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Generating punctured surface triangulations with degree at least 4

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Abstract

As a sequel of a previous paper by the authors, we present here a generating theorem for the family of triangulations of an arbitrary punctured surface with vertex degree ≥ 4. The method is based on a series of reversible operations termed reductions which lead to a minimal set of triangulations in such a way that all intermediate triangulations throughout the reduction process remain within the family. Besides contractible edges and octahedra, the reduction operations act on two new configurations near the surface boundary named quasi-octahedra and N-components. It is also observed that another configuration called M-component remains unaltered under any sequence of reduction operations. We show that one gets rid of M-components by flipping appropriate edges.

1 Introduction

By a *triangulation* of a surface F^2 we mean a simple graph G (i.e., a graph without loops and multiple edges) embedded in F^2 so that each face is bounded by a 3-cycle and any two faces share at most one edge. In other words, the vertices, edges and faces of G form a simplicial complex whose underlying space is F^2. Two triangulations G and G' of F^2 are *equivalent* if there is a

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homeomorphism $\varphi : F^2 \to F^2$ with $\varphi(G) = G'$. In this paper surfaces are supposed to be compact and connected and possibly with boundary. Surfaces without boundary will be termed *closed surfaces*. Here, we distinguish between triangulations only up to equivalence.

The enumeration of triangulated surfaces with and without boundary is applied to computation and physics (see [12] and the references therein). Three major methods to generate triangulations are presently available in the literature (see [22]). One of these methods is based on finding out a family of irreducible triangulations and obtaining from them all triangulations under the desired conditions by means of generating theorems.

A variety of generating theorems can be found in the literature (see [15, 16, 17, 18, 20, 21] among others), these theorems provide certain sets of operations deviced to construct all triangulations in a given class $\mathcal{M}$ from a subclass $\mathcal{M}_0 \subseteq \mathcal{M}$ by sequences of such operations. The subclass $\mathcal{M}_0$ can be regarded a generating set for all triangulations in $\mathcal{M}$. The operations which yield the whole class $\mathcal{M}$ from $\mathcal{M}_0$ are generally termed expansions. Most of the generating theorems also give operations, termed reductions, which act as the inverses of expansions, so that by sequences of reductions we get the "minimal" subclass $\mathcal{M}_0$ starting with the class $\mathcal{M}$ (see [2]).

The classical reduction operation is a contraction of edges and its inverse a vertex splitting. Recall that an edge of a triangulation G of F^2 is *contractible* if the vertices of the edge can be identified and the result is still a triangulation of F^2. A triangulation is said to be *irreducible* if it has no contractible edges (see [1] and [3]).

As a contribution to this research area, we state and prove here a generating theorem for *punctured surfaces* (i.e., surfaces obtained by deleting the interior of a disk in closed surfaces). It is well known that any irreducible triangulation of a non-spherical closed surface has minimum degree ≥ 4, [19]. This is no longer true for punctured surfaces; in fact, it is readily checked that all irreducible triangulations of a punctured surface (other than the disk) F^2 are elements of the class $\mathcal{F}^2_\circ(4)$ consisting of all triangulations of F with minimum degree ≥ 3 on the boundary and degree≥ 4 for all inner vertices. Particular examples of irreducible triangulations with 3-valent boundary vertices can be found in [4] and [13].

A generating theorem for the class $\mathcal{F}^2_\circ(4)$ is given in [6]. As a sequel, in this paper we introduce a set of six reversible internal operations in the subfamily $\mathcal{F}^2(4) \subseteq \mathcal{F}^2_\circ(4)$ consisting of all triangulations with minimum degree ≥ 4.

The minimum degree at least 4 condition is particularly relevant in order to obtain 4-connected triangulations of surfaces. Several works concerning this property on closed surfaces can be found (see [15, 17], for instance).

Moreover, the 4-connectivity of a triangulation is also closely related to the hamiltonicity property. This fact has been shown by many papers since the seminal Whitney's result [24] and Grümbaun's conjecture [9] appeared (see [8, 10, 23] and the references therein).

The main result of this paper (Theorem 3) states that, given a punctured surface F, any triangulation of F in $\mathcal{F}^2(4)$ can be obtained from a 4-minimal triangulation by a sequence of operations that preserve the degree 4 condition during the whole procedure. This result can be regarded as an extension to punctured surfaces of the main theorems by Nakamoto and Negami for closed surfaces [18]. In fact, the operations termed 4-contractions and removals of octahedra in [18] coincide, respectively, with the operations R_1 and R_2 in this paper. Recently, the operation R_2 has been used in [20] under the name of R-reduction for even triangulations of closed surfaces. Furthermore, the other three operations in [20], called (P,T,Q)-reductions, are the composite of 4-contractions and their inverses. The configurations on which Q-reductions act are termed N-components in this paper. Notwithstanding N-components here always involve the boundary of a punctured surface.

In contrast with the case of closed surfaces, the minimal triangulations of a punctured surface F obtained by the use of such operations in $\mathcal{F}^2(4)$ may contain contractible edges whose contraction produce 3-valent boundary vertices. We prove that such contractible edges are necessarily located in two particular configurations (see Theorem 1), that persist during the whole reduction process. In order to achieve the irreducible triangulation within $\mathcal{F}^2(4)$, we consider diagonal flips of edges and state another generating theorem (Theorem 6).

Recently, irreducible triangulations of the Möbius band from [4] have been used in [7] to give a hint of the width of the gap between the simplicial Lusternik-Schnirelmann (L-S) category of a triangulated surface and the minimum number of critical elements of its Morse functions. The width of such a gap is far from being estimated yet. It might be expected that the present work jointly with its companion [6] enlighten the ongoing research concerning this problem.

2 New reductions/expansions for the family $\mathcal{F}^2(4)$

With the same terminology as used in [2], in this section we introduce the reduction/expansion operations involved in the main results of the paper (Theorems 1 and 3) others than classical edge contraction and octahedron removal and its inverses, vertex splitting and octahedron addition, respectively.

Throughout this paper F^2 will denote a surface with connected (possibly empty) boundary. If G is a triangulation of the surface F^2, let $\partial G \subset G$ denote the subgraph triangulating the boundary ∂F^2. The vertices and edges of ∂G

will be called *boundary vertices* and *boundary edges* of G, respectively. The vertices and edges of $G - \partial G$ will be called *inner vertices* and *inner edges* of G, respectively. The link of a vertex $x \in G$, denoted $link(x) = x_1 x_2 ... x_n$, is the set of edges $x_i x_{i+1}$ in G which jointly with the vertex x form a triangle $x_i x_{i+1} x$ in G for $1 \leq i \leq n-1$. Observe that if x is an inner vertex, $x_n = x_1$.

In addition, let us introduce further terminology concerning edge contraction. Henceforth, G/e will denote the contraction of the edge $e = v_1 v_2$ in the graph G. Notice that the new vertex $v = v_1 = v_2$ in G/e satisfies $deg(v) = deg(v_1) + deg(v_2) - 3$ when e is a boundary edge of G, and $deg(v) = deg(v_1) + deg(v_2) - 4$ otherwise. Besides, if xv_1v_2 is a face of G, then $deg(x)$ diminishes by one after the contraction of e. Here $deg(v)$ denotes the degree of the vertex v.

The vertex v is said to be *k-valent* if $deg(v) = k$. Given a triangulation G with minimum degree $\geq k$, an edge e is said to be *k-contractible* (*kc-edge* for short) if the minimum degree of G/e is at least k. The contraction of such an edge is termed a *k-contraction*.The corresponding vertex splitting will be called a *k-splitting*. In this paper by a *cn4c-edge* we mean a contractible edge which is not 4-contractible.

Remark 1. Notice that for any interior $4c$-edge e, the vertices in G sharing a face with e have $degree \geq 5$.

Remark 2. The two following locations of edge e are obstructions to contractibility of e. By a *critical 3-cycle* we mean a 3-cycle whose three edges do not bound a face of G.

(1) e belongs to a critical cycle of G. This is the case when e lies on a boundary of length 3.

(2) e is an inner edge but its two vertices belong to ∂G.

Since $\mathcal{F}^2(4)$ is a subfamily of $\mathcal{F}^2_\circ(4)$ results from [6] also apply to surfaces in $\mathcal{F}^2(4)$. In particular, we will use [6], Proposition 3.3. Recall that the distance from an edge e to ∂G, denoted $d(e, \partial G)$, is defined to be the minimum number of edges needed to connected e and ∂G.

Lemma 1. [6] *Let $G \in \mathcal{F}^2_\circ(4)$ be a triangulation of the punctured surface F^2. Assume that ab is a cn4c-edge in G, and let abx be a face with $deg(x) \leq 4$. If G is different from the disk and $d(ab, \partial G) \leq 1$, then either a $4c$-edge or a subgraph $H \subseteq G$ in the family*

$$\mathcal{A} = \{\textit{octahedron component},\ \textit{triode detecting edge},\ \textit{flag}\}$$

can be found at distance at most 1 from ab.

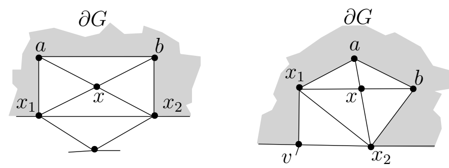

Figure 1: Flags.

In this lemma the following terminology from [6] is used.

Definition 1. A 3c-edge e of G is said to be a *triode detecting edge* if the posible vertices of degree 3 in G/e belong to the boundary.

The configurations termed flags in Lemma 1 are ruled out in the family $\mathcal{F}^2(4)$ since they contain two vertices of degree 3 on ∂G.

It readily follows from Definition 1 that ab is a triode detecting edge whenever abx is a face such that ab is a contractible boundary edge, x lies in the boundary and $deg(x) = 4$.

However triode detecting edges can appear. By focusing on this possibility we find new subgraphs involving triode detecting edges on G that give rise to new configurations and therefore the necessity of new reduction operations to remove then within the class $\mathcal{F}^2(4)$.

The first configuration, termed a quasi-octahedron component, is a variation of the well known notion of octahedron given in [18] and [6]. Let us start by recalling the latter.

Definition 2. A graph $H \subseteq G$ (possibly $H \cap \partial G \neq \emptyset$) of vertices set $\{a_1, a_2, a_3, v_1, v_2, v_3\}$ is said to be an *octahedron component* centered at the 3-cycle $v_1v_2v_3$ if $deg(v_i) = 4$ in G (for $1 \leq i \leq 3$) and the edges set of H is $\{v_iv_j,\ a_ia_j \text{ for } 1 \leq i, j \leq 3\} \cup \{v_ia_j \text{ for } i \neq j\}$. Octahedron components are denoted by $\mathcal{O}$.

An octahedron component of G is said to be *external* if two edges a_ia_j, a_ja_k lie in ∂G (in particular, $deg(a_j) = 4$). Notice that any edge of $\mathcal{O} - \partial G$ is a $cn4c$-edge. If $G \in \mathcal{F}^2(4)$ and $G' = G - \{v_1, v_2, v_3\}$ remains in $\mathcal{F}^2(4)$ (or equivalently $deg(a_i) \geq 6$, for $1 \leq i \leq 3$), we say that $\mathcal{O}$ is 4-*removable* (or, alternatively that G is the *addition of* $\mathcal{O}$ *to* G').

Remark 3. Let us first suppose that $\mathcal{O}$ is not 4-removable and let us consider $deg(a_2) = 5$. If no edge of $\mathcal{O}$ lies in the boundary, then $deg(a_1), deg(a_3) \geq 6$ and there exist the two faces a_1a_2v and a_2a_3v in G such that the common edge a_2v is a $4c$-edge and lies in $G - \mathcal{F}^2(4)$.

The previous reasoning is similar to the case of closed surfaces although some vertices a_i may lie in the ∂G (see [18]).

Notice that some face $v_1v_2v_3$ in G with its three vertices of degree 4 may have some triode detecting edges and not be the center of an octahedron component. This occurs in the configuration defined as follows.

Definition 3. The subgraph H in Definition 2 will be termed a *quasi-octahedron component* of G centered at $v_1v_2v_3$ and remaining vertices a_1, a_2, a_3 if precisely one v_i belongs to $\in \partial G$ and either the edge a_ja_k $(j, k \neq i)$ does not exist or, otherwise, the cycle $a_1a_2a_3$ is not a face in G and $\partial G \neq a_1a_2a_3$. Quasi-octahedron components will be denoted by $\widehat{\mathcal{O}}$ (see Figure 3 (right)).

Let us remark that the edges a_ia_3 (for $i = 1, 2$) necessarily are inner edges. Moreover, $deg(a_3) \geq 5$ since otherwise, this quasi-octahedron becomes an octahedron. If, in addition, $a_3 \in \partial G$, it is clear that $deg(a_3) \geq 6$.

For the sake of simplicity, we henceforth assume that $v_3 \in \partial G$ in any quasi-octahedron component.

Remark 4. Let us consider a quasi-octahedron component $\widehat{\mathcal{O}}$. If $deg(a_i) = 4$, for some $i = 1, 2$, then there exists a boundary vertex t such that a_it is a boundary $4c$-edge.

If $a_3 \in G - \partial G$ and $deg(a_3) = 5$, since a_ia_3 is an inner edge for $i = 1, 2$, there must exist a vertex t defining two faces a_ia_3t $(i = 1, 2)$ and the edge a_3t is 4-contractible. In both cases after contracting the $4c$-edge a_it (for $i = 1, 2$), we observe that the quasi-octahedron remains unaltered and $deg(a_3) \geq 6$ becomes after a finite number of similar edge contractions in the new triangulation.

After the previous observations, without loss of generality, if no edge incident with a_i, $(i = 1, 2, 3)$ is 4-contractible, we may suppose that $deg(a_i) \geq 5$ for $i = 1, 2$ and $deg(a_3) \geq 6$.

The new reduction operations needed to deal with configurations containing triode detecting edges will be defined in the following subsections.

2.1 New reduction/expansion operations involving octahedron and quasi-octahedron components

In order to ease the reading, in an external octahedron component the vertex a_3 will be assumed to be of degree 4. Similarly in any quasi-octahedron component the vertex v_3 will be assumed to be the only vertex v_i in ∂G.

Let $\mathcal{O}$ be an external octahedron component such that $deg(a_1) = 6$ or $deg(a_2) = 6$. Then $\mathcal{O}$ is not 4-removable although it is redundant from the topological point of view (see Figure 2(left)). To get rid of such components we introduce the following operation. By *folding the octahedron* $\mathcal{O}$ *onto the face* a_1a_2v we mean the removal of vertices a_3, v_1, v_2, v_3 from G followed by the addition of an octahedron to the face a_1a_2v (Figure 2). The inverse operation is called *unfolding an octahedron* with respect to the boundary of G.

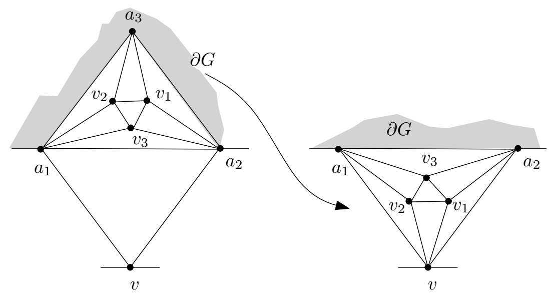

Figure 2: Folding the octahedron $\mathcal{O}$ onto the face a_1a_2v.

Another obstruction to reduce an octahedron component $\mathcal{O}$ within the class $\mathcal{F}^2(4)$ arises when $\mathcal{O}$ hits the boundary in exactly one edge a_1a_2 and such that no edge a_iv is 4-contractible, and $deg(a_2) = 5$. For this configurations we will introduce a further reduction operation as follows. Let v be the only neighbour of a_2 outside $\mathcal{O}$ (Figure 3). The *replacement of the boundary octahedron* $\mathcal{O}$ by a quasi-octahedron $\widehat{\mathcal{O}}$ is defined to be the removal of the edge a_1a_2 followed by the contraction of the edge a_2v in G. The inverse operation is called the *replacement of the quasi-octahedron* $\widehat{\mathcal{O}}$ by a boundary octahedron $\mathcal{O}$.

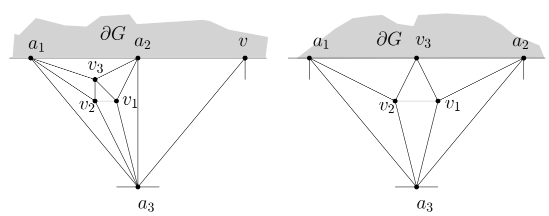

Figure 3: A replacement of a boundary octahedron by a quasi-octahedron $\widehat{\mathcal{O}}$.

The replacement of $\mathcal{O}$ by $\widehat{\mathcal{O}}$ can be regarded as removing the edge a_1a_2 and them contracting the edge a_2v. Notice that the edge a_1a_2 turns to be a $4c$-edge after deleting a_1a_2.

Definition 4. A *quasi-octahedron component* of G, $\widehat{\mathcal{O}}$, is said to be *removable in* $\mathcal{F}^2(4)$ (or *4-removable*, for short) if one of the following conditions holds:

(1) The graph $G' = G - \{v_1, v_2, v_3\}$ yields a triangulation in $\mathcal{F}^2(4)$.

(2) If replacing the quasi-octahedron component $\mathcal{O}$ by the face $a_1a_2a_3$ yields a triangulation in $\mathcal{F}^2(4)$.

In both cases, we will simply say that G' is obtained by *removing a quasi-octahedron* from G. Conversely, if (1) happens, we say that G is obtained from G' by *adding a quasi-octahedron* along two consecutive boundary edges of G'. In (2) we say that G is obtained from G' by *embedding a quasi-octahedron* in a face of G' sharing one edge with $\partial G'$.

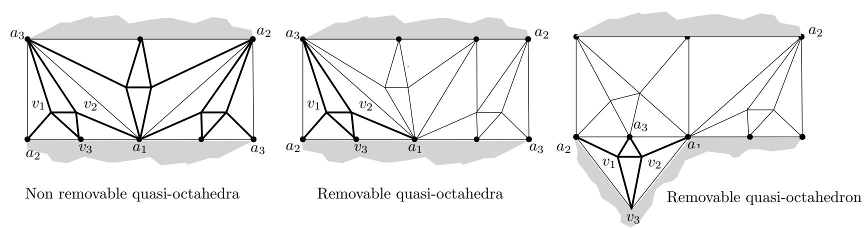

Figure 4: Triangulations for the Möbius strip with some quasi-octahedra components.

In both cases G' is obtained from G by a sequence of three successive edge contractions. As a consequence, if G contains a quasi-octahedron component $\widehat{\mathcal{O}}$, then G is reducible. Indeed, the interior edges v_iv_j and a_iv_j of $\widehat{\mathcal{O}}$ are readily checked to be *cn*4*c*-edges. Let us observe other facts with regard to removing quasi-octahedra. Conditions (1) and (2) above are not mutually exclusive. Indeed, in Figure 3 (right) both ways of removing the quasi-octahedron can be carried out whenever a_3 is an inner vertex, the edge a_1a_2 does not exist and $deg(a_i) \geq 6$ for $i = 1, 2, 3$. Other possibilities for removing a quasi-octahedron may appear as it is illustrated in Figure 4. The removable quasi-octahedron in Figure 4 (center) verifies only condition (2), while the removable quasi-octahedron in Figure 4 (right) verifies only condition (1).

We can establish the following characterization of a non-removable quasi-octahedron.

Proposition 1. *Let $G \in \mathcal{F}^2(4)$ be a triangulation of the surface F^2 and $\widehat{\mathcal{O}}$ be a quasi-octahedron component of G. Then, $\widehat{\mathcal{O}}$ is not removable in $\mathcal{F}^2(4)$ if and only if one of the following conditions holds:*

(a.1) $a_3 \in G - \partial G$ *and* $deg(a_i) = 4$ *for some* $i = 1, 2$.

(a.2) $a_3 \in G - \partial G$ *and* $deg(a_3) \leq 5$.

(b.1) $a_3 \in \partial G$ *and* a_1a_2 *is an inner edge of* G, *(or, equivalently,* v_3a_1 *and* v_3a_2 *are non-contractible edges).*

(b.2) $a_3 \in \partial G$ *and* $deg(a_i) = 4$ *for some* $i = 1, 2$.

Proof. If a_3 is an inner vertex, deleting the quasi-octahedron via (1) provides a triangulation of the same surface. Hence, (1) holds if and only if $deg(a_i) \geq 6$ ($i = 1, 2, 3$). Otherwise, condition (2) holds if and only if a_1 and a_2 are not adjacent and $deg(a_i) \geq 5$ ($i = 1, 2$) and $deg(a_3) \geq 6$.

Therefore, in this case, $\widehat{\mathcal{O}}$ is not removable in $\mathcal{F}^2(4)$ if and only if (a.1) or (a.2) is verified.

If a_3 is a boundary vertex, then by Remark 4 $deg(a_3) \geq 6$ holds. If a_1a_2 is an inner edge (or, equivalently, the edges v_3a_1 and v_3a_2 are non-contractible), the quasi-octahedron is not removable since by removing it by condition (1) a singular boundary point occurs and removing it by condition (2) provides a double edge a_1a_2. Therefore, if $a_3 \in \partial G$ (2) holds if and only if $deg(a_1), deg(a_2) \geq 5$ and the edges v_3a_1 and v_3a_2 are contractible.

Hence, in this case, $\widehat{\mathcal{O}}$ is not removable in $\mathcal{F}^2(4)$ if and only if (b.1) or (b.2) is verified. □

2.2 A new reduction/expansion operations involving triode detecting edges

Quasi-octahedron components do not exhaust all possible appearance of triode detecting edges in triangulations in $\mathcal{F}^2(4)$ (see Figure 5, left and center).

Pursuing our goal of finding minimal triangulation in the family $\mathcal{F}^2(4)$, we detect a new configuration in G and define a new operation to reduce it within $\mathcal{F}^2(4)$ to reach a minimum number of unavoidable triode detecting edges in G.

Definition 5. An *N-component* of a triangulation $G \in \mathcal{F}^2(4)$ of the surface F^2 consists of a subgraph $\mathcal{N}$ of G determined by two faces sharing an edge, where at least two non-incident edges are *cn*4*c*-edges and at least one of them lies in ∂G (Figure 5).

An N-component $\mathcal{N} \subset G$ is termed *contractible* if both non-incident contractible edges lie in the boundary or else some inner vertex in $\mathcal{N}$ has degree ≥ 5. In such configurations, the simultaneous contractions of the two non-incident contractible edges in $\mathcal{N}$ yields a triangulation in $\mathcal{F}^2(4)$. This double contraction will be called *contracting an N-component.*

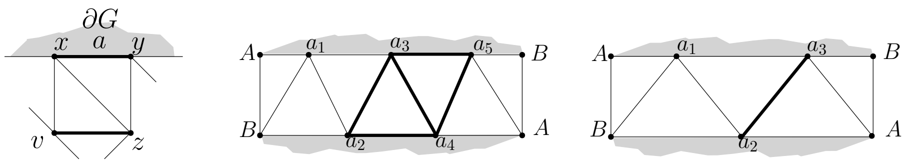

Figure 5: Two $N-$components on the left and center. The double contraction of the edges a_2a_4, a_3a_5 in the triangulation of the Möbius strip on the center provides the triangulation on the right.

Remark 5. Observe that an N-component is contractible if and only if it is not contained in a quasi-octhaedron component because of the degree condition of its inner vertices.

3 A generating theorem for the class $\mathcal{F}^2(4)$

In this section we state and prove a generating theorem (Theorem 3) for triangulations of degree at least 4 of a punctured surface. The reduction / expansion operations involved in the theorem are summarized in Table 1. The two operations introduced by Nakamoto and Negami in [18] are among reductions and they are the only ones which are defined in absence of boundary. In particular, the triangulations of closed surfaces which are minimal for such reductions coincides with the irreducible triangulations in [18]. This way we generalize Theorems 1 and 2 in [18]. In sharp contrast with the class of closed surfaces, for a punctured surface, the minimal triangulations obtained by such reductions may contain contractible edges whose contraction produce 3-valent vertices. For this case, we prove in Theorem 1 that those possible contractible edges are located in two particular configurations, the quasi-octahedron component and the M-component given in Definition 7 below. The special case of the disk is also considered in Theorem 2.

	4-reductions		**4-expansions**	**Figure**
R_1	edge 4-contraction	E_1	vertex 4-splitting	
R_2	octahedron removal	E_2	octahedron addition	
R_3	folding an octahedron	E_3	unfolding an octahedron	Figure 2
R_4	quasi-octahedron removal	E_4	quasi-octahedron addition	Figure 4
R_5	boundary octahedron replacement	E_5	quasi-octahedron replacement	Figure 3
R_6	contracting a N-component	E_6	double splitting of vertices	Figure 5

Table 1: Reduction / expansion operations in $\mathcal{F}^2(4)$.

By a 4-reduction (4-expansion, respectively) we mean any reduction (expansion, respectively) in Table 1. Operations R_1, R_2 and E_1, E_2 were introduced in [18].

By the use of the operations of Table 1 we eventually get a class which is minimal in $\mathcal{F}^2(4)$ in the following sense.

Definition 6. A triangulation $G \in \mathcal{F}^2(4)$ of the surface F^2 is said to be *minimal in* $\mathcal{F}^2(4)$ (or 4-*minimal**, for short) if G does not admit any further 4-reduction.

Some 4-minimal triangulations of punctured surfaces can be found in [5] (Example 32, Figures 13, 14 and 15).

Unfortunately, for punctured surfaces not all *cn*4*c*-edges can be removed from a 4-minimal triangulation. Notwithstanding such edges can be located in two special components of any triangulation G which reduces to G_0.

Given any triangulation $G \in \mathcal{F}^2(4)$, a configuration $H \subseteq G$ is termed 4-*fixed* if it remains unaltered under any sequence of 4-reduction performed on G.

Next we will show that there exist exactly two families of 4-fixed components: a special type of quasi-octahedron component described in Proposition 3 below and the M-component defined as follows.

Definition 7. Let $G \in \mathcal{F}^2(4)$ be a triangulation of the surface F^2. Let abx be a face with x a 4-valent vertex and ab a boudary *cn*4*c*-edge. An *M-component* centered at abx in G consists of a subgraph $\mathcal{M} \subseteq G$ determined by three faces $\{xab, xax_1, xbx_2\}$ such that xx_1, xx_2 lie in the boundary and x_1x_2 is an inner edge. Notice that xx_1x_2x is a critical 3-cycle (see Figure 6).

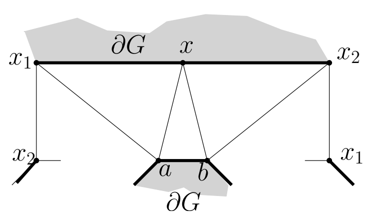

Figure 6: M-component centered at abx.

Remark 6. In any M-component centered at abx in a triangulation $G \in \mathcal{F}^2(4)$, the boundary edge ab is a triode detecting edge and $deg(x_i) \geq 5$ for

*This term appears in [14] with a different meaning.

$i = 1, 2$. Hence there are vertices p, q so that apx_1 and bqx_2 are faces in G. This way, $\{q, x_2, p, a, x\} \subseteq V(link(x_1))$ and $\{p, x_1, q, b, x\} \subseteq V(link(x_2))$. Moreover,

(1) If $deg(a) = 4$ (or $deg(b) = 4$), then, ap (bq respectively) is a boundary $4c$-edge.

(2) Otherwise, there are faces apw, bqr. If $deg(a) = 5$ (or $deg(b) = 5$) then the edge $aw \subset \partial G$ and it is not a triode detecting edge although aw may be contractible.

Let us observe that in case that all 3-cycles x_1x_2p, x_1ap, x_1x_2q and x_2bq are faces of the triangulation, the M-component centered at abx coincides with the triangulation obtained by the splitting of a 5-valent boundary vertex in the irreducible triangulation of the Möbius strip M_2 collected in [4].

On the other hand, it is not difficult to see that the set of vertices $\{a, b, x_1, x_2, x\}$ in the M-component are principal vertices of a subdivision of the complete graph K_5 in G. Hence, M is not present in any triangulation of the disk.

The interest of the M-component is pointed out by the following proposition.

Proposition 2. *Let $G \in \mathcal{F}^2(4)$ be a triangulation of the punctured surface F^2. Any M-component $\mathcal{M} \subset G$ remains unaltered after performing any reduction R_i ($i = 1, \ldots, 6$).*

Proof. Let $\mathcal{M}$ be an M-component centered at abx. The edge ab is the only contractible one in $\mathcal{M}$ (in fact, it is a $cn4c$-edge), hence no reduction R_1 can be applied to $\mathcal{M}$.

Furthermore, the only possible octahedron or quasi-octahedron components containing ab must be centered at abx (since $deg(x_i) \geq 5$ for $i = 1, 2$, by Remark 6 (1)). However, such a quasi-octahedron component cannot exist since $a, b \in \partial G$. Similarly no octahedron components exists since otherwise ∂G reduces to abx. Therefore, no reduction R_2 to R_5 may be performed to $\mathcal{M}$.

Finally, the existence of an N-component containing ab is ruled out by the existence of the edge x_1x_2. Then, reduction R_6 does not affect $\mathcal{M}$. This finishes the proof. □

Next proposition gives a sufficient condition for a quasi-octahedron component to be 4-fixed.

Proposition 3. *Let $G \in \mathcal{F}^2(4)$ be a triangulation of the punctured surface F^2. Any quasi-octahedron component in G under conditions (b.1) in Proposition 1 is 4-fixed.*

Besides, if G is 4-minimal, any quasi-octahedron component $\widehat{\mathcal{O}}$ in G verifies conditions (b.1) in Proposition 1.

Proof. In Proposition 1 (b.1) $a_3 \in \partial G$ and a_1a_2 is an inner edge, and therefore the quasi-octahedron is not removable.

Since a_1a_2 is an inner edge, $a_1a_2a_3$ and $a_1v_3a_2$ are critical 3-cycles and none of their edges is contractible. Although other edges incident to a_i may be contractible, their contractions do not alter the quasi-octahedron, hence it is 4-fixed.

Next, let us consider a non-removable quasi-octahedron component $\widehat{\mathcal{O}}$ in a 4-minimal triangulation G. Observe that, from Remark 4, $deg(a_i) \geq 5$ for $i = 1, 2$ and $deg(a_3) \geq 6$. Then, (*a.1*) and (*a.2*) are ruled out if $a_3 \in G - \partial G$ and so is (*b.2*) if $a_3 \in \partial G$ □

Moreover Propositions 2 and 3 provide the only 4-fixed components for punctured surfaces. This is proved in the following theorem.

Theorem 1. *Let F^2 be a punctured surface different from the disk. Then a triangulation $G \in \mathcal{F}^2(4)$ is 4-minimal if and only if each contractible edge in G (if any) is located in either an M-component or a 4-fixed quasi-octahedron.*

Theorem 2. *The only 4-minimal triangulation of the disk is the octahedron.*

Let us observe that irreducible triangulations within $\mathcal{F}^2(4)$ form a subset of 4-minimal triangulations family, as the following resut shows.

Proposition 4. *Let G be a 4-minimal triangulation of a punctured surface different from the disk. Then G is irreducible if and only if G contains neither quasi-octahedron component nor M-component.*

Proof. As quasi-octahedron components and M-components have contractible edges, if G is irreducible no such components appear in G. Conversely, if G is reducible, it has a contractible edge, which must be placed at a quasi-octahedron component or a M-component, by Theorem 1. □

As a corollary we can state the following generating theorem.

Theorem 3. *Let F^2 be a punctured surface. Any triangulation in $\mathcal{F}^2(4)$ can be obtained from a 4-minimal triangulation by a sequence of 4-expansions.*

Theorems 1 and 2 are the versions for punctured surfaces of Theorems 1 and 2 in [18]. In fact, for the non-spherical closed surface F^2, the 4-minimal triangulations in $\mathcal{F}^2(4)$ coincide with the usual irreducible ones since operations R_i and E_i, for $i \geq 3$ in Table 1 make sense only when F^2 has boundary. This way, Theorems 1 and 2 in [18] can be restated jointly as follows.

Theorem 4. *Let F^2 be a closed surface. Any triangulation in $\mathcal{F}^2(4)$ can be obtained from a 4-minimal triangulation by a sequence of 4-expansions (namely, 4-splittings and addition of octahedra). In particular, if F^2 is the sphere the only 4-minimal triangulation of F^2 is the octahedron.*

In order to prove Theorems 1 and 2 we will need the following technical lemma where the operations R_i in Table 1 are used.

Lemma 2. *Any octahedron component of a triangulation in $\mathcal{F}^2(4)$ of the surface F^2 can be removed by applying R_2 after one of the reductions R_1 or R_3 or else by applying reduction R_5 of Table 1.*

Proof. Let $G \in \mathcal{F}^2(4)$ be a triangulation of the surface F^2 such that G contains an octahedron component $\mathcal{O}$.

First of all, observe that $V(\mathcal{O}) \cap \partial G = \{a_1, a_2, a_3\}$ and $E(\mathcal{O}) \cap \partial G = \emptyset$ implies that $\mathcal{O}$ is 4-removable (since $deg(a_i) \geq 6$ for $i = 1, 2, 3$).

Next, let us consider $\mathcal{O}$ to be non-4-removable, then $deg(a_i) \leq 5$ for some $i = 1, 2, 3$. We distinguish several cases according to $E(\mathcal{O}) \cap \partial G$ and $V(\mathcal{O}) \cap \partial G$.

1. If $E(\mathcal{O}) \cap \partial G = \emptyset$ and $\partial G \cap V(\mathcal{O}) = \emptyset$, let us suppose $deg(a_2) = 5$. Then Remark 3 assures that there is at least one 4c-edge a_2t with $a_2 \in \mathcal{O}$, $t \notin \mathcal{O}$ and $\mathcal{O}$ turns to be 4-removable after contracting a_2t.

2. If $E(\mathcal{O}) \cap \partial G = \emptyset$ and the intersection $V(\mathcal{O}) \cap \partial G$ reduces to a single vertex, say a_2 then $deg(a_2) \geq 6$ holds and since $\mathcal{O}$ is non-removable, $deg(a_i) = 5$ for $i = 1$ or $i = 3$. Let us suppose $deg(a_1) = 5$ and let t be the boundary vertex adjacent to a_2 and a_1. Therefore ta_1a_2 and ta_1a_3 are faces of G, and since a_1 is not a boundary vertex, it readily follows that a_1t is a 4c-edge of G. Again, after contracting it, the octahedron becomes 4-removable.

3. If $E(\mathcal{O}) \cap \partial G = \emptyset$ and $\emptyset \neq \partial G \cap V(\mathcal{O}) \subsetneq \{a_1, a_2, a_3\}$, again there exists precisely one vertex a_j such that $deg(a_j) = 5$ and there is a 4c-edge incident with a_j.

4. If $E(\mathcal{O}) \cap \partial G = \{a_1a_2\}$ and $deg(a_2) = 5$, then, there is precisely one vertex $v \in V(link(a_2)) - V(\mathcal{O})$ and by applying an operation R_5 to G, the new triangulation G' belongs to $\mathcal{F}^2(4)$.

5. If $E(\mathcal{O}) \cap \partial G = \{a_2a_3, a_1a_3\}$ and $\delta(a_1) = 6$ or $\delta(a_2) = 6$, then by applying an operation R_3 to G, the new triangulation G' belongs to $\mathcal{F}^2(4)$.
6. If $E(\mathcal{O}) \cap \partial G = \{v_1a_2, a_2a_3, v_1a_3\}$ and $\delta(a_2) = 5$, there is at least one $4c$-edge a_2t with $t \notin \mathcal{O}$ and $\mathcal{O}$ turns to be 4-removable after contracting a_2t.

□

As a consequence of Lemma 2 we get that any octahedron component is not 4-fixed. Namely,

Proposition 5. *No octahedron component appears in a 4-minimal triangulation in $\mathcal{F}^2(4)$ of the surface F^2.*

The following lemma informs about possible configurations around a *cn*4*c*-edge. It is the corresponding analogue of Lemma 1 (Proposition 3.3 in [6]) for the class $\mathcal{F}^2(4)$.

Lemma 3. *Let $G \in \mathcal{F}^2(4)$ be a triangulation of the surface F^2. If ab is a cn4c-edge in G so that $d(ab, \partial G) \leq 1$, and abx is a face with $deg(x) = 4$, then one of the following configurations can be found at distance at most 1 from ab:*

1. *A 4c-edge*
2. *A subgraph in the family*
   $\mathcal{B} = \{$*octahedron component, quasi-octahedron component, N-component*$\}$
3. *An M-component centered at abx.*

The proof consists of the exhaustive analysis of all possible local configurations around a *cn*4*c*-edge ab that lies in a face abx with $deg(x) = 4$. To ease the reading of the paper we will postpone the proof to appendix A.

Proof of Theorems 1 and 2: Let ab be a contractible edge in G. As G is 4-minimal, ab is a *cn*4*c*-edge. Moreover, if $d(ab, \partial G)) \geq 2$ then the same arguments given in Lemma 1 of [18] for closed surfaces allows us to find a 4*c*-edge or an octahedron component at distance ≤ 1 from ab. This contradicts the 4-minimality of G. Thus, necessarily, $d(ab, \partial G) \leq 1$ and Lemma 3, Proposition 5 and Remark 5 yield that ab lies in a quasi-octahedron component or an M-component. Now, by Proposition 3, the quasi-octahedron component must be 4-fixed.

Conversely, by hypothesis the only contractible edges are *cn*4*c*-edges. If a *cn*4*c*-edge ab belongs to a non-removable quasi-octahedron component $\widehat{\mathcal{O}}$ then it is not a $4c$-edge since $\widehat{\mathcal{O}}$ does not contain such edges. Moreover, $\widehat{\mathcal{O}}$ cannot be extended to an octahedron in G by Definition 3. Finally, no N-component contained in $\widehat{\mathcal{O}}$ can be reduced by Remark 5. Hence, no reduction R_i can be applied to remove ab.

On the other hand, if ab belongs to an M-component $\mathcal{M} \subset G$, we know by Proposition 2 that $\mathcal{M}$ is stable under reductions R_i $(i = 1, \ldots, 6)$. This finishes the proof of Theorem 1.

Let us consider the case of the triangulated disk. From Remark 6 no M-component may appear in a triangulation of the disk. Besides, a quasi-octahedron component $\widehat{\mathcal{O}}$ will be always removable according to Definition 4. In fact, it is clear that the degree ≥ 4 condition expels the quasi-octahedron from the set of disk triangulations. Moreover, according to Definition 3, vertex a_3 must have degree ≥ 5. Let a_3t be an edge with t outside $\widehat{\mathcal{O}}$. Observe that $deg(a_3) = 5$ leads to the contractibility of at, which contradicts the minimality of G, hence $deg(a_3) \geq 6$. Besides, $deg(a_i) \geq 5$ for $i = 1, 2$ since otherwise a 4-contractible edge incident at a_i appears, which is impossible. Therefore, $\widehat{\mathcal{O}}$ can be removed by applying Definition 4 (1) if $deg(a_i) \geq 6$ for $i = 1, 2$ and $a_3 \in \partial G$ or Definition 4 (2) otherwise. This finishes the proof of Theorem 2.

□

4 Further developments

According to Proposition 4, it may occur that given a triangulation G in $\mathcal{F}^2(4)$ all possible 4-minimal triangulations obtained from G by applying the 4-reductions in Table 1 are reducible. To bridge this gap, it is natural to ask for new 4-reduction operations to be defined in $\mathcal{F}^2(4)$, such that the corresponding triangulations are irreducible.

Alternatively, one may look for further operations (not increasing the number of vertices and edges) to be added to the family of R_i-operations in order to reach the same goal.

With regard to the latter, let us observe that any 4-minimal triangulation admits further reductions by allowing diagonal flips. Actually, diagonal flips have been already considered in relation with irreducible triangulations of closed surfaces in [11] and [19]; in fact, the Q-reduction operation described in [20] can be regarded as the composite of a diagonal flip and an edge contraction. Concerning this problem we can prove the following result, which gives a way of turning 4-minimal triangulations into irreducible.

Theorem 1 shows that 4-reductions do not suffice to get all irreducible triangulations within the class $\mathcal{F}^2(4)$. If, similarly as in [11] for closed surfaces, we allow diagonal flips that preserve the 4-degree condition, then we get the following theorem.

Theorem 5. *If diagonal flips are added to 4-reductions as admissible operations in the family $\mathcal{F}^2(4)$ of triangulations of a given punctured surface F^2, then the 4-minimal triangulations are exactly the irreducible triangulations in $\mathcal{F}^2(4)$.*

Proof. The diagonal flip operation is a way of getting rid of quasi-octahedra and M-components in 4-minimal triangulations. For instance, if we flip the edge x_1a in an M-configuration when $\deg(a) \geq 5$ (similarly, flip x_2b when $\deg(b) \geq 5$) we still have a triangulation in $\mathcal{F}^2(4)$ but now the edge ab is 4-contractible. Notice that $deg(x_1) \geq 5$ by definition of an M-configuration and, moreover, that some 4-contractible edge is detected whenever $\deg(a) = 4$ ($deg(b) = 4$, respectively) (see Remark 6(1)).

On the other hand, by flipping an edge a_ia_3 of a quasi-octahedron component, new 4-contractible edges are available to perform further 4-reductions and dismantle the original quasi-octahedron component. □

As a consequence, we conclude with another generating theorem with the same flavour as Theorem 3.

Theorem 6. *Let F^2 be a punctured surface. Any triangulation in $\mathcal{F}^2(4)$ can be obtained from an irreducible triangulation by a sequence of 4-expansions and diagonal flips.*

Appendix: Proof of Lemma 3

Let us start by fixing some notation. Besides the edge ab and the vertex x given by Lemma 3, we will denote by x_1 and x_2 the vertices adjacent to x for which $link(x) = x_1abx_2x_1$ if $x \notin \partial G$ or $link(x) = x_1abx_2$ if $x \in \partial G$. Recall that a vertex v is said to be independent of degree k if all neighbours of v have degree $\neq k$.

Lemma 1 establishes that the edge ab is at distance at most 1 from a subgraph H of G wich is isomorphic to a $4c$-edge or an ocathedron component or a triode detecting edge or a flag. Moreover, if G triangulates the disk, then H may reduce to a flag or an octahedron.

Since we are dealing with $G \in \mathcal{F}^2(4)$, H cannot be a flag. Hence we can take advantage of the other cases given by Lemma 1 and focus on the situation in which H is a triode detecting edge located within a quasi-octahedron, or an N-component, or an M-component at distance less than or equal to 1 from ab.

After the previous observations, all remaining cases correspond to the ones depicted in Figure 7.

We will next analyze these cases by following the Roman numbering in Figure 7.

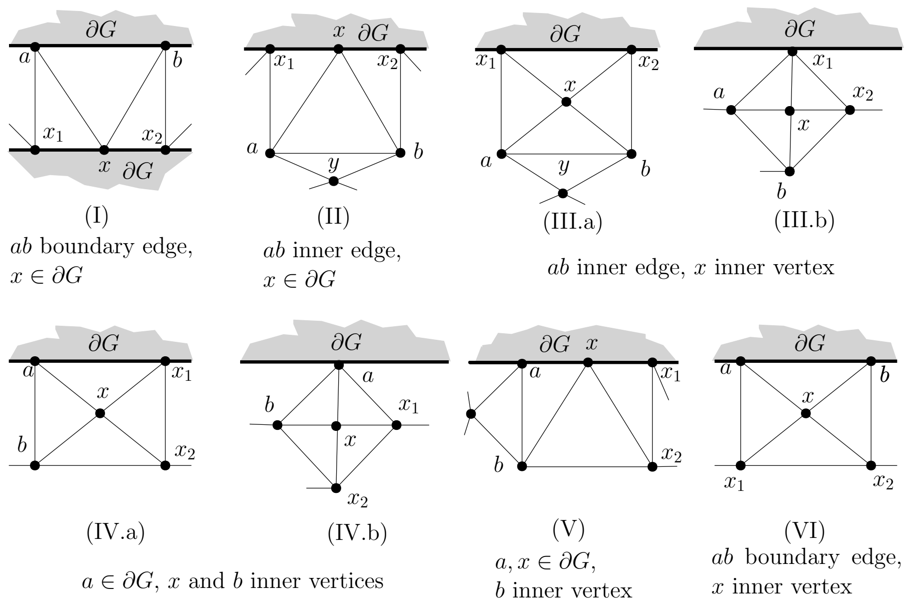

Figure 7: Different configurations for $link(x)$, with $deg(x) = 4$ and distance at most 1 from ∂G.

(I) $x \in \partial G$ and $ab \subset \partial G$.

If abx is the center of an M-component, statement 3 holds. Otherwise, x_1 and x_2 do not define an edge and xx_i is a contractible edge, for $i = 1, 2$. We distinguish two cases according to $deg(a)$ and $deg(b)$.

If $deg(a) \geq 5$ (or $deg(b) \geq 5$), then the edge xx_1 turns to be a $4c$-edge. Otherwise ($deg(a) = 4$ and $deg(b) = 4$), there is an N-component with parallel edges xx_1, ab.

Notice that x and ab do not lie simultaneously in ∂G except for Case (I). Let $m \geq 4$ denote the minimum degree of the vertices of $link(x)$. If $m \geq 5$, then it is not difficult to check that a $4c$-edge incident in x must appear. A similar situation occurs if $m = 4$ and only one vertex of $link(x)$ has degree 4.

Next we can deal cases (II) - (VI) under the following assumption.

(A) $m = 4$ with at least two vertices $\{u, v\} \subset V(link(x))$ having degree m.

(II) $ab \subseteq G - \partial G$, $x \in \partial G$.

If abx is the center of an octahedron component or a quasi-octahedron component, we are done. If xx_1x_2 is the center of an octahedron, we are done. Otherwise we distinguish two cases according to the existence of the inner edge x_1x_2 or not. Observe that with these conditions we get $deg(a) \geq 5$ or $deg(b) \geq 5$.

If x_1x_2 does not exist, then xx_i is a $4c$-edge (for $i = 1$ or 2)

If x_1x_2 does exist, it must be an inner edge. Since ab is contractible, we can suppose $b \in G - \partial G$ and xb contractible edge. Moreover, $deg(a) \geq 5$ since $deg(a) = 4$ implies the existence of the edge bx_1 and this contradicts the contractibility of ab. Observe that in this case $deg(x_2) \geq 5$, since otherwise x_2bx_1 must define a face of G, contradicting again the contractibility of ab. Therefore, xb is a $4c$-edge and this case is finished.

(III.a) $ab \subseteq G - \partial G$, $x \in G - \partial G$, $x_1x_2 \subseteq \partial G$.

If abx is the center of an octahedron component we are done. If axx_1 (analogously bxx_2) is the center of an octahedron component or a quasi-octahedron component, we are done. Otherwise, the 4-valent vertices of $link(x)$ can not be adjacent, except possibly x_1 and x_2.

Let us suppose $deg(a) = deg(x_2) = 4$ ($deg(b) = deg(x_1) = 4$ is analogous), then $deg(b) \geq 5$ and $deg(x_1) \geq 5$ implies xb and xx_1 are $4c$-edges.

If $deg(x_1) = deg(x_2) = 4$, then $deg(a) \geq 5$ and $deg(b) \geq 5$, If x_1x_2 is not contractible, it must be because of the existence of an octahedron component centered at xx_1x_2. Otherwise, x_1x_2 is contractible and there exists an N-component with parallel edges xa and x_1x_2.

(III.b) $ab \subseteq G - \partial G$, $x \in G - \partial G$, $x_1x_2 \subseteq G - \partial G$, $x_1 \in \partial G$. If abx or bx_2x is the center of an octahedron component, we are done. Otherwise, by assumption (A) $deg(b) \geq 5$ and ax and xx_2 are $4c$-edges.

(IV.a) $ab \subseteq G - \partial G$, $x, b \in G - \partial G$, $a \in \partial G$, $ax_1 \subseteq \partial G$. If one of the triangles meeting x is the center of an octahedron or quasi-octahedron component, we are done. By assumption (A), there are at least two vertices of degree 4 in $V(link(x))$. If $deg(a) = deg(x_2) = 4$ (analogous for $deg(b) = deg(x_1) = 4$), then xa is $4c$-edge if $deg(x_1) \geq 5$ (since $deg(b) \geq 5$). If $deg(x_1) = 4$ and ax_1 is contractible, then there exists an N-component with parallel edges ax_1 and xx_2. If ax_1 is not contractible, then ∂G has length 3 and axx_1 is the center of an octahedron component.

(IV.b) $ab \subseteq G - \partial G$, $x, b \in G - \partial G$, $a \in \partial G$, $ax_1 \subseteq G - \partial G$.

It is not difficult to see that all edges incident in x are contractible. If xx_1x_2 or xbx_2 is the center of an octahedron, we are done. Otherwise, by assumption (A), $deg(x_2) \geq 5$ and xx_1 and xb are $4c$-edges.

(V) $ab \subseteq G - \partial G$, $x, a \in \partial G$.

If xbx_2 is the center of an octahedron or quasi-octahedron component, we are done. If axx_1 is the center of a boundary octahedron component we are done. Otherwise $deg(b) \geq 5$ or $deg(x_2) \geq 5$. Observe that xb is a contractible edge. We distinguish two cases: there exists inner edge ax_1 or not.

If ax_1 is an inner edge, then $deg(a) \geq 5$ since $deg(a) = 4$ implies the existence of the edge bx_1 contradicting the contractibility of ab. If $deg(x_2) \geq 5$, then xb is a $4c$-edge. If $deg(x_2) = 4$, then $deg(b) \geq 5$ and it is not difficult to check that $x_2 \in G - \partial G$ ($x_2 \in \partial G$ implies x_1x_2 boundary edge, a contradiction). Therefore xx_2 is also a contractible edge. Now, notice that $deg(x_1) \geq 5$ since $deg(x_1) = 4$ implies the existence of the edge ax_2 contradicting the contractibility of ab. Hence, xx_2 is a $4c$-edge.

If ax_1 is not an edge, then xa and xx_1 are contractible and one of them must be a $4c$-edge since b and x_2 can not be 4-valent vertices simultaneously.

(VI) $ab \subseteq \partial G$, $x \in G - \partial G$. If one of the triangles meeting x is the center of an octahedron or quasi-octahedron component, we are done. Otherwise, no pair of adjacent vertices are 4-valent, except possibly a and b. If $deg(a) = deg(b) = 4$, then an N-component with parallel edges ab, xx_2 is found. If $deg(a) \geq 5$ and $deg(x_2) \geq 5$, then xx_1 and xb are $4c$-edges. If $deg(a) \geq 5$ and $deg(x_2) = 4$, then $deg(x_1) \geq 5$ and $deg(b) \geq 5$ (otherwise an octahaedron or quasi-octahedron appear), and xx_2 and xa are $4c$-edges.

$\square$

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