Compactness in quasi-Banach function spaces with applications to L^1 of the semivariation of a vector measure

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Abstract

We characterize the relatively compact subsets of the order continuous part E_a of a quasi-Banach function space E showing that the strong connection between compactness, uniform absolute continuity, uniform integrability, almost order boundedness and L-weak compactness that appears in the classical setting of Lebesgue spaces remains almost invariant in this new context under mild assumptions. We also present a de la Vallée–Poussin type theorem in this context that allows us to locate each compact subset of E_a as a compact subset of a smaller quasi-Banach Orlicz space E^{Φ}. Our results generalize the previous known results for the Banach function spaces $L^1(m)$ and $L^1_w(m)$ associated to a vector measure m and moreover they can also be applied to the quasi-Banach function space $L^1(\|m\|)$ associated to the semivariation of m.

Keywords Orlicz spaces · Vector measure · Semivariation · Uniform integrability · Uniform absolute continuity · Compactness · De la Vallée–Poussin's theorem

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1 Introduction

Different kinds of spaces of scalar integrable functions associated to a vector measure m, with values in a Banach space, have become basic tools in several aspects of the study of function spaces and operators among them, as the representation of Banach lattices by using function spaces or the description of the optimal domain of certain operators between function spaces. One of these function spaces is the quasi-Banach space $L^1(\|m\|)$ of integrable scalar functions (in the Choquet sense) with respect to the semivariation $\|m\|$ of the vector measure m. This semivariation is an special type of what is known as a capacity, for which the Choquet integration was built. The space $L^1(\|m\|)$ is the cornerstone in the construction of the Lorentz spaces with respect to the vector measure m (see [4,11]) and plays a similar role to the one made by the Lebesgue space L^1 for classical Lorentz spaces in the case of a positive scalar measure.

Some properties as reflexivity, σ-order continuity, sequential Fatou property or denseness of simple functions have been already studied in [5] for the space $L^1(\|m\|)$ and its p-th powers $L^p(\|m\|)$, with $p > 1$, when the measure m is defined over a δ-ring of subsets. Here we will deal with the characterization of the relatively compact subsets of $L^1(\|m\|)$, where the measure m is defined now over a σ-algebra of subsets.

The first major goal of this work (see Corollary 3.3) is the characterization of the relative compactness on $L^1(\|m\|)$ by means of the relative compactness in the topology of *convergence in measure* together with another additional property that can be the *almost order boundedness*, the *L-weak compactness*, the *uniform integrability* or the *uniform absolute continuity*. This result will be obtained as a consequence of a more general study (see Proposition 3.2 and Corollary 3.1) that we will do about the characterization of the relative compactness in general quasi-Banach function spaces. As it is well known, relative compactness in metric spaces is characterized by *convergence of sequences*. Therefore, the basis of this study consists of characterizing the convergence of sequences by means of a weaker convergence (convergence in measure) together with an additional property such as those mentioned above. This is what we do in Proposition 3.2 that can be considered as an abstract version of the classical Vitali convergence theorem that characterizes the convergence in Lebesgue L^p-spaces, with $1 \leq p < \infty$, in terms of the convergence in measure (or even pointwise μ-a.e.) and some additional conditions. In particular, for a positive finite measure μ, this result (see [10, Theorem III.3.6]) states that, for a sequence of functions $(f_n)_n$ in $L^p(\mu)$ and a function $f : \Omega \longrightarrow \mathbb{R}$, the following two conditions are equivalent:

i) $f \in L^p(\mu)$ and $\|f_n - f\|_{L^p(\mu)} \to 0$, as $n \to \infty$,
ii) $(f_n)_n$ converges in measure to f and $\lim_{\mu(A)\to 0} \sup_{n\geq 1} \|f_n \chi_A\|_{L^p(\mu)} = 0$.

Also we would like to note that equivalence 1) ⇔ 5) in Proposition 3.2 and Corollary 3.1 has been proved by Caetano et al. (see [3, Lemma 3.15 and Theorem 3.17]) under some different conditions. They adopt the Bennett–Sharpley definition for a quasi-Banach function space for a σ-finite measure (see [3, Definition 3.2]). Consequently its ambient space always has implicitly the sequential Fatou property (see [3, Lemma 3.5]). It is worth noting that we do not use this property as an hypothesis in our results to establish that equivalence.

Probably one of the first results that relates compactness to uniform absolute continuity in a Banach function space appeared in [16, Theorem 7.1] (see also [2, Chap. 1 Exercise 8]). At the same time, it is well known the relationship between weak compactness of a set and the notion of uniform integrability provided by the Dunford–Pettis theorem (see, for example, [8]). Note that weak compactness may not make sense in the context of quasi-Banach spaces, but it can be replaced by L-weak compactness in a broad class of spaces. In this context, the equivalence between L-weak compactness and uniform integrability can be as useful as the Dunford–Pettis theorem. This equivalence and some others are presented in Proposition 3.1 which is the key to obtain Proposition 3.2 and Corollary 3.1.

The second major goal of this work is to *locate* the relatively compact subsets of $L^1(\|m\|)$ inside another smaller Orlicz-type spaces (see Corollary 5.3). To accomplish this it would be necessary to obtain a *de la Vallée–Poussin* type theorem characterizing (from another point of view) uniformly integrable subsets of $L^1(\|m\|)$ similar to [6, Theorem 4.1]. Recall that the classical result of the de la Vallée–Poussin states that a set $H \subseteq L^1(\mu)$ is uniformly integrable (for a positive finite measure μ) if and only if there is a non-decreasing convex function $\Phi : [0, \infty) \longrightarrow [0, \infty)$ with $\lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$ such that $\sup_{f\in H} \|\Phi(|f|)\|_{L^1(\mu)} < \infty$ or, equivalently, if and only if H is (norm) bounded in some classical Orlicz space $L^\Phi(\mu)$. In fact, we will establish this result for a general quasi-Banach function space E (see Theorem 5.1). In our case the main spaces taking part now are the Orlicz spaces E^Φ associated to E instead of $L^\Phi(\mu)$ that have been considered and studied in [7] (see also [13] and [17]). Some results on compactness and inclusions of these Orlicz spaces will be also necessary for our study and they will be considered in Sect. 4.

As far as we know our Theorem 5.1 is the first completely general de la Vallée–Poussin type result for quasi-Banach function spaces, even in the framework of Banach function spaces. However, its connection with similar results which can be found in the literature, as [12, Proposition 6.6] and [15, Theorem 4], is clear. These two quoted results are particular cases of our theorem.

2 Notation and preliminaries

Throughout this paper, we will always assume that Ω is a nonempty set, Σ is a σ-algebra of subsets of Ω, and μ is a positive finite measure defined on Σ. Let $L^0(\mu)$ be the space of (μ-a.e. equivalence classes of) real valued measurable functions defined on Ω. The natural topology on $L^0(\mu)$ is given by the complete metric $d(f, g) := \int_\Omega \frac{|f-g|}{1+|f-g|} d\mu$, for all $f, g \in L^0(\mu)$. It is folklore that convergence of sequences in this topology is exactly the *convergence in measure*, that is, a sequence $(f_n)_n$ of measurable functions converges under d to a measurable function f if and only if $\lim_{n\to\infty} \mu(\{w \in \Omega : |f_n(w) - f(w)| \geq \varepsilon\}) = 0$ for all $\varepsilon > 0$.

In what follows we will denote by $[|f_n - f| \geq \varepsilon]$ the measurable set

$$\{w \in \Omega : |f_n(w) - f(w)| \geq \varepsilon\}.$$

This notation and similar ones will be used frequently throughout this work.

Recall that a (real) *quasi-Banach space* E is a complete metrizable real vector space whose topology is given by a quasi-norm $\|\cdot\|_E$ satisfying

$$\begin{aligned} \|x\|_E &> 0 \quad (x \in E, x \neq 0) \\ \|ax\|_E &= |a|\|x\|_E \quad (a \in \mathbb{R}, x \in E) \\ \|x + y\|_E &\leq K\left(\|x\|_E + \|y\|_E\right) \quad (x, y \in E), \end{aligned} \tag{2.1}$$

where $K \geq 1$ is a constant depending only of E. The smallest of all those constants will be denoted by K_E and is called the *quasi-norm constant* of E. If in addition E is a vector lattice and $\|x\|_E \leq \|y\|_E$ whenever $|x| \leq |y|$ we say that E is a *quasi-Banach lattice*. A quasi-Banach space $E \subseteq L^0(\mu)$ is called a *quasi-Banach function space* with respect to μ if it has the following properties:

(a) E is an order ideal of $L^0(\mu)$ and a quasi-Banach lattice with respect to the μ-a.e. order, that is, if $f \in L^0(\mu)$, $g \in E$ and $|f| \le |g|$ μ-a.e., then $f \in E$ and $\|f\|_E \le \|g\|_E$.
(b) The characteristic function of Ω, denoted by χ_Ω, belongs to E.

If E is a Banach space instead of a quasi-Banach space we say that E is a Banach function space with respect to μ (see [20, p. 23]). For a quasi-Banach space E we will denote by $B_E := \{x \in E : \|x\|_E \le 1\}$ its unit ball.

The following result will be useful in the sequel. It proof can be found in [20, Lemma 2.7].

Lemma 2.1 *Let E be a quasi-Banach function space with respect to μ. Then the inclusion $E \subseteq L^0(\mu)$ is continuous. In particular,* $\lim_{\|\chi_A\|_E \to 0} \mu(A) = 0$, *that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \varepsilon$ whenever $\|\chi_A\|_E < \delta$.*

A quasi-Banach lattice E is said to be σ*-order continuous* if, for every decreasing sequence $(x_n)_n$ in E with $\inf_n x_n = 0$ we have $\|x_n\|_E \to 0$. If the above condition holds for *nets* instead of sequences, then E is said to be *order continuous*. In the case of quasi-Banach function spaces both concepts coincide (see [20, Remark 2.5]). Note that a quasi-Banach function space E is σ*-order continuous* if and only if for any positive increasing sequence $(f_n)_n$ in E such that $f_n \to f \in E$, μ-a.e., then $\|f - f_n\|_E \to 0$. Besides, we say that a quasi-Banach function space E has the *sequential Fatou property* if for any positive increasing sequence $(f_n)_n$ in E with $\sup_n \|f_n\|_E < \infty$ and $f_n \to f \in L^0(\mu)$ μ-a.e., then $f \in E$ and $\|f\|_E = \sup_n \|f_n\|_E$.

An element x of a quasi-Banach lattice E is σ*-order continuous* if it has the property that $\|x_n\|_E \to 0$ for any decreasing sequence $(x_n)_n \subseteq E$ satisfying that $\inf_n x_n = 0$ and $0 \le x_n \le |x|$. The collection of all σ-order continuous elements of E is denoted by E_a and it is the (closed) maximal σ-order continuous order ideal of E. It is called the *order continuous part* of E.

3 Compactness in quasi-Banach function spaces

In the first part of this section we present general results about the characterization of relatively compact subsets of a quasi-Banach function space E over a finite positive measure μ. In the second part we introduce the space $L^1(\|m\|)$ associated to a vector measure m and particularize these compactness results for this space.

Now we recall the definitions of the four main concepts that we will use to characterize relative compact subsets of a quasi-Banach function space. It is well known that uniform absolute continuity and uniform integrability are closely connected in the setting of Banach function spaces. Moreover these two notions have their counterpart (namely, almost order bounded and L-weak compactness, see [18, Definition 3.6.1]) in the context of Banach and quasi-Banach lattices.

Definition 3.1 Let E be a quasi-Banach function space over a finite positive measure μ.

(1) A subset $H \subseteq E$ is said to be *uniformly absolutely continuous* if

$$\lim_{\mu(A)\to 0} \sup_{f\in H} \|f\chi_A\|_E = 0,$$

that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|f\chi_A\|_E < \varepsilon$ for all $f \in H$ and $A \in \Sigma$, with $\mu(A) < \delta$.

(2) A subset $H \subseteq E$ is said to be *uniformly integrable* if $\lim_{c\to\infty} \sup_{f\in H} \left\| f\chi_{[|f|>c]} \right\|_E = 0$.

Let E be a quasi-Banach lattice.

(3) A (quasi-norm) bounded subset $H \subseteq E$ is said to be *L-weakly compact* if $\|f_n\|_E \to 0$ for every disjoint sequence $(f_n)_n$ in the solid hull of H.
(4) A subset $H \subseteq E$ is said to be *almost order bounded* if for every $\varepsilon > 0$ there exists an element $0 < g \in E_a$ such that $H \subseteq [-g, g] + \varepsilon B_E$.

Remark 3.1 Note that every almost order bounded subset of E is also (quasi-norm) bounded, and by definition, an L-weakly compact subset is (quasi-norm) bounded too. Also, every uniformly integrable set H is (quasi-norm) bounded. In fact, for a large enough c such that $\left\| f\chi_{[|f|>c]} \right\|_E \le 1$ for every $f \in H$ we have

$$\begin{aligned}\|f\|_E &= \left\| f\chi_{[|f|>c]} + f\chi_{[|f|\le c]} \right\|_E \le K_E \left\| f\chi_{[|f|>c]} \right\|_E + K_E \left\| f\chi_{[|f|\le c]} \right\|_E \\ &\le K_E + cK_E \left\| \chi_{[|f|\le c]} \right\|_E \le K_E \left(1 + c\,\|\chi_\Omega\|_E\right).\end{aligned}$$

The fact that uniform absolute continuity does not imply (quasi-norm) boundedness belongs to the folklore and, in fact, one atom suffices to construct an example. If Ω is a singleton, $\Sigma := \{\Omega, \varnothing\}$ and μ is defined by $\mu(\Omega) = 1$ and $\mu(\varnothing) = 0$, then $L^1(\mu) \equiv \mathbb{R}$ is uniformly absolutely continuous. On the other hand, let E be a quasi-Banach function space over a non-atomic (finite) measure μ. Then every uniformly absolutely continuous set $H \subseteq E$ is (quasi-norm) bounded. Indeed, choose $\delta > 0$ such that $\sup_{f\in H} \|f\chi_A\|_E \le 1$ whenever $\mu(A) \le \delta$. Since μ is non-atomic, there is a finite partition $\{A_1, \dots, A_n\}$ of Ω such that $A_r \in \Sigma$ and $\mu(A_r) \le \delta$ for all $r = 1, \dots, n$. Now, for $f \in H$ we have

$$\|f\|_E = \left\| \sum_{r=1}^{n} f\chi_{A_r} \right\|_E \le \sum_{r=1}^{n} K_E^r \left\| f\chi_{A_r} \right\|_E \le M := \sum_{r=1}^{n} K_E^r.$$

Remark 3.2 If E is a Banach lattice, then a set $H \subseteq E$ is *L-weakly compact* if and only if H is *almost order bounded*. This equivalence is not known to be true in the context of quasi-Banach lattices. As far as we know, its proof for Banach lattices (see [18, Proposition 3.6.2] and [19, Satz II.2]) does not seem to be easily adaptable for quasi-Banach lattices. However, as we will see in the next Proposition 3.1, this equivalence ends up being true for subsets of a σ-order continuous quasi-Banach function space.

Remark 3.3 On the other hand, an extremely special case of the equivalence between L-weak compactness and almost order boundedness of subsets of a general quasi-Banach lattice E is known to be true. Indeed, for every $0 < x \in E$, the following assertions are equivalent:

i) *Every order increasing sequence in* $[0, x]$ *is convergent.*
ii) *Every disjoint sequence in* $[0, x]$ *converges to* 0.

The proof of $i) \Leftrightarrow ii)$ follows (up to some changes) the same argument of the well-known case of Banach lattices proved first by Fremlin and Meyer–Nieberg (see [1, Theorem 12.12] and [1, Theorem 12.13]).

In any case, every element $0 < x \in E$ satisfying $i)$ or $ii)$ must belong to E_a. This allows to deduce that *if* H *is an L-weakly compact subset of* E, *then* $H \subseteq E_a$. Indeed, if $0 \ne x \in H$ and H is L-weakly compact, then $[0, |x|]$ is L-weakly compact and hence $ii)$ is satisfied.

Uniform absolute continuity can be thought as a kind of uniform σ-order continuity in quasi-Banach function spaces. In fact, the (uniform) absolute continuity of a singleton $\{g\}$ is equivalent to the σ-order continuity of the element g. Moreover, it can be also described in terms of *decreasing* sequences or *disjoint* sequences of sets.

Lemma 3.1 *Let E be a quasi-Banach function space with respect to μ and $g \in E$. The following conditions are equivalent.*

1) $g \in E_a$.
2) $\lim_{\mu(A)\to 0} \|g\chi_A\|_E = 0$.
3) $\lim_{n\to\infty} \left\|g\chi_{A_n}\right\|_E = 0$ *for every decreasing sequence* $(A_n)_n \subseteq \Sigma$ *with* $\mu(A_n) \to 0$.
4) $\lim_{n\to\infty} \left\|g\chi_{B_n}\right\|_E = 0$ *for every disjoint sequence* $(B_n)_n \subseteq \Sigma$.

Proof 1) $\Rightarrow$ 2) Suppose that 2) is false. Then there exist $\varepsilon > 0$ and a sequence $(C_n)_n \subseteq \Sigma$ with $\mu(C_n) < \frac{1}{2^n}$ and $\left\|g\chi_{C_n}\right\|_E \geq \varepsilon$ for all $n = 1, 2, \ldots$ Take the subsets $A_n := \bigcup_{k\geq n} C_k$ and the functions $g_n := |g|\chi_{A_n}$ for all $n = 1, 2, \ldots$ Note that $(A_n)_n$ is decreasing and $\mu(A_n) \leq \frac{1}{2^{n-1}} \to 0$, as $n \to \infty$. Therefore, $(g_n)_n$ is decreasing, $g_n \to 0$ μ-a.e. and $g_n \leq |g|$, but $\|g_n\|_E = \left\|g\chi_{A_n}\right\|_E \geq \left\|g\chi_{C_n}\right\|_E \geq \varepsilon$ for all $n = 1, 2, \ldots$ which contradicts 1).
2) $\Rightarrow$ 3) Evident.
3) $\Rightarrow$ 4) Let $(B_n)_n \subseteq \Sigma$ a disjoint sequence. Take $A_n := \bigcup_{k\geq n} B_k$ for all $n = 1, 2, \ldots$ Thus $(A_n)_n$ is a decreasing sequence in Σ and $\mu(A_n) \to 0$ as $n \to \infty$. Moreover $\left\|g\chi_{B_n}\right\|_E \leq \left\|g\chi_{A_n}\right\|_E \to 0$, and the conclusion follows.
4) $\Rightarrow$ 1) Let $(g_n)_n$ be a disjoint sequence in $[0, |g|]$. Put $B_n := [|g_n| \neq 0]$ for all $n = 1, 2, \ldots$ Evidently $(B_n)_n$ is a disjoint sequence and $0 \leq g_n = g_n\chi_{B_n} \leq |g|\chi_{B_n}$ for all $n = 1, 2, \ldots$ Thus $\|g_n\|_E \leq \left\|g\chi_{B_n}\right\|_E \to 0$, which implies that $\|g_n\|_E \to 0$. According to Remark 3.3 this means that $g \in E_a$. □

Actually, uniform absolute continuity can be described in terms of *decreasing* sequences or *disjoint* sequences of sets for any set H and not only for singletons.

Lemma 3.2 *Let E be a quasi-Banach function space with respect to μ and $H \subseteq E$. The following conditions are equivalent.*

1) *H is uniformly absolutely continuous.*
2) $\lim_{n\to\infty} \sup_{f\in H} \left\|f\chi_{A_n}\right\|_E = 0$ *for every decreasing sequence* $(A_n)_n \subseteq \Sigma$ *with* $\mu(A_n) \to 0$.
3) $\lim_{n\to\infty} \sup_{f\in H} \left\|f\chi_{B_n}\right\|_E = 0$ *for every disjoint sequence* $(B_n)_n \subseteq \Sigma$.

Moreover, if any of the above conditions is satisfied, then $H \subseteq E_a$.

Proof First, note that Lemma 3.1 guarantees that $H \subseteq E_a$ provided it satisfies any of the above conditions.
1) $\Rightarrow$ 2) Evident.
2) $\Rightarrow$ 1) Suppose that 1) is false. Then there exist $\varepsilon > 0$, a sequence $(f_n)_n \subseteq H$ and another sequence $(C_n)_n \subseteq \Sigma$ with $\mu(C_n) < \frac{1}{2^n}$ and $\left\|f_n\chi_{C_n}\right\|_E \geq \varepsilon$ for all $n = 1, 2, \ldots$ Take $A_n := \bigcup_{k\geq n} C_k$ for all $n = 1, 2, \ldots$ Note that $(A_n)_n$ is decreasing and $\mu(A_n) \leq \frac{1}{2^{n-1}} \to 0$, as $n \to \infty$. On the other hand, $|f_n|\chi_{A_n} \geq |f_n|\chi_{C_n}$ for all $n = 1, 2, \ldots$ and then $\left\|f_n\chi_{A_n}\right\|_E \geq \left\|f_n\chi_{C_n}\right\|_E \geq \varepsilon$ for all $n = 1, 2, \ldots$ which contradicts 2).
2) $\Rightarrow$ 3) Proceed as in the implication 3) $\Rightarrow$ 4) of Lemma 3.1.
3) $\Rightarrow$ 2) Assume that 2) is false. Then, there exist $\varepsilon > 0$, a sequence $(f_n)_n \subseteq H$ and a decreasing sequence $(A_n)_n \subseteq \Sigma$ with $\mu(A_n) \to 0$, such that $\left\|f_n\chi_{A_n}\right\|_E > \varepsilon$ for all $n = 1, 2, \ldots$ By Lemma 3.1, $\left\|f_1\chi_{A_n}\right\|_E \to 0$. Then there exists $n_1 \geq 1$ such that

$$\left\|f_1\chi_{A_{n_1}}\right\|_E < \frac{1}{K_E}\left(\left\|f_1\chi_{A_1}\right\|_E - \varepsilon\right) > 0.$$

Put $B_1 := A_1 \setminus A_{n_1}$. Then $f_1 \chi_{B_1} = f_1 \chi_{A_1} - f_1 \chi_{A_{n_1}}$ and having in mind that K_E is the quasi-norm constant of E we get

$$\| f_1 \chi_{B_1} \|_E \geq \frac{1}{K_E} \| f_1 \chi_{A_1} \|_E - \left\| f_1 \chi_{A_{n_1}} \right\|_E > \frac{\varepsilon}{K_E}.$$

Again, Lemma 3.1 says that $\| f_{n_1} \chi_{A_n} \|_E \to 0$. Then there exists $n_2 > n_1$ such that

$$\left\| f_{n_1} \chi_{A_{n_2}} \right\|_E < \frac{1}{K_E} \left(\left\| f_{n_1} \chi_{A_{n_1}} \right\|_E - \varepsilon \right) > 0.$$

Put $B_2 := A_{n_1} \setminus A_{n_2}$. Then $f_{n_1} \chi_{B_2} = f_{n_1} \chi_{A_{n_1}} - f_{n_1} \chi_{A_{n_2}}$, and consequently

$$\| f_{n_1} \chi_{B_2} \|_E \geq \frac{1}{K_E} \left\| f_{n_1} \chi_{A_{n_1}} \right\|_E - \left\| f_{n_1} \chi_{A_{n_2}} \right\|_E > \frac{\varepsilon}{K_E}.$$

By continuing in this way we construct a sequence $(f_{n_k})_k \subseteq H$ and a disjoint sequence $(B_k)_k \subseteq \Sigma$ such that $\| f_{n_k} \chi_{B_{k+1}} \|_E > \frac{\varepsilon}{K_E}$ for all $k = 1, 2, \ldots$ which contradicts the hypothesis 3). □

We are now in a position to clarify the relationships that the four concepts introduced in Definition 3.1 possess in quasi-Banach function spaces.

Proposition 3.1 (see [20, Lemma 2.37]) *Let E be a quasi-Banach function space with respect to μ and let $H \subseteq E$. Consider the following conditions:*

1) *H is almost order bounded.*
2) *H is (quasi-norm) bounded and uniformly absolutely continuous.*
3) *H is L-weakly compact.*
4) *H is uniformly integrable.*

Then 1) $\Rightarrow$ 2) $\Leftrightarrow$ 3) $\Rightarrow$ 4). *Moreover, if $L^\infty(\mu) \subseteq E_a$, then* 4) $\Rightarrow$ 1). *In particular, if E is σ-order continuous then all conditions* 1) $-$ 4) *are equivalent.*

Proof 1) $\Rightarrow$ 2) As we have said in Remark 3.1, H is a (quasi-norm) bounded set. To show that it is also uniformly absolutely continuous take any $\varepsilon > 0$ and let $0 < g \in E_a$ be such that $H \subseteq [-g, g] + \frac{\varepsilon}{2K_E} B_E$. Now, by using Lemma 3.1, there exists $\delta > 0$ such that $\| g \chi_A \|_E < \frac{\varepsilon}{2K_E}$ for all $A \in \Sigma$ with $\mu(A) < \delta$. On the other hand, for every $f \in H$ there exists $h \in B_E$ such that $\left| f - \frac{\varepsilon}{2K_E} h \right| \leq g$. Thus, if $A \in \Sigma$ and $\mu(A) < \delta$, we have

$$\begin{aligned} \| f \chi_A \|_E &\leq K_E \left\| f \chi_A - \frac{\varepsilon}{2K_E} h \chi_A \right\|_E + K_E \left\| \frac{\varepsilon}{2K_E} h \chi_A \right\|_E \\ &\leq K_E \| g \chi_A \|_E + K_E \frac{\varepsilon}{2K_E} \| h \chi_A \|_E \leq \varepsilon. \end{aligned}$$

2) $\Rightarrow$ 3) Let $(f_n)_n$ be a disjoint sequence in the solid hull of H. Then there exist $g_n \in H$ such that $|f_n| \leq |g_n|$ for all $n = 1, 2, \ldots$ Put $B_n := [|f_n| \neq 0]$ for all $n = 1, 2, \ldots$ We have $|f_n| = |f_n| \chi_{B_n} \leq |g_n| \chi_{B_n}$ for all $n = 1, 2, \ldots$ and hence

$$\| f_n \|_E \leq \| g_n \chi_{B_n} \|_E \leq \sup_{g \in H} \| g \chi_{B_n} \|_E \to 0$$

by Lemma 3.2, which proves 3).
3) $\Rightarrow$ 2) Take a disjoint sequence $(B_n)_n$ in Σ. For every $n = 1, 2, \ldots$ choose $f_n \in H$ such that $\sup_{f \in H} \| f \chi_{B_n} \|_E \leq \| f_n \chi_{B_n} \|_E + \frac{1}{n}$. Then the conclusion follows by Lemma 3.2

because the disjoint sequence $\left(f_n\chi_{B_n}\right)_n$ belongs to the solid hull of H and so, $\left\|f_n\chi_{B_n}\right\|_E \to 0$ as $n \to \infty$.

2) $\Rightarrow$ 4) Suppose that H is (quasi-norm) bounded and uniformly absolutely continuous. Let us put $M := \sup_{f\in H}\|f\|_E < \infty$. Now, by taking into account that the inequality $c\chi_{[|f|>c]} \le |f|\chi_{[|f|>c]}$ holds for all $c>0$ and $f \in E$, we get

$$c\left\|\chi_{[|f|>c]}\right\|_E \le \left\|f\chi_{[|f|>c]}\right\|_E \le \|f\|_E \le M$$

for all $c>0$ and all $f \in H$. Therefore

$$\left\|\chi_{[|f|>c]}\right\|_E \le \frac{M}{c} \tag{3.1}$$

for all $c>0$ and all $f \in H$. Moreover, given $\varepsilon>0$ there exists $\delta>0$ such that $\|f\chi_A\|_E<\varepsilon$ for all $f\in H$ and $A\in\Sigma$, with $\mu(A)<\delta$. Then, taking into account Lemma 2.1, for this $\delta>0$ there exists $\delta_1>0$ such that $\mu(A)<\delta$ for all $A\in\Sigma$ with $\|\chi_A\|_E<\delta_1$. Now choose $c_1>0$ such that $\frac{M}{c_1}<\delta_1$. If $c\ge c_1$, according to (3.1), for each $f\in H$ we have

$$\left\|\chi_{[|f|>c]}\right\|_E \le \frac{M}{c} \le \frac{M}{c_1} < \delta_1,$$

and accordingly $\mu\left([|f|>c]\right)<\delta$. Thus $\left\|f\chi_{[|f|>c]}\right\|_E<\varepsilon$ for all $f\in H$ and $c\ge c_1$ and 4) is proved.

4) $\Rightarrow$ 1) Let us suppose now that $L^\infty(\mu)\subseteq E_a$. If H is uniformly integrable, given $\varepsilon>0$ there exists $c>0$ such that $\left\|f\chi_{[|f|>c]}\right\|_E<\varepsilon$ for all $f\in H$. Then, for any $f\in H$ we have $f=f\chi_{[|f|\le c]}+f\chi_{[|f|>c]}$. But $f\chi_{[|f|\le c]}\in[-c\chi_\Omega, c\chi_\Omega]$ and $f\chi_{[|f|>c]}\in\varepsilon B_E$. Thus H is almost order bounded because $\chi_\Omega\in E_a$. □

Now we are going to relate the four concepts mentioned above to compactness. The first link is a direct consequence of the concept of relative compactness in quasi-Banach lattices.

Lemma 3.3 *Let E be a quasi-Banach lattice. Every relatively compact set H in E_a is almost order bounded.*

Proof Let $\varepsilon>0$. There exist $x_1,\ldots,x_n$ in H such that $H\subseteq\bigcup_{i=1}^n(x_i+\varepsilon B_E)$. Taking $x:=\max\{|x_1|,\ldots,|x_n|\}$, it follows that $H\subseteq[-x,x]+\varepsilon B_E$ and $x\in E_a$ since E_a is an order ideal. Therefore H is almost order bounded. □

The next Proposition 3.2 and Corollary 3.1 are partially inspired by [3]. In particular, the equivalence 1) $\Leftrightarrow$ 5) was proved there (see [3, Lemma 3.15 and Theorem 3.17]) under the implicit assumption that E has the sequential Fatou property. In the context of Banach function spaces with the Fatou property, the equivalence 1) $\Leftrightarrow$ 5) is also described in [2, Chap. 1, Exercise 8].

Proposition 3.2 *Let E be a quasi-Banach function space with respect to μ. Let $(f_n)_n\subseteq E_a$, and $f\in L^0(\mu)$. Consider the following conditions:*

1) *$f\in E$, and $\|f_n-f\|_E\to 0$.*
2) *f_n converges to f in $L^0(\mu)$ and $\{f_n : n\ge 1\}$ is almost order bounded.*
3) *f_n converges to f in $L^0(\mu)$ and $\{f_n : n\ge 1\}$ is L-weakly compact.*
4) *f_n converges to f in $L^0(\mu)$ and $\{f_n : n\ge 1\}$ is uniformly integrable.*
5) *f_n converges to f in $L^0(\mu)$ and $\{f_n : n\ge 1\}$ is uniformly absolutely continuous.*

Then 5) $\Rightarrow$ 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4). *Moreover, if $L^\infty(\mu)\subseteq E_a$, then all conditions* 1) $-$ 5) *are equivalent.*

Proof 5) $\Rightarrow$ 1) It is enough to prove the following

Fact. Any subsequence of $(f_n)_n$ *has another Cauchy subsequence.*

Let us take a subsequence $(g_n)_n$ of $(f_n)_n$. Then g_n converges to f in $L^0(\mu)$ and therefore it has another subsequence $(h_n)_n$ such that h_n converges to f μ-a.e. Let us check that $(h_n)_n$ is a Cauchy sequence in E. Note that the set $\{h_n : n \geq 1\}$ is uniformly absolutely continuous. Then, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|h_n \chi_A\|_E < \frac{\varepsilon}{3K_E^2} \tag{3.2}$$

for all $n = 1, 2, \ldots$ and all $A \in \Sigma$, with $\mu(A) < \delta$. Since μ is a finite measure, the Egoroff theorem ensures that there exists a measurable set A_δ, with $\mu(A_\delta) < \delta$, such that $h_n \to f$ uniformly on $\Omega \setminus A_\delta$. Then there exists $n_0 \geq 1$ such that

$$|h_n - h_k| \chi_{\Omega \setminus A_\delta} \leq \frac{\varepsilon}{3K_E \|\chi_\Omega\|_E} \chi_{\Omega \setminus A_\delta} \quad \text{for all } n, k \geq n_0. \tag{3.3}$$

Now, if $n, k \geq n_0$, by using (3.2) and (3.3) we obtain

$$\begin{aligned}\|h_n - h_k\|_E &\leq K_E \left\|(h_n - h_k)\chi_{A_\delta}\right\|_E + K_E \left\|(h_n - h_k)\chi_{\Omega \setminus A_\delta}\right\|_E \\ &\leq K_E^2 \left\|h_n \chi_{A_\delta}\right\|_E + K_E^2 \left\|h_k \chi_{A_\delta}\right\|_E + K_E \frac{\varepsilon}{3K_E \|\chi_\Omega\|_E} \left\|\chi_{\Omega \setminus A_\delta}\right\|_E < \varepsilon.\end{aligned}$$

From the *Fact* and Lemma 2.1 it follows that $f \in E$ and also that any subsequence $(g_n)_n$ of $(f_n)_n$ has another subsequence $(h_n)_n$ such that $\|h_n - f\|_E \to 0$ as $n \to \infty$. This is enough to prove that $\|f_n - f\|_E \to 0$ as $n \to \infty$.
1) $\Rightarrow$ 2) The convergence of $(f_n)_n$ to f in E implies that the set $\{f_n : n \geq 1\}$ in E_a is relatively compact in E and hence almost order bounded by Lemma 3.3. Moreover, Lemma 2.1 ensures that $f_n \to f$ in $L^0(\mu)$.
The implications 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 5) follow from Proposition 3.1. To deduce the implication 4) $\Rightarrow$ 5) from Proposition 3.1 we need to assume that $L^\infty(\mu) \subseteq E_a$. □

Corollary 3.1 *Let* E *be a quasi-Banach function space with respect to* μ*, and let* $H \subseteq E_a$*. Consider the following conditions:*

1) H *is relatively compact in* E.
2) H *is relatively compact in* $L^0(\mu)$ *and almost order bounded.*
3) H *is relatively compact in* $L^0(\mu)$ *and L-weakly compact.*
4) H *is relatively compact in* $L^0(\mu)$ *and uniformly integrable.*
5) H *is relatively compact in* $L^0(\mu)$ *and uniformly absolutely continuous.*

Then 5) $\Rightarrow$ 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4)*. Moreover, if* $L^\infty(\mu) \subseteq E_a$*, then all conditions* 1) $-$ 5) *are equivalent.*

Proof 5) $\Rightarrow$ 1) Take a sequence $(f_n)_n \subseteq H$. Since H is relatively compact in $L^0(\mu)$ there exists a subsequence $(f_{n_k})_k$ of $(f_n)_n$ and a function $f \in L^0(\mu)$ such that $f_{n_k} \to f$ in $L^0(\mu)$. Clearly this sequence $(f_{n_k})_k$ is uniformly absolutely continuous. Then, Proposition 3.2 tells us that $f \in E$ and $\left\|f_{n_k} - f\right\|_E \to 0$. That is, H is relatively compact in E.
1) $\Rightarrow$ 2) Apply Lemmas 2.1 and 3.3.
The implications 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 5) follow from Proposition 3.1. To deduce the implication 4) $\Rightarrow$ 5) from Proposition 3.1 we need to assume that $L^\infty(\mu) \subseteq E_a$. □

Let $m : \Sigma \to X$ be a countably additive vector measure with values in a real Banach space X. The *semivariation* of m is the subadditive set function defined on Σ by $\|m\|(A) := \sup\{|\langle m, x^*\rangle|(A) : x^* \in B_{X^*}\}$, where $|\langle m, x^*\rangle|$ denotes the variation of the scalar measure $\langle m, x^*\rangle : \Sigma \to \mathbb{R}$ given by $\langle m, x^*\rangle(A) := \langle m(A), x^*\rangle$ for all $A \in \Sigma$, and X^* is the continuous dual of X. A set $A \in \Sigma$ is called *m-null* if $\|m\|(A) = 0$. On the space $L^0(m)$ of (m-a.e. equivalence classes of) measurable functions $f : \Omega \longrightarrow \mathbb{R}$ we will consider the topology of *convergence in measure* with respect to $\|m\|$, that is, a sequence $(f_n)_n$ of measurable functions converges to a measurable function f if and only if $\lim_{n\to\infty} \|m\| ([|f_n - f| > \varepsilon]) = 0$ for all $\varepsilon > 0$. A measure $\mu := |\langle m, x^*\rangle|$, where $x^* \in B_{X^*}$, that is equivalent to m (in the sense that $\|m\|(A) \to 0$ if and only if $\mu(A) \to 0$) is called a *Rybakov control measure* for m. Such a measure always exists (see [9, Theorem 2, p. 268]). Note that $L^0(m) = L^0(\mu)$ holds, and also that the corresponding topologies of convergence in measure coincide.

Given a measurable function $f : \Omega \longrightarrow \mathbb{R}$, we will consider its *distribution function* (with respect to the semivariation $\|m\|$)

$$\|m\|_f : t \in [0, \infty) \longrightarrow \|m\|_f(t) := \|m\| ([|f| > t]) \in [0, \infty).$$

This distribution function $\|m\|_f$ has similar properties as the distribution function with respect to a positive scalar measure (see [11]). For instance, it is bounded, non-increasing and right-continuous. Denote by $L^1(\|m\|)$ the space of all functions $f \in L^0(m)$ such that the (Lebesgue) integral $\int_0^\infty \|m\|_f(t)dt < \infty$. Then $L^1(\|m\|)$, with the *lattice quasi-norm* given by

$$\|f\|_{L^1(\|m\|)} := \int_0^\infty \|m\|_f(t)dt$$

and the usual m-a.e. order, becomes a quasi-Banach function space with respect to any Rybakov control measure of m. Moreover $L^1(\|m\|)$ has the *sequential Fatou property* (see [5, Proposition 3.1]), and is σ*-order continuous* (see [5, Proposition 3.6]). It is straightforward to see that the quasi-norm constant of $L^1(\|m\|)$ is less than or equal to two. Finally note that the following inclusions

$$L^\infty(m) \subseteq L^1(\|m\|) \subseteq L^0(m) \tag{3.4}$$

are both continuous.

We finish this section by collecting together all the information that our general Propositions 3.1 and 3.2 provide about relatively compact subsets of $L^1(\|m\|)$.

Corollary 3.2 *Let $m : \Sigma \to X$ be a vector measure. The following assertions are equivalent for every subset $H \subseteq L^1(\|m\|)$:*

1) *H is almost order bounded.*
2) *H is (quasi-norm) bounded and uniformly absolutely continuous.*
3) *H is L-weakly compact.*
4) *H is uniformly integrable.*

Corollary 3.3 *Let $m : \Sigma \to X$ be a vector measure. The following assertions are equivalent for every subset $H \subseteq L^1(\|m\|)$:*

1) *H is relatively compact in $L^1(\|m\|)$.*
2) *H is relatively compact in $L^0(m)$ and almost order bounded.*
3) *H is relatively compact in $L^0(m)$ and L-weakly compact.*
4) *H is relatively compact in $L^0(m)$ and uniformly integrable.*
5) *H is relatively compact in $L^0(m)$ and uniformly absolutely continuous.*

4 Orlicz spaces associated to a quasi-Banach function space

The Orlicz spaces associated to a quasi-Banach function space were introduced and studied in [7]. In this section we investigate inclusions between such spaces. They possess a certain compactness property which will be very useful in the next section.

We recall that a *Young function* is any function $\Phi : [0, \infty) \longrightarrow [0, \infty)$ which is strictly increasing, convex, $\Phi(0) = 0$, and $\lim_{x\to\infty} \Phi(x) = \infty$. A Young function Φ satisfies the following useful inequalities: for $x \geq 0$, we have

$$\Phi(\alpha x) \leq \alpha\, \Phi(x) \quad \text{if} \quad 0 \leq \alpha \leq 1, \tag{4.1}$$

$$\Phi(\alpha x) \geq \alpha\, \Phi(x) \quad \text{if} \quad \alpha \geq 1. \tag{4.2}$$

We call a Young function Φ an *N-function* (in that case we will write $\Phi \in \mathcal{N}$) if it satisfies the limit conditions $\lim_{x\to 0} \frac{\Phi(x)}{x} = 0$ and $\lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$. Note that $\Phi_p(x) := x^p$ are Young functions for all $p \geq 1$, but they are N-functions only if $p > 1$.

A Young function Φ has the Δ_2*-property* if and only if there exists a real number $C > 0$ such that $\Phi(2x) \leq C\Phi(x)$ for all $x \geq 0$. In such a case we will write $\Phi \in \Delta_2$. Note that $\Phi_p(x) = x^p$ has trivially the Δ_2-property for all $p \geq 1$.

Let Φ be a Young function and E be a quasi-Banach function space with respect to μ. The *Orlicz space* E^Φ consists of those functions $f \in L^0(\mu)$ for which the *Luxemburg quasi-norm* $\|f\|_{E^\Phi} < \infty$, where

$$\|f\|_{E^\Phi} := \inf\left\{c > 0 : \Phi\left(\frac{|f|}{c}\right) \in E \text{ with } \left\|\Phi\left(\frac{|f|}{c}\right)\right\|_E \leq 1\right\}. \tag{4.3}$$

The Orlicz space E^Φ equipped with the Luxemburg quasi-norm is actually a quasi-Banach function space with respect to μ. Its main properties have been studied in [7]. For our purposes we only need to remind the following ones:

Proposition 4.1 (see [7]) *Let E be a quasi-Banach function space with respect to μ and let Φ be a Young function.*

1) *For all $A \in \Sigma$ with $\mu(A) > 0$, $\|\chi_A\|_{E^\Phi} = \dfrac{1}{\Phi^{-1}\left(\frac{1}{\|\chi_A\|_E}\right)}$.*
2) *If $f \in E^\Phi$ with $\|f\|_{E^\Phi} < 1$, then $\Phi(|f|) \in E$ and $\|\Phi(|f|)\|_E \leq \|f\|_{E^\Phi}$.*
3) *If $\Phi \in \Delta_2$ and E is σ-order continuous, then E^Φ is σ-order continuous.*

Using Lemma 3.1 and Proposition 4.1, we can establish the relationships between the order continuous parts of E and E^Φ.

Proposition 4.2 *Let E be a quasi-Banach function space with respect to μ and let Φ be a Young function.*

1) *$L^\infty(\mu) \subseteq \left(E^\Phi\right)_a$ if and only if $L^\infty(\mu) \subseteq E_a$.*
2) *If $L^\infty(\mu) \subseteq E_a$, then $\left(E^\Phi\right)_a \subseteq (E_a)^\Phi$.*
3) *If $L^\infty(\mu) \subseteq E_a$ and $\Phi \in \Delta_2$, then $\left(E^\Phi\right)_a = (E_a)^\Phi$.*

Proof 1) We have to check that $\chi_\Omega \in \left(E^\Phi\right)_a$ if and only if $\chi_\Omega \in E_a$, or equivalently (see Lemma 3.1) that $\lim_{\mu(A)\to 0} \|\chi_A\|_{E^\Phi} = 0$ if and only if $\lim_{\mu(A)\to 0} \|\chi_A\|_E = 0$. But this is immediate from 1) of Proposition 4.1 since $\lim_{x\to\infty} \Phi^{-1}(x) = \lim_{x\to\infty} \Phi(x) = \infty$.
2) First, note that $L^\infty(\mu) \subseteq E_a$ guarantees that E_a is a quasi-Banach function space. If $f \in \left(E^\Phi\right)_a$, then $\lim_{\mu(A)\to 0} \|f\chi_A\|_{E^\Phi} = 0$ by applying Lemma 3.1. Keeping

in mind 2) of Proposition 4.1, it follows that $\lim_{\mu(A)\to 0} \|\Phi(|f|\chi_A)\|_E = 0$, that is, $\lim_{\mu(A)\to 0} \|\Phi(|f|)\chi_A\|_E = 0$. Thus, by applying again Lemma 3.1, we have $\Phi(|f|) \in E_a$ which implies that $f \in (E_a)^\Phi$.
3) First note that by 2) it only remains to check the inclusion $(E_a)^\Phi \subseteq \left(E^\Phi\right)_a$ and that $(E_a)^\Phi$ is a (closed) order ideal of E^Φ. Evidently $(E_a)^\Phi \subseteq E^\Phi$. Moreover, by 3) of Proposition 4.1, $(E_a)^\Phi$ is σ-order continuous. Hence, $(E_a)^\Phi \subseteq \left(E^\Phi\right)_a$. □

It is possible to consider different partial ordering relations between Young functions and they are useful in dealing with embeddings of Orlicz spaces. Here are some of these relations (see [21, Section 2.2]).

Definition 4.1 We will write for any two Young functions Φ_0 and Φ_1

a) $\Phi_1 \prec \Phi_0$ if there exist $\varepsilon > 0$ and $x_0 \geq 0$ such that $\Phi_1(x) \leq \Phi_0(\varepsilon x)$, for all $x \geq x_0$.
b) $\Phi_1 \prec\prec \Phi_0$ if for each $\varepsilon > 0$, there exists $x_\varepsilon \geq 0$ such that $\Phi_1(x) \leq \Phi_0(\varepsilon x)$, for all $x \geq x_\varepsilon$.

Observe that if Φ is an N-function, then $\Phi \prec \Phi$ is always satisfied but $\Phi \prec\prec \Phi$ is never possible (see [21, §2.2 Theorem 2] for complete characterizations of relations $\prec$ and $\prec\prec$). The following inclusion result will be crucial in what follows.

Proposition 4.3 *Let E be a quasi-Banach function space with respect to μ and let Φ_0 and Φ_1 be two Young functions.*

1) *If $\Phi_1 \prec \Phi_0$, then $E^{\Phi_0} \subseteq E^{\Phi_1}$ and this inclusion is continuous.*
2) *If $\Phi_1 \prec\prec \Phi_0$ and $L^\infty(\mu) \subseteq E_a$, then $E^{\Phi_0} \subseteq E^{\Phi_1}$ and this inclusion is L-weakly compact, that is, every (quasi-norm) bounded subset of E^{Φ_0} is an L-weakly compact subset of E^{Φ_1}.*

Proof 1) By hypothesis, there exist $\varepsilon > 0$ and $x_0 \geq 0$ such that $\Phi_1(x) \leq \Phi_0(\varepsilon x)$, for all $x \geq x_0$. We are going to prove that

$$\|f\|_{E^{\Phi_1}} \leq 2M\varepsilon \, \|f\|_{E^{\Phi_0}}$$

for all $f \in E^{\Phi_0}$, where $M := K_E(\Phi_1(x_0)\|\chi_\Omega\|_E + 1) \geq 1$, whereby the continuous inclusion $E^{\Phi_0} \subseteq E^{\Phi_1}$ follows.

Take $f \in E^{\Phi_0}$ and let $c > 0$ such that $\left\|\Phi_0\left(\frac{|f|}{c}\right)\right\|_E \leq 1$. Consider the measurable set $A := [|f| < \varepsilon x_0 c]$ and note that $\frac{|f|\chi_A}{\varepsilon c} \leq x_0$ and similarly $\frac{|f|\chi_{\Omega\setminus A}}{\varepsilon c} \geq x_0\chi_{\Omega\setminus A}$. It follows that

$$\Phi_1\left(\frac{|f|\,\chi_A}{\varepsilon c}\right) \leq \Phi_1\left(x_0\right), \tag{4.4}$$

$$\Phi_1\left(\frac{|f|\,\chi_{\Omega\setminus A}}{\varepsilon c}\right) \leq \Phi_0\left(\frac{|f|\,\chi_{\Omega\setminus A}}{c}\right) \leq \Phi_0\left(\frac{|f|}{c}\right). \tag{4.5}$$

Then, by using the convexity of Φ_1 and according to the inequalities (4.1), (4.4) and (4.5), we obtain

$$\begin{aligned}\left\|\Phi_1\left(\frac{|f|}{2M\varepsilon c}\right)\right\|_E &\leq \frac{1}{M}\left\|\Phi_1\left(\frac{|f|}{2\varepsilon c}\right)\right\|_E = \frac{1}{M}\left\|\Phi_1\left(\frac{|f|\,\chi_A}{2\varepsilon c} + \frac{|f|\,\chi_{\Omega\setminus A}}{2\varepsilon c}\right)\right\|_E \\ &\leq \frac{K_E}{2M}\left\|\Phi_1\left(\frac{|f|\,\chi_A}{\varepsilon c}\right)\right\|_E + \frac{K_E}{2M}\left\|\Phi_1\left(\frac{|f|\,\chi_{\Omega\setminus A}}{\varepsilon c}\right)\right\|_E\end{aligned}$$

$$\leq \frac{K_E}{2M}\Phi_1(x_0)\|\chi_\Omega\|_E + \frac{K_E}{2M}\left\|\Phi_0\left(\frac{|f|}{c}\right)\right\|_E$$
$$\leq \frac{K_E}{2M}\left(\Phi_1(x_0)\|\chi_\Omega\|_E + 1\right) = \frac{1}{2} \leq 1.$$

Thus, from definition (4.3), we conclude that $\|f\|_{E^{\Phi_1}} \leq 2M\varepsilon c$ and hence

$$\|f\|_{E^{\Phi_1}} \leq 2M\varepsilon \|f\|_{E^{\Phi_0}},$$

as we wanted to see.

2) Let H be a (quasi-norm) bounded subset of E^{Φ_0} and let $M := \sup_{f\in H}\|f\|_{E^{\Phi_0}} < \infty$. We must show that H is an L-weakly compact subset of E^{Φ_1} or, equivalently, almost order bounded by Proposition 3.1 and 1) of Proposition 4.2. Of course, it is enough to prove that $H' := \left\{\frac{h}{M+1} : h \in H\right\}$ is almost order bounded, that is, for every $\varepsilon > 0$ there exists $0 < g \in \left(E^{\Phi_1}\right)_a$ such that $H' \subseteq [-g, g] + \varepsilon B_{E^{\Phi_1}}$.

Thus, given $\varepsilon > 0$, by applying the hypothesis $\Phi_1 \prec\prec \Phi_0$ there exists $y_\varepsilon > 0$ such that $\Phi_1\left(\frac{y}{\varepsilon}\right) \leq \Phi_0(y)$, for all $y \geq y_\varepsilon$. Take $g := y_\varepsilon \chi_\Omega$ and note that

$$0 < g \in L^\infty(\mu) \subseteq \left(E^{\Phi_1}\right)_a$$

by 1) of Proposition 4.2. Now, let us take any $f \in H'$, in which case, we have $\|\Phi_0(|f|)\|_E \leq 1$ by 2) of Proposition 4.1. Moreover, we can write

$$f = f\chi_{[|f|\leq y_\varepsilon]} + f\chi_{[|f|>y_\varepsilon]}. \tag{4.6}$$

On one hand, $\left|f\chi_{[|f|\leq y_\varepsilon]}\right| \leq y_\varepsilon\chi_{[|f|\leq y_\varepsilon]} \leq g$, and so

$$f\chi_{[|f|\leq y_\varepsilon]} \in [-g, g]. \tag{4.7}$$

On the other hand, $\Phi_1\left(\frac{|f|\chi_{[|f|>y_\varepsilon]}}{\varepsilon}\right) \leq \Phi_0\left(|f|\chi_{[|f|>y_\varepsilon]}\right) \leq \Phi_0(|f|)$, and so

$$\left\|\Phi_1\left(\frac{\left|f\chi_{[|f|>y_\varepsilon]}\right|}{\varepsilon}\right)\right\|_E \leq \|\Phi_0(|f|)\|_E \leq 1.$$

Definition (4.3) tells us that $\left\|f\chi_{[|f|>y_\varepsilon]}\right\|_{E^{\Phi_1}} \leq \varepsilon$ and so

$$f\chi_{[|f|>y_\varepsilon]} \in \varepsilon B_{E^{\Phi_1}}. \tag{4.8}$$

Finally, from (4.6), (4.7) and (4.8) the conclusion follows. □

Remark 4.1 We already knew that the inclusion $E^\Phi \subseteq E$ was continuous for every quasi-Banach function space E and every Young function Φ (see [7, Proposition 4.4 and Remark 4.7]). Furthermore, if $L^\infty(\mu) \subseteq E_a$ and $\Phi \in \mathcal{N}$, this inclusion is in fact L-weakly compact. Indeed, since Φ is an N-function, given $\varepsilon > 0$ there exists $x_\varepsilon > 0$ such that $\frac{\Phi(\varepsilon x)}{\varepsilon x} \geq \frac{1}{\varepsilon}$ for all $x \geq x_\varepsilon$, which means $\Psi \prec\prec \Phi$, where Ψ is the Young function given by $\Psi(x) := x$ for all $x \geq 0$. By applying 2) of Proposition 4.3 we conclude that the inclusion $E^\Phi \subseteq E^\Psi = E$ is L-weakly compact.

5 A de la Vallée–Poussin theorem for quasi-Banach function spaces

We start this section by establishing a de la Vallée–Poussin theorem for general quasi-Banach function spaces. This version generalizes the one obtained in [6, Theorem 4.1] when applied to the Banach function space $L^1_w(m)$ of scalarly integrable functions with respect to a vector measure m (see [7, Proposition 5.1]). Moreover, it can also be applied to the quasi-Banach function space $L^1(\|m\|)$ with respect to the semivariation of m. We need a previous result borrowed from [7] about (quasi-norm) boundedness.

Proposition 5.1 (see [7, Lemma 4.9 and 4.10]) *Let E be a quasi-Banach function space with respect to μ and let $H \subseteq L^0(\mu)$.*

1) *If Φ is a Young function and $\{\Phi(|f|) : f \in H\}$ is (quasi-norm) bounded in E, then H is (quasi-norm) bounded in E^Φ.*
2) *If Ψ is a Young function and H is (quasi-norm) bounded in E^Ψ, then there exists a Young function Φ such that $\{\Phi(|f|) : f \in H\}$ is (quasi-norm) bounded in E.*

Note that the Young function Φ of the previous item 2) can be chosen in $\mathcal{N}$ whenever $\Psi \in \mathcal{N}$, as it follows immediately from the proof of [7, Lemma 4.10].

Theorem 5.1 (de la Vallée–Poussin) *Let E be a quasi-Banach function space with respect to μ and let $H \subseteq E$. The following conditions are equivalent:*

1) *H is uniformly integrable in E.*
2) *There exists a non-decreasing, convex function $\Phi : [0,\infty) \longrightarrow [0,\infty)$ with $\lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$, such that $\{\Phi(|f|) : f \in H\}$ is a (quasi-norm) bounded subset of E.*
3) *There exists an N-function Ψ such that H is a (quasi-norm) bounded subset of E^Ψ.*

Proof 2) $\Rightarrow$ 1) Let $M := \sup_{f\in H} \|\Phi(|f|)\|_E < \infty$. For a given $\varepsilon > 0$ let $c \geq 0$ be such that $\Phi(x) \geq \frac{M}{\varepsilon}x$ for $x > c$. Then for every $f \in H$

$$\Phi(|f|) \geq \Phi(|f|\chi_{[|f|>c]}) \geq \frac{M}{\varepsilon}|f|\chi_{[|f|>c]},$$

that is, $|f|\chi_{[|f|>c]} \leq \frac{\varepsilon}{M}\Phi(|f|)$. By taking quasi-norm, we get

$$\left\| f\chi_{[|f|>c]} \right\|_E \leq \frac{\varepsilon}{M} \|\Phi(|f|)\|_E \leq \varepsilon$$

for every $f \in H$. Note that the convexity of Φ is not needed in this implication.
1) $\Rightarrow$ 2) From the hypothesis we can select an increasing sequence $0 < c_1 < c_2 < \cdots \uparrow \infty$ such that

$$\left\| f\chi_{[|f|>c_n]} \right\|_E \leq \frac{1}{K_E^{n+1} 2^n}, \tag{5.1}$$

for all $n \geq 1$, and all $f \in H$. Let us define $\Phi(x) := \sum_{k=1}^\infty (x - c_k)^+$ for $x \geq 0$, where

$$(x-c)^+ = \begin{cases} x - c, & x > c, \\ 0, & x \leq c. \end{cases}$$

Then $\Phi : [0,\infty) \longrightarrow [0,\infty)$ is convex, non-decreasing and for $x \geq 2c_n$ we have

$$\frac{\Phi(x)}{x} \geq \sum_{k=1}^n \left(1 - \frac{c_k}{x}\right)^+ \geq \frac{n}{2},$$

and so Φ meets the requirements of 2). We will check that $\Phi(|f|) \in E$ and $\|\Phi(|f|)\|_E \leq 1$ for every $f \in H$. Note that, for all $n \geq 1$ and every function $f \in H$ we have

$$(|f| - c_n)^+ \leq |f|\chi_{[|f|>c_n]} \in E.$$

Therefore, $(|f| - c_n)^+ \in E$ and by taking into account (5.1), we get that

$$\sum_{n=1}^{\infty} K_E^n \left\|(|f| - c_n)^+\right\|_E \leq \sum_{n=1}^{\infty} K_E^n \left\|f \chi_{[|f|>c_n]}\right\|_E \leq \frac{1}{K_E} \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{K_E} < \infty.$$

According to [7, Theorem 3.1] the series $\sum_{n=1}^{\infty} (|f| - c_n)^+$ converges in the quasi-norm topology of E to some $g \in E$ and

$$\|g\|_E \leq K_E \sum_{n=1}^{\infty} K_E^n \left\|(|f| - c_n)^+\right\|_E \leq 1.$$

By Lemma 2.1, the series $\sum_{n=1}^{\infty} (|f| - c_n)^+$ also converges in measure to g. But by the definition of Φ one has $\sum_{n=1}^{\infty} (|f| - c_n)^+ = \Phi(|f|)$ μ-a.e. and so $g = \Phi(|f|)$.
2) $\Rightarrow$ 3) We know by applying [14, Theorem 3.3] that Φ is the *principal part* of some N-function Ψ, which means that there exists $x_0 \geq 0$ such that $\Phi(x) = \Psi(x)$, for all $x \geq x_0$. To prove that H is a (quasi-norm) bounded subset of E^{Ψ} it is enough to check that $\{\Psi(|f|) : f \in H\}$ is a (quasi-norm) bounded subset of E, by 1) of Proposition 5.1. Note that $\Psi(|f|) \leq \Phi(|f|)\chi_{[|f|\geq x_0]} + \Psi(x_0)\chi_{[|f|<x_0]}$, and then

$$\begin{aligned}\|\Psi(|f|)\|_E &\leq K_E \left\|\Phi(|f|)\chi_{[|f|\geq x_0]}\right\|_E + K_E \left\|\Psi(x_0)\chi_{[|f|<x_0]}\right\|_E \\ &\leq K_E \|\Phi(|f|)\|_E + K_E \Psi(x_0) \|\chi_\Omega\|_E .\end{aligned}$$

Thus, $\{\Psi(|f|) : f \in H\}$ is (quasi-norm) bounded because $\{\Phi(|f|) : f \in H\}$ so is by hypothesis.
3) $\Rightarrow$ 2) It follows from 2) of Proposition 5.1. □

We present now some consequences of the above result. They ensure that any compact or L-weakly compact subset of the order continuous part of a quasi-Banach function space E is located into the space E^{Φ}, for a certain N-function $\Phi \in \Delta_2$. A key point in the arguments is the next lemma which follows from Propositions 3.3, 3.4 and 3.7 of [6].

Lemma 5.1 *For every N-function Ψ there exists another N-function Φ with the Δ_2-property such that $\Phi \prec\prec \Psi$.*

Corollary 5.1 *Let E be a quasi-Banach function space with respect to μ such that $L^{\infty}(\mu) \subseteq E_a$. A subset $H \subseteq E_a$ is relatively compact if and only if there exists an N-function $\Phi \in \Delta_2$ such that H is relatively compact in E^{Φ}.*

Proof One implication is trivial because the inclusion $E^{\Phi} \subseteq E$ is continuous. On the other hand, if $H \subseteq E_a$ is relatively compact, then H is uniformly integrable in E and relatively compact in measure by Corollary 3.1. Thus, the de la Vallée–Poussin Theorem 5.1 produces an N-function Ψ such that H is (quasi-norm) bounded in E^{Ψ}. Now by applying Lemma 5.1 we know that there exists another N-function $\Phi \in \Delta_2$ such that $\Phi \prec\prec \Psi$ and so $E^{\Psi} \subseteq E^{\Phi}$ and moreover this inclusion is L-weakly compact by 2) of Proposition 4.3. Thus H is L-weakly compact in E^{Φ} and relatively compact in measure. Note that $L^{\infty}(\mu) \subseteq \left(E^{\Phi}\right)_a$ by Proposition 4.2 and $H \subseteq \left(E^{\Phi}\right)_a$ by Remark 3.3. Hence, Corollary 3.1 guarantees that H is relatively compact in E^{Φ}. □

Corollary 5.2 *Let E be a quasi-Banach function space with respect to μ such that $L^{\infty}(\mu) \subseteq E_a$. A subset $H \subseteq E$ is L-weakly compact if and only if there exists an N-function $\Phi \in \Delta_2$ such that H is L-weakly compact in E^{Φ}.*

Proof The *if part* follows from Remark 4.1 because the inclusion $E^{\Phi} \subseteq E$ is L-weakly compact. In fact, it is enough to assume that H is (quasi-norm) bounded in E for this implication. The assumption that $\Phi \in \Delta_2$ is not needed for the above implication. On the other hand, if $H \subseteq E$ is L-weakly compact, then H is uniformly integrable by Proposition 3.1. Now, the argument of the proof of Corollary 5.1 ensures the existence of an N-function $\Phi \in \Delta_2$ such that H is an L-weakly compact subset of E^{Φ}. □

Note that Corollary 5.1 extends [6, Corollary 4.7]. Moreover, when particularized to the quasi-Banach function space $L^1(\|m\|)$, it provides the promised location of the relatively compact subsets of $L^1(\|m\|)$ in a suitable Orlicz space

$$L^{\Phi}(\|m\|) := L^1(\|m\|)^{\Phi}$$

and the same occurs with Corollary 5.2 and L-weak compactness.

Corollary 5.3 *Let $m : \Sigma \to X$ be a vector measure and $H \subseteq L^0(m)$.*

1) *H is relatively compact in $L^1(\|m\|)$ if and only if there exists an N-function $\Phi \in \Delta_2$ such that H is relatively compact in $L^{\Phi}(\|m\|)$.*
2) *H is L-weakly compact in $L^1(\|m\|)$ if and only if there exists an N-function $\Phi \in \Delta_2$ such that H is L-weakly compact in $L^{\Phi}(\|m\|)$.*

By applying Corollary 5.1 to singletons we have

$$E = \bigcup_{\Phi \in \Delta_2 \cap \mathcal{N}} E^{\Phi}$$

for any quasi-Banach function space E such that $L^{\infty}(\mu) \subseteq E_a$. In particular, this yields [6, Corollary 4.2] and furthermore

$$L^1(\|m\|) = \bigcup_{\Phi \in \Delta_2 \cap \mathcal{N}} L^{\Phi}(\|m\|).$$

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