c be integers with $k, n \geq 1$. Let (L) be the equation corresponding to (1) with these parameters, i.e. $\alpha \cdot x = -c$ for $\alpha = (1, \ldots, 1, -1) \in \mathbb{Z}^{k+1}$. Then $R_k(n, c) = \infty$ if and only if $dor(L) < n$, i.e. if and only if (L) is not n-regular.*

Moreover, if $R_k(n, c)$ is finite, then $R_k(n, c)$ is the smallest positive integer N such that (L) is n-regular over $A = [1, N]$, i.e. such that $dor_A(L) \geq n$.

Proof. Straightforward from the definitions, together with a standard compactness argument according to which (L) is n-regular over $\mathbb{N}_+$ if and only if there exists $N \in \mathbb{N}_+$ such that (L) is n-regular over $[1, N]$. $\square$

2.4. Comparing $dor_{\mathbb{N}_+}(L)$ and $dor_{\mathbb{Z}}(L)$

We now show that, under suitable conditions, inequality (2) is in fact an equality. As will be seen in the applications, this turns out to be very helpful in efforts to determine the degree of regularity of Eq. (1) for some values of the parameters k, n, c.

Theorem 2.5. *Let $k \geq 1$, $\alpha \in \mathbb{Z}^{k+1}$ and $c \in \mathbb{Z}$. Let (L) be the equation $\alpha \cdot x = -c$, and let (L_0) be its homogeneous counterpart $\alpha \cdot x = 0$. Assume that equation (L_0) is regular, and that the coordinate sum of α is nonzero. Then $dor_{\mathbb{N}_+}(L) = dor_{\mathbb{Z}}(L)$.*

Proof. By (2), it remains to prove the inequality $dor_{\mathbb{Z}}(L) \leq dor_{\mathbb{N}_+}(L)$. If $dor_{\mathbb{N}_+}(L)$ is infinite, there is nothing to do. Assume now that this number is finite. Set $dor_{\mathbb{N}_+}(L) = n - 1$. Since (L) is not n-regular, there exists an n-coloring of $\mathbb{N}_+$, say $\Delta : \mathbb{N}_+ \longrightarrow \{1, \dots, n\}$, such that there exists no Δ-monochromatic solution in $\mathbb{N}_+^{k+1}$ to equation (L).

Let $K \in \mathbb{N}_+$. Let $V = [-K, K] \subseteq \mathbb{Z}$. We shall prove the inequality $dor_V(L) \leq n - 1$ by establishing the existence of a specific n-coloring $\overline{\Delta} : V \longrightarrow \{1, \dots, n\}$ with the property that V^{k+1} *contains no monochromatic solution to equation* (L). Since K is arbitrary, this will imply $dor_{\mathbb{Z}}(L) \leq n - 1$ by a standard compactness argument, as desired.

The main idea is to first consider the n^{2K+1}-coloring

$$\Delta^* : \mathbb{N}_+ \setminus [1, K] \longrightarrow \{1, \dots, n\}^{2K+1}$$

defined by $\Delta^*(m) = (\Delta(m - K), \Delta(m - K + 1), \dots, \Delta(m + K))$ for all $m \geq K + 1$. We extend it to a $(n^{2K+1} + K)$-coloring of the whole of $\mathbb{N}_+$, $\Delta^* : \mathbb{N}_+ \longrightarrow \{1, \dots, n\}^{2K+1} \sqcup \{q_1, \dots, q_K\}$, by setting $\Delta^*(m) = q_m$ for all $m \in [1, K]$, where $q_1, \dots, q_K$ are K new pairwise distinct colors.

Since the homogeneous equation (L_0) is assumed to be regular, there exists a Δ^*-monochromatic solution $s = (s_1, \dots, s_{k+1}) \in \mathbb{N}_+^{k+1}$ to this equation, i.e. satisfying $\alpha \cdot s = 0$.

We claim that $s_j \geq K + 1$ for all $1 \leq j \leq k + 1$, i.e. that $s \in (\mathbb{N}_+ \setminus [1, K])^{k+1}$. For otherwise, if some entry s_j of s belonged to $[1, K]$, then since s is monochromatic, all entries of s would have the same unique color q_j, whence all entries of s would be equal to s_j, i.e. $s = s_j(1, 1, \dots, 1)$. But then, $\alpha \cdot s$ would equal s_j times the coordinate sum of α, and hence would be nonzero by hypothesis, in contradiction with $\alpha \cdot s = 0$. This proves the claim.

Therefore, since $s \in (\mathbb{N}_+ \setminus [1, K])^{k+1}$ and s is Δ^*-monochromatic, it follows that for all $i \in V = [-K, K]$, we have $\Delta(s_1 + i) = \Delta(s_2 + i) = \cdots = \Delta(s_{k+1} + i)$. Denote by $\overline{\Delta}(i)$ this common color, i.e.

$$\overline{\Delta}(i) = \Delta(s_1 + i) = \Delta(s_2 + i) = \cdots = \Delta(s_{k+1} + i). \tag{3}$$

This is the announced n-coloring $\overline{\Delta} : V \longrightarrow \{1, \dots, n\}$ with the desired property. Indeed, we claim that V^{k+1} contains no $\overline{\Delta}$-monochromatic solution to equation (L).

Assume for a contradiction that, on the contrary, there exists $\delta \in V^{k+1}$ satisfying (L), i.e. such that $\alpha \cdot \delta = -c$, and which is monochromatic under $\overline{\Delta}$, say of color $t \in \{1, \dots, n\}$. Let E denote the set of distinct coordinates of δ. We then have $\overline{\Delta}(i) = t$ for all $i \in E$. It follows from (3) that $\Delta(s_j + i) = t$ for all $1 \leq j \leq k + 1$ and all $i \in E$. Therefore the vector $s + \delta$ is Δ-monochromatic of color t. But now, $s+\delta$ belongs to $\mathbb{N}_+^{k+1}$, and it satisfies $\alpha \cdot (s+\delta) = -c$. Thus, $s+\delta$ is a Δ-monochromatic solution to equation (L) in $\mathbb{N}_+^{k+1}$, thereby contradicting our hypothesis on Δ. This proves our claim about $\overline{\Delta}$.

It follows that $dor_V(L) \leq n - 1$ as desired, and this concludes the proof of the theorem. $\square$

3. An associated hypergraph

We now associate to equation (L) a certain hypergraph whose chromatic number is closely related to the degree of regularity of (L).

Let us first recall a few basic notions. Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph, with vertex set V and hyperedge set $\mathcal{E}$. A *proper n-coloring* of $\mathcal{H}$ is a coloring $\Delta : V \longrightarrow \{1, \dots, n\}$ of its vertices such that none of its hyperedges is monochromatic. The *chromatic number* $\chi(\mathcal{H})$ of $\mathcal{H}$ is the least positive integer n such that $\mathcal{H}$ admits a proper n-coloring. Finally, for any subset $W \subseteq V$, let $\mathcal{H}|_W = (W, \mathcal{E}|_W)$ denote the *restriction* of $\mathcal{H}$ to W. By definition, the vertex set of $\mathcal{H}|_W$ is W, and its hyperedges are all hyperedges of $\mathcal{H}$ which are contained in W. That is, for $E \in \mathcal{E}$, we have $E \in \mathcal{E}|_W \iff E \subseteq W$. Note that the chromatic number is monotonic with respect to restriction. That is, if $W \subseteq W'$, then $\chi(\mathcal{H}|_W) \leq \chi(\mathcal{H}|_{W'})$.

Notation 3.1. *Let $z = (z_1, \dots, z_m) \in \mathbb{Z}^m$. We denote by $U(z)$ the underlying set of the coordinates of z, i.e. $U(z) = \{z_1, \dots, z_m\}$.*

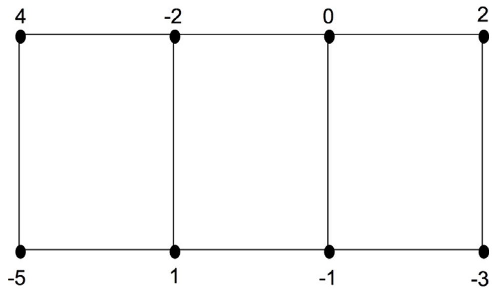

Fig. 1. The graph G_0.

For example, if $z = (2, -1, 2, 1, 1) \in \mathbb{Z}^5$, then $U(z) = \{-1, 1, 2\}$. We are now ready to associate a suitable hypergraph $\mathcal{H} = \mathcal{H}(L)$ to equation (L). It is defined as follows.

Definition 3.2. The set of vertices of $\mathcal{H}(L)$ is $\mathbb{Z}$, and a subset $E \subseteq \mathbb{Z}$ is a hyperedge in $\mathcal{H}(L)$ if and only if $E = U(\delta)$ for some solution $\delta \in \mathbb{Z}^{k+1}$ to equation (L).

Here is the relationship, to be used in the next section, between the chromatic number of $\mathcal{H}$ and the degree of regularity of (L) over any subset of $\mathbb{Z}$.

Proposition 3.3. *Let $\mathcal{H} = \mathcal{H}(L)$ be the hypergraph associated to equation (L), and let $A \subseteq \mathbb{Z}$. Then*

$$dor_A(L) = \chi(\mathcal{H}|_A) - 1. \tag{4}$$

Proof. Let $n = dor_A(L) + 1$. Then there exists an n-coloring $\Delta : A \longrightarrow \{1, \dots, n\}$ under which (L) admits no monochromatic solution with entries in A. Hence, for any solution $\delta \in A^{k+1}$ to (L), the underlying set $U(\delta)$ is not monochromatic either. Therefore no hyperedge of $\mathcal{H}|_A$ is monochromatic, showing that Δ is a proper n-coloring of $\mathcal{H}|_A$. This yields $dor_A(L) + 1 \geq \chi(H|_A)$. The inequality in the other direction is obvious. $\square$

4. Applications

We now apply the above results and show that $R_5(3, 2)$ and $R_6(4, 1)$ are both finite. Combined with the results of [1], this settles Conjecture 1.2 for all $k \leq 7$.

4.1. *Finiteness of* $R_5(3, 2)$

Here we focus on equation (L) with parameters $k = 5$, $\alpha = (1, 1, 1, 1, 1, -1) \in \mathbb{Z}^6$ and $c = 2$, i.e.

$$x_1 + x_2 + x_3 + x_4 + x_5 - x_6 = -2. \tag{5}$$

Our purpose is to show that $R_5(3, 2)$ is finite, i.e. that there exists an integer N such that (L), i.e. Eq. (5), is 3-regular over the interval $[1, N]$.

We do not know how to prove this directly. However, since the associated homogeneous equation $\alpha \cdot x = 0$ is regular by Rado's Theorem 2.2, the vector α satisfies the hypotheses of Theorem 2.5. It follows that $dor_{\mathbb{N}_+}(L) = dor_{\mathbb{Z}}(L)$. This fact, allowing us to use negative entries, will prove very effective in our quest for the value of $dor_{\mathbb{N}_+}(L)$. Indeed, we now show that $dor_{\mathbb{Z}}(L) = 3$. By $dor_{\mathbb{N}_+}(L) = dor_{\mathbb{Z}}(L)$ and Proposition 2.4, this will imply $R_5(3, 2) < \infty$, as desired.

Let $\mathcal{H} = \mathcal{H}(L)$ denote the hypergraph associated to (L). Consider the subset $V = \{-5, -3, -2, -1, 0, 1, 2, 4\}$ of $\mathbb{Z}$. We shall show that $\chi(\mathcal{H}|_V) \geq 4$, which by (4) will imply $dor_V(L) \geq 3$ and hence $dor_{\mathbb{Z}}(L) \geq 3$. This already suffices to get $R_5(3, 2) < \infty$. Note that, in order to obtain the exact value $dor_{\mathbb{Z}}(L) = 3$, it would remain to show that $dor_{\mathbb{Z}}(L) < 4$. But this easily follows from the 4-coloring of the integers according to the class mod 4.

Recall that a *stable set* in a graph is a subset of its vertices which are pairwise non-adjacent. Of course, under a proper vertex coloring, any monochromatic subset of vertices is stable. Also, by *triple*, we shall mean a 3-element subset.

Proposition 4.1. *Using the above notation, we have $\chi(\mathcal{H}|_V) \geq 4$.*

Proof. Let G_0 be the graph on the vertex set V shown in Fig. 1.

Claim 1. *The graph G_0 is a subhypergraph of $\mathcal{H}|_V$.*

Table 1
Ten vectors δ with $\alpha \cdot \delta = -2$.

$\{\lambda, \lambda'\}$	$\delta = \delta(\lambda, \lambda')$
$\{4, -2\}$	$(-2, -2, -2, 4, 4, 4)$
$\{-2, 0\}$	$(-2, 0, 0, 0, 0, 0)$
$\{0, 2\}$	$(0, 0, 0, 0, 0, 2)$
$\{-5, 1\}$	$(-5, 1, 1, 1, 1, 1)$
$\{1, -1\}$	$(-1, -1, -1, 1, 1, 1)$
$\{-1, -3\}$	$(-1, -1, -1, -1, -1, -3)$
$\{4, -5\}$	$(-5, -5, 4, 4, 4, 4)$
$\{-2, 1\}$	$(-2, -2, 1, 1, 1, 1)$
$\{0, -1\}$	$(-1, -1, 0, 0, 0, 0)$
$\{2, -3\}$	$(-3, -3, 2, 2, 2, 2)$

Table 2
Ten more vectors δ with $\alpha \cdot \delta = -2$.

Stable triple X	$\delta = \delta(X)$
$\{-5, -2, -1\}$	$(-2, -2, -1, -1, -1, -5)$
$\{-5, -2, 2\}$	$(-5, -5, 2, 2, 2, -2)$
$\{-5, -1, 2\}$	$(-5, -1, 2, 2, 2, 2)$
$\{-3, 0, 1\}$	$(-3, 1, 0, 0, 0, 0)$
$\{-3, 0, 4\}$	$(-3, -3, 4, 0, 0, 0)$
$\{-3, 0, -5\}$	$(-5, 0, 0, 0, 0, -3)$
$\{-3, 1, 4\}$	$(-3, -3, -3, 4, 4, 1)$
$\{-2, -1, 2\}$	$(-2, -1, -1, 2, 2, 2)$
$\{-1, 2, 4\}$	$(4, -1, -1, -1, -1, 2)$
$\{0, 1, 4\}$	$(1, 1, 0, 0, 0, 4)$

Proof of Claim 1. We must show that every edge $\{\lambda, \lambda'\}$ of G_0 is a hyperedge of $\mathcal{H}|_V$. This amounts to exhibit a vector $\delta = \delta(\lambda, \lambda')$ in $\mathbb{Z}^6$, with entries in the pair $\{\lambda, \lambda'\}$ exclusively, satisfying $\alpha \cdot \delta = -2$.

The graph G_0 has 10 edges $\{\lambda, \lambda'\}$. The occurrence of each one in the hypergraph $\mathcal{H}|_V$ is testified by the vector $\delta = \delta(\lambda, \lambda')$ given in Table 1.

This settles Claim 1.

Note that G_0 is a bipartite graph. Indeed, its vertex set V admits a partition into two stable sets, namely $V_0 = \{4, 1, 0, -3\}$ and $V_1 = \{-5, -2, -1, 2\}$.

We shall see that *most stable triples in G_0 belong to $\mathcal{H}|_V$*. Let us first count them.

Claim 2. *The graph G_0 contains exactly 12 stable triples.*

Proof of Claim 2. The top row of G_0 contains three stable pairs, and each one of them can be extended in two distinct ways to a stable triple by adding a third vertex from the bottom row. This gives 6 stable triples. Symmetrically, there are 6 more stable triples with 2 vertices in the bottom row and 1 vertex in the top row. This settles Claim 2.

Claim 3. *All twelve stable triples in G_0, with the possible two exceptions of $\{4, 2, 1\}$ and $\{-5, -3, -2\}$, belong to $\mathcal{H}|_V$.*

Proof of Claim 3. Let $X \subseteq V$ be any subset. As in the proof of Claim 1, in order to prove that X belongs to $\mathcal{H}|_V$, it suffices to exhibit a vector $\delta = \delta(X)$ of length 6, with entries in X exclusively, satisfying $\alpha \cdot \delta = -2$.

Table 2 provides the required vector $\delta = \delta(X)$ for each of the 10 relevant stable triples $X \subseteq V$. This settles Claim 3.

We are now ready to conclude the proof of the proposition. Assume for a contradiction that there exists a proper 3-coloring of $\mathcal{H}|_V$, say $\Delta : V \longrightarrow \{a, b, c\}$. In particular, Δ is a proper 3-coloring of the subgraph G_0. Since there are 8 vertices colored with 3 colors, then either

(1) some color occurs with multiplicity at least 4 in V,
(2) or else two colors occur with multiplicity 3 each.

Option (1) is impossible. Indeed, in the subgraph G_0, no color can occur more than twice in either the top or the bottom row. Moreover, no color can simultaneously occur twice in both rows, for otherwise we would get four monochromatic stable triples, in contradiction with Claim 3 and the fact that Δ is a proper 3-coloring of $\mathcal{H}|_V$.

Therefore option (2) holds: there are two colors occurring each with multiplicity 3, the third color being then of multiplicity 2. Since, by Claim 3, there are at most two stable triples which do not necessarily belong to $\mathcal{H}|_V$ and hence are not forbidden to be monochromatic, in fact now they *must be* monochromatic. These stable triples being $\{4, 2, 1\}$ and $\{-5, -3, -2\}$, it follows, up to color permutation, that $\Delta(4) = \Delta(2) = \Delta(1) = a, \quad \Delta(-5) = \Delta(-3) = \Delta(-2) = b$.

Therefore, the remaining two vertices, namely -1 and 0, must be colored c. But $\{-1, 0\}$ is an edge in G_0, hence in $\mathcal{H}|_V$, and Δ is a proper vertex coloring of $\mathcal{H}|_V$. This contradiction concludes the proof of the proposition. □

As already commented above, this result implies the finiteness of $R_5(3, 2)$.

Remark 4.2. This proof does not yield a realistic estimate for $R_5(3, 2)$, as is often the case in Ramsey theory. However, with some computer help, we have determined this number and obtained the exact value $R_5(3, 2) = 259$. The inequality $R_5(3, 2) \geq 259$ follows from the following 3-coloring of $[1, 258]$, where $X_i \subset [1, 258]$ denotes the subset of elements of color i for $1 \leq i \leq 3$:

$$\begin{aligned}
X_1 &= [1, 6] \cup [37, 42] \cup [217, 222] \cup [253, 258],\\
X_2 &= [7, 36] \cup [223, 252],\\
X_3 &= [43, 216].
\end{aligned}$$

As easily checked, none of the X_i's contains a monochromatic solution to (L). Therefore $R_5(3, 2) \geq 259$, as stated. For the reverse inequality, we reduced equation (L) to its 4-variable version

$$x_1 + x_2 + 3x_3 - x_6 = -2 \tag{6}$$

by identifying $x_3 = x_4 = x_5$. The SAT solver *march_rw* [7], running on a standard desktop computer, then established in about 13 000 s that, for every 3-coloring of $[1, 259]$, there is a monochromatic solution to the reduced equation (6); and hence to (L) itself, from which $R_5(3, 2) \leq 259$ follows.

Actually, one can establish $R_5(3, 2) \leq 259$ in a dramatically shorter computing time, as follows. Consider the subset $X \subset [1, 259]$ of cardinality 51:

$$\begin{aligned}
X = \{&1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 17, 19, 21, 23, 25, 27, 29,\\
&31, 33, 35, 37, 39, 41, 43, 47, 49, 51, 55, 69, 71, 75, 81, 83, 87, 93, 99,\\
&109, 121, 127, 139, 163, 175, 187, 193, 217, 223, 247, 259\}.
\end{aligned}$$

Then, here again, every 3-coloring of X admits a monochromatic solution to Eq. (6). This has been established with *march_rw* in just 1 s on the same desktop computer as above. To be on the safe side, this property of X has also been established with a completely different method, namely by backtrack programming in C, in about 25 min on a comparable desktop computer. Finally, let us mention that X is minimal for this property: after removing any of its elements, the resulting set admits 3-colorings without any monochromatic solution to (6).

4.2. Finiteness of $R_6(4, 1)$

We now focus on equation (L) with parameters $k = 6$, $\alpha = (1, 1, 1, 1, 1, 1, -1) \in \mathbb{Z}^7$ and $c = 1$, i.e.

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 - x_7 = -1. \tag{7}$$

Using the same approach as in the preceding section, we shall show that $R_6(4, 1)$ is finite or, in other terms, that $dor_{\mathbb{N}_+}(L) \geq 4$. However, here, we shall need to rely on some computer help.

Theorem 2.5 applies again and yields $dor_{\mathbb{N}_+}(L) = dor_{\mathbb{Z}}(L)$.

Let $\mathcal{H} = \mathcal{H}(L)$ be the hypergraph associated to (L). By $dor_{\mathbb{N}_+}(L) = dor_{\mathbb{Z}}(L)$ and Proposition 3.3, we must show $\chi(\mathcal{H}) \geq 5$. As in the preceding section, since the chromatic number is monotonic with respect to restriction, it suffices to find a finite restriction of $\mathcal{H}$ with chromatic number at least 5. Computer experiments show that restricting $\mathcal{H}$ to the interval $[-12, 14] \subset \mathbb{Z}$ already suffices for this purpose, as we now explain. More precisely, let

$$V = \{-12, -11, -8, -7, -6, -5, -3, -2, -1, 0, 1, 2, 3, 7, 8, 9, 10, 11, 13, 14\}.$$

Let us now consider the hyperedges of $\mathcal{H}|_V$. Generally speaking, the constraint contributed by a hyperedge to the chromatic number tends to fade away with its cardinality. Therefore, in $\mathcal{H}|_V$, we only considered those hyperedges of cardinality 2 or 3. We have found, by computer, that $\mathcal{H}|_V$ contains 16 edges of cardinality 2 and 218 hyperedges of cardinality 3.

Recall that, in order to testify that some subset $E \subseteq V$ is a hyperedge of $\mathcal{H}|_V$, it suffices to exhibit a solution $\delta \in \mathbb{Z}^7$ to equation (L) whose set of distinct coordinates is equal to E. For instance, both $\{-11, -2\}$ and $\{-12, -11, 0\}$ are hyperedges of $\mathcal{H}|_V$, as witnessed by the following two solutions to (L):

$$(-2, -2, -2, -2, -2, -2, -11) \quad \text{and} \quad (-12, 0, 0, 0, 0, 0, -11).$$

Here are the promised 16 edges of $\mathcal{H}|_V$:

$\{-11, -2\}$, $\{-11, 7\}$, $\{-5, -1\}$, $\{-5, 1\}$, $\{-5, 3\}$, $\{-5, 7\}$, $\{-2, 1\}$, $\{-2, 7\}$, $\{-1, 0\}$, $\{-1, 1\}$, $\{-1, 3\}$, $\{0, 1\}$, $\{1, 7\}$, $\{-7, 10\}$, $\{-3, 11\}$, $\{2, 13\}$.

As for the 218 hyperedges of cardinality 3 of $\mathcal{H}|_V$, it turns out that a certain subset of 95 of them suffices to guarantee a chromatic number of 5. Here is this subset:

$\{-12, -11, 0\}$, $\{-11, -8, 1\}$, $\{-11, -7, -1\}$, $\{-11, -5, 9\}$, $\{-11, -5, 10\}$, $\{-11, -5, 13\}$, $\{-11, -3, -1\}$, $\{-11, -3, 0\}$, $\{-11, -1, 11\}$, $\{-11, -1, 13\}$, $\{-11, 0, 2\}$, $\{-11, 1, 11\}$, $\{-8, -2, -1\}$, $\{-8, 0, 7\}$, $\{-7, -2, -1\}$, $\{-7, 0, 3\}$, $\{-7, 1, 3\}$, $\{-7, 3, 7\}$, $\{-6, -5, 0\}$, $\{-6, -2, -1\}$, $\{-6, 1, 3\}$, $\{-5, -3, 0\}$, $\{-5, -2, 2\}$, $\{-5, -2, 10\}$, $\{-5, -2, 13\}$, $\{-5, 0, 2\}$, $\{-3, -2, 0\}$, $\{-3, -2, 3\}$, $\{-3, -1, 7\}$, $\{-3, 1, 3\}$, $\{-3, 3, 7\}$, $\{-2, -1, 2\}$, $\{-2, -1, 8\}$, $\{-2, 0, 9\}$, $\{-2, 2, 3\}$, $\{-1, 2, 7\}$, $\{-1, 7, 11\}$, $\{0, 2, 3\}$, $\{0, 2, 7\}$, $\{0, 3, 10\}$, $\{0, 3, 13\}$, $\{0, 7, 8\}$, $\{1, 3, 9\}$, $\{1, 3, 11\}$, $\{1, 3, 13\}$, $\{-12, -3, -2\}$, $\{-12, -3, 2\}$, $\{-12, -3, 13\}$, $\{-12, -2, 11\}$, $\{-12, -1, 2\}$, $\{-12, 0, 11\}$, $\{-12, 2, 3\}$, $\{-12, 3, 10\}$, $\{-11, -8, 13\}$, $\{-11, -7, 14\}$, $\{-11, -6, 2\}$, $\{-11, -3, 8\}$, $\{-8, -7, 0\}$, $\{-8, -7, 2\}$, $\{-8, -7, 11\}$, $\{-8, -1, 10\}$, $\{-8, 1, 2\}$, $\{-8, 2, 3\}$, $\{-7, -6, 0\}$, $\{-7, -6, 1\}$, $\{-7, -5, 8\}$, $\{-6, -2, 11\}$, $\{-6, 2, 7\}$, $\{-6, 7, 10\}$, $\{-3, -2, 9\}$, $\{-1, 8, 13\}$, $\{-1, 9, 13\}$, $\{-1, 10, 14\}$, $\{0, 13, 14\}$, $\{1, 2, 8\}$, $\{1, 2, 9\}$, $\{2, 3, 14\}$, $\{-12, -11, 8\}$, $\{-12, -6, 1\}$, $\{-12, -1, 14\}$, $\{-12, 7, 9\}$, $\{-11, 1, 3\}$, $\{-8, -6, 3\}$, $\{-8, -3, 14\}$, $\{-8, 1, 8\}$, $\{-7, -3, 7\}$, $\{-7, 1, 2\}$, $\{-5, -2, 0\}$, $\{-3, 1, 2\}$, $\{-2, 0, 3\}$, $\{-1, 2, 10\}$, $\{0, 3, 7\}$, $\{0, 8, 9\}$, $\{0, 10, 11\}$, $\{1, 2, 10\}$.

Restricting ourselves to those $16 + 95$ (hyper)edges, and using a SAT solver [7], we have obtained the following result.

Proposition 4.3. *Using the above notation, we have* $\chi(\mathcal{H}|_V) = 5$.

Proof. Besides SAT solvers, tools to determine the chromatic number of a hypergraph are available on the web and may be used to confirm this statement. $\square$

As commented earlier, Proposition 4.3 implies that $R_6(4, 1)$ is indeed finite.

4.3. Conclusion

Combining the finiteness of $R_5(3, 2)$ and $R_6(4, 1)$ with the results of [1], it follows that Conjecture 1.2 holds for all $k \leq 7$. More explicitly, the following result holds.

Theorem 4.4. *For all integers* k, n, c *such that* $2 \leq k \leq 7$, $n \geq 1$ *and* $c \geq 0$, *we have that* $R_k(n, c)$ *is finite if and only if every divisor* $d \leq n$ *of* $k - 1$ *also divides* c. $\square$

The smallest open case of Conjecture 1.2 is now $R_8(6, 1)$, which should be finite according to the conjecture.

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