3-color Schur numbers

L. Boza, J.M. Marín, M.P. Revuelta, M.I. Sanz *

Departamento de Matemática Aplicada I, Universidad de Sevilla, Avenida Reina Mercedes s/n, C.P. 41012, Sevilla, Spain

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A B S T R A C T

Let $k \geq 3$ be an integer, the Schur number $S_k(3)$ is the least positive integer, such that for every 3-coloring of the integer interval $[1, S_k(3)]$ there exists a monochromatic solution to the equation $x_1 + \cdots + x_k = x_{k+1}$, where x_i, $i = 1, \ldots, k$ need not be distinct.

In 1966, a lower bound of $S_k(3)$ was established by Znám (1966). In this paper, we determine the exact formula of $S_k(3) = k^3 + 2k^2 - 2$, finding an upper bound which coincides with the lower bound given by Znám (1966). This is shown in two different ways: in the first instance, by the exhaustive development of all possible cases and in the second instance translating the problem into a Boolean satisfiability problem, which can be handled by a SAT solver.

1. Introduction

For integers $a \leq b$, we shall denote $[a, b]$ the *integer interval*. A set A of integers is said to be *k-sum-free* if it contains no $k + 1$ elements $x_1, x_2, \ldots, x_{k+1} \in A$ satisfying $x_1 + \cdots + x_k = x_{k+1}$, where x_i, $i = 1, \ldots, k$ are not necessarily distinct.

Schur [11] in 1916 proved that, given a positive integer $n \geq 1$, there exists a greatest positive integer $S_2(n)$, with the property that the integer interval $[1, S_2(n) - 1]$ can be partitioned into n sets which are *2-sum-free*. These numbers $S_2(n)$ are called the Schur numbers.

In terms of coloring, the Schur number $S_2(n)$ [11] is the least positive integer such that for every n-coloring of $[1, S_2(n)]$:

$$\Delta : [1, S_2(n)] \longrightarrow \{1, 2, \ldots, n\}$$

there exists a monochromatic solution to the equation $x_1 + x_2 = x_3$, where x_1, x_2 and x_3 need not be distinct.

Schur numbers are known only for a few small values of n. For instance, one has $S_2(1) = 2$ and $S_2(2) = 5$. A partition of $[1, 4]$ into 2 *sum-free* sets is provided by $[1, 4] = \{1, 4\} \sqcup \{2, 3\}$. Two more exact values of $S_2(n)$ are currently known, namely $S_2(3) = 14$ and $S_2(4) = 45$. While the value $S_2(3)$ can still be settled by hand, the value $S_2(4)$ [2] relies on an exhaustive computer search. Recently $S_2(6) = 161$ has been obtained by Heule [6].

In 2000, Fredricksen and Sweet [4] obtained $S_2(6) \geq 537$ and $S_2(7) \geq 1681$ by considering symmetric sum-free partitions.

Schur proved for every $n \geq 1$, the following lower and upper general bounds [11]:

$$(3^n + 1)/2 \;\leq\; S_2(n) \;\leq\; \lfloor n!e \rfloor.$$

Furthermore, he established

$$S_2(n) \geq 3S_2(n-1) - 1.$$

* Corresponding author.
E-mail addresses: boza@us.es (L. Boza), jmarin@us.es (J.M. Marín), pastora@us.es (M.P. Revuelta), isanz@us.es (M.I. Sanz).

In 1933, Rado [8] obtained in particular the following generalization: given two positive integers, n and $k \geq 2$, there exists a greatest positive integer, $S_k(n)$, such that the integer interval $[1, S_k(n) - 1]$ can be partitioned into n sets which are *k-sum-free*. In 1966, Znám [12] established a lower bound on the numbers $S_k(n)$:

$$S_k(n) \geq \frac{k-1}{k}((k+1)^n - 1) + 1. \tag{1}$$

An upper bound given in 1955 [5] was improved by Irving [7] in 1973:

$$S_k(n) \leq \left\lfloor \sqrt[k-1]{e}\, n!(k-1)^n \right\rfloor.$$

In 1982, Beutelspacher and Brestovansky [3] proved that for two k-sum-free sets the following equality is verified:

$$S_k(2) = k^2 + k - 1, k \geq 2.$$

Although there has been progress in establishing new bounds in the last years, there have been no advances for over 35 years in the calculation of exact formulas.

In 2010 [9], the last author obtained the exact value of $S_3(3) = 43$. Independently, Ahmed and Schaal [1] in 2016 obtain the values of $S_k(3)$ for $k = 3, 4, 5$ and also present a conjecture that coincides with the exact formula of $S_k(3)$ given in this paper.

We determine the exact formula of $S_k(3) = k^3+2k^2-2$ for all $k \geq 3$, finding an upper bound that coincides with the lower bound given by Znám [12]. This is shown in two different ways: first, by the exhaustive development of all possible cases; and second by translating the problem into a Boolean satisfiability problem, which can be handled by a SAT solver [10].

2. Exact formula of $S_k(3)$

We now prove that the exact formula of $S_k(3) = k^3 + 2k^2 - 2$. The lower bound was established in 1966 [12], so we focus on the upper bound.

We shall prove that for every 3-coloring of the integer interval $[1, k^3 + 2k^2 - 2]$, there exists a monochromatic solution to the equation: $x_1 + \cdots + x_k = x_{k+1}$. We start, by analyzing 27 cases and achieving a contradiction in each of them. The proof consists in checking in all 27 cases that all the coloring obtained have monochromatic solutions.

We shall prove that for every 3-coloring:

$$\Delta : [1, k^3 + 2k^2 - 2] \longrightarrow \{1, 2, 3\}$$

there exists a monochromatic solution to the equation $x_1 + \cdots + x_k = x_{k+1}$.

Let sets A_1, A_2 and A_3 be such that $\Delta(A_1) = 1$, $\Delta(A_2) = 2$ and $\Delta(A_3) = 3$, so that $[1, k^3 + 2k^2 - 2] = A_1 \sqcup A_2 \sqcup A_3$. Given a subset $X \subseteq [1, k^3 + 2k^2 - 2]$, we denote

$$f(X) = X \dot{+} \cdots \dot{+} X = \{x_1 + \cdots + x_k \mid x_1, \ldots, x_k \in X\} \cap [1, k^3 + 2k^2 - 2].$$

By hypothesis on Δ, for $1 \leq i \leq 3$, we have

$$A_i \cap f(A_i) = \emptyset. \tag{2}$$

The proof rests on the following claims, which are both direct consequences of (2). For all integers i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$, we have

- **Claim I.** $f(A_i) \cap f(A_j) \subseteq A_k$.
- **Claim II.** $f(A_i) \cap f(A_j) \cap f(A_k) = \emptyset$.

Without loss of generality we may assume that $\Delta(1) = 1$ and $\Delta(k) = 2$. To avoid the monochromatic solution in A_2, since

$$\overbrace{k + \cdots + k}^{k} = k^2,$$

we consider two cases depending on the color of k^2. These two cases are discussed in Sections 3 and 4.

We begin with a scheme of the cases to study, to facilitate the exhaustive development of this proof.

$$\Delta(k^2) = 1 \begin{cases} \Delta(k^2+k-1) = 2 \begin{cases} \Delta(k^2+k) = 1 \\ \Delta(k^2+k) = 2 \end{cases} \\ \Delta(k^2+k-1) = 3 \begin{cases} \Delta(k^3) = 2 \\ \Delta(k^3) = 3 \begin{cases} \Delta(k^3+k^2-k) = 1 \begin{cases} \Delta(k+1) = 2 \\ \Delta(k+1) = 3 \end{cases} \\ \Delta(k^3+k^2-k) = 2 \begin{cases} \Delta(k^3+2k^2-2k) = 1 \\ \Delta(k^3+2k^2-2k) = 3 \begin{cases} \Delta(2k-1) = 1 \\ \Delta(2k-1) = 2 \end{cases} \end{cases} \end{cases} \end{cases} \end{cases}$$

$$
\Delta(k^2)=3\left\{\begin{array}{l}
\Delta(k^3)=1\left\{\begin{array}{l}
\Delta(k^3+k-1)=2\left\{\begin{array}{l}
\Delta(k+1)=2\\
\Delta(k+1)=3\left\{\begin{array}{l}
\Delta(k^3+k^2+k-2)=2\left\{\begin{array}{l}\Delta(k^2-k+1)=2\\ \Delta(k^2-k+1)=3\end{array}\right.\\
\Delta(k^3+k^2+k-2)=3\left\{\begin{array}{l}\Delta(k^2+k-1)=1\\ \Delta(k^2+k-1)=2\end{array}\right.
\end{array}\right.
\end{array}\right.\\
\Delta(k^3+k-1)=3\left\{\begin{array}{l}
\Delta(k^2+k-1)=1\left\{\begin{array}{l}\Delta(k^2+2k-2)=2\\ \Delta(k^2+2k-2)=3\end{array}\right.\\
\Delta(k^2+k-1)=2\left\{\begin{array}{l}
\Delta(2k^2-1)=1\left\{\begin{array}{l}\Delta(2k-1)=1\\ \Delta(2k-1)=3\end{array}\right.\\
\Delta(2k^2-1)=3
\end{array}\right.
\end{array}\right.
\end{array}\right.\\
\Delta(k^3)=2\left\{\begin{array}{l}
\Delta(k^2+k)=1\left\{\begin{array}{l}
\Delta(k^3+k^2-k)=1\left\{\begin{array}{l}\Delta(k^2+k-1)=2\\ \Delta(k^2+k-1)=3\end{array}\right.\\
\Delta(k^3+k^2-k)=3\left\{\begin{array}{l}
\Delta(k^3-k+1)=2\left\{\begin{array}{l}\Delta(k+1)=2\\ \Delta(k+1)=3\end{array}\right.\\
\Delta(k^3-k+1)=3
\end{array}\right.
\end{array}\right.\\
\Delta(k^2+k)=3\left\{\begin{array}{l}\Delta(k^2+k-1)=2\\ \Delta(k^2+k-1)=3\end{array}\right.
\end{array}\right.
\end{array}\right.
$$

3. Assigning the color 1 to k^2

We suppose $\Delta(k^2) = 1$, therefore $A_1 \supseteq \{1, k^2\}$, $A_2 \supseteq \{k\}$. We have that $k^2 + \overbrace{1 + \cdots + 1}^{k-1} = k^2 + k - 1$ so to avoid a monochromatic solution in A_1, we assume that $\Delta(k^2 + k - 1) = 2$ or $\Delta(k^2 + k - 1) = 3$.

3.1. Assigning the color 2 to $k^2 + k - 1$

We consider $\Delta(k^2 + k - 1) = 2$ so that $A_1 \supseteq \{1, k^2\}$ and $A_2 \supseteq \{k, k^2 + k - 1\}$. As $1 + (k - 1) \times (k + 1) = k^2$ would be a monochromatic solution in A_1 and $1 \times k + (k - 1) \times (k + 1) = k^2 + k - 1$ would be a monochromatic solution in A_2, applying Claim I produces $k + 1 \in A_3$. To avoid $k \times (k + 1) = k^2 + k$ being a monochromatic solution in A_3, we assume that $\Delta(k^2 + k) = 1$ or $\Delta(k^2 + k) = 2$.

Case 1: $A_1 \supseteq \{1, k^2, k^2 + k\}, A_2 \supseteq \{k, k^2 + k - 1\}, A_3 \supseteq \{k + 1\}$.

Consider $1 + (k - 1) \times (k^2 + k) = k^3 - k + 1$ and $k + (k - 1) \times (k^2 + k - 1) = k^3 - k + 1$ and apply Claim I to get $\Delta(k^3 - k + 1) = 3$.

Now, one of the following equations must be monochromatic $k^2+(k-1)\times(k^2+k) = k^3+k^2-k$; $k^3+k^2-k = k\times(k^2+k-1)$; and $k^3 + k^2 - k = k^3 - k + 1 + (k - 1) \times (k + 1)$. We show $k^3 + k^2 - k \notin A_i$, $i = 1, 2, 3$, for otherwise $k^3 + k^2 - k \subseteq A_i \cap f(A_i)$, a contradiction.

Case 2: $A_1 \supseteq \{1, k^2\}, A_2 \supseteq \{k, k^2 + k - 1, k^2 + k\}, A_3 \supseteq \{k + 1\}$.

We have $k \times k^2 = k^3$ and $k + (k - 1) \times (k^2 + k) = k^3$ so we deduce that $\Delta(k^3) = 3$. Apply Claim I to $k^2 + k - 1 + (k - 1) \times (k^2 + k) = k^3 + k^2 - 1$ and $k^3 + (k - 1) \times (k - 1) = k^3 + k^2 - 1$ to get $\Delta(k^3 + k^2 - 1) = 1$. Since $(2k^2 - 1) + (k - 1) \times k^2 = k^3 + k^2 - 1$ and $k^2 + k - 1 + (k - 1) \times k = 2k^2 - 1$, applying Claim I so we deduce that $\Delta(2k^2 - 1) = 3$.

We now obtain a contradiction by noting that one of the following equations must be monochromatic, $1 + (3k^2 - 2) + (k - 2) \times k^2 = k^3 + k^2 - 1$; $3k^2 - 2 = 2 \times (k^2 + k - 1) + (k - 2) \times k$; and $3k^2 - 2 = 2k^2 - 1 + (k - 1) \times (k + 1)$. We show $3k^2 - 2 \notin A_i$, $i = 1, 2, 3$, for otherwise $3k^2 - 2 \subseteq A_i \cap f(A_i)$.

3.2. Assigning the color 3 to $k^2 + k - 1$

We consider $\Delta(k^2 + k - 1) = 3$, so that $A_1 \supseteq \{1, k^2\}$, $A_2 \supseteq \{k\}$ and $A_3 \supseteq \{k^2 + k - 1\}$. We have $k \times k^2 = k^3$, we assume $\Delta(k^3) \in \{2, 3\}$.

Case 1: $A_1 \supseteq \{1, k^2\}$, $A_2 \supseteq \{k, k^3\}$ and $A_3 \supseteq \{k^2 + k - 1\}$. Apply Claim I to $k^3 + (k - 1) \times k = k^3 + k^2 - k$ and $k \times (k^2 + k - 1) = k^3 + k^2 - k$, to get $\Delta(k^3 + k^2 - k) = 1$. Since $k^2 + (k - 1) \times (k^2 + k) = k^3 + k^2 - k$ and $k + (k - 1) \times (k^2 + k) = k^3$, applying Claim I we deduce $\Delta(k^2+k)=3$. Consider $1+(k-1)\times(k+1)=k^2$ and $k\times(k+1)=k^2+k$ and apply Claim I to get $\Delta(k+1)=2$.

We now obtain a contradiction by noting that one of the following equations must be monochromatic, $k^3+k^2-k+(k-1)\times 1=k^3+k^2-1$; $k^3+(k-1)\times(k+1)=k^3+k^2-1$; and $k^2+k-1+(k-1)\times(k^2+k)=k^3+k^2-1$. We show $k^3+k^2-1\notin A_i$, $i=1,2,3$, for otherwise $k^3+k^2-1\subseteq A_i\cap f(A_i)$.

Case 2: $A_1\supseteq\{1,k^2\}$, $A_2\supseteq\{k\}$ and $A_3\supseteq\{k^2+k-1,k^3\}$. We have $k\times(k^2+k-1)=k^3+k^2-k$ to avoid the monochromatic solution in A_3, we consider $\Delta(k^3+k^2-k)\in\{1,2\}$.

Case 2.1: $A_1\supseteq\{1,k^2,k^3+k^2-k\}$, $A_2\supseteq\{k\}$ and $A_3\supseteq\{k^2+k-1,k^3\}$. We have $1+(k-1)\times(k+1)=k^2$, so to avoid a monochromatic solution in A_1, we assume $\Delta(k+1)\in\{2,3\}$.

Case 2.1.1: $\Delta(k+1)=2$. Apply Claim I to $k^2+(k-1)\times(k^2+k)=k^3+k^2-k$ and $k\times(k+1)=k^2+k$ to get $\Delta(k^2+k)=3$. Since $(k^3+k^2-k)+(k-1)\times 1=k^3+k^2-1$ and $(k^2+k-1)+(k-1)\times(k^2+k)=k^3+k^2-1$, we have $\Delta(k^3+k^2-1)=2$. After applying Claim I to $1+(k-1)\times(k^2+2k+1)=k^3+k^2-1$ and $k+(k-1)\times(k^2+2k+1)=k^3+k^2-1$ we obtain $\Delta(k^2+2k+1)=3$.

We now obtain a contradiction by noting that one of the following equations must be monochromatic, $k^3+k^2-k+(k-2)\times 1+k^2=k^3+2k^2-2$; $(k^3+k^2-1)+(k-1)\times(k+1)=k^3+2k^2-2$; and $(k^2+k-1)+(k-1)\times(k^2+2k+1)=k^3+2k^2-2$. We show $k^3+2k^2-2\notin A_i$, $i=1,2,3$, for otherwise $k^3+2k^2-2\subseteq A_i\cap f(A_i)$.

Case 2.1.2: $\Delta(k+1)=3$. Consider $k^2+(k-1)\times(k^2+k)=k^3+k^2-k$ and $k\times(k+1)=k^2+k$ and apply Claim I to get $\Delta(k^2+k)=2$. We have $2\times k^2+(k-2)\times 1=2k^2+k-2$ and $(k^2+k-1)+(k-1)\times(k+1)=2k^2+k-2$ so we deduce that $\Delta(2k^2+k-2)=2$. Since $(k^3+k^2-k)+(k-1)\times 1=k^3+k^2-1$ and $k^3+(k-1)\times(k+1)=k^3+k^2-1$ applying Claim I we have $\Delta(k^3+k^2-1)=2$. Once we apply Claim I to $k^2+(k^3+k^2-k)+(k-2)\times 1=k^3+2k^2-2$ and $(2k^2+k-2)+(k-1)\times(k^2+k)=k^3+2k^2-2$ we obtain $\Delta(k^3+2k^2-2)=3$.

We now obtain a contradiction by noting that one of the following equations must be monochromatic, $1+(k-1)\times(k^2+2k+1)=k^3+k^2-k$; $1\times k+(k-1)\times(k^2+2k+1)=k^3+k^2-1$; and $(k^2+k-1)+(k-1)\times(k^2+2k+1)=k^3+2k^2-2$. We show $k^2+2k+1\notin A_i$, $i=1,2,3$, for otherwise $k^2+2k+1\subseteq A_i\cap f(A_i)$.

Case 2.2: $A_1\supseteq\{1,k^2\}$, $A_2\supseteq\{k,k^3+k^2-k\}$ and $A_3\supseteq\{k^2+k-1,k^3\}$. We have $(k^3+k^2-k)+(k-1)\times k=k^3+2k^2-2k$, so to avoid a monochromatic solution in A_1, we assume $\Delta(k^3+2k^2-2k)\in\{1,3\}$.

Case 2.2.1: $\Delta(k^3+2k^2-2k)=1$. Apply Claim I to $k^2+(k-1)\times(k^2+2k)=k^3+2k^2-2k$ and $k+(k-1)\times(k^2+2k)=k^3+k^2-k$ to get $\Delta(k^2+2k)=3$. Since $(k^3+2k^2-2k)+(k-1)\times 1=k^3+2k^2-k-1$ and $(k^2+k-1)+(k-1)\times(k^2+2k)=k^3+2k^2-k-1$, applying Claim I we deduce $\Delta(k^3+2k^2-k-1)=2$. We have $1+(k-1)\times(k+1)=k^2$ and $(k^3+k^2-k)+(k-1)\times(k+1)=k^3+2k^2-k-1$ so we obtain $\Delta(k+1)=3$. Once we apply Claim I to $2\times k^2+(k-2)\times 1=2k^2+k-2$ and $(k^2+k-1)+(k-1)\times(k+1)=2k^2+k-2$ we deduce $\Delta(2k^2+k-2)=2$. Since $k\times(k^2+2k-2)=k^3+2k^2-2k$ and $(k^2+2k-2)+(k-1)\times k=2k^2+k-2$ we have $\Delta(k^2+2k-2)=3$. Consider $(2k^2+k-2)+(k-1)\times(k^2-2)=k^3+k^2-k$ and $(k^2+2k-2)+(k-1)\times(k^2-2)=k^3$ and apply Claim I to get $\Delta(k^2-2)=1$. Since $2\times 1+(k-2)\times(k+2)=k^2-2$ and $k\times(k+2)=k^2+2k$ we deduce $\Delta(k+2)=2$.

We now obtain a contradiction by noting that one of the following equations must be monochromatic, $k^2+(k-1)\times(k^2-2)=k^3-2k+2$; $(k^3-2k+2)+(k-1)\times(k+2)=k^3+k^2-k$; and $2\times(k+1)+(k-2)\times(k^2+2k)=k^3-2k+2$. We show $k^3-2k+2\notin A_i$, $i=1,2,3$, for otherwise $k^3-2k+2\subseteq A_i\cap f(A_i)$.

Case 2.2.2: $\Delta(k^3+2k^2-2k)=3$. We have $(2k-1)+(k-1)\times(k^2+k-1)=k^3$, so to avoid a monochromatic solution in A_3, we assume $\Delta(2k-1)\in\{1,2\}$.

1. $\Delta(2k-1)=1$. We have $(k^2-k+1)+(k-1)\times 1=k^2$, we assume $\Delta(k^2-k+1)\in\{2,3\}$.
   (a) $\Delta(2k-1)=1$. Consider $(2k-1)+(k-1)\times(k-1)=k^2$ and $k+(k-1)\times(k-1)=k^2-k+1$ and apply Claim I to get $\Delta(k-1)=3$. Since $k\times(2k-1)=2k^2-k$ and $(k^2+k-1)+(k-1)\times(k-1)=2k^2-k$ applying Claim I we obtain $\Delta(2k^2-k)=2$. We have $k+(k-1)\times 2k=2k^2-k$ and $k^3+(k-1)\times 2k=k^3+2k^2-2k$ so we deduce that $\Delta(2k)=1$. Since $1+(k-1)\times(k+1)=k^2$ and $(k^2-k+1)+(k-1)\times(k+1)=2k^2-k$, we have $\Delta(k+1)=3$. Apply Claim I to $(2k-1)+(k-1)\times 2k=2k^2-1$ and $(2k^2-1)+(k-1)\times(k^2+k-1)=k^3+2k^2-2k$ to get $\Delta(2k^2-1)=2$. Consider $1+(k-1)\times 2k=2k^2-2k+1$ and $(k^2-k+1)+(k-1)\times k=2k^2-2k+1$ and apply Claim I to obtain $\Delta(2k^2-2k+1)=3$.
   We now obtain a contradiction by noting that one of the following equations must be monochromatic, $k^2+(k-1)\times 2k=3k^2-2k$; $(2k^2-k)+(k-1)\times k=3k^2-2k$; and $(2k^2-2k+1)+(k-1)\times(k+1)=3k^2-2k$. We show $3k^2-2k\notin A_i$, $i=1,2,3$, for otherwise $3k^2-2k\subseteq A_i\cap f(A_i)$..
   (b) $\Delta(2k-1)=1$. Apply Claim I to $1+(k-1)\times 2=2k-1$ and $(k^2-k+1)+(k-1)\times 2=k^2+k-1$, to get $\Delta(2)=2$. Since $k\times 2=2k$ and $k^3+(k-1)\times 2k=k^3+2k^2-2k$ we have $\Delta(2k)=1$. Once we apply Claim I to $(2k-1)+(k-1)\times 2k=2k^2-1$ and $(2k^2-1)+(k-1)\times(k^2+k-1)=k^3+2k^2-2k$ we obtain $\Delta(2k^2-1)=2$.
   We now obtain a contradiction by noting that one of the following equations must be monochromatic, $1+(k-1)\times 2k=2k^2-2k+1$; $(2k^2-2k+1)+(k-1)\times 2=2k^2-1$; and $(2k^2-2k+1)+(k-1)\times(k^2-k+1)=k^3$. We show $2k^2-2k+1\notin A_i$, $i=1,2,3$, for otherwise $2k^2-2k+1\subseteq A_i\cap f(A_i)$.

2. $\Delta(2k-1)=1$. Apply Claim I to $2\times(2k-1)+(k-2)\times k=k^2+2k-2$, and $k\times(k^2+2k-2)=k^3+2k^2-2k$, to get $\Delta(k^2+2k-2)=1$. Since $(2k-1)+(k-1)\times(k^2+2k-1)=k^3+k^2-k$ and $(k^2+k-1)+(k-1)\times(k^2+2k-1)=k^3+2k^2-2k$, we have $\Delta(k^2+2k-1)=1$. We have $(k^2+2k-2)+(k-1)\times(k^2+2k-1)=k^3+2k^2-k-1$ and $(k^3+k^2-k)+(2k-1)+(k-2)\times k=k^3+2k^2-k-1$ apply so we deduce $\Delta(k^3+2k^2-k-1)=3$. Since $k+(k-1)\times(k^2+2k)=k^3+k^2-k$ and $(k^2+k-1)+(k-1)\times(k^2+2k)=k^3+2k^2-k-1$, we obtain $\Delta(k^2+2k)=1$. We have $2\times k^2+(k-2)\times 1=2k^2+k-2$ and $(2k^2+k-2)+(k-1)\times(k^2+k-1)=k^3+2k^2-k-1$ applying Claim I we deduce $\Delta(2k^2+k-2)=2$. Since $(2k-1)+(k-1)\times(2k+1)=2k^2+k-2$ and $k^3+(k-1)\times(2k+1)=k^3+2k^2-k-1$ we obtain $\Delta(2k+1)=1$. After applying Claim I to $1+(k-1)\times(2k+1)=2k^2-k$ and $k\times(2k-1)=2k^2-k$, we have $\Delta(2k^2-k)=3$. Consider $1+(2k+1)+(k-2)\times(k^2+2k)=k^3-2k+2$ and $(k^3-2k+2)+2\times(2k-1)+(k-3)\times k=k^3+k^2-k$ we have $\Delta(k^3-2k+2)=3$. Since $k^2+(k-1)\times 2=k^2+2k-2$ and $(k^3-2k+2)+(k-1)\times 2=k^3$ we deduce $\Delta(2)=2$.
   We now obtain a contradiction by noting that one of the following equations must be monochromatic $(k^2+k)+(k-1)\times 1=k^2+2k-1$; $(2k^2+k-2)+2+(k-2)\times(k^2+k)=k^3+k^2-k$; and $(2k^2-k)+(k-1)\times(k^2+k)=k^3+2k^2-2k$. We show $k^2+k\notin A_i$, $i=1,2,3$, for otherwise $k^2+k\subseteq A_i\cap f(A_i)$.

4. Assigning the color 3 to k^2

We suppose $\Delta(k^2)=3$, therefore $A_1\supseteq\{1\}$, $A_2\supseteq\{k\}$ and $A_3\supseteq\{k^2\}$. We have that $k\times k^2=k^3$ so to avoid a monochromatic solution in A_1, we assume that $\Delta(k^3)\in\{1,2\}$.

4.1. Assigning the color 1 to k^3

We consider $\Delta(k^3)=1$, so that $A_1\supseteq\{1,k^3\}$, $A_2\supseteq\{k\}$ and $A_3\supseteq\{k^2\}$. To avoid $k^3+(k-1)\times 1=k^3+k-1$, being a monochromatic solution in A_1, we assume that $\Delta(k^3+k-1)\in\{2,3\}$.

Case 1: $A_1\supseteq\{1,k^3\}$, $A_2\supseteq\{k,k^3+k-1\}$ and $A_3\supseteq\{k^2\}$. Apply Claim I to $1+(k-1)\times(k^2+k+1)=k^3$ and $k+(k-1)\times(k^2+k+1)=k^3+k-1$, to get $\Delta(k^2+k+1)=3$. Since $(k^3+k-1)+(k-1)\times k=k^3+k^2-1$ and $k^2+(k-1)\times(k^2+k+1)=k^3+k^2-1$ we have $\Delta(k^3+k^2-1)=1$. To avoid $k^3+(k-1)\times(k+1)=k^3+k^2-1$, being a monochromatic solution in A_1, we consider that $\Delta(k+1)\in\{2,3\}$.

Case 1.1: $A_1\supseteq\{1,k^3,k^3+k^2-1\}$, $A_2\supseteq\{k,k^3+k-1,k+1\}$ and $A_3\supseteq\{k^2,k^2+k+1\}$. Apply Claim I to $(k^3+k^2-1)+(k-1)\times 1=k^3+k^2+k-2$ and $(k^3+k-1)+(k-1)\times(k+1)=k^3+k^2+k-2$ we have $\Delta(k^3+k^2+k-2)=3$. Consider $k\times(k+1)=k^2+k$ and $2\times(k^2+k)+(k-2)\times(k^2+k+1)=k^3+k^2+k-2$ we deduce $\Delta(k^2+k)=1$.

We now obtain a contradiction by noting that one of the following equations must be monochromatic $(k^2+k-1)+(k-1)\times(k^2+k)=k^3+k^2-1$; $k+(k-1)\times(k+1)=k^2+k-1$; and $(k^2+k-1)+(k-1)\times(k^2+k+1)=k^3+k^2+k-2$. We show $k^2+k-1\notin A_i$, $i=1,2,3$, for otherwise $k^2+k-1\subseteq A_i\cap f(A_i)$.

Case 1.2: $A_1\supseteq\{1,k^3,k^3+k^2-1\}$, $A_2\supseteq\{k,k^3+k-1\}$ and $A_3\supseteq\{k^2,k^2+k+1,k+1\}$. To avoid $(k^3+k^2-1)+(k-1)\times 1=k^3+k^2+k-2$ being a monochromatic solution in A_1, we assume that $\Delta(k^3+k^2+k-2)\in\{2,3\}$.

Case 1.2.1: $\Delta(k^3+k^2+k-2)=2$. Apply Claim I to $1+(k-1)\times(k^2+2k+2)=k^3+k^2-1$ and $k+(k-1)\times(k^2+2k+2)=k^3+k^2+k-2$, so we deduce that $\Delta(k^2+2k+2)=3$. Since $(k^3+k^2+k-2)+(k-1)\times k=k^3+2k^2-2$ and $k^2+(k-1)\times(k^2+2k+2)=k^3+2k^2-2$ we have $\Delta(k^3+2k^2-2)=1$. To avoid $k^3+(k^2-k+1)+(k-2)\times 1=k^3+k^2-1$, being a monochromatic solution in A_1, we consider that $\Delta(k^2-k+1)\in\{2,3\}$.

1. $\Delta(k^2-k+1)=2$. Apply Claim I to $k^3+(2k^2-k)+(k-2)\times 1=k^3+2k^2-2$ and $(2k^2-k)+(k-1)\times(k^2-k+1)=k^3+k-1$, to get $\Delta(2k^2-k)=3$. Since $(k^3+k-1)+(2k-1)+(k-2)\times k=k^3+k^2+k-2$ and $k\times(2k-1)=2k^2-k$ we have $\Delta(2k-1)=1$. Consider $(k^3+k^2-k)+(k-1)\times 1=k^3+k^2-1$ and $(2k^2-k)+(k-1)\times k=k^3+k^2-k$ we deduce $\Delta(k^3+k^2-k)=2$.
   We now obtain a contradiction by noting that one of the following equations must be monochromatic $1+(k-1)\times 2=2k-1$; $(k^3+k^2-k)+(k-1)\times 2=k^3+k^2+k-2$; and $1k^2+2+(k-2)\times(k+1)=2k^2-k$. We show $2k^2-k\notin A_i$, $i=1,2,3$, for otherwise $2k^2-k\subseteq A_i\cap f(A_i)$.
2. $\Delta(k^2-k+1)=3$. Apply Claim I to $(k^3-k+1)+(k-1)\times 1=k^3$ and $(k^2-k+1)+(k-1)\times k^2=k^3-k+1$ we have $\Delta(k^3-k+1)=2$. To avoid $(k^3-k+1)+(k-1)\times 2=k^3+k-1$, being a monochromatic solution in A_2, we consider that $\Delta(2)\in\{1,3\}$.
   (a) $\Delta(2k-1)=1$. Apply Claim I to $k^3+(k-1)\times 2=k^3+2k-2$ and $1(k^3+2k-2)+(k-1)\times k=k^3+k^2+k-2$, to get $\Delta(k^3+2k-2)=3$.
   We now obtain a contradiction by noting that one of the following equations must be monochromatic $1+(k-1)\times 2=2k-1$; $(k^3+k-1)+(2k-1)+(k-2)\times k=k^3+k^2+k-2$; and $(2k-1)+(k-1)\times(k^2+k+1)=k^3+2k-2$. We show $2k-1\notin A_i$, $i=1,2,3$, for otherwise $2k-1\subseteq A_i\cap f(A_i)$.

(b) $\Delta(2k-1)=1$. Apply Claim I to $(k^3+k-1)+(2k-1)+(k-2)\times k = k^3+k^2+k-2$ and $(2k-1)+2\times 2+(k-3)\times(k+1)=k^2$, we have $\Delta(2k-1)=1$.
We now obtain a contradiction by noting that one of the following equations must be monochromatic $(2k-1)+(k-1)\times(k^2+k-1)=k^3$; $k+(k-1)\times(k^2+k-1)=k^3-k+1$; and $(k^2-k+1)+(k-1)\times 2=k^2+k-1$. We show $k^2+k-1\notin A_i$, $i=1,2,3$, for otherwise $k^2+k-1\subseteq A_i\cap f(A_i)$.

Case 1.2.2: $\Delta(k^3+k^2+k-2)=3$. To avoid $(k^2+k-1)+(k-1)\times(k^2+k+1)=k^3+k^2+k-2$, being a monochromatic solution in A_3, we consider that $\Delta(k^2+k-1)\in\{1,2\}$.

1. $\Delta(2k-1)=1$. After applying Claim I to $1(k^2+k-1)+(k-1)\times(k^2+k)=k^3+k^2-1$ and $k\times(k+1)=k^2+k$, we obtain $\Delta(k^2+k)=2$. Since $1+(k-1)\times(k+2)=k^2+k-1$ and $(k+2)+(k-1)\times(k+1)=k^2+k+1$, we deduce that $\Delta(k+2)=2$. We have $(2k-1)+(k-1)\times(k^2+k-1)=k^3$ and $(2k-1)+(k-1)\times(k^2+k)=k^3+k-1$ so we obtain $\Delta(2k-1)=3$.
We now obtain a contradiction by noting that one of the following equations must be monochromatic $(k^2+k-1)+(k-1)\times 1=k^2+2k-2$; $k+(k-1)\times(k+2)=k^2+2k-2$; and $(2k-1)+(k-1)\times(k+1)=k^2+2k-2$. We show $k^2+2k-2\notin A_i$, $i=1,2,3$, for otherwise $k^2+2k-2\subseteq A_i\cap f(A_i)$.
2. $\Delta(2k-1)=1$. Apply Claim I to $(k^3-k+1)+(k-1)\times 1=k^3$ and $k+(k-1)\times(k^2+k-1)=k^3-k+1$, to get $\Delta(k^3-k+1)=3$.
We now obtain a contradiction by noting that one of the following equations must be monochromatic $(k^3+k^2-k)+(k-1)\times 1=k^3+k^2-1$; $k\times(k^2+k-1)=k^3+k^2-k$; and $(k^3-k+1)+(k-1)\times(k+1)=k^3+k^2-k$. We show $k^3+k^2-k\notin A_i$, $i=1,2,3$, for otherwise $k^3+k^2-k\subseteq A_i\cap f(A_i)$.

Case 2: $A_1\supseteq\{1,k^3\}$, $A_2\supseteq\{k\}$ and $A_3\supseteq\{k^2,k^3+k-1\}$. To avoid $(k^2+k-1)+(k-1)\times k^2=k^3+k-1$, being a monochromatic solution in A_3, we consider that $\Delta(k^2+k-1)\in\{1,2\}$.

Case 2.1: $A_1\supseteq\{1,k^3,k^2+k-1\}$, $A_2\supseteq\{k\}$ and $A_3\supseteq\{k^2,k^3+k-1\}$. To avoid $(k^2+k-1)+(k-1)\times 1=k^2+2k-2$, being a monochromatic solution in A_1, we assume $\Delta(k^2+2k-2)\in\{2,3\}$.

Case 2.1.1 $A_1\supseteq\{1,k^3,k^2+k-1\}$, $A_2\supseteq\{k,k^2+2k-2\}$ and $A_3\supseteq\{k^2,k^3+k-1\}$. Apply Claim I to $1+(k-1)\times(k+2)=k^2+k-1$ and $k+(k-1)\times(k+2)=k^2+2k-2$, we have $\Delta(k+2)=3$. Since $(2k-1)+(k-1)\times(k^2+k-1)=k^3$ and $2\times(2k-1)+(k-2)\times k=k^2+2k-2$, we deduce $\Delta(2k-1)=3$. We have $(k^2+2k-2)+(k-1)\times k=2k^2+k-2$ and $k^2+(k-1)\times(k+2)=2k^2+k-2$, so we obtain $\Delta(2k^2+k-2)=1$.

We now obtain a contradiction by noting that one of the following equations must be monochromatic $1+2\times(2k^2+k-2)+(k-3)\times(k^2+k-1)$; $k\times(k^2+2k-2)=k^3+2k^2-2k$; and $(k^3+k-1)+(k-1)\times(2k-1)=k^3+2k^2-2k$. We show $k^3+2k^2-2k\notin A_i$, $i=1,2,3$, for otherwise $k^3+2k^2-2k\subseteq A_i\cap f(A_i)$.

Case 2.1.2: $A_1\supseteq\{1,k^3,k^2+k-1\}$, $A_2\supseteq\{k\}$ and $A_3\supseteq\{k^2,k^3+k-1,k^2+2k-2\}$. Apply Claim I to $(k^2+k-1)+(k-1)\times(k^2-1)=k^3$ and $(k^2+2k-2)+(k-1)\times(k^2-1)=k^3+k-1$, to get $\Delta(k^2-1)=2$. To avoid, $k\times(k^2+k-1)=k^3+k^2-k$, being a monochromatic solution in A_1, we consider that $\Delta(k^3+k^2-k)\in\{2,3\}$.

1. $\Delta(2k-1)=1$. Apply Claim I to $1(k^3+k^2-k)+(k-1)\times k=k^3+2k^2-2k$ and $k\times(k^2+2k-2)=k^3+2k^2-2k$, we have $\Delta(k^3+2k^2-2k)=1$. Since $(2k^2-1)+(k-1)\times(k^2+k-1)=k^3+2k^2-2k$ and $(2k^2-1)+(k-1)\times(k^2-1)=k^3+k^2-k$ we deduce $\Delta(2k^2-1)=3$. We have $k^3+(k^2+k-1)+(k-2)\times 1=k^3+k^2+2k-3$ and $(k^2+2k-2)+(2k^2-1)+(k-2)\times k^2=k^3+k^2+2k-3$, so we deduce $\Delta(k^3+k^2+2k-3)=2$. Apply Claim I to $(k-1)+(k-1)\times k=k^2-1$ and $(k^2+2k-2)+(k-1)\times(k-1)=2k^2-1$, to get $\Delta(k-1)=1$. Since $(k^3+k^2-1)+(k-1)\times(k-1)=k^3+2k^2-2k$ and $(2k^2-1)+(k-1)\times k^2=k^3+k^2-1$, we deduce $\Delta(k^3+k^2-1)=2$. Consider $(k^3+k^2-1)+(k-1)\times 2=k^3+k^2+2k-3$ and $k^2+(k-1)\times 2=k^2+2k-2$ we obtain $\Delta(2)=1$. Apply Claim I to $2+(k-1)\times 1=k+1$ and $k^2+(k-1)\times(k+1)=2k^2-1$, to get $\Delta(k+1)=2$.
We now obtain a contradiction by noting that one of the following equations must be monochromatic $(k^3+2k^2-2k)+(k-1)\times 2=k^3+2k^2-2$; $(k^3+k^2-1)+(k-1)\times(k+1)=k^3+2k^2-2$; and $2\times(2k^2-1)+(k-2)\times k^2=k^3+2k^2-2$. We show $k^3+2k^2-2\notin A_i$, $i=1,2,3$, for otherwise $k^3+2k^2-2\subseteq A_i\cap f(A_i)$.
2. $\Delta(2k-1)=1$. Apply Claim I to $(k-1)+(k-1)\times k=k^2-1$ and $\times(k^3+k-1)+(k-1)\times(k-1)=k^3+k^2-k$, we have $\Delta(k-1)=1$. Since $(k^2+k-1)+(k-1)\times(k-1)=2k^2-k$ and $(2k^2-k)+(k-1)\times k^2=k^3+k^2-k$ we deduce $\Delta(2k^2-k)=2$. Once we apply Claim I to $(2k-1)+(k-1)\times(k^2+k-1)=k^3$ and $k\times(2k-1)=2k^2-k$ we have $\Delta(2k-1)=3$.
We now obtain a contradiction by noting that one of the following equations must be monochromatic $k^3+(k-1)\times(k-1)=k^3+k^2-2k+1$; $(2k^2-k)+(k-1)\times(k^2-1)=k^3+k^2-2k+1$; and $(2k-1)+(k-1)\times(k^2+2k-2)=k^3+k^2-2k+1$. We show $k^3+k^2-2k+1\notin A_i$, $i=1,2,3$, for otherwise $k^3+k^2-2k+1\subseteq A_i\cap f(A_i)$.

Case 2.2: $A_1\supseteq\{1,k^3\}$, $A_2\supseteq\{k,k^2+k-1\}$ and $A_3\supseteq\{k^2,k^3+k-1\}$. Apply Claim I to $(k^3-k+1)+(k-1)\times 1=k^3$ and $k+(k-1)\times(k^2+k-1)=k^3-k+1$ we have $\Delta(k^3-k+1)=3$. To avoid $(k^2+k-1)+(k-1)\times k=2k^2-1$ being a monochromatic solution in A_2 we considered that $\Delta(2k^2-1)\in\{1,3\}$.

Case 2.2.1) $A_1\supseteq\{1,k^3,2k^2-1\}$, $A_2\supseteq\{k,k^2+k-1\}$ and $A_3\supseteq\{k^2,k^3-k+1\}$. Apply Claim I to $(k^3-k+1)+(k-1)\times 1=k^3$ and $k+(k-1)\times(k^2+k-1)=k^3-k+1$ to get $\Delta(k^3-k+1)=3$. To avoid $(2k-1)+(k-1)\times k=k^2+k-1$ being a monochromatic solution in A_2 we assume $\Delta(2k-1)\in\{1,3\}$.

1. $\Delta(2k-1)=1$. Apply Claim I to $1+(k-1)\times 2=2k-1$ and $(k^3-k+1)+(k-1)\times 2=k^3+k-1$ to get $\Delta(2)=2$. Since $(k^2-k+1)+(k-1)\times 2=k^2+k-1$ and $(k^2-k+1)+(k-1)\times k^2=k^3-k+1$, we have $\Delta(k^2-k+1)=1$. Consider $(2k-1)+(k-1)\times 2k=2k^2-1$ and $k\times 2=2k$, so we deduce that $\Delta(2k)=3$. After applying Claim I to $(2k^2-1)+(k^2-k+1)+(k-2)\times 1=3k^2-2$ and $2\times(k^2+k-1)+(k-2)\times k=3k^2-2$, we obtain $\Delta(3k^2-2)=3$. We have $(2k-1)+(k^2-k+1)+(k-2)\times 1=k^2+2k-2$, $(k^2+2k-2)+(k-1)\times 2k=3k^2-2$, applying Claim I we deduce $\Delta(k^2+2k-2)=2$. Since $2+(k^2+2k-2)+(k-2)\times k=2k^2$, $k\times 2k=2k^2$, we have $\Delta(2k^2)=1$. Apply Claim I to $(k+1)+2\times(k^2-k+1)+(k-3)\times 1=2k^2$ and $k+(k-1)\times(k+1)=k^2+k-1$, to get $\Delta(k+1)=3$. We now obtain a contradiction by noting that one of the following equations must be monochromatic $k^3+(k^2-k+1)+(k-2)\times 1=k^3+k^2-1$; $(k^2+2k-2)+(k-1)\times(k^2+k-1)=k^3+k^2-1$; and $(k^3-k+1)+2k+(k-2)\times(k+1)=k^3+k^2-1$. We show $k^3+k^2-1\notin A_i$, $i=1,2,3$, for otherwise $k^3+k^2-1\subseteq A_i\cap f(A_i)$.
2. $\Delta(2k-1)=1$. Apply Claim I to $k\times(2k-1)=2k^2-k$ and $k\times(2k-1)=2k^2-k$, we have $\Delta(2k^2-k)=2$. Since $(k^2+k-1)+(k-1)\times(k-1)=2k^2-k$ and $(2k-1)+(k-1)\times(k-1)=k^2$, we deduce $\Delta(k-1)=1$.
We now obtain a contradiction by noting that one of the following equations must be monochromatic $(2k-1)+(k^2+k-2)+(k-2)\times(k-1)=2k^2-1$; $2+(k^2+k-2)+(k-2)\times k=2k^2-k$; and $(2k-1)+(k-1)\times(k^2+k-2)=k^3-k+1$. We show $k^2+k-2\notin A_i$, $i=1,2,3$, for otherwise $k^2+k-2\subseteq A_i\cap f(A_i)$.

Case 2.2.2: $A_1\supseteq\{1,k^3\}$, $A_2\supseteq\{k,k^2+k-1\}$ and $A_3\supseteq\{k^2,k^3-k+1,2k^2-1\}$. Apply Claim I to $k+(k-1)\times(k+1)=k^2+k-1$ and $k^2+(k-1)\times(k+1)=2k^2-1$, we have $\Delta(k+1)=1$. Consider $2+(k-1)\times 1=k+1$ and $(k^3-k+1)+(k-1)\times 2=k^3+k-1$, we deduce $\Delta(2)=2$. Since $(k+1)+(k-1)\times 1=2k$ and $k\times 2=2k$, so we obtain $\Delta(2k)=3$. Apply Claim I to $k^3+(k-1)\times(k+1)=k^3+k^2-1$ and $(2k^2-1)+(k-1)\times k^2=k^3+k^2-1$, to get $\Delta(k^3+k^2-1)=2$.

We now obtain a contradiction by noting that one of the following equations must be monochromatic $k\times(k+1)=k^2+k$; $(k^2+k-1)+(k-1)\times(k^2+k)=k^3+k^2-1$; and $(2k^2-1)+2\times 2k+(k-3)\times(k^2+k)=k^3+k-1$. We show $k^2+k\notin A_i$, $i=1,2,3$, for otherwise $k^2+k\subseteq A_i\cap f(A_i)$.

4.2. Assigning the color 2 to k^3

We consider $\Delta(k^3)=2$, so that $A_1\supseteq\{1\}$, $A_2\supseteq\{k,k^3\}$ and $A_3\supseteq\{k^2\}$. To avoid $k+(k-1)\times(k^2+k)=k^3$, being a monochromatic solution in A_2, we considered $\Delta(k^2+k)\in\{1,3\}$.

Case 1: $A_1\supseteq\{1,k^2+k\}$, $A_2\supseteq\{k,k^3\}$ and $A_3\supseteq\{k^2\}$. To avoid $k^3+(k-1)\times k=k^3+k^2-k$, we assume $\Delta(k^3+k^2-k)\in\{1,3\}$.

Case 1.1: $A_1\supseteq\{1,k^2+k,k^3+k^2-k\}$, $A_2\supseteq\{k,k^3\}$ and $A_3\supseteq\{k^2\}$. To avoid $k\times(k^2+k-1)=k^3+k^2-k$ being a monochromatic solution in A_1 we considered $\Delta(k^2+k-1)\in\{2,3\}$.

Case 1.1.1: $A_1\supseteq\{1,k^2+k,k^3+k^2-k\}$, $A_2\supseteq\{k,k^3,k^2+k-1\}$ and $A_3\supseteq\{k^2\}$. Apply Claim I to $k\times(k+1)=k^2+k$ and $k+(k-1)\times(k+1)=k^2+k-1$, to get $\Delta(k+1)=3$. Since $1+(k-1)\times(k^2+k)=k^3-k+1$ and $k+(k-1)\times(k^2+k-1)=k^3-k+1$ we deduce $\Delta(k^3-k+1)=3$. We have $1(k^2+k-1)+(k-1)\times(k^2-1)=k^3$ and $k^2+(k-1)\times(k^2-1)=k^3-k+1$, so we obtain $\Delta(k^2-1)=1$.

We now obtain a contradiction by noting that one of the following equations must be monochromatic $(2k^2-1)+(k-1)\times(k^2-1)=k^3+k^2-k$; $(k^2+k-1)+(k-1)\times k=2k^2-1$; and $k^2+(k-1)\times(k+1)=2k^2-1$. We show $2k^2-1\notin A_i$, $i=1,2,3$, for otherwise $2k^2-1\subseteq A_i\cap f(A_i)$.

Case 1.1.2: $A_1\supseteq\{1,k^2+k,k^3+k^2-k\}$, $A_2\supseteq\{k,k^3\}$ and $A_3\supseteq\{k^2,k^2+k-1\}$. Apply Claim I to $(k^3+k^2-2k+1)+(k-1)\times 1=k^3+k^2-k$ and $k^2+(k-1)\times(k^2+k-1)=k^3+k^2-2k+1$, we have $\Delta(k^3+k^2-2k+1)=2$. Consider $(k^2+k)+(k-1)\times 1=k^2+2k-1$ and $k+(k-1)\times(k^2+2k-1)=k^3+k^2-2k+1$, we deduce $\Delta(k^2+2k-1)=3$. Since $1+(k-1)\times(k^2+k)=k^3-k+1$ and $(k^3-k+1)+(k-1)\times k=k^3+k^2-2k+1$, we have $\Delta(k^3-k+1)=3$. Apply Claim I to $k^3+(k-1)\times(k-1)=k^3+k^2-2k+1$ and $k^2+(k^2+2k-1)+(k-1)+(k-3)\times(k^2+k-1)=k^3-k+1$, we have $\Delta(k-1)=1$. Since $1+(k-1)\times(k-1)=k^2-2k+2$ and $(k^2+k-1)+(k^2-2k+2)+(k-2)\times k^2=k^3-k+1$, we deduce $\Delta(k^2-2k+2)=2$. After Applying Claim I to $2\times(2k-1)+(k-2)\times(k-1)=k^2+k$ and $k^2+(2k-1)+(k-2)\times(k^2+k-1)=k^3-k+1$, we obtain $\Delta(2k-1)=2$. Consider $(k^3+k^2-k)+(k-1)\times 1=k^3+k^2-1$ and $k^3+(2k-1)+(k-2)\times k=k^3+k^2-1$, we have $\Delta(k^3+k^2-1)=3$.

We now obtain a contradiction by noting that one of the following equations must be monochromatic $(2k^2-k)+(k-1)\times(k^2+k)=k^3+2k^2-2k$; $k\times(2k-1)=2k^2-k$; and $(k^2+k-1)+(2k^2-k)+(k-2)\times k^2=k^3+k^2-1$. We show $2k^2-k\notin A_i$, $i=1,2,3$, for otherwise $2k^2-k\subseteq A_i\cap f(A_i)$.

Case 1.2: $A_1\supseteq\{1,k^2+k\}$, $A_2\supseteq\{k,k^3\}$ and $A_3\supseteq\{k^2,k^3+k^2-k\}$. To avoid $1+(k-1)\times(k^2+k)=k^3-k+1$ being a monochromatic solution in A_1 we considered $\Delta(k^3-k+1)\in\{2,3\}$.

Case 1.2.1: $A_1\supseteq\{1,k^2+k\}$, $A_2\supseteq\{k,k^3,k^3-k+1\}$ and $A_3\supseteq\{k^2,k^3+k^2-k\}$. Apply Claim I to $k+(k-1)\times(k^2+k-1)=k^3-k+1$ and $k\times(k^2+k-1)=k^3+k^2-k$ to get $\Delta(k^2+k-1)=1$. To avoid $k\times(k+1)=k^2+k$, we considered $\Delta(k+1)\in\{2,3\}$.

1. $\Delta(2k-1)=1$. Apply Claim I to $(k^2+k-1)+(k-1)\times 1=k^2+2k-2$ and $k+(k-1)\times(k+2)=k^2+2k-2$, to get $\Delta(k^2+2k-2)=3$. Since $(k^2+k-1)+(k-1)\times(k^2+k)=k^3+k^2-1$ and $(k^3-k+1)+(k-1)\times(k+2)=k^3+k^2-1$, we have $\Delta(k^3+k^2-1)=3$. To avoid $k^2+(k-1)\times 2=k^2+2k-2$, being monochromatic solution en A_3, we assume $\Delta(2)\in\{1,2\}$.

(a) $\Delta(2k-1)=1$. Apply Claim I to $(k^2+k)+2+(k-2)\times 1=k^2+2k$ and $k\times(k+2)=k^2+2k$, we have $\Delta(k^2+2k)=3$. Since $1+(k-1)\times 2=2k-1$ and $(2k-1)+(k-1)\times(k^2+2k)=k^3+k^2-1$, we deduce $\Delta(2k-1)=2$. Consider $k+(k-1)\times(2k-1)=2k^2-2k+1$ and $(k^2+2k-2)+(2k^2-2k+1)+(k-2)\times k^2=k^3+k^2-1$, we have $\Delta(2k^2-2k+1)=1$.
We now obtain a contradiction by noting that one of the following equations must be monochromatic $(2k^2-2k+1)+(k-1)\times 1=2k^2-k$; $k\times(2k-1)=2k^2-k$; and $(2k^2-k)+(k-1)\times k^2=k^3+k^2-k$. We show $2k^2-k\notin A_i$, $i=1,2,3$, for otherwise $2k^2-k\subseteq A_i\cap f(A_i)$.

(b) $\Delta(2k-1)=1$. Apply Claim I to $k+2+(k-2)\times(k+2)=k^2+k-2$ and $(k^2+2k-2)+(k-1)\times(k^2+k-2)=k^3+k^2-k$, we have $\Delta(k^2+k-2)=1$. Since $k^3+(k-1)\times 2=k^3+2k-2$ and $(k^2+2k-2)+(k-1)\times k^2=k^3+2k-2$, we deduce $\Delta(k^3+2k-2)=1$. Consider $2\times(k^2+k-1)+(k-2)\times(k^2+k)=k^3+k^2-2$ and $k^3+2+(k-2)\times(k+2)=k^3+k^2-2$, we obtain $\Delta(k^3+k^2-2)=3$. Once Apply Claim I to $(k^2+k-1)+(k-1)\times(k^2+k-2)=k^3+k^2-2k+1$ and $(k^3-k+1)+(k-1)\times k=k^3+k^2-2k+1$, we deduce $\Delta(k^3+k^2-2k+1)=3$. Since $k\times(k+1)=k^2+k$ and $(k^3-2k+2)+(k-1)\times(k+1)=k^3+k^2-2k+1$, we have $\Delta(k+1)=2$. Consider $k+2\times 2+(k-3)\times(k+2)=k^2-2$ and $k^2+(k-1)\times(k^2-2)=k^3-2k+2$, we deduce $\Delta(k^2-2)=1$.
We now obtain a contradiction by noting that one of the following equations must be monochromatic $(k^2-2)+(k-1)\times 1=k^2+k-3$; $3\times k+(k-3)\times(k+1)=k^2+k-3$; and $(k^2+2k-2)+(k-1)\times(k^2+k-3)=k^3+k^2-2k+1$. We show $k^2+k-3\notin A_i$, $i=1,2,3$, for otherwise $k^2+k-3\subseteq A_i\cap f(A_i)$.

2. $\Delta(2k-1)=1$. Apply Claim I to $1+(k-1)\times(k^2+k-1)=k^3-2k+2$ and $(k^3-2k+2)+(k-1)\times(k+2)=k^3+k^2-k$, to get $\Delta(k^3-2k+2)=2$. Consider $(k^3-2k+2)+(k-1)\times 2=k^3$ and $2\times 2+(k-2)\times(k+2)=k^2$, we have $\Delta(2)=1$. Since $(k^2+k)+2+(k-2)\times 1=k^2+2k$ and $k\times(k+2)=k^2+2k$, so we deduce that $\Delta(k^2+2k)=2$. After applying Claim I to $k\times(k+1)=k^2+k$ and $2\times(k+1)+(k-2)\times(k^2+2k)=k^3-2k+2$, we have $\Delta(k+1)=3$. Since $(k^3-k+1)+(k^2+2k)+(k-2)\times k=k^3+2k^2-k+1$ and $(k^3+k^2-k)+2\times(k+2)+(k-3)\times(k+1)=k^3+2k^2-k+1$, we obtain $\Delta(k^3+2k^2-k+1)=1$. Apply Claim I to $(k^2+k-1)+(k-1)\times 1=k^2+2k-2$ and $2\times(k+1)+(k-2)\times(k+2)=k^2+2k-2$, we have $\Delta(k^2+2k-2)=2$.
We now obtain a contradiction by noting that one of the following equations must be monochromatic $(2k^2+k-2)+2\times(k^2+k)+(k-3)\times(k^2+k-1)=k^3+2k^2-k+1$; $(k^2+2k-2)+(k-1)\times k=2k^2+k-2$; and $k^2+(k-1)\times(k+2)=2k^2+k-2$. We show $2k^2+k-2\notin A_i$, $i=1,2,3$, for otherwise $2k^2+k-2\subseteq A_i\cap f(A_i)$.

Case 1.2.2: $A_1\supseteq\{1,k^2+k\}$, $A_2\supseteq\{k,k^3\}$ and $A_3\supseteq\{k^2,k^3+k^2-k,k^3-k+1\}$. Apply Claim I to $k\times(k+1)=k^2+k$ and $(k^3-k+1)+(k-1)\times(k+1)=k^3+k^2-k$, we have $\Delta(k+1)=2$. Consider $k+(k-1)\times(k+1)=k^2+k-1$ and $k\times(k^2+k-1)=k^3+k^2-k$, we deduce $\Delta(k^2+k-1)=1$. Since $(k^2+1)+(k-1)\times 1=k^2+k$ and $1(k+1)+(k-1)\times k=k^2+1$, so we obtain $\Delta(k^2+1)=3$. After Applying Claim I to $(k^2+k)+(k-1)\times(k^2+k-1)=k^3+k^2-k+1$ and $k^3+(k+1)+(k-2)\times k=k^3+k^2-k+1$, we have $\Delta(k^3+k^2-k+1)=3$. Since $2\times(k^2+k-1)+(k-2)\times(k^2+k)=k^3+k^2-2$ and $k+k^3+(k-2)\times(k+1)=k^3+k^2-2$, we deduce $\Delta(k^3+k^2-2)=3$. Apply Claim I to $(k^2+k-1)+(k-1)\times(k^2+k)=k^3+k^2-1$ and $k^3+(k-1)\times(k+1)=k^3+k^2-1$, to get $\Delta(k^3+k^2-1)=3$. Consider $1+(k-1)\times(k+2)=k^2+k-1$ and $(k^3-k+1)+(k-1)\times(k+2)=k^3+k^2-1$, we have $\Delta(k+2)=2$. Since $(k+2)+(k-1)\times(k+1)=k^2+k+1$ and $k^2+(k-1)\times(k^2+k+1)=k^3+k^2-1$, we deduce $\Delta(k^2+k+1)=1$. We have $2\times(k+2)+(k-2)\times(k+1)=k^2+k+2$ and $2\times(k^2+1)+(k-2)\times(k^2+k+2)=k^3+k^2-2$, so we obtain $\Delta(k^2+k+2)=1$. Since $(k^2+k)+(k-1)\times 1=k^2+2k-1$ and $(k+1)+(k-1)\times(k+2)=k^2+2k-1$, we deduce $\Delta(k^2+2k-1)=3$. Consider $k\times(k^2+k)=k^3+k^2$ and $k^3+(k+2)+(k-2)\times(k+1)=k^3+k^2$, we have $\Delta(k^3+k^2)=3$. Once we apply claim I to $1+2\times(k^2+k+2)+(k-3)\times(k^2+k+1)=k^3+2$ and $2\times(k^2+1)+(k-2)\times k^2=k^3+2$, we obtain $\Delta(k^3+2)=2$. Apply Claim I to $2\times k+(k-2)\times(k+1)=k^2+k-2$ and $(k^2+2k-1)+(k-1)\times(k^2+k-2)=k^3+k^2-k+1$, to get $\Delta(k^2+k-2)=1$. Since $(k^2-1)+(k-1)\times 1=k^2+k-2$ and $k^2+(k-1)\times(k^2-1)=k^3-k+1$, we have $\Delta(k^2-1)=2$. Consider $k+(2k^2)+(k-2)\times(k^2-1)=k^3+2$ and $(2k^2)+(k-1)\times k^2=k^3+k^2$, we deduce $\Delta(2k^2)=1$.

We now obtain a contradiction by noting that one of the following equations must be monochromatic $(2k^2-k+1)+(k-1)\times 1=2k^2$; $(k+2)+(k^2-1)+(k-2)\times k=2k^2-k+1$; and $(2k^2-k+1)+(k-1)\times k^2=k^3+k^2-k+1$. We show $2k^2-k+1\notin A_i$, $i=1,2,3$, for otherwise $2k^2-k+1\subseteq A_i\cap f(A_i)$.

Case 2: $A_1\supseteq\{1\}$, $A_2\supseteq\{k,k^3\}$ and $A_3\supseteq\{k^2,k^2+k\}$. Apply Claim I to $k^3+(k-1)\times k=k^3+k^2-k$ and $k^2+(k-1)\times(k^2+k)=k^3+k^2-k$, to get $\Delta(k^3+k^2-k)=1$. To avoid $k\times(k^2+k-1)=k^3+k^2-k$, being a monochromatic solution in A_1, we consider that $\Delta(k^2+k-1)\in\{2,3\}$.

Case 2.1: $A_1\supseteq\{1\}$, $A_2\supseteq\{k,k^3,k^2+k-1\}$ and $A_3\supseteq\{k^2,k^2+k\}$. Apply Claim I to $k+(k-1)\times(k+1)=k^2+k-1$ and $k\times(k+1)=k^2+k$ we deduce $\Delta(k+1)=1$. Consider $(k^3-k+1)+(k-1)\times(k+1)=k^3+k^2-k$ and $k+(k-1)\times(k^2+k-1)=k^3-k+1$, we have $\Delta(k^3-k+1)=3$. Since $(k^3+k^2-k)+(k-1)\times(k+1)=k^3+2k^2-k-1$ and $k^3+(k^2+k-1)+(k-2)\times k=k^3+2k^2-k-1$, we obtain $\Delta(k^3+2k^2-k-1)=3$. Apply claim I to $(k^2+k-1)+(k-1)\times k=2k^2-1$ and $(2k^2-1)+(k-1)\times(k^2+k)=k^3+2k^2-k-1$, we have $\Delta(2k^2-1)=1$.

We now obtain a contradiction by noting that one of the following equations must be monochromatic $(2k^2-1)+(k-1)\times(k^2-1)=k^3+k^2-k$; $(k^2+k-1)+(k-1)\times(k^2-1)=k^3$; and $k^2+(k-1)\times(k^2-1)=k^3-k+1$. We show $k^2-1\notin A_i$, $i=1,2,3$, for otherwise $k^2-1\subseteq A_i\cap f(A_i)$.

Case 2.2: $A_1\supseteq\{1\}$, $A_2\supseteq\{k,k^3\}$ and $A_3\supseteq\{k^2,k^2+k,k^2+k-1\}$. Apply Claim I to $(k^3+k^2-k)+(k-1)\times 1=k^3+k^2-1$ and $(k^2+k-1)+(k-1)\times(k^2+k)=k^3+k^2-1$, to get $\Delta(k^3+k^2-1)=2$. Since $(k-1)\times(k^2+2k+1)=k^3+k^2-k$

and $k+(k-1)\times(k^2+2k+1)=k^3+k^2-1$, we have $\Delta(k^2+2k+1)=3$. Consider $k^3+(k-1)\times(k+1)=k^3+k^2-1$ and $k\times(k+1)=k^2+k$, we deduce $\Delta(k+1)=1$.

We now obtain a contradiction by noting that one of the following equations must be monochromatic $(k^3+k^2-k)+(k-1)\times(k+1)=k^3+2k^2-k-1$; $(k^3+k^2-1)+(k-1)\times k=k^3+2k^2-k-1$; and $k^2+(k-1)\times(k^2+2k+1)=k^3+2k^2-k-1$. We show $k^3+2k^2-k-1\notin A_i$, $i=1,2,3$, for otherwise $k^3+2k^2-k-1\subseteq A_i\cap f(A_i)$.

4.3. Reformulation as a SAT problem

To prove that the equation $x_1+\cdots+x_k=x_{k+1}$ has a monochromatic solution for every 3-coloring of the integer interval $[1,k^3+2k^2-2]$, it is necessary to prove the following result.

Lemma 4.1. *Let $k\geq 3$ and the set*
$\mathcal{X}_k=\{1,2,k,k+1,k+2,2k,k^2-k+1,k^2-1,k^2,k^2+1,k^2+k-1,k^2+k,k^2+k+1,2k^2-2k+1,2k^2-k,2k^2-k+1,2k^2-1,2k^2+k-2,3k^2-2k,3k^2-k-1,3k^2-2,k^3,k^3+1,k^3+k-1,k^3+k,k^3+k^2-k,k^3+k^2-1,k^3+k^2+k-2,k^3+2k^2-k-1,k^3+2k^2-2\}$, *then it is verified that:*

1. *For every $x\in\mathcal{X}_k$, $1\leq x\leq k^3+2k^2-2$.*
2. *For every 3-coloring of the set $\mathcal{X}_k$ there exists a monochromatic solution to the equation $x_1+x_2+x_3+(k-3)x_4=x_5$.*

Proof.
1. This is trivial.
2. We have checked the result transforming the problem into a Boolean satisfiability problem and solving it with a SAT solver [10].

$\mathcal{Y}=\{y_1,\ldots,y_{30}\}=\{(a,b,c,d):ak^3+bk^2+ck+d\in\mathcal{X}_k$ for any $k\geq 3\}$, i.e., $\mathcal{Y}=\{(0,0,0,1),(0,0,0,2),(0,0,1,0),(0,0,1,1),(0,0,1,2),(0,0,2,0),(0,1,-1,1),(0,1,0,-1),(0,1,0,0),(0,1,0,1),(0,1,1,-1),(0,1,1,0),(0,1,1,1),(0,2,-2,1),(0,2,-1,0),(0,2,-1,1),(0,2,0,-1),(0,2,1,-2),(0,3,-2,0),(0,3,-1,-1),(0,3,0,-2),(1,0,0,0),(1,0,0,1),(1,0,1,-1),(1,0,1,0),(1,1,-1,0),(1,1,0,-1),(1,1,1,-2),(1,2,-1,-1),(1,2,0,-2)\}$.

Let $\Delta:[1,k^3+2k^2-2]\longrightarrow\{1,2,3\}$ be a 3-coloration of $[1,k^3+2k^2-2]$.

We consider two Boolean variables $\phi,\psi:[1,30]\longrightarrow\{True,False\}$ defined as follow:

$$\phi(n)=\begin{cases}True & \text{if}\quad \Delta(y_n)=1\text{ or }2,\\ False & \text{if}\quad \Delta(y_n)=3.\end{cases}$$

$$\psi(n)=\begin{cases}True & \text{if}\quad \Delta(y_n)=1\text{ or }3,\\ False & \text{if}\quad \Delta(y_n)=2.\end{cases}$$

Thus, for any $n\in[1,30]$ we have that $\phi(n)$ is True or $\psi(n)$ is True and

$$\mathcal{D}=\bigwedge_{n=1}^{30}(\phi(n)\vee\psi(n))$$

is satisfiable.

Let $\mathcal{S}$ be the set of $(n_1,n_2,n_3,n_4,n_5)\in[1,30]^5$, with $n_1\leq n_2\leq n_3$, $y_{n_i}=(a_i,b_i,c_i,d_i)$, $a_4=0$ and $y_{n_5}=y_{n_1}+y_{n_2}+y_{n_3}+(b_4,c_4,d_4,0)-3y_{n_4}=(a_1+a_2+a_3+b_4,b_1+b_2+b_3+c_4-3b_4,c_1+c_2+c_3+d_4-3c_4,d_1+d_2+d_3-3d_4)$.

For any $s=(n_1,n_2,n_3,n_4,n_5)\in\mathcal{S}$, we consider three clauses:

$$p(s)=\bigvee_{i=1}^{5}(\neg\phi(n_i)\vee\neg\psi(n_i)),$$

$$q(s)=\bigvee_{i=1}^{5}(\neg\phi(n_i)\vee\psi(n_i)),\ \text{and}$$

$$r(s)=\bigvee_{i=1}^{5}(\phi(n_i)\vee\neg\psi(n_i)).$$

Then, $p(s)$ is satisfiable if and only if $\Delta(x_n)\neq 1$ for some $n\in s$, $q(s)$ is satisfiable if and only if $\Delta(x_n)\neq 2$ for some $n\in s$; and $r(s)$ is satisfiable if and only if $\Delta(x_n)\neq 3$ for some $n\in s$. Thus Δ does not induce on s a monochromatic solution of the equation $x_1+x_2+x_3+(k-3)x_4=x_5$ if and only if $p(s)\wedge q(s)\wedge r(s)$ is satisfiable.

Let $\mathcal{C}=\bigwedge_{s\in\mathcal{S}}(p(s)\wedge q(s)\wedge r(s))$.

Clearly $\mathcal{C}\wedge\mathcal{D}$ is satisfiable if and only if the restriction of Δ to $\mathcal{X}_k$ is a 3-coloring without monochromatic solution to the equation$x_1+x_2+x_3+(k-3)x_4=x_5$.

The SAT-Solver shows that $\mathcal{C} \wedge \mathcal{D}$ is not satisfiable, therefore there does not exist a 3-coloring of the sets $\mathcal{X}_k$ (and hence the integer interval $[1, k^3+2k^2-2]$) without monochromatic solution of the $x_1+x_2+x_3+(k-3)x_4 = x_5$. A monochromatic solution of $x_1+x_2+x_3+(k-3)x_4 = x_5$ induces a monochromatic solution of $x_1+\cdots+x_k = x_{k+1}$. Hence $S_k(3) \le k^3+2k^2-2$. ■

Therefore, we conclude with the following result:

Theorem 4.2. *We have* $S_k(3) = k^3 + 2k^2 - 2$, *for* $k \ge 3$.

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