Numerical semigroups of Szemerédi type

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A B S T R A C T

Given any length $k \geq 3$ and density $0 < \delta \leq 1$, we introduce and study the set $\mathrm{Sz}(k, \delta)$ consisting of all positive integers n such that every subset of $\{1, 2, \dots, n\}$ of density at least δ contains an arithmetic progression of length k. A famous theorem of Szemerédi guarantees that this set is not empty. We show that $\mathrm{Sz}(k, \delta) \cup \{0\}$ is a numerical semigroup and we determine it for $(k, \delta) = (4, 1/2)$ and for more than thirty pairs $(3, \delta)$ with $\delta > 1/5$.

1. Introduction

Denote $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$. Given integers $a \leq b$, we denote by $[a, b] = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$ the *integer interval* they span, and $[a, \infty[\ = \{z \in \mathbb{Z} \mid z \geq a\}$.

A famous theorem of Szemerédi states that any subset of $\mathbb{N}_+$ of positive upper density contains arbitrary long arithmetic progressions. Informally, an equivalent finitary version states that given any length $k \geq 3$, any sufficiently dense subset of a sufficiently large integer interval contains an arithmetic progression of length k.

Our purpose in this paper is to introduce and study a closely related set $\mathrm{Sz}(k, \delta)$, parametrized by a desired length $k \geq 3$ and density $0 < \delta \leq 1$. That set, introduced in Section 2, consists of all positive integers n satisfying Szemerédi's theorem relative to k and δ. In Section 3, we prove that $\mathrm{Sz}(k, \delta) \cup \{0\}$ is a numerical semigroup. Section 4 displays value tables of closely related functions $r_3(n), r_4(n)$. In the last Section 5, we completely determine $\mathrm{Sz}(k, \delta)$ for $(k, \delta) = (4, 1/2)$ and for more than thirty pairs $(3, \delta)$ with $\delta > 1/5$. Interestingly, in a majority of these examples, it occurs that $\mathrm{Sz}(k, \delta)$ contains some integer n but not $n + 1$. Said otherwise, the *conductor* of the numerical semigroup $\mathrm{Sz}(k, \delta) \cup \{0\}$ does not necessarily coincide with its *multiplicity*. The simplest occurrence of this phenomenon is the case $\mathrm{Sz}(3, 1/3)$, which contains 50 but not 51; more precisely, the corresponding multiplicity and conductor equal 49 and 55, respectively.

2. The set $\mathrm{Sz}(k, \delta)$

Definition 2.1. Given any length $k \geq 3$ and density $0 < \delta \leq 1$, let $\mathrm{Sz}(k, \delta)$ denote the set consisting of all $n \in \mathbb{N}_+$ satisfying the following property: *every subset $X \subseteq [1, n]$ of density $|X|/n \geq \delta$ contains an arithmetic progression of length k.*

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That the set $\mathrm{Sz}(k, \delta)$ is not empty follows from a famous theorem of Szemerédi, stated here in its finitary version [9].

Theorem 2.2 (*Szemerédi*)**.** *Given any integer $k \geq 3$ and real number $0 < \delta \leq 1$, there exists $n(k, \delta) \in \mathbb{N}_+$ such that for any integer $n \geq n(k, \delta)$, every subset $A \subseteq [1, n]$ of density $|A|/n \geq \delta$ contains an arithmetic progression of length k.*

We shall denote $M(k, \delta) = \min \mathrm{Sz}(k, \delta)$. Observe that $M(k, \delta)$ owes its existence to Szemerédi's theorem above as already noted, and that[2] if $\delta \leq (k-1)/k$, then

$$M(k, \delta) \geq k + 1$$

since no proper subset of $[1, k]$ contains an arithmetic progression of length k. As for $\delta = 1$, it is clear that $\mathrm{Sz}(k, 1) = [k, \infty[$ for all $k \in \mathbb{N}_+$.

2.1. The function $r_k(n)$

Closely linked to Szemerédi's theorem is the function $r_k(n)$, defined as the maximal cardinality of a subset $A \subseteq [1, n]$ containing no arithmetic progression of length k. Indeed, Szemerédi's theorem is equivalent to the asymptotic bound $r_k(n) = o(n)$. This function can be used to reformulate membership in $\mathrm{Sz}(k, \delta)$, as follows.

Lemma 2.3. *Let $k \geq 3$ be an integer and let $0 < \delta \leq 1$. Then, for every positive integer n, we have $n \in Sz(k, \delta)$ if and only if $r_k(n)/n < \delta$.*

Proof. Assume $n \in \mathrm{Sz}(k, \delta)$. Let $A \subseteq [1, n]$ be a subset of cardinality $r_k(n)$ containing no arithmetic progression of length k. Then $r_k(n)/n = |A|/n < \delta$ since $n \in \mathrm{Sz}(k, \delta)$. Conversely, assume $r_k(n)/n < \delta$. Let $A \subseteq [1, n]$ be a subset of density $|A|/n \geq \delta$. Then $|A| \geq n\delta > r_k(n)$. Hence, by definition of $r_k(n)$, the subset A contains an arithmetic progression of length k. Therefore $n \in \mathrm{Sz}(k, \delta)$. □

2.2. Comparison with the van der Waerden numbers

Not much explicit information about $M(k, \delta)$ seems to be currently available in the literature. For $\delta = 1/r$ with $r \in \mathbb{N}_+$, the number $M(k, 1/r)$ is bounded below by the corresponding van der Waerden number $W(k, r)$. Given integers $k, r \geq 2$, recall that $W(k, r)$ denotes the least integer M such that, for every r-coloring of $[1, M]$, there is a monochromatic arithmetic progression of length k in $[1, M]$. To show

$$M(k, 1/r) \geq W(k, r), \tag{1}$$

let $N = M(k, 1/r)$, and consider any r-coloring of $[1, N]$. Then some color class $X \subseteq [1, N]$ is of density $|X|/N \geq 1/r$, and hence X contains an arithmetic progression of length k which is monochromatic by construction. This settles (1), as desired.

The only exactly known van der Waerden numbers at the time of writing are given in the following table. See e.g. [10], a web page which also displays lower bounds on $W(k, r)$ for many more pairs (k, r).

$W(3, 2) = 9$	$W(3, 3) = 27$	$W(3, 4) = 76$
$W(4, 2) = 35$	$W(4, 3) = 293$	
$W(5, 2) = 178$		
$W(6, 2) = 1132$		

In subsequent sections, we shall prove that $\mathrm{Sz}(k, \delta) \cup \{0\}$ is a numerical semigroup and shall determine it for several pairs (k, δ). We first recall some basic notions regarding numerical semigroups.

3. $\mathrm{Sz}(k, \delta) \cup \{0\}$ as a numerical semigroup

A *numerical semigroup* is a cofinite submonoid of $\mathbb{N}$. That is, a subset $S \subset \mathbb{N}$ containing 0, stable under addition and with finite complement $\mathbb{N} \setminus S$. Equivalently, it is a subset of $\mathbb{N}$ of the form $S = \langle a_1, \ldots, a_n \rangle = a_1\mathbb{N} + \cdots + a_n\mathbb{N}$ for some globally coprime positive integers $a_1, \ldots, a_n$.

Given a numerical semigroup $S \subseteq \mathbb{N}$, the *multiplicity* of S is $m = \min(S \setminus \{0\})$, its *Frobenius number* is $F = \max(\mathbb{Z} \setminus S)$, that is its largest gap, and its *conductor* is $c = F + 1$ or, equivalently, the smallest integer c such that $[c, \infty[\subseteq S$.

If $S = \langle a_1, \ldots, a_n \rangle = a_1\mathbb{N} + \cdots + a_n\mathbb{N}$ with the a_i increasing and globally coprime, the multiplicity of S is $m = a_1$. But determining the Frobenius number of S from the sole generators a_i is a notoriously difficult problem for $n \geq 3$. See e.g. [2,4]. As for $n = 2$, Sylvester proved long ago [8] that the Frobenius number of $\langle a_1, a_2 \rangle$ equals $(a_1 - 1)(a_2 - 1) - 1$. See [5,6] for extensive information on numerical semigroups.

Our objective in this section is to prove that $\mathrm{Sz}(k, \delta) \cup \{0\}$ is a numerical semigroup, using only a weakened version of Szemerédi's theorem.

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2 Thanks are due to Pierre Catoire, an undergraduate math student in Calais, for pointing out an error in a preliminary version of this statement.

3.1. *Stability under addition*

Our first task is to prove that $\mathrm{Sz}(k, \delta)$ is stable under addition. We shall need the following elementary lemma [1].

Lemma 3.1. *Let A, E be nonempty finite sets such that $A \subseteq E$. Denote $\delta = |A|/|E|$ the density of A in E. Let $E = E_1 \sqcup \ldots \sqcup E_r$ be a partition of E into r nonempty parts. Then there exists an index $i \leq r$ such that $|A \cap E_i|/|E_i| \geq \delta$.*

Proof. If $|A \cap E_i|/|E_i| < \delta$ for all i, then $\sum_i |A \cap E_i| < \delta \sum_i |E_i|$. Since $\sum_i |A \cap E_i| = |A|$ and $\sum_i |E_i| = |E|$, this implies $|A| < \delta|E| = |A|$, a contradiction. □

Proposition 3.2. *For any integer $k \geq 3$ and $0 < \delta \leq 1$, the set $\mathrm{Sz}(k, \delta) \cup \{0\}$ is stable under addition.*

Proof. Let $n_1, n_2 \in \mathrm{Sz}(k, \delta)$. Let $E = [1, n_1 + n_2]$, and consider the partition $E = E_1 \sqcup E_2$ with $E_1 = [1, n_1]$ and $E_2 = [n_1 + 1, n_1 + n_2]$. Thus $|E_i| = n_i$ for $i = 1, 2$. Let $X \subseteq E$ be of density $|X|/(n_1 + n_2) \geq \delta$. We must show that X contains an arithmetic progression of length k. Let $X_i = X \cap E_i$ for $i = 1, 2$. By the above lemma, either $|X_1|/n_1 \geq \delta$ or $|X_2|/n_2 \geq \delta$. It follows that either X_1 or X_2 contains an arithmetic progression of length k, whence X also does. Thus $n_1 + n_2 \in \mathrm{Sz}(k, \delta)$, as stated. □

3.2. *Cofiniteness in $\mathbb{N}$*

It directly follows from Szemerédi's Theorem 2.2 that $\mathbb{N} \setminus \mathrm{Sz}(k, \delta)$ is finite since it implies that, for some $n(k, \delta) \in \mathbb{N}_+$, every integer $n \geq n(k, \delta)$ belongs to $\mathrm{Sz}(k, \delta)$. However, that statement can also be deduced by elementary arguments from the following weaker version of Szemerédi's theorem.

Theorem 3.3. *Given an integer $k \geq 3$ and $0 < \delta \leq 1$, there exists a positive integer $n = n(k, \delta)$ such that every subset $A \subseteq [1, n]$ of density $|A|/n \geq \delta$ contains an arithmetic progression of length k.*

This version 'only' states that $\mathrm{Sz}(k, \delta)$ is nonempty, and hence that the number $M(k, \delta) = \min \mathrm{Sz}(k, \delta)$ exists, whereas the original version states that $\mathbb{N} \setminus \mathrm{Sz}(k, \delta)$ is finite. As shown here, these statements are equivalent. Indeed, below we shall only use the existence of $M(k, 1/r)$ for $k, r \in \mathbb{N}_+$ to deduce the cofiniteness of $\mathrm{Sz}(k, \delta)$ in $\mathbb{N}$ in general. That is, we shall deduce Theorem 2.2 from its weaker version Theorem 3.3.

Let us start by observing that $\mathrm{Sz}(k, \delta)$ is monotonous in the parameter δ.

Lemma 3.4. *Let $k \geq 3$ be an integer, and let $0 < \delta_1 \leq \delta_2 \leq 1$. Then $\mathrm{Sz}(k, \delta_1) \subseteq \mathrm{Sz}(k, \delta_2)$.*

Proof. Let $n \in \mathrm{Sz}(k, \delta_1)$. Every subset of $[1, n]$ of density at least δ_2 has density at least δ_1, whence contains an arithmetic sequence of length k. Therefore $n \in \mathrm{Sz}(k, \delta_2)$. □

We shall first prove the cofiniteness of $\mathrm{Sz}(k, \delta)$ when $\delta = 1/r$ with $r \in \mathbb{N}_+$, and shall then use the above lemma for δ arbitrary. The case $\delta = 1/r$ relies upon the following intermediary result.

Proposition 3.5. *Let k, r, n be integers with $k \geq 3$ and $r, n \geq 2$. If $n \in \mathrm{Sz}(k, 1/r)$ and $n \not\equiv 1 \bmod r$, then $n - 1 \in \mathrm{Sz}(k, 1/r)$.*

Proof. By Euclidean division by r with remainder in $[1, r]$, there are integers q, t such that $n = qr + t$ with $1 \leq t \leq r$. We have $t \geq 2$ since $n \not\equiv 1 \bmod r$. Let $X \subseteq [1, n - 1]$ be any subset of density $|X|/(n - 1) \geq 1/r$. We claim that $|X|/n \geq 1/r$. Indeed, we have $r|X| \geq n - 1$, whence $|X| \geq q + (t - 1)/r$. But since $2 \leq t \leq r$, we have $1/r \leq (t - 1)/r < 1$. Since $|X|$ is an integer, it follows that $|X| \geq q + 1$, whence $r|X| \geq qr + r \geq qr + t = n$, whence $|X|/n \geq 1/r$. Thus X is still of density at least $1/r$ in $[1, n]$. It follows that X contains an arithmetic progression of length k. Therefore $n - 1 \in \mathrm{Sz}(k, 1/r)$, as claimed. □

Corollary 3.6. *Let k, r be integers with $k \geq 3$ and $r \geq 2$. Then $M(k, 1/r) \equiv 1 \bmod r$.*

Proof. Let $n = M(k, 1/r) = \min \mathrm{Sz}(k, 1/r)$. Since $n - 1 \notin \mathrm{Sz}(k, 1/r)$, the above proposition implies $n \equiv 1 \bmod r$, as desired. □

Proposition 3.7. *For any integers $k \geq 3$ and $r \geq 2$, the complement $\mathbb{N} \setminus \mathrm{Sz}(k, 1/r)$ is finite.*

Proof. Set $n = M(k, 1/r)$. Then $n \equiv 1 \bmod r$ as seen above. Moreover, we have $2n \in \mathrm{Sz}(k, 1/r)$, and $2n \equiv 2 \bmod r$. It follows that $2n - 1$ also belongs to $\mathrm{Sz}(k, 1/r)$. Therefore $\mathrm{Sz}(k, 1/r)$ contains the numerical semigroup $\langle 2n - 1, 2n \rangle$, and in particular it contains all integers greater than or equal to the conductor of the latter semigroup, namely $(2n - 2)(2n - 1)$ as given by the old result of Sylvester recalled above [8]. □

3.3. Completing the proof

We may now reach the objective of this section.

Theorem 3.8. *For every integer $k \geq 3$ and $0 < \delta \leq 1$, the set $\mathrm{Sz}(k, \delta) \cup \{0\}$ is a numerical semigroup.*

Proof. The stability of $\mathrm{Sz}(k, \delta)$ under addition is given by Proposition 3.2. It remains to prove that its complement in $\mathbb{N}$ is finite, without invoking the full force of Theorem 2.2. There exists $r \in \mathbb{N}_+$ such that $1/r \leq \delta$. Since $\mathrm{Sz}(k, 1/r) \subseteq \mathrm{Sz}(k, \delta)$ by Lemma 3.4, and since the complement of $\mathrm{Sz}(k, 1/r)$ in $\mathbb{N}$ is finite by Proposition 3.7, the same holds for $\mathrm{Sz}(k, \delta)$. □

We propose to call *numerical semigroups of Szemerédi type* those numerical semigroups S of the form $S = \mathrm{Sz}(k, \delta) \cup \{0\}$ for $k \geq 3$ and $0 < \delta \leq 1$.

3.4. The number $C(k, \delta)$

Recall that $M(k, \delta) = \min \mathrm{Sz}(k, \delta)$. Thus, in the standard terminology of numerical semigroups, the number $M(k, \delta)$ is the *multiplicity* of $\mathrm{Sz}(k, \delta) \cup \{0\}$. We now introduce a notation for the conductor of that numerical semigroup.

Notation 3.9. Let $k \geq 3$ and $0 < \delta \leq 1$. We shall denote $C(k, \delta)$ the *conductor* of the numerical semigroup $\mathrm{Sz}(k, \delta) \cup \{0\}$.

We have

$$M(k, \delta) \leq C(k, \delta), \tag{2}$$

since the multiplicity of any numerical semigroup is smaller than or equal to its conductor. Of course, $M(k, 1) = C(k, 1) = k$. Here is yet another consequence of Proposition 3.5, similar in content and proof to Corollary 3.6.

Corollary 3.10. *Let k, r be integers with $k \geq 3$ and $r \geq 2$. Then $C(k, 1/r) \equiv 1 \bmod r$.*

Proof. The conductor $n = C(k, 1/r)$ of $\mathrm{Sz}(k, 1/r)$ satisfies $n - 1 \notin \mathrm{Sz}(k, 1/r)$. Hence $n \equiv 1 \bmod r$ by Proposition 3.5. □

We shall need the following characterization of the conductor.

Lemma 3.11. *Let S be a numerical semigroup with multiplicity m. Then the conductor of S is the smallest integer $c \in S$ such that S contains m consecutive integers starting from c.*

Proof. Indeed, if S contains $[c, c + m - 1]$, then by successively adding multiples of m, it will contain all of $[c, \infty[$. □

The following statement was suggested by one of the referees.

Corollary 3.12. *Let k, r be integers with $k \geq 3$ and $r \geq 2$. Let $M = M(k, 1/r)$ and $C = C(k, 1/r)$. Then $C \leq (M - 1)^2 + 1$.*

Proof. Denote $S = \mathrm{Sz}(k, 1/r)$. By Lemma 3.11, it suffices to show that S contains M consecutive integers starting from $(M - 1)^2 + 1$, i.e. that

$$\left[(M - 1)^2 + 1, (M - 1)^2 + M\right] \subset S. \tag{3}$$

Recall from Proposition 3.5 that if $n \in S$ and $n \not\equiv 1 \bmod r$, then $n - 1 \in S$. Hence, if $n \in S$ and $n \equiv a \bmod r$ with $1 \leq a \leq r - 1$, then $[n - a + 1, n] \subset S$. In particular, we have $M \equiv 1 \bmod r$ by Corollary 3.6.

Let $J_i = [iM - i + 1, iM]$ for any positive integer $i \geq 1$. Then $\mathrm{card}(J_i) = i$. We claim that $J_i \subset S$ for all $i \geq 1$. Indeed, this holds for $i = 1$ since $J_1 = \{M\}$. For $i \geq 2$ assume, by induction hypothesis, that the claim holds for $i - 1$, i.e. that $J_{i-1} \subset S$. Then $J_{i-1} + M \subset S$, i.e. $[iM - i + 2, iM] \subset S$. Now, since $iM - i + 2 \equiv 2 \bmod M$, it follows that $iM - i + 1 \in S$. Hence S contains $[iM - i + 1, iM] = J_i$, as claimed.

For $i = M - 1$, the claim yields $[(M - 1)^2 + 1, M^2 - M] \subset S$. Moreover, since $\min(J_m) = M^2 - M + 1$, the claim for $i = M$ implies $M^2 - M + 1 \in S$. Therefore $[(M - 1)^2 + 1, M^2 - M + 1] \subset S$, whence (3) holds. □

Note that the conclusion of Corollary 3.12 may not necessarily hold for densities δ other than $1/r$ with $r \geq 2$ an integer. For instance, with $k = 3$ and $\delta = 3/4$, we have $\mathrm{Sz}(3, 3/4) = \{3\} \cup [6, \infty[$, so that $M = 3$ and $C = 6$, whence $C > (M - 1)^2 + 1$.

4. Exact values of $r_k(n)$

Exact values of the functions $r_3(n)$ and $r_4(n)$ defined in Section 2.1 are currently known for $n \leq 187$ and $n \leq 112$, respectively. They are partly listed in the two tables below, which were read off from [3] and [7], respectively. In the next section, we shall use these values, in conjunction with Lemma 2.3, to determine $\mathrm{Sz}(k, \delta)$ in many instances.

n	$r_3(n)$	n	$r_3(n)$	n	$r_3(n)$	n	$r_3(n)$
1	1	26 – 29	11	71 – 73	21	121	31
2 – 3	2	30 – 31	12	74 – 81	22	122 – 136	32
4	3	32 – 35	13	82 – 83	23	137 – 144	33
5 – 8	4	36 – 39	14	84 – 91	24	145 – 149	34
9 – 10	5	40	15	92 – 94	25	150 – 156	35
11 – 12	6	41 – 50	16	95 – 99	26	157 – 162	36
13	7	51 – 53	17	100 – 103	27	163 – 164	37
14 – 19	8	54 – 57	18	104 – 110	28	165 – 168	38
20 – 23	9	58 – 62	19	111 – 113	29	169 – 173	39
24 – 25	10	63 – 70	20	114 – 120	30	174 – 187	40

n	$r_4(n)$	n	$r_4(n)$
53	27	84 – 86	39
54 – 57	28	87 – 90	40
58 – 59	29	91 – 92	41
60 – 63	30	93 – 96	42
64 – 65	31	97 – 98	43
66 – 67	32	99 – 100	44
68 – 69	33	101 – 103	45
70 – 73	34	104	46
74 – 76	35	105 – 106	47
77 – 78	36	107 – 111	48
79 – 81	37	112	49
82 – 83	38		

5. Determining Sz(k, δ)

We now determine Sz$(3, \delta)$ for various values of $\delta < 1$, and Sz$(4, 1/2)$, using Lemma 2.3 and the two tables above. In each case, we give the multiplicity $M(k, \delta)$, the conductor $C(k, \delta)$ and the full set Sz(k, δ).

In most cases, the set Sz$(3, \delta)$ can be directly read off from the displayed values of $r_3(n)$ for $n \le 187$. Yet sometimes, we need upper bounds on $r_3(n)$ for several $n > 187$. For that, we use the easy and well-known inequality

$$r_k(n + i) \le r_k(n) + r_k(i) \tag{4}$$

for all $k, n, i \ge 1$.

Determining the multiplicity $M(k, \delta) = \min \mathrm{Sz}(k, \delta)$ is straightforward from Lemma 2.3 and the tables in Section 4. As for the conductor $C(k, \delta)$, it may be determined using Lemma 3.11. Tables 1 and 2 give Sz(k, δ) for more than 30 pairs (k, δ).

For illustration purposes, we now prove one case in detail.

Proposition 5.1. *We have* Sz$(3, 1/4) = [129, \infty[$.

Proof. Looking at the values of $r_3(n)$ in Section 4, we see that the smallest $n \ge 1$ satisfying $r_3(n) < n/4$ is 129. Hence $M(3, 1/4) = 129$ by Lemma 2.3. Let us now prove $C(3, 1/4) = 129$. By Lemma 3.11, is suffices to show that Sz$(3, 1/4)$

Table 1
The sets Sz$(3, \delta)$ for selected values of δ.

(k, δ)	$M(k, \delta)$	$C(k, \delta)$	Sz(k, δ)
(3, 1/2)	17	17	$[17, \infty[$
(3, 1/3)	49	55	$\{49, 50, 52, 53\} \cup [55, \infty[$
(3, 1/4)	129	129	$[129, \infty[$
(3, 2/3)	7	7	$[7, \infty[$
(3, 2/5)	23	33	$\{23, 28, 29, 31\} \cup [33, \infty[$
(3, 2/7)	78	85	$[78, 83] \cup [85, \infty[$
(3, 2/9)	181	?	?
(3, 3/4)	3	6	$\{3\} \cup [6, \infty[$
(3, 3/5)	7	7	$[7, \infty[$
(3, 3/7)	19	22	$\{19\} \cup [22, \infty[$
(3, 3/8)	35	43	$\{35, 38, 39\} \cup [43, \infty[$
(3, 3/10)	67	67	$[67, \infty[$
(3, 3/11)	81	96	$\{81\} \cup [89, 94] \cup [96, \infty[$
(3, 3/13)	144	170?	$\{144, 148, 149\} \cup [152, 168] \cup [170, 187] \cup$?

Table 2
More instances of sets $\mathrm{Sz}(3, \delta)$, and $\mathrm{Sz}(4, 1/2)$.

(k, δ)	$M(k, \delta)$	$C(k, \delta)$	$\mathrm{Sz}(k, \delta)$
(3, 4/5)	3	6	$\{3, 4\} \cup [6, \infty[$
(3, 4/7)	8	15	$[8, 13] \cup [15, \infty[$
(3, 4/9)	19	21	$\{19\} \cup [21, \infty[$
(3, 4/11)	39	45	$\{39\} \cup [45, \infty[$
(3, 4/13)	62	66	$\{62\} \cup [66, \infty[$
(3, 4/15)	91	106	$\{91, 94, 98, 99, 102, 103\} \cup [106, \infty[$
(3, 4/17)	141	141?	$[141, \infty[$?
(3, 5/6)	3	3	$[3, \infty[$
(3, 5/7)	3	6	$\{3\} \cup [6, \infty[$
(3, 5/8)	7	7	$[7, \infty[$
(3, 5/9)	8	15	$\{8\} \cup [10, 13] \cup [15, \infty[$
(3, 5/11)	18	18	$[18, \infty[$
(3, 5/12)	22	27	$\{22, 23, 25\} \cup [27, \infty[$
(3, 5/13)	29	42	$\{29, 34, 35\} \cup [37, 40] \cup [42, \infty[$
(3, 5/14)	45	45	$[45, \infty[$
(3, 5/16)	61	65	$\{61, 62\} \cup [65, \infty[$
(3, 5/17)	69	75	$\{69, 70, 72, 73\} \cup [75, \infty[$
(3, 5/18)	80	87	$\{80, 81, 83\} \cup [87, \infty[$
(3, 5/19)	99	115	$\{99, 103\} \cup [107, 113] \cup [115, \infty[$
(3, 5/21)	135	139?	$\{135, 136\} \cup [139, 200] \cup$?
(4, 1/2)	57	61	$\{57, 59\} \cup [61, \infty[$

contains the whole of $[129, 257]$, i.e. that

$$r_3(n) < n/4 \tag{5}$$

for all $129 \leq n \leq 257$. The relevant table in Section 4 shows that (5) holds for all $129 \leq n \leq 187$. It remains to see that $187 + i$ satisfies (5) for all $1 \leq i \leq 70$. This follows from the inequality $r_k(n + i) \leq r_k(n) + r_k(i)$ recalled in (4). Indeed, we have $r_3(187) = 40$, and

$$r_3(187) + r_3(i) < (187 + i)/4$$

for all $1 \leq i \leq 70$, as checked by scanning the values of $r_3(i)$ in this range. More precisely, the smallest difference $(187 + i)/4 - (r_3(187) + r_3(i))$ for $1 \leq i \leq 70$ comes at $i = 41$ and equals $228/4 - (40 + 16) = 1$. □

5.1. Concluding questions

The present results raise some obvious questions regarding numerical semigroups of Szemerédi type. For instance, all cases met so far satisfy $C(k, \delta)/M(k, \delta) \leq 2$. Is this true in general? More generally, what is the value of $\max C(k, \delta)/M(k, \delta)$ as $k \geq 3$ and $0 < \delta \leq 1$ vary? Is it finite or infinite? If finite, is it attained?

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