On the n-Color Weak Rado Numbers for the Equation $x_1 + x_2 + \cdots + x_k + c = x_{k+1}$

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ABSTRACT

For integers k, n, c with $k, n \geq 1$, and $c \geq 0$, the *n-color weak Rado number* $WR_k(n, c)$ is defined as the least integer N, if it exists, such that for every n-coloring of the integer interval $[1, N]$, there exists a monochromatic solution $x_1, \ldots, x_k, x_{k+1}$ in that interval to the equation

$$x_1 + x_2 + \cdots + x_k + c = x_{k+1},$$

with $x_i \neq x_j$, when $i \neq j$. If no such N exists, then $WR_k(n, c)$ is defined as infinite.

In this paper, we determine the exact value of some of these numbers for $n = 2$ and $n = 3$, namely $WR_3(2, c) = 5c + 24$, $WR_4(2, c) = 6c + 52$ for all $c \geq 0$ and $WR_2(3, c) = 13c + 22$ for all $c > 0$. Our method consists in translating the problem into a Boolean satisfiability problem, which can then be handled by a SAT solver or by backtrack programming in the language C.

KEYWORDS

Schur numbers; sum-free sets; weak Schur numbers; weakly sum-free sets; Rado numbers; weak Rado numbers

MATHEMATICS SUBJECT CLASSIFICATION

05C55; 05D10; 05-04; 05A17

1. Introduction

For integers $a \leq b$, we shall denote $[a, b]$ the *integer interval* consisting of all $t \in \mathbb{N}_+ = \{1, 2, \ldots\}$ such that $a \leq t \leq b$. A function

$$\Delta : [1, N] \longrightarrow \{d_1, \ldots, d_n\},$$

where $d_1, \ldots, d_n \in \mathbb{N}_+$ represent different colors, is a n-coloring of the interval $[1, N]$.

Given a n-coloring Δ and the equation $x_1 + \cdots + x_k = x_{k+1}$ in $k + 1$ variables, then we say that a solution $x_1, \ldots, x_k, x_{k+1}$ to the equation is monochromatic if and only if $\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_{k+1})$.

For integers k, n, c with $k, n \geq 1$, and $c \geq 0$, the *n-color weak Rado number* $WR_k(n, c)$ is defined as the least integer N, if it exists, such that for every n-coloring of the integer interval $[1, N]$, there exists a monochromatic solution $x_1, \ldots, x_k, x_{k+1}$ in that interval to the equation

$$x_1 + x_2 + \cdots + x_k + c = x_{k+1},$$

with $x_i \neq x_j$ when $i \neq j$. If no such N exists, then $WR_k(n, c)$ is defined as infinite.

1.1. Schur numbers and weak Schur numbers

A set A of integers is called *sum-free* if it contains no elements $x_1, x_2, x_3 \in A$ satisfying $x_1 + x_2 = x_3$, where x_1, x_2 need not be distinct. It is called *weakly sum-free* if it contains no pairwise distinct elements $x_1, x_2, x_3 \in A$ satisfying $x_1 + x_2 = x_3$.

[Schur 16] proved that, given a positive integer n, there exists a greatest positive integer $S_2(n) = N$ with the property that the integer interval $[1, N - 1]$ can be partitioned into n *sum-free* sets. The numbers $S_2(n)$ are called Schur numbers. The current knowledge on these numbers for $1 \leq n \leq 7$ is given in Table 1.

The exact value of $S_2(4)$ was obtained by [Baumert 61]. The lower and upper bounds on $S_2(5)$ are due to [Exoo 94] and [Sanz 10], respectively. Finally, the lower bounds on $S_2(6)$ and $S_2(7)$ were obtained by [Fredricksen and Sweet 00] by considering symmetric sum-free partitions.

Many generalizations of Schur numbers have appeared since their introduction. We denote by $WS_2(n)$, the greatest integer N, for which the integer interval $[1, N - 1]$ can be partitioned into n weakly sum-free sets $\{A_1, A_2, \ldots, A_n\}$.

The numbers $WS_2(n)$ are called the *weak Schur numbers*. The known weak Schur numbers are given in Table 2.

The current state of knowledge concerning $WS_2(n)$ is quite confused.

The problem seems to have been first considered in [Walker 52], which is Walker's solution to Problem E985 proposed a year earlier, in 1951, by Moser. Walker considered the cases $n = 3, 4$, and 5, and claimed the values $WS_2(3) = 24$, $WS_2(4) = 67$ and $WS_2(5) = 197$. Unfortunately, the short account written by Moser on Walker's

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Table 1. The first few Schur numbers $S_2(n)$.

n	1	2	3	4	5	6	7
$S_2(n)$	2	5	14	45	$161 \le \cdots \le 306$	≥ 537	≥ 1681

solution only gives suitable partitions of $[1, 23]$ for $n = 3$, and no details at all for the cases $n = 4$ and 5. Walker's claimed values of $WS_2(3)$ and $WS_2(4)$ were later confirmed by [Blanchard et al. 06]. The lower bound $WS_2(5) \ge 197$ has been confirmed in [Eliahou et al. 12]. Whether equality holds in still an open problem. A lower bound on $WS_2(6)$ was obtained by [Eliahou et al. 12] and later improved to $WS_2(6) \ge 583$ in [Eliahou 13].

1.2. Rado numbers and weak Rado numbers

In terms of coloring, the Schur number $S_2(n)$ [Schur 16] is the least positive integer N such that for every n-coloring of $[1, N]$,

$$\Delta : [1, N] \longrightarrow \{d_1, \ldots, d_n\},$$

where $d_1, \ldots, d_n$ represent n different colors, there exists a monochromatic solution to the equation $x_1 + x_2 = x_3$, such that $\Delta(x_1) = \Delta(x_2) = \Delta(x_3)$ where x_1 and x_2 need not be distinct.

In 1933, [Rado 33, Rado 36] generalized the work of Schur to arbitrary systems of linear equations. Given a system of linear equations L and a natural number n, the least integer N (if it exists) such that for every coloring of the integer interval $[1, N]$ with n colors there is a monochromatic solution to L, is called the n-color Rado number for L. If no such integer N exists, then the n-color Rado number for the system L is taken to be infinite.

After those first results of Rado, very little progress has been obtained for some systems of linear equations. [Burr and Loo 92] were able to determine the 2-color Rado number for the equations $x_1 + x_2 + c = x_3$ and $x_1 + x_2 = kx_3$ for every integer c and for every positive integer k.

In 1993, [Schaal 93] determined the 2-color Rado number $R_k(2, c)$ for the equation $x_1 + x_2 + \cdots + x_k + c = x_{k+1}$. He also obtained [Schaal 95] the 3-color Rado number $R_2(3, c)$ for the equation $x_1 + x_2 + c = x_3$. There are several results due to Schaal and other authors concerning 2-color and 3-color Rado numbers for particular equations, see [Jones and Schaal 04], [Kosek and Schaal 01], [Rendall and Schaal 06], and other authors [Guo and Sun 08]. In addition, recently we have studied when $R_k(n, c)$ is finite or infinite and we have obtained new exact values [Adhikari 16, Adhikari 17].

Table 2. The first few weak Schur numbers $WS_2(n)$.

n	1	2	3	4	5	6
$WS_2(n)$	3	9	24	67	≥ 197	≥ 583

For every integer $c \ge 0$, $n \ge 1$, let $WR_2(n, c)$ be the least integer N (if it exists) such that, for every coloring of the integer interval $[1, N]$ with n colors, there exists a monochromatic solution to the equation $x_1 + x_2 + c = x_3$, where $x_1 \ne x_2$. The numbers $WR_2(n, c)$ are called the *weak Rado numbers*.

The number $WR_2(n, c)$ can be defined equivalently as the greatest N such that the integer interval $[1, N - 1]$ can be partitioned into n sets $A_1, A_2, \ldots, A_n$ which are free of solutions to the equation $x_1 + x_2 + c = x_3$ with $x_1 \ne x_2$.

Recently, Schaal et al. [Flint 13] have obtained the number $WR_2(2, c)$ for every integer c.

1.3. Contents

In Section 2, we determine the exact value of the 3-color weak Rado number for the equation $x_1 + x_2 + c = x_3$.

Computational Theorem 2.1. *For every $c > 0$, we have $WR_2(3, c) = 13c + 22$.*

In Section 3, we verify the exact values of the 2-color weak Rado numbers for $k = 3, 4$.

Computational Theorem 3.1. *For every $c \ge 0$, we have*

$$WR_3(2, c) = \begin{cases} \infty & \text{if } c \text{ odd,} \\ 5c + 24 & \text{if } c \text{ even.} \end{cases}$$

Computational Theorem 3.2. *For every $c \ge 0$, we have $WR_4(2, c) = 6c + 52$.*

In addition, we prove $WR_5(2, 2) = 109$ and $WR_5(2, 4) = 123$.

These exact values were obtained in two independent ways. One of them, by transforming the problem into a Boolean satisfiability problem and solving it with a SAT solver [Heule 11], and the other one using backtrack programming in the language C [Helsgaun 95].

In Sections 4 and 5, the two computational procedures used in the proofs are shown.

2. Exact value of the weak Rado numbers $WR_2(3, c)$

In this section, we shall prove that $WR_2(3, c) = 13c + 22$ for every positive integer $c > 0$.

2.1. Lower bound

We now prove the lower bound.

Lemma 2.1. *We have $WR_2(3, c) \ge 13c + 22$ for any integer $c > 0$.*

Proof. Let $c > 0$ be a positive integer. We shall prove $WR_2(3, c) \ge 13c + 22$. Let Δ be a 3-coloring:

$$\Delta : [1, 13c + 22] \longrightarrow \{d_1, d_2, d_3\},$$

where d_1, d_2, d_3 represent 3 different colors. Let $A_i = \Delta^{-1}(d_i)$ for $i = 1, 2, 3$ thus $[1, 13c + 22] = A_1 \sqcup A_2 \sqcup A_3$. □

Consider the following partition of the integer interval $[1, 13c + 21]$:

$$\begin{cases} A_1 = [1, c+2] \cup [3c+7, 4c+7] \\ \qquad \cup [9c+17, 10c+17] \cup [12c+21, 13c+21], \\ A_2 = [c+3, 3c+6] \cup [10c+18, 12c+20], \\ A_3 = [4c+8, 9c+16]. \end{cases}$$

Hence $\{A_1, A_2, A_3\}$ is a partition of $[1, 13c + 21]$.

We now prove that for each i, $1 \leq i \leq 3$, if $x_1, x_2 \in A_i$ with $x_1 \neq x_2$ then $x_1 + x_2 + c \notin A_i$. We assume, without any loss of generality, that $x_1 < x_2$.

Case 1: $x_1, x_2 \in A_1$

- If $x_2 \leq c + 2$, then $c + 3 \leq x_1 + x_2 + c \leq 3c + 3$, therefore $x_1 + x_2 + c \notin A_1$.
- If $3c + 7 \leq x_2 \leq 4c + 7$ then $4c + 8 \leq x_1 + x_2 + c \leq 9c + 13$, therefore $x_1 + x_2 + c \notin A_1$.
- If $9c + 17 \leq x_2 \leq 10c + 17$, we have:
  - If $x_1 \leq c + 2$ then $10c + 18 \leq x_1 + x_2 + c \leq 12c + 19$, therefore $x_1 + x_2 + c \notin A_1$.
  - If $3c + 7 \leq x_1$ then $13c + 24 \leq x_1 + x_2 + c$, therefore $x_1 + x_2 + c \notin A_1$.
- If $x_2 \geq 12c + 21$ then $x_1 + x_2 + c \geq 13c + 22$, therefore $x_1 + x_2 + c \notin A_1$.

Case 2: $x_1, x_2 \in A_2$ and $x_1 \geq c + 3$

- If $x_2 \leq 3c + 6$, then $3c + 7 \leq x_1 + x_2 + c \leq 7c + 11$, therefore $x_1 + x_2 + c \notin A_2$.
- If $x_2 \geq 10c + 18$ then $12c + 21 \leq x_1 + x_2 + c$, therefore $x_1 + x_2 + c \notin A_2$.

Case 3: $x_1, x_2 \in A_3$

Since $9c + 17 \leq x_1 + x_2 + c$, then $x_1 + x_2 + c \notin A_3$.

2.2. Upper bound

Let $c > 0$ be a positive integer. We shall prove $WR_2(3, c) \leq 13c + 22$. This upper bound was established in the doctoral thesis [Sanz 10] through an exhaustive analysis of nearly 500 cases. We provide here a sketch of that proof, to this end, we shall prove that for every 3-coloring of the integer interval $[1, 13c + 22]$, there exists a monochromatic solution to the equation $x_1 + x_2 + c = x_3, x_1 \neq x_2$.

Assume, for a contradiction, that there exists a 3-coloring:

$$\Delta : [1, 13c + 22] \longrightarrow \{d_1, d_2, d_3\},$$

where d_1, d_2, d_3 represent three different colors, without any monochromatic solution of the equation $x_1 + x_2 + c = x_3, x_1 \neq x_2$.

Let $A_i = \Delta^{-1}(d_i)$ for $i = 1, 2, 3$ thus $[1, 13c + 22] = A_1 \sqcup A_2 \sqcup A_3$.

We considered five main cases, depending on the colors assigned to the numbers 1, 2 and 3:

Case 1. $A_1 \supseteq \{1, 2, 3\}$.
Case 2. $A_1 \supseteq \{1, 2\}$ and $A_2 \supseteq \{3\}$.
Case 3. $A_1 \supseteq \{1, 3\}$ and $A_2 \supseteq \{2\}$.
Case 4. $A_1 \supseteq \{1\}$ and $A_2 \supseteq \{2, 3\}$.
Case 5. $A_1 \supseteq \{1\}, A_2 \supseteq \{2\}$ and $A_3 \supseteq \{3\}$.

Given a subset $X \subseteq [1, 13c + 22]$, we denote

$$\begin{aligned} f(X) &= (X \dot{+} X + c) \cap [1, 13c + 22] \\ &= (\{x_1 + x_2 + c \mid x_1, x_2 \in X, x_1 \neq x_2\}) \\ &\quad \cap [1, 13c + 22]. \end{aligned}$$

By hypothesis on Δ, for $1 \leq i \leq 3$, we have

$$A_i \cap f(A_i) = \emptyset. \tag{1}$$

The proof rests on the following claims, which are both direct consequences of (1). For every integers i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$, we have:

- **Claim I.** $f(A_i) \cap f(A_j) \subseteq A_k$.
- **Claim II.** $f(A_i) \cap f(A_j) \cap f(A_k) = \emptyset$

We now start our analysis with Case 1 and explore various subcases.

Case 1: $A_1 \supseteq \{1, 2, 3\}$

As $c + 3 = 1 + 2 + c$, without any loss of generality, we may assume that $\Delta(c + 3) = d_2$. In addition, since $c + 4 = 1 + 3 + c$ then $\Delta(c + 4) \neq d_1$, and therefore $\Delta(c + 4) = d_2$ or $\Delta(c + 4) = d_3$.

Case 1.1: $\Delta(c + 4) = d_2$

$A_1 \supseteq \{1, 2, 3\}, A_2 \supseteq \{c + 3, c + 4\}$.

Since {3c+7, c+3, c+4} would be a monochromatic solution in A_2, we must have $\Delta(3c + 7) = d_1$ or $\Delta(3c + 7) = d_3$.

Case 1.1.1: $\Delta(3c + 7) = d_1$

$A_1 \supseteq \{1, 2, 3, 3c + 7\}, A_2 \supseteq \{c + 3, c + 4\}$.

As {4c+10, 3c+7, 3} would be a monochromatic solution in A_1, we must have $\Delta(4c + 10) = d_2$ or $\Delta(4c + 10) = d_3$.

Case 1.1.1a: $\Delta(4c + 10) = d_2$.

Hence $A_1 \supseteq \{1, 2, 3, 3c + 7\}$, $A_2 \supseteq \{c + 3, c + 4, 4c + 10\}$. We now show $2c + 6 \in A_3$. Indeed, we cannot have $2c + 6 \in A_1$, for otherwise we would have $3c + 7 \in A_1 \cap f(\{1, 2c + 6\}) \subseteq A_1 \cap f(A_1)$, a contradiction since $A_1 \cap f(A_1) = \emptyset$ by (1). Similarly, we cannot have $2c + 6 \in A_2$, for otherwise we would have $4c + 10 \in A_2 \cap f(\{c + 4, 2c + 6\}) \subseteq A_2 \cap f(A_2)$, a contradiction again. It follows that $2c + 6 \in A_3$, i.e. $\Delta(2c + 6) = d_3$, as claimed.

The element $2c + 5$ does not belong to A_1, since otherwise $3c + 7 \in A_1 \cap f(\{2, 2c + 5\}) \subseteq A_1 \cap f(A_1) = \emptyset$. Hence $\Delta(2c + 5) = d_2$ or $\Delta(2c + 5) = d_3$.

Case 1.1.1a1: $\Delta(2c + 5) = d_2$

Hence $A_1 \supseteq \{1, 2, 3, 3c + 7\}$, $A_2 \supseteq \{c + 3, c + 4, 4c + 10, 2c + 5\}$ and $A_3 \supseteq \{2c + 6\}$. We now show

$4c+9 \in A_3$. In fact, we cannot have $4c+9 \in A_1$, for otherwise we would have $4c+9 \in A_1 \cap f(\{2, 3c+7\}) \subseteq A_1 \cap f(A_1)$, a contradiction since $A_1 \cap f(A_1) = \emptyset$ by (1). The same way, the element $4c+9$ does not belong to A_2, for otherwise $4c+9 \in A_2 \cap f(\{c+4, 2c+5\}) \subseteq A_2 \cap f(A_2)$, a contradiction again. It follows that $4c+9 \in A_3$, i.e. $\Delta(4c+9) = d_3$, as claimed.

Therefore, $A_1 \supseteq \{1, 2, 3, 3c+7\}$, $A_2 \supseteq \{c+3, c+4, 4c+10, 2c+5\}$, and $A_3 \supseteq \{2c+6, 4c+9\}$. We now show $7c+15 \in A_1$. Indeed, it does not hold that $7c+15 \in A_2$, for otherwise we would have $7c+15 \in A_2 \cap f(\{4c+10, 2c+5\}) \subseteq A_2 \cap f(A_2)$, a contradiction since $A_2 \cap f(A_2) = \emptyset$ by (1). Similarly, we cannot have $7c+15 \in A_3$, for otherwise we would have $7c+15 \in A_3 \cap f(\{2c+6, 4c+9\}) \subseteq A_3 \cap f(A_3)$, a contradiction again. It follows that $7c+15 \in A_1$, i.e. $\Delta(7c+15) = d_1$, as claimed.

Accordingly, $A_1 \supseteq \{1, 2, 3, 3c+7, 7c+15\}$, $A_2 \supseteq \{c+3, c+4, 4c+10, 2c+5\}$, and $A_3 \supseteq \{2c+6, 4c+9\}$. We now show $6c+14 \in A_3$. Certainly, we cannot have $6c+14 \in A_1$, for otherwise we would have $7c+15 \in A_1 \cap f(\{1, 6c+14\}) \subseteq A_1 \cap f(A_1)$, a contradiction since $A_1 \cap f(A_1) = \emptyset$ by (1). Analogously, the element $6c+14$ does not belong to A_2, for otherwise we would have $6c+14 \in A_2 \cap f(\{c+4, 4c+10\}) \subseteq A_2 \cap f(A_2)$, a contradiction again. It follows that $6c+14 \in A_3$, i.e. $\Delta(6c+14) = d_3$, as claimed.

Hence, $A_1 \supseteq \{1, 2, 3, 3c+7, 7c+15\}$, $A_2 \supseteq \{c+3, c+4, 4c+10, 2c+5\}$, and $A_3 \supseteq \{2c+6, 4c+9, 6c+14\}$. We now show $c+5 \notin A_1$, $c+5 \notin A_2$ and $c+5 \notin A_3$. In fact, it does not hold that $c+5 \in A_1$, for otherwise we would have $c+5 \in A_1 \cap f(\{2, 3\}) \subseteq A_1 \cap f(A_1)$, a contradiction since $A_1 \cap f(A_1) = \emptyset$ by (1). We cannot have $c+5 \in A_2$, for otherwise we would have $4c+10 \in A_2 \cap f(\{2c+5, c+5\}) \subseteq A_2 \cap f(A_2)$, a contradiction since $A_2 \cap f(A_2) = \emptyset$ by (1). We cannot have $c+5 \in A_3$, for otherwise we would have $6c+14 \in A_3 \cap f(\{4c+9, c+5\}) \subseteq A_3 \cap f(A_3)$, a contradiction since $A_3 \cap f(A_3) = \emptyset$ by (1).

This subcase is over.

Here is an outline of the proof in **Case 1**:

Case 1.1

$$
\begin{cases}
\Delta(3c+7) = d_1 \begin{cases} \Delta(4c+10) = d_2 \begin{cases} \Delta(2c+5) = d_2 \\ \Delta(2c+5) = d_3 \end{cases} \\ \Delta(4c+10) = d_3 \begin{cases} \Delta(4c+9) = d_2 \\ \Delta(4c+9) = d_3 \end{cases} \end{cases} \\
\Delta(3c+7) = d_3 \begin{cases} \Delta(c+5) = d_2 \begin{cases} \Delta(3c+9) = d_1 \\ \Delta(3c+9) = d_3 \end{cases} \\ \Delta(c+5) = d_3 \begin{cases} \Delta(5c+12) = d_1 \\ \Delta(5c+12) = d_2 \end{cases} \end{cases}
\end{cases}
$$

Case 1.2

$$
\begin{cases}
\Delta(c+5) = d_1 \begin{cases} \Delta(3c+8) = d_1 \begin{cases} \Delta(4c+11) = d_2 \\ \Delta(4c+11) = d_3 \end{cases} \\ \Delta(3c+8) = d_3 \begin{cases} \Delta(5c+12) = d_1 \\ \Delta(5c+12) = d_2 \end{cases} \end{cases} \\
\Delta(c+5) = d_3 \begin{cases} \Delta(3c+9) = d_1 \begin{cases} \Delta(4c+10) = d_2 \\ \Delta(4c+10) = d_3 \end{cases} \\ \Delta(3c+9) = d_2 \begin{cases} \Delta(5c+12) = d_1 \\ \Delta(5c+12) = d_3 \end{cases} \end{cases}
\end{cases}
$$

The other four cases were obtained in a similar way.

We now present two independent computational proofs of the upper bound.

Computational Lemma 2.1. Let $c > 1$ and $\mathcal{X}_c = \{1, 2, 3, c+2, c+3, c+4, 2c+4, 2c+5, 2c+6, 3c+5, 3c+6, 3c+7, 4c+7, 4c+8, 4c+9, 5c+9, 5c+10, 5c+11, 6c+10, 6c+11, 6c+12, 7c+13, 8c+14, 8c+15, 9c+16, 9c+17, 10c+18, 10c+19, 11c+20, 12c+21, 13c+22\}$ then:

(1) We have $\mathcal{X}_c \subseteq [1, 13c+22]$ and $|\mathcal{X}_c| = 31$.

(2) For every partition of $\mathcal{X}_c$ into three subsets A_1, A_2, A_3, some A_i contains a monochromatic solution of $x_1 + x_2 + c = x_3, x_1 \neq x_2$.

Proof.

1. This is trivial.
2. We have checked the result transforming the problem into a Boolean satisfiability problem and solving it with a SAT solver [Heule XX] and using CBack [Helsgaun 95]. □

Computational Lemma 2.2. For $c = 1$, let $\mathcal{X} = \{1, 2, \ldots, 18, 20, 22, 25, 26, 28, 29, 31, 33, 35\}$then:

1. We have $\mathcal{X} \subseteq [1, 35]$.
2. For every partition of $\mathcal{X}$ into three subsets A_1, A_2, A_3, some A_i contains a monochromatic solution of $x_1 + x_2 + c = x_3, x_1 \neq x_2$.

The proof is similar to Lemma 2.1.

In Section 4, the proof of the following result is given in detail.

Computational Theorem 2.1. For every $c > 0$, we have $WR_2(3, c) = 13c + 22$.

3. Exact values of weak Rado numbers $WR_k(2, c)$ for some $k > 2$

In this section, we prove that $WR_3(2, c) = 5c + 24$ if c is even, $WR_4(2, c) = 6c + 52$, $WR_5(2, 2) = 109$, and $WR_5(2, 4) = 123$. In addition, we formulate Corollary 3.1, which relates $WR_k(2, c)$ and a lower bound on the weak Schur number $WS_k(2)$, leading us to formulate Conjecture 3.1.

3.1. The weak Rado numbers $WR_3(2, c)$

For $c = 0$, [Blanchard et al. 06] obtained the weak Schur number $WS_3(2) = WR_3(2, 0) = 24$. A partition which is free of monochromatic solutions to the equation $x_1 + x_2 + x_3 + c = x_4$ is $A_1 = [1, 5] \cup [21, 23]$ and $A_2 = [6, 20]$.

For $c \geq 0$ and odd, $WR_3(2, c) \geq R_3(2, c) = \infty$ [Schaal 93].

Let us first consider the lower bound for any $c \geq 0$ and even.

Lemma 3.1. *We have $WR_3(2, c) \geq 5c + 24$ for any $c \geq 0$ and even.*

Proof. For every even integer $c \geq 0$, it is easy to verify that the 2-coloring

$$\Delta : [1, 5c + 23] \longrightarrow \{d_1, d_2\},$$

where d_1, d_2 represent 2 different colors, defined by

$$\Delta(x) = \begin{cases} d_1 & \text{if } 1 \leq x \leq c + 5, \\ d_2 & \text{if } c + 6 \leq x \leq 4c + 20, \\ d_1 & \text{if } 4c + 21 \leq x \leq 5c + 23 \end{cases}$$

has no monochromatic solutions to the equation $x_1 + x_2 + x_3 + c = x_4$ such that $x_i \neq x_j$ when $i \neq j$. □

Computational Lemma 3.1. We have $WR_3(2, 2) = 34$.

A partition which is free of monochromatic solutions to the equation $x_1 + x_2 + x_3 + c = x_4$ is $A_1 = [1, 7] \cup [29, 33]$ and $A_2 = [8, 28]$.

In order to prove the upper bounds, we shall use the following result:

Computational Lemma 3.2. Let $c \geq 4$ and even. If $l = c/2$, then the set $\mathcal{Y}_l = \{1, 2, 3, 4, 2 + l, 3 + l, 4 + l, 3 + 2l, 4 + 2l, 5 + 2l, 6 + 2l, 7 + 2l, 8 + 2l, 6 + 3l, 7 + 3l, 8 + 3l, 6 + 4l, 9 + 4l, 11 + 5l, 10 + 6l, 12 + 6l, 13 + 7l, 14 + 8l, 15 + 8l, 16 + 8l, 18 + 8l, 21 + 8l, 23 + 10l, 24 + 10l\}$ verifies:

1. We have $\mathcal{Y}_l \subseteq [1, 24 + 10l]$.
2. For every partition of $\mathcal{Y}_l$ into two subsets A_1, A_2, some A_i contains a monochromatic solution of $x_1 + x_2 + x_3 + c = x_4, x_i \neq x_j$, with $i \neq j$.

In the proof of Lemma 3.2, we proceed similarly to Lemma 2.1.

Therefore, we conclude with the following result:

Computational Theorem 3.1. For every $c \geq 0$, we have

$$WR_3(2, c) = \begin{cases} \infty & \text{if } c \text{ odd}, \\ 5c + 24 & \text{if } c \text{ even}. \end{cases}$$

3.2. The weak Rado numbers $WR_4(2, c)$

For $c = 0$, the weak Schur number $WS_4(2) = WR_4(2, 0) = 52$ was obtained [Sanz 10]. A partition which is free of monochromatic solutions to the equation $x_1 + x_2 + x_3 + x_4 + c = x_5$ is $A_1 = [1, 9] \cup [46, 51]$ and $A_2 = [10, 45]$.

We now consider the lower bound for any $c \geq 0$.

Lemma 3.2. *We have $WR_4(2, c) \geq 6c + 52$ for any $c \geq 0$.*

Proof. For every integer $c \geq 0$, it is easy to verify that the 2-coloring

$$\Delta : [1, 6c + 51] \longrightarrow \{d_1, d_2\}$$

defined by

$$\Delta(x) = \begin{cases} d_1 & \text{if } 1 \leq x \leq c + 9, \\ d_2 & \text{if } c + 10 \leq x \leq 5c + 45, \\ d_1 & \text{if } 5c + 46 \leq x \leq 6c + 51 \end{cases}$$

has no monochromatic solutions to the equation $x_1 + x_2 + x_3 + x_4 + c = x_5$ such that $x_i \neq x_j$ when $i \neq j$. □

In order to prove the opposite inequality, we shall use the following result:

Computational Lemma 3.3. Let $c \geq 1$. The set $\mathcal{Z}_c = \{1, 2, 3, 4, 5, 6, 8, c + 9, c + 10, c + 11, c + 12, c + 13, c + 14, c + 15, 2c + 16, 2c + 17, 2c + 18, 2c + 19, 2c + 20, 2c + 21, 2c + 22, 2c + 24, 3c + 32, 4c + 33, 5c + 43, 5c + 44, 5c + 45, 5c + 46, 6c + 52\}$ verifies:

1. $\mathcal{Z}_c \subseteq [1, 6c + 52]$.
2. For every partition of $\mathcal{Z}_c$ into two subsets A_1, A_2, some A_i contains a monochromatic solution of $x_1 + x_2 + x_3 + x_4 + c = x_5, x_i \neq x_j$, with $i \neq j$.

In the proof of Lemma 3.3, we use similar reasonings to those established in Lemma 2.1 and Lemma 3.2.

Therefore, we conclude with the following result:

Computational Theorem 3.2. For every $c \geq 0$, we have $WR_4(2, c) = 6c + 52$.

3.3. The weak Rado numbers $WR_5(2, 2)$ and $WR_5(2, 4)$

The weak Rado numbers $WR_5(2, 2)$ and $WR_5(2, 4)$ have been obtained through backtrack programming [Helsgaun 95] and by transforming the problem into a Boolean satisfiability problem and solving it with a SAT solver. In the Sections 4.4, 5.4, and 5.5, the results are shown.

In the case of $WR_5(2, 2)$, backtrack programming shows three partitions, which are free of monochromatic solutions to the equation $x_1 + x_2 + x_3 + x_4 + 2 = x_5$,

$x_i \neq x_j$, with $i \neq j$. These are

$$A_1 = [1, 16] \cup [97, 108], \ A_2 = [17, 96].$$
$$A_1 = [1, 20] \cup [94, 108], \ A_2 = \{2\} \cup [21, 93].$$
$$A_1 = \{1\} \cup [21, 92], \ A_2 = [2, 20] \cup [93, 108].$$

Therefore, we have that 109 is the lower bound of $WR_5(2, 2)$. In Section 4.4, we show that $WR_5(2, 2) \leq 109$.

In the case of $WR_5(2, 4)$, a partition free of monochromatic solutions to the equation $x_1 + x_2 + x_3 + x_4 + 4 = x_5$, $x_i \neq x_j$, with $i \neq j$, is:

$$A_1 = [1, 18] \cup [109, 122], \ A_2 = [19, 108].$$

Therefore, we have that 123 is the lower bound of $WR_5(2, 4)$. In Section 5.4, we show that $WR_5(2, 4) \leq 123$.

Hence, we conclude with the following results:

Computational Theorem 3.3. We have $WR_5(2, 2) = 109$.

Computational Theorem 3.4. We have$WR_5(2, 4) = 123$.

3.4. Weak Schur numbers $WS_5(2)$ and lower bounds

In this subsection, we obtain the weak Schur number $WS_5(2) = 101$ and we show a lower bound for the weak Schur numbers $WS_k(2)$.

To obtain the lower bound $WS_5(2) \geq 101$, the following partition of $[1, 100]$ is considered $A_1 = \{1\} \cup [20, 86]$ and $A_2 = [2, 19] \cup [87, 100]$.

In order to obtain the upper bound $WS_5(2) \leq 101$, we shall use the following result:

Computational Lemma 3.4. The set $\mathcal{U} = [1, 7] \cup \{9, 11, 13\} \cup [15, 17] \cup [19, 23] \cup [25, 27] \cup \{29, 31, 35, 39\} \cup [43, 45] \cup \{51, 75, 87, 101\} \subseteq [1, 101]$ verifies that for every partition of $\mathcal{U}$ into two subsets A_1, A_2, some A_i contains a monochromatic solution of $x_1 + x_2 + x_3 + x_4 + x_5 = x_6$, $x_i \neq x_j$, with $i \neq j$.

In the proof of Lemma 3.4, we use similar reasonings to those established in Lemma 2.1 and Lemma 3.2.

Therefore, we conclude with the following result:

Computational Theorem 3.5. We have $WS_5(2) = 101$.

Here, below can be seeing a new lower bound for the weak Schur numbers $WS_k(2)$.

Lemma 3.3. *We have* $WS_k(2) \geq (k+2)T_k - 2k$, *with* $T_k = \frac{(1+k)}{2}k$.

Proof. It is easy to verify that the 2-coloring

$$\Delta : [1, (k+2)T_k - 2k - 1] \longrightarrow \{d_1, d_2\}$$

defined by

$$\Delta(x) = \begin{cases} d_1 & \text{if } 1 \leq x \leq T_k - 1, \\ d_2 & \text{if } T_k \leq x \leq (k+1)T_k - k - 1, \\ d_1 & \text{if } (k+1)T_k - k \leq x \leq (k+2)T_k - 2k - 1 \end{cases}$$

has no monochromatic solution to the equation $x_1 + x_2 + \cdots + x_k = x_{k+1}$ such that $x_i \neq x_j$ when $i \neq j$. □

Consider the lower bound $LWS_k(2) = (k+2)T_k - 2k$. We formulate the following Corollaries that relate the weak Rado numbers $WR_k(2, c)$ with the lower bound $LWS_k(2)$.

Corollary 3.1. *Let c be a integer with $c \geq 0$ and $k = 2, 3, 4$. Then, $WR_k(2, c) = (k+2)c + LWS_k(2)$.*

Corollary 3.2. *Let $c = 2$ or $c = 4$ and $k = 5$. Then, $WR_k(2, c) = (k+2)c + LWS_k(2)$.*

The exact values $WR_2(2, c)$, $WR_3(2, c)$, $WR_4(2, c)$, $WR_5(2, 2)$, and $WR_5(2, 4)$ have been obtained. All of them verify the following Conjecture 3.1.

Conjecture 3.1. Let c and k be integers with $c \geq 0$ and $k \geq 2$, we have $WR_k(2, c) = (k+2)c + LWS_k(2)$, when c or k is even.

4. Reformulation as a SAT problem

Our idea for constructing the above partitions is to express the corresponding combinatorial constraints as Boolean satisfiability problems, to be then fed into a SAT solver. See [Dransfield et al. 04, Eliahou et al. 12, Herwig et al. 07, Kouril and Paul 08, Robilliard et al. 10] for earlier successful uses of SAT solvers in combinatorial number theory. The specific SAT solver used here, is the March rw, the gold medal winner of the 2011 International SAT Competition [Heule XX]. Recall that a logical expression over Boolean variables $x_1, \ldots, x_n$ is said to be *satisfiable* if there is an assignment of the x_i's to True or False in such a way that the value evaluates to True.

4.1. Seeking $WR_2(3, c)$ by computer

Let $c > 1$ and consider the set $\mathcal{X}_c$ of Lemma 2.1. That is, $\mathcal{X}_c = \{1, 2, 3, c+2, c+3, c+4, 2c+4, 2c+5, 2c+6, 3c+5, 3c+6, 3c+7, 4c+7, 4c+8, 4c+9, 5c+9, 5c+10, 5c+11, 6c+10, 6c+11, 6c+12, 7c+13, 8c+14, 8c+15, 9c+16, 9c+17, 10c+18, 10c+19, 11c+20, 12c+21, 13c+22\}$.

Let Δ_c be a 3-coloring of $[1, 13c+22]$:

$$\Delta_c : [1, 13c+22] \longrightarrow \{d_1, d_2, d_3\},$$

and let $X^* = \{(a, b) : ac + b \in \mathcal{X}_c \text{ for any } c \geq 1\}$, i.e. $X^* = \{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (1, 4), (2, 4),$

$(2,5)$, $(2,6)$, $(3,5)$, $(3,6)$, $(3,7)$, $(4,7)$, $(4,8)$, $(4,9)$, $(5,9)$, $(5,10)$, $(5,11)$, $(6,10)$, $(6,11)$, $(6,12)$, $(7,13)$, $(8,14)$, $(8,15)$, $(9,16)$, $(9,17)$, $(10,18)$, $(10,19)$, $(11,20)$, $(12,21)$, $(13,22)\}$.

For any $(a,b) \in X^*$, we consider two Boolean variables $\phi((a,b))$ and $\psi((a,b))$ defined as follow:

$$\phi((a,b)) = \begin{cases} True & \text{if } \Delta_c(ac+b) = d_1 \text{ or } d_2, \\ False & \text{if } \Delta_c(ac+b) = d_3. \end{cases}$$

$$\psi((a,b)) = \begin{cases} True & \text{if } \Delta_c(ac+b) = d_1 \text{ or } d_3, \\ False & \text{if } \Delta_c(ac+b) = d_2. \end{cases}$$

Thus, for any $n \in X^*$ we have that $\phi(n)$ is True or $\psi(n)$ is True.

Let $\mathcal{S} = \{(n_1,n_2,n_3) \mid n_i = (a_i,b_i) \in X^*,$ verifying that $a_1+b_1 < a_2+b_2,\ a_1+a_2+1 = a_3,\ b_1+b_2 = b_3\}$.

For any $s = (n_1,n_2,n_3) \in \mathcal{S}$, we consider three clauses:

$$\begin{aligned} p(s) &= (\neg\phi(n_1) \vee \neg\psi(n_1) \vee \neg\phi(n_2) \vee \neg\psi(n_2) \\ &\quad \vee\neg\phi(n_3) \vee \neg\psi(n_3)), \\ q(s) &= (\neg\phi(n_1) \vee \psi(n_1) \vee \neg\phi(n_2) \vee \psi(n_2) \\ &\quad \vee\neg\phi(n_3) \vee \psi(n_3)), \text{ and} \\ r(s) &= (\phi(n_1) \vee \neg\psi(n_1) \vee \phi(n_2) \vee \neg\psi(n_2) \\ &\quad \vee\phi(n_3) \vee \neg\psi(n_3)). \end{aligned}$$

Then, $p(s)$ is satisfiable if and only if $\Delta_c(n) \neq d_1$ for some $n \in s$, $q(s)$ is satisfiable if and only if $\Delta_c(n) \neq d_2$ for some $n \in s$ and $r(s)$ is satisfiable if and only if $\Delta_c(n) \neq d_3$ for some $n \in s$, thus $p(s) \wedge q(s) \wedge r(s)$ are satisfiable if and only if Δ_c does not induce on s a monochromatic solution of the equation $x_1 + x_2 + c = x_3$.

$$\text{Let } \mathcal{C} = \bigwedge_{s\in\mathcal{S}} (p(s) \wedge q(s) \wedge r(s)) \text{ and}$$
$$\mathcal{D} = \bigwedge_{n\in X^*} (\phi(n) \vee \psi(n)).$$

Clearly, $\mathcal{C} \wedge \mathcal{D}$ is satisfiable if and only if the restriction of Δ_c to $\mathcal{X}_c$ is a 3-coloring without monochromatic solution of the equation.

The SAT-Solver shows that $\mathcal{C} \wedge \mathcal{D}$ is not satisfiable, therefore there does not exist a 3-coloring of the sets $\mathcal{X}_c$ and $[1, 13c+22]$ without monochromatic solution. Thus, $WR_2(3,c) \leq 13c+22$.

4.2. *Seeking $WR_3(2,c)$ by computer*

Let $c = 2l \geq 4$ and the set $\mathcal{Y}_l$ of Lemma 3.2. That is, $\mathcal{Y}_l = \{1, 2, 3, 4, 2+l, 3+l, 4+l, 3+2l, 4+2l, 5+2l, 6+2l, 7+2l, 8+2l, 6+3l, 7+3l, 8+3l, 6+4l, 9+4l, 11+5l, 10+6l, 12+6l, 13+7l, 14+8l, 15+8l, 16+8l, 18+8l, 21+8l, 23+10l, 24+10l\}$.

Let Δ_l be a 2-coloring of $[1, 24+10l]$,

$$\Delta_l : [1, 24+10l] \longrightarrow \{d_1, d_2\},$$

and let $Y^* = \{(a,b) : al+b \in \mathcal{Y}_l \text{ for any } l \geq 2\}$, i.e. $Y^* = \{(0,1)$, $(0,2)$, $(0,3)$, $(0,4)$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,3)$, $(2,4)$, $(2,5)$, $(2,6)$, $(2,7)$, $(2,8)$, $(3,6)$, $(3,7)$, $(3,8)$, $(4,6)$, $(4,9)$, $(5,11)$, $(6,10)$, $(6,12)$, $(7,13)$, $(8,14)$, $(8,15)$, $(8,16)$, $(8,18)$, $(8,21)$, $(10,23)$, $(10,24)\}$.

For any $(a,b) \in Y^*$, we consider a Boolean variable $\phi((a,b))$ defined as follows:

$$\phi((a,b)) = \begin{cases} True & \text{if } \Delta_l(2al+b) = d_1, \\ False & \text{if } \Delta_l(2al+b) = d_2. \end{cases}$$

Let $\mathcal{S}' = \{(n_1,\ldots,n_4) \mid n_i = (a_i,b_i) \in Y^*,$ verifying that $4a_1+b_1 < 4a_2+b_2 < 4a_3+b3,\ a_1+a_2+a_3+2 = a_4,\ b_1+b_2+b_3 = b_4\}$ For any $s = (n_1,\ldots,n_4) \in \mathcal{S}'$, we consider two clauses:

$$p(s) = (\phi(n_1) \vee \phi(n_2) \vee \phi(n_3) \vee \phi(n_4))$$

and

$$q(s) = (\neg\phi(n_1) \vee \neg\phi(n_2) \vee \neg\phi(n_3) \vee \neg\phi(n_4)).$$

Then $p(s)$ is satisfiable if and only if $\Delta_l(n) \neq d_2$ for some $n \in s$ and $q(s)$ is satisfiable if and only if $\Delta_l(n) \neq d_1$ for some $n \in s$, thus $p(s) \wedge q(s)$ are satisfiable if and only if Δ_l does not induce on s a monochromatic solution of the equation $x_1 + x_2 + x_3 + x_4 + c = x_5$.

$$\text{Let } \mathcal{C}' = \bigwedge_{s\in\mathcal{S}'} (p(s) \wedge q(s)).$$

Clearly $\mathcal{C}'$ is satisfiable if and only if the restriction of Δ_l to $\mathcal{Y}_l$ is a 2-coloring without monochromatic solution of the equation.

The SAT-Solver shows that $\mathcal{C}'$ is not satisfiable, therefore there does not exist a 2-coloring of the sets $\mathcal{Y}_l$ and $[1, 24+10l]$ without monochromatic solution. Thus $WR_3(2, 2l) \leq 24+10l$.

4.3. *Seeking $WR_4(2,c)$ by computer*

Let $\mathcal{Z}_c$ be the set of Lemma 3.3. That is, $\mathcal{Z}_c = \{1, 2, 3, 4, 5, 6, 8, c+9, c+10, c+11, c+12, c+13, c+14, c+15, 2c+16, 2c+17, 2c+18, 2c+19, 2c+20, 2c+21, 2c+22, 2c+24, 3c+32, 4c+33, 5c+43, 5c+44, 5c+45, 5c+46, 6c+52\}$ Let Δ_c be a 2-coloring of $[1, 6c+52]$,

$$\Delta_c : [1, 6c+52] \longrightarrow \{d_1, d_2\}$$

and let $Z^* = \{(a, b) : ac + b \in \mathcal{Z}_c \text{ for any } c \geq 0\}$, i.e. $Z^* = \{(0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (0, 8), (1, 9), (1, 10), (1, 11), (1, 12), (1, 13), (1, 14), (1, 15), (2, 16), (2, 17), (2, 18), (2, 19), (2, 20), (2, 21), (2, 22), (2, 24), (3, 32), (4, 33), (5, 43), (5, 44), (5, 45), (5, 46), (6, 52)\}$. For any $(a, b) \in Z^*$ we consider a Boolean variable $\phi((a, b))$ defined as follow:

$$\phi((a, b)) = \begin{cases} \mathit{True} & \text{if } \Delta_c(ac + b) = d_1, \\ \mathit{False} & \text{if } \Delta_c(ac + b) = d_2. \end{cases}$$

Let $\mathcal{S}'' = \{(n_1, \ldots, n_5) \mid n_i = (a_i, b_i) \in Z^*,$ verifying that $b_1 < b_2 < b_3 < b_4,\ a_1 + a_2 + a_3 + a_4 + 1 = a_5,\ b_1 + b_2 + b_3 + b_4 = b_5\}$.

For any $s = (n_1, \ldots, n_5) \in \mathcal{S}''$, we consider two clauses:

$$p(s) = (\phi(n_1) \vee \phi(n_2) \vee \phi(n_3) \vee \phi(n_4) \vee \phi(n_5))$$

and

$$q(s) = (\neg\phi(n_1) \vee \neg\phi(n_2) \vee \neg\phi(n_3) \vee \neg\phi(n_4) \vee \neg\phi(n_5)).$$

Then, $p(s)$ is satisfiable if and only if $\Delta_c(n) \neq d_2$ for some $n \in s$ and $q(s)$ is satisfiable if and only if $\Delta_c(n) \neq d_1$ for some $n \in s$, thus $p(s) \wedge q(s)$ are satisfiable if and only if Δ_c does not induce on s a monochromatic solution of the equation $x_1 + x_2 + x_3 + x_4 + c = x_5$.

$$\text{Let } \mathcal{C}'' = \bigwedge_{s \in \mathcal{S}''} (p(s) \wedge q(s)).$$

Clearly, $\mathcal{C}''$ is satisfiable if and only if the restriction of Δ_c to $\mathcal{Z}_c$ is a 2-coloring without monochromatic solution of the equation.

The SAT-Solver shows that $\mathcal{C}''$ is not satisfiable, therefore there does not exist a 2-coloring of the sets $\mathcal{Z}_c$ and $[1, 6c + 52]$ without monochromatic solution. Thus $WR_4(2, c) \leq 6c + 52$.

4.4. Seeking $WR_5(2, 2)$ by computer

Let $\mathcal{T}_2 = [1, 109]$. Let Δ be a 2-coloring of $\mathcal{T}_2$. For any $n \in \mathcal{T}_2$, we consider a Boolean variable $\phi(n)$ defined as follow:

$$\phi(n) = \begin{cases} \mathit{True} & \text{if } \Delta(n) = d_1, \\ \mathit{False} & \text{if } \Delta(n) = d_2. \end{cases}$$

Let $\mathcal{S}''' = \{(n_1, \ldots, n_6) \mid 1 \leq n_1 < n_2 < \cdots < n_6 \leq 109, \text{ and } n_1 + n_2 + n_3 + n_4 + n_5 + 2 = n_6\}$.

For any $s = (n_1, \ldots, n_6) \in \mathcal{S}'''$, we consider two clauses:

$$p(s) = (\phi(n_1) \vee \phi(n_2) \vee \phi(n_3) \vee \phi(n_4) \vee \phi(n_5) \vee \phi(n_6))$$

and

$$q(s) = (\neg\phi(n_1) \vee \neg\phi(n_2) \vee \neg\phi(n_3) \vee \neg\phi(n_4) \vee \neg\phi(n_5) \vee \neg\phi(n_6)).$$

Then, $p(s)$ is satisfiable if and only if $\Delta(n) \neq d_2$ for some $n \in s$ and $q(s)$ is satisfiable if and only if $\Delta(n) \neq d_1$ for some $n \in s$, thus $p(s) \wedge q(s)$ are satisfiable if and only if Δ does not induce on s a monochromatic solution of the equation $x_1 + x_2 + x_3 + x_4 + x_5 + 2 = x_6$.

$$\text{Let } \mathcal{C}''' = \bigwedge_{s \in \mathcal{S}'''} (p(s) \wedge q(s)).$$

Clearly, $\mathcal{C}'''$ is satisfiable if and only if Δ is a 2-coloring without monochromatic solution of the equation.

The SAT-Solver shows that $\mathcal{C}'''$ is not satisfiable, therefore there does not exist a 2-coloring of the set $\mathcal{T}_2$ without monochromatic solution. Thus, $WR_5(2, 2) \leq 109$.

This result can be generalized to prove $WR_5(2,4) \leq 123$.

5. Backtrack programming in language C

5.1. Seeking $WR_2(3, c)$ by computer

```
#include "CBack.c"
int i, j, k, l, N, Count, Solu;
FILE *fp;
void PrintSol()
{ fprintf(fp,"N = %d  is the maximum with %d solutions. \n",Count,Solu); }

int Problem()
{
int r, t, c, a, rr, tt;
int R[4][600]={0};
```

```
int T[4][600]={0};
int L[4]={0};
int VR[31]={0,0,0,1,1,1,2,2,2,3,3,3,4,4,4,5, 5, 5, 6, 6, 6, 7, 8, 8, 9, 9,10,
10,11,12,13};
int VT[31]={1,2,3,2,3,4,4,5,6,5,6,7,7,8,9,9,10,11,10,11,12,13,14,15,16,17,18,
19,20,21,22};
Solu=0;

Fiasco=PrintSol;
N=Select(30,31);
for (r = 0; r <= N-1; r++)
{
  c = Choice(3);
  for (i = 0; i <= L[c]-1; i++)
   for (j = 0; j < i; j++)
     {
     if (VR[r]==R[c][i]+R[c][j]+1&&VT[r]==T[c][i]+T[c][j])
     Backtrack();
     }
     R[c][L[c]] = VR[r];
     T[c][L[c]] = VT[r];
     L[c]++;
}
Count=0;
Solu++;
for (c = 1; c <= 3; c++)
{
 Count+=L[c];
 for (r = 0; r <= L[c]-1; r++)
    {
    if (R[c][r]==0) fprintf(fp,"(%d)",T[c][r]);
    else
     if (R[c][r]==1) fprintf(fp,"(%c%c%d)",'a','+',T[c][r]);
     else fprintf(fp,"(%d%c%c%d)",R[c][r],'a','+',T[c][r]);
    }
/*    fprintf(fp,"((%d))\n",L[c]); */
    fprintf(fp,"\n");
}

    fprintf(fp,"%c",'\n');
    printf(" Solutions : %d \n",Solu);
    Backtrack();
}

main(int argc, char *argv[])
{
char str[80];
strcpy (str,argv[0]);
strcat (str,".txt");
fp = fopen(str,"w");
Backtracking(Problem())
fclose(fp);
}
```

5.2. Seeking $WR_3(2, c)$ by computer

```
#include "CBack.c"
int i, j, k, l, N, Count, Solu;
FILE *fp;
void PrintSol()

{ fprintf(fp,"N = %d  is the maximum with %d solutions. \n",Count,Solu); }

int Problem()
{
int r, t, c, a, rr, tt;
int R[4][600]={0};
int T[4][600]={0};
int L[4]={0};
int VR[29]={0,0,0,0,1,1,1,2,2,2,2,2,2,3,3,3,4,4, 5, 6, 6, 7, 8, 8, 8, 8, 8,
          10,10};
int VT[29]={1,2,3,4,2,3,4,3,4,5,6,7,8,6,7,8,6,9,11,10,12,13,14,15,16,18,21,23,
          24};

Solu=0;

Fiasco=PrintSol;
N=Select(28,29);
for (r = 0; r <= N-1; r++)
{
  c = Choice(2);
  for (i = 0; i <= L[c]-1; i++)
   for (j = 0; j < i; j++)
    for (k = 0; k < j; k++)
     {
     if (VR[r]==R[c][i]+R[c][j]+R[c][k]+2&&VT[r]==T[c][i]+T[c][j]+T[c][k])
     Backtrack();
     }
     R[c][L[c]] = VR[r];
     T[c][L[c]] = VT[r];
     L[c]++;
}
Count=0;
Solu++;
for (c = 1; c <=2; c++)
{
 Count+=L[c];
 for (r = 0; r <= L[c]-1; r++)
    {
    if (R[c][r]==0) fprintf(fp,"(%d)",T[c][r]);
    else
     if (R[c][r]==1) fprintf(fp,"(%c%c%d)",'a','+',T[c][r]);
     else fprintf(fp,"(%d%c%c%d)",R[c][r],'a','+',T[c][r]);
    }
/*    fprintf(fp,"((%d))\n",L[c]); */
    fprintf(fp,"\n");
```

```
}
    fprintf(fp,"%c",'\n');
    printf(" Solutions : %d \n",Solu);
    Backtrack();
}

main(int argc, char *argv[])
{
char str[80];
strcpy (str,argv[0]);
strcat (str,".txt");
fp = fopen(str,"w");
Backtracking(Problem())
fclose(fp);
}
```

5.3. Seeking $WR_4(2, c)$ by computer

```
#include "CBack.c"
int i, j, k, l, N, Count, Solu;
FILE *fp;
void PrintSol()
{ fprintf(fp,"N = %d  is the maximum with %d solutions. \n",Count,Solu); }

int Problem()
{
int r, t, c, a, rr, tt;
int R[4][600]={0};
int T[4][600]={0};
int L[4]={0};
//int VR[30]={0,0,0,0,0,0,0,1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3,
            4, 5, 5, 5, 5, 6};
//int VT[30]={1,2,3,4,5,6,7,9,10,11,12,13,14,15,15,16,17,18,19,20,21,22,24,32,
            33,43,44,45,46,52};

int VR[29]={0,0,0,0,0,0,0,1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3,
             4, 5, 5, 5, 5, 6};
int VT[29]={1,2,3,4,5,6,7,9,10,11,12,13,14,15,15,16,17,18,19,20,21,22,24,32,
             33,43,44,45,46,52};

Solu=0;

Fiasco=PrintSol;
N=Select(30,30);
for (r = 0; r <= N-1; r++)
{
  c = Choice(2);
  for (i = 0; i <= L[c]-1; i++)
   for (j = 0; j < i; j++)
    for (k = 0; k < j; k++)
     for (l = 0; l < k; l++)
```

```
      {
       if (VR[r]==R[c][i]+R[c][j]+R[c][k]+R[c][l]+1&&VT[r]==
       T[c][i]+T[c][j]+T[c][k]+T[c][l])
       Backtrack();
      }
      R[c][L[c]] = VR[r];
      T[c][L[c]] = VT[r];
      L[c]++;
}
Count=0;
Solu++;
for (c = 1; c <=2; c++)
{
 Count+=L[c];
 for (r = 0; r <= L[c]-1; r++)
    {
    if (R[c][r]==0) fprintf(fp,"(%d)",T[c][r]);
    else
     if (R[c][r]==1) fprintf(fp,"(%c%c%d)",'a','+',T[c][r]);
     else fprintf(fp,"(%d%c%c%d)",R[c][r],'a','+',T[c][r]);
    }
/*    fprintf(fp,"((%d))\n",L[c]); */
    fprintf(fp,"\n");
}
    fprintf(fp,"%c",'\n');
    printf(" Solutions : %d \n",Solu);
//    if (Solu<10)
    Backtrack();
}

main(int argc, char *argv[])
{
char str[80];
strcpy (str,argv[0]);
strcat (str,".txt");
fp = fopen(str,"w");
Backtracking(Problem())
fclose(fp);
}
```

5.4. Seeking $WR_5(2, 2)$ by computer

```
#include "CBack.c"
int i, j, k, l, m, N, Count, Solu;
FILE *fp;
void PrintSol()
{ fprintf(fp,"N = %d  is the maximum with %d solutions. \n",Count,Solu); }


int Problem()
{
```

```
int r, c, a;
int R[4][600]={0};
int L[4]={0};
Solu=0;
Fiasco=PrintSol;
N=Select(108,109);
a=2;
for (r = 1; r <= N; r++)
{
  c = Choice(2);
  for (i = 0; i <= L[c]-1; i++)
   for (j = 0; j < i; j++)
    for (k = 0; k < j; k++)
     for (l = 0; l < k; l++)
      for (m = 0; m < l; m++)
      {
      if (r-a==R[c][i]+R[c][j]+R[c][k]+R[c][l]+R[c][m])
      Backtrack();
      }
      R[c][L[c]] = r;
      L[c]++;
}
Count=0;
Solu++;
for (c = 1; c <= 2; c++)
{
 Count+=L[c];
 for (r = 0; r <= L[c]; r++)
    {
    fprintf(fp,"(%d)",R[c][r]);
    }
    fprintf(fp,"((%d))\n",L[c]);
}
    fprintf(fp,"%c",'\n');
//    if (Solu < 2)
    Backtrack();
}

main(int argc, char *argv[])
{
char str[80];
strcpy (str,argv[0]);
strcat (str,".txt");
fp = fopen(str,"w");
Backtracking(Problem())
fclose(fp);
}
```

5.5. Seeking $WR_5(2, 4)$ by computer

```
#include "CBack.c"
int i, j, k, l, m, N, Count, Solu;
```

```
FILE *fp;
void PrintSol()

{ fprintf(fp,"N = %d  is the maximum with %d solutions. \n",Count,Solu); }

int Problem()
{
int r, t, c, a, rr, tt;
int R[4][600]={0};
int L[4]={0};
int VR[123]={
int VR[123]={1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,
            25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,
            47,48,49,50,51,52,53,54,55,56,57,58,59,60,61,62,63,64,65,66,67,68,
            69,70,71,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,87,88,89,90,
            91,92,93,94,95,96,97,98,99,100,101,102,103,104,105,106,107,108,109,
            110,111,112,113,114,115,116,117,118,119,120,121,122,123};

Solu=0;

Fiasco=PrintSol;
N=Select(122,123);
a=4;
for (r = 0; r <= N-1; r++)
{
  c = Choice(2);
  for (i = 0; i <= L[c]-1; i++)
   for (j = 0; j < i; j++)
    for (k = 0; k < j; k++)
     for (l = 0; l < k; l++)
      for (m = 0; m < l; m++)
      {
      if (VR[r]-a==R[c][i]+R[c][j]+R[c][k]+R[c][l]+R[c][m])
      Backtrack();
      }
      R[c][L[c]] = VR[r];
      L[c]++;
}
Count=0;
Solu++;
for (c = 1; c <=2; c++)
{
 Count+=L[c];
 for (r = 0; r <= L[c]; r++)
    {
    fprintf(fp,"(%d)",R[c][r]);
    }
    fprintf(fp,"((%d))\n",L[c]);
}
    fprintf(fp,"%c",'\n');
//    if (Solu < 2)
    Backtrack();
}
```

```
main(int argc, char *argv[])
{
char str[80];
strcpy (str,argv[0]);
strcat (str,".txt");
fp = fopen(str,"w");
Backtracking(Problem())
fclose(fp);
}
```

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