The de la Vallée-Poussin theorem and Orlicz spaces associated to a vector measure ☆

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A B S T R A C T

For a vector measure m, with values in a Banach space, we analyze the localization of uniformly integrable subsets of scalar integrable functions with respect to m in a suitable Orlicz space associated to m. As tools we consider the properties of Orlicz spaces associated to intermediate functions of a given pair of Young functions. These properties allow us to obtain compactness properties of the inclusion operators.

1. Introduction

This article is devoted to some measure and topological aspects of uniform integrability, a basic concept on function spaces. Let $\mathcal{D}$ be the class of non-decreasing functions $\Phi : [0, \infty) \to [0, \infty)$ such that $\lim\limits_{x\to\infty} \dfrac{\Phi(x)}{x} = \infty$. This class contains for instance the convex functions $x \mapsto x^p$ with $p > 1$ and $x \mapsto x \log(x + 1)$. Let us fix a finite measure space (Ω, Σ, μ), and denote by $L^\Phi(\mu)$ the set (Orlicz class) of measurable functions f such that $\Phi(|f|)$ belongs to the Lebesgue space $L^1(\mu)$. To avoid trivial cases, we will assume that the σ-algebra Σ is infinite. Then we have $L^\Phi(\mu) \subsetneq L^1(\mu)$. Clearly, if H is bounded in $L^\Phi(\mu)$ then H is bounded in $L^1(\mu)$. Moreover, for any bounded subset $H \subseteq L^1(\mu)$, the following three properties are equivalent.

1. H is *uniformly integrable*, that is, $\lim\limits_{c\to\infty} \sup\limits_{f\in H} \displaystyle\int_{[|f|>c]} |f|\, d\mu = 0$.

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2. H is *equiintegrable*, that is, $\lim_{\mu(A)\to 0} \sup_{f\in H} \|f\chi_A\|_{L^1(\mu)} = 0$.
3. There exists $\Phi \in \mathfrak{D}$ such that $\sup_{f\in H} \|\Phi(|f|)\|_{L^1(\mu)} < \infty$.

For equivalence between 1. and 2. see [6, Theorem 1] and for the equivalence between 1. and 3. see [13, Theorem II.T22]. The third and last property is a boundedness property in $L^\Phi(\mu)$ and is due to *de la Vallée-Poussin* (1866–1962). From this theorem we can extract interesting results about the structure of the space $L^1(\mu)$. For example, note that a singleton subset $H := \{f\}$ of $L^1(\mu)$ is clearly uniformly integrable. Thus, the de la Vallée-Poussin theorem allows us to prove that $L^1(\mu) = \bigcup_{\Phi\in\mathfrak{D}} L^\Phi(\mu)$. Let us mention that it is well known that $\bigcup_{p>1} L^p(\mu) \subsetneq L^1(\mu)$. On the other hand, the uniform integrability is strongly connected with compactness in the space $L^1(\mu)$ thanks to the Dunford–Pettis theorem. It states that for every subset $H \subseteq L^1(\mu)$, the following propositions are equivalent.

1. H is *uniformly integrable*.
2. H is relatively compact for the weak topology of $L^1(\mu)$.
3. H is relatively sequentially compact for the weak topology of $L^1(\mu)$.

The de la Vallée-Poussin theorem has been considered in different settings. For instance, in the space of measures with values into a Banach space which are countably additive, of bounded variation and μ-continuous, endowed with the variation norm [17]. In [2] the de la Vallée-Poussin theorem is applied to obtain a characterization of the countably additivity of the Dunford integral of vector functions, and also they characterize those strongly measurable vector functions that are Pettis integrable through the compactness of a certain set of scalar functions in a certain space of Orlicz. Also an abstract version of the theorem for uniform integrability in real interpolation spaces is given in [12]. More recently the theorem has been considered in [4] to obtain improved results for tightness and Cesàro uniform integrability-type conditions.

The purpose of the present article is to study the de la Vallée-Poussin theorem in the context of spaces $L^1(m)$ of scalar integrable functions with respect to a vector measure m and then, by using this result, find some consequences related to the compactness in these spaces in the style of Dunford–Pettis theorem which will allow us to locate each compact subset of $L^1(m)$ as a compact subset of a smaller Orlicz space $L^\Phi(m)$ associated to the measure m. This is carried out in Sect. 4 (Theorem 4.1, Corollary 4.6 and Corollary 4.7), after having established the necessary preliminaries in Sect. 3. In Sect. 2 we will introduce the spaces of functions with which we will work.

2. Lebesgue and Orlicz spaces associated to vector measures

Throughout this paper, we shall always assume that Ω is a nonempty set, Σ is a σ-algebra of subsets of Ω, μ is a finite positive measure defined on Σ and $L^0(\mu)$ is the space of (μ-a.e. equivalence classes of) measurable functions $f : \Omega \to \mathbb{R}$. The natural topology on $L^0(\mu)$ is given by the complete metric $d(f,g) := \int_\Omega \frac{|f-g|}{1+|f-g|} d\mu$, for all $f, g \in L^0(\mu)$. It is folklore that convergence of sequences in this topology is exactly the *convergence in measure*, that is, a sequence $(f_n)_n$ of measurable functions converges under d to a measurable function f if $\lim_{n\to\infty} \mu(\{w \in \Omega : |f_n(w) - f(w)| \geq \varepsilon\}) = 0$ for all $\varepsilon > 0$. In what follows we will denote by $[|f_n - f| \geq \varepsilon]$ the measurable set $\{w \in \Omega : |f_n(w) - f(w)| \geq \varepsilon\}$.

A Banach space $X \subseteq L^0(\mu)$ is called a *Banach function space* (B.f.s. for short) with respect to μ if it has the following properties:

(a) X is an ideal of $L^0(\mu)$ and a Banach lattice with respect to the μ-a.e. order, that is, if $f \in L^0(\mu)$, $g \in X$ and $|f| \le |g|$ μ-a.e., then $f \in X$ and $\|f\|_X \le \|g\|_X$.
(b) The characteristic function of Ω, χ_Ω, belongs to X.
(c) X is continuously included into $L^1(\mu)$.

We say that a B.f.s. X has the *Fatou property* if for any positive increasing sequence $(f_n)_n$ in X with $\sup_n \|f_n\|_X < \infty$ and $f_n \to f \in L^0(\mu)$ pointwise μ-a.e., then $f \in X$ and $\|f\|_X = \sup_n \|f_n\|_X$. And a B.f.s. X is said to be *σ-order continuous* if for any positive increasing sequence $(f_n)_n$ in X such that $f_n \to f \in X$, pointwise μ-a.e., then $\|f - f_n\|_X \to 0$.

2.1. Lebesgue spaces

Let $m : \Sigma \to X$ be a countably additive vector measure with values in a real Banach space X and consider the vector space $L^0(m)$ of (m-a.e. equivalence classes of) measurable functions $f : \Omega \to \mathbb{R}$. The *semivariation* of m is the subadditive set function defined on Σ by $\|m\|(A) := \sup\{|\langle m, x^*\rangle|(A) : x^* \in B_{X^*}\}$, where $|\langle m, x^*\rangle|$ denotes the variation of the scalar measure $\langle m, x^*\rangle : \Sigma \to \mathbb{R}$ given by $\langle m, x^*\rangle(A) := \langle m(A), x^*\rangle$ for all $A \in \Sigma$, and B_{X^*} is the unit ball of X^*, the continuous dual of X. A set $A \in \Sigma$ is called *m-null* if $\|m\|(A) = 0$.

A measure $\mu := |\langle m, x^*\rangle|$, where $x^* \in B_{X^*}$, that is equivalent to m (in the sense that $\|m\|(A) \to 0$ if and only if $\mu(A) \to 0$) is called a *Rybakov control measure* for m. Such a measure always exists (see [7, Theorem 2, p. 268]). We refer to [7] for this notion and basic results on vector measures.

A measurable function $f : \Omega \to \mathbb{R}$ is called *weakly integrable* (with respect to m) if f is integrable with respect to $|\langle m, x^*\rangle|$ for all $x^* \in X^*$. A weakly integrable function f is said to be *integrable* (with respect to m) if, for each $A \in \Sigma$ there exists an element (necessarily unique) $\int_A f\,dm \in X$, satisfying

$$\left\langle \int_A f\,dm, x^* \right\rangle = \int_A f\,d\langle m, x^*\rangle, \quad x^* \in X^*.$$

The space $L^1_w(m)$ of all (m-a.e. equivalence classes of) weakly integrable functions becomes a Banach function space, with respect to any Rybakov control measure for m, with the Fatou property when endowed with the m-a.e. order and the norm

$$\|f\|_{L^1_w(m)} := \sup\left\{ \int_\Omega |f|\,d|\langle m, x^*\rangle| : x^* \in B_{X^*} \right\}.$$

Moreover, the space $L^1(m)$ of all (m-a.e equivalence classes of) integrable functions is a closed subspace and an order continuous ideal of $L^1_w(m)$. In fact, it is the closure of $\mathcal{S}(\Sigma)$, the space of simple functions supported on Σ.

Let $1 \le p < \infty$. A measurable function $f : \Omega \to \mathbb{R}$ is called *p-integrable* (with respect to m) if $|f|^p \in L^1(m)$. We denote by $L^p(m)$ the space of (m-a.e. equivalence classes of) p-integrable functions, and by $L^p_w(m)$ the space of (m-a.e. equivalence classes of) weakly p-integrable functions. Obviously we have $L^p(m) \subseteq L^p_w(m)$. The natural norm for these spaces is given by

$$\|f\|_{L^p_w(m)} := \sup\left\{ \left(\int_\Omega |f|^p\,d\,|\langle m, x^*\rangle| \right)^{\frac{1}{p}} : x^* \in B_{X^*} \right\}, \quad f \in L^p_w(m).$$

If μ is a Rybakov control measure for m then $L^p(m)$ and $L^p_w(m)$ are B.f.s. on (Ω, Σ, μ) for all $p \geq 1$. These spaces were introduced and began to be studied in [8]. See also [15] for more information about them. In particular, recall that every bounded function belongs to $L^p(m)$ for all $p \geq 1$.

2.2. Orlicz spaces

We recall that a *Young function* is any function $\Phi : [0, \infty) \to [0, \infty)$ which is strictly increasing, convex, $\Phi(0) = 0$, and $\lim_{x\to\infty} \Phi(x) = \infty$. If moreover $\lim_{x\to 0} \frac{\Phi(x)}{x} = 0$ and $\lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$ it is said that Φ is an *N-function*. In such a case we shall write $\Phi \in \mathcal{N}$. Note that $\Phi_p(x) := x^p$ are Young functions for all $p \geq 1$, but they are N-functions only if $p > 1$.

A Young function Φ has the Δ_2-property if there exists a real number $C > 0$ such that $\Phi(2x) \leq C\Phi(x)$ for all $x \geq 0$. In such a case we shall write $\Phi \in \Delta_2$. We will also put $\mathcal{N}_2 := \mathcal{N} \cap \Delta_2$. Note that $\Phi_p(x) = x^p$ has trivially the Δ_2-property for all $p \geq 1$.

Let us fix a positive finite measure μ and let Φ be an N-function. The *Orlicz space* $L^\Phi(\mu)$ consists of those (μ-a.e. equivalence classes of) functions $f \in L^0(\mu)$ for which $\|f\|_{L^\Phi(\mu)} < \infty$, where

$$\|f\|_{L^\Phi(\mu)} := \inf \left\{ k > 0 : \int_\Omega \Phi\left(\frac{|f|}{k}\right) d\mu \leq 1 \right\}, \quad f \in L^0(\mu)$$

is the *Luxemburg norm* associated to Φ. Note that $\|f\|_{L^\Phi(\mu)} \leq 1$ if and only if $\|\Phi(|f|)\|_{L^1(\mu)} \leq 1$. In $L^\Phi(\mu)$ we can consider another norm, the *Orlicz norm*,

$$\|f\|^o_{L^\Phi(\mu)} := \sup \left\{ \int_\Omega |fg| \, d\mu : \|g\|_{L^{\hat{\Phi}}(\mu)} \leq 1 \right\}, \quad f \in L^0(\mu),$$

where $\hat{\Phi}$ is the *conjugated N-function of* Φ, defined as $\hat{\Phi}(y) := \sup_{x\geq 0}\{xy - \Phi(x)\}$, for all $y \geq 0$. From the definition of $\hat{\Phi}$ it is clear that Φ and $\hat{\Phi}$ satisfy the *Young inequality*:

$$xy \leq \Phi(x) + \hat{\Phi}(y), \quad x, y \geq 0. \tag{1}$$

The Orlicz norm is equivalent to the Luxemburg norm. In fact,

$$\|f\|_{L^\Phi(\mu)} \leq \|f\|^o_{L^\Phi(\mu)} \leq 2\|f\|_{L^\Phi(\mu)}, \quad f \in L^\Phi(\mu). \tag{2}$$

The *Orlicz class* corresponding to Φ is defined by

$$O^\Phi(\mu) := \{f :\in L^0(\mu) : \Phi(|f|) \in L^1(\mu)\}.$$

It holds that $O^\Phi(\mu) \subseteq L^\Phi(\mu)$ but the Orlicz class and the Orlicz space are not equal in general. However, if $\Phi \in \mathcal{N}_2$ then $O^\Phi(\mu) = L^\Phi(\mu)$. Detailed information about Orlicz spaces can be found in the classic books [9] and [16] or in the most recent monographs [11] and [19].

The *weak Orlicz space* with respect to a vector measure m and an N-function Φ can be introduced as the following linear space

$$L^\Phi_w(m) := \left\{ f \in L^0(m) : \|f\|_{L^\Phi_w(m)} < \infty \right\},$$

where $\|f\|_{L^\Phi_w(m)} := \sup\left\{\|f\|_{L^\Phi(|\langle m,x^*\rangle|)} : x^* \in B_{X^*}\right\}$, for all $f \in L^0(m)$, and coincides with the intersection of all scalar Orlicz spaces $L^\Phi(|\langle m, x^*\rangle|)$ with $x^* \in X^*$. In addition, the *Orlicz space* with respect to the vector measure m is defined as the closure of simple functions $\mathcal{S}(\Sigma)$ in $L^\Phi_w(m)$ and will be denoted by $L^\Phi(m)$. It can be proved that $L^\Phi_w(m)$ is a B.f.s. (with respect to any Rybakov control measure for m) having the Fatou property which is continuously included in $L^1_w(m)$, and $L^\Phi(m)$ is a σ-order continuous B.f.s. (with respect to any Rybakov control measure for m) which is continuously included in $L^1(m)$. The corresponding *Orlicz classes* are given by

$$O^\Phi_w(m) := \{f :\in L^0(m) : \Phi(|f|) \in L^1_w(m)\},$$
$$O^\Phi(m) := \{f :\in L^0(m) : \Phi(|f|) \in L^1(m)\}.$$

The Orlicz spaces and Orlicz classes for a vector measure were introduced in [18] and subsequently studied in [5]. In general we only have $O^\Phi_w(m) \subseteq L^\Phi_w(m)$ and $O^\Phi(m) \subseteq L^\Phi(m)$, but if $\Phi \in \Delta_2$ then $O^\Phi_w(m) = L^\Phi_w(m)$ and $O^\Phi(m) = L^\Phi(m)$. Furthermore, there exists a close relation between the quantities $\|f\|_{L^\Phi_w(m)}$ and $\|\Phi(|f|)\|_{L^1_w(m)}$. We finish this section with the next result (see [3, Lemma 2.1]) that we shall need later.

Lemma 2.1. *Let $f \in L^0(m)$ be and let $C > 0$. If $\|\Phi(|f|)\|_{L^1_w(m)} \leq C$, then $\|f\|_{L^\Phi_w(m)} \leq C+1$, for any $\Phi \in \mathcal{N}$.*

Proof. Assume that $\|\Phi(|f|)\|_{L^1_w(m)} \leq C$, and fix $x^* \in B_{X^*}$. Given a function $g \in L^{\hat{\Phi}}(|\langle m, x^*\rangle|)$ with $\|g\|_{L^{\hat{\Phi}}(|\langle m,x^*\rangle|)} \leq 1$, we get $\|\hat{\Phi}(|g|)\|_{L^1(|\langle m,x^*\rangle|)} \leq 1$. According to (1), we have $|fg| \leq \Phi(|f|) + \hat{\Phi}(|g|)$. Therefore,

$$\begin{aligned}\int_\Omega |fg| d\,|\langle m, x^*\rangle| &\leq \int_\Omega \Phi(|f|) d\,|\langle m, x^*\rangle| + \int_\Omega \hat{\Phi}(|g|) d\,|\langle m, x^*\rangle| \\ &= \int_\Omega \Phi(|f|) d\,|\langle m, x^*\rangle| + \|\hat{\Phi}(|g|)\|_{L^1(|\langle m,x^*\rangle|)} \\ &\leq \int_\Omega \Phi(|f|) d\,|\langle m, x^*\rangle| + 1.\end{aligned}$$

Taking supremum in g and using (2), it follows that

$$\|f\|_{L^\Phi(|\langle m,x^*\rangle|)} \leq \|f\|^o_{L^\Phi(|\langle m,x^*\rangle|)} \leq \int_\Omega \Phi(|f|) d\,|\langle m, x^*\rangle| + 1, \tag{3}$$

and again taking supremum in (3) with $x^* \in B_{X^*}$ we deduce that

$$\|f\|_{L^\Phi_w(m)} \leq \sup_{x^* \in B_{X^*}} \int_\Omega \Phi(|f|) d\,|\langle m, x^*\rangle| + 1 = \|\Phi(|f|)\|_{L^1_w(m)} + 1 \leq C + 1. \quad \square$$

3. Inclusions between Orlicz spaces associated to vector measures

It is possible to consider different partial ordering relations between Young functions and they are useful in dealing with embeddings of Orlicz spaces. Here are some of these relations (see [16, Section 2.2]).

Definition 3.1. We shall write:

- $\Phi_1 \prec \Phi_0$ if there exist $\varepsilon > 0$ and $x_0 \geq 0$ such that $\Phi_1(x) \leq \Phi_0(\varepsilon x)$, for all $x \geq x_0$.
- $\Phi_1 \prec\prec \Phi_0$ if for each $\varepsilon > 0$, there exists $x_\varepsilon \geq 0$ such that $\Phi_1(x) \leq \Phi_0(\varepsilon x)$, for all $x \geq x_\varepsilon$.

Observe that if Φ is an N-function, then $\Phi \prec \Phi$ is always satisfied but $\Phi \prec\prec \Phi$ is never possible. The following inclusion result (essentially established in [3] for N-functions with the Δ_2-property) will be of interest for us in what follows. Recall that a subset H of a σ-order continuous B.f.s. X is said to be *L-weakly compact* (see [14, Proposition 3.6.2]) if for every $\varepsilon > 0$ there exists a function $0 < g \in X$ such that $H \subseteq [-g, g] + \varepsilon B_X$. Note that an L-weakly compact set is always weakly compact (see [14, Proposition 3.6.5]).

Proposition 3.2. *Let Φ_0, Φ_1 be Young functions.*

1) *If $\Phi_1 \prec \Phi_0$, then $L_w^{\Phi_0}(m) \subseteq L_w^{\Phi_1}(m)$.*
2) *If $\phi_1 \prec\prec \phi_0$ and $\phi_0, \phi_1 \in \backslash_2$, then $l_w^{\phi_0}(m) \subseteq l^{\phi_1}(m)$ and this inclusion is l-weakly compact, that is, every bounded subset of $l_w^{\phi_0}(m)$ is an l-weakly compact subset of $l^{\phi_1}(m)$.*

Proof. Keeping in mind that $L_w^\Phi(m) := \bigcap_{x^* \in X^*} L^\Phi\left(|\langle m, x^*\rangle|\right)$, part 1) follows from [16, Theorem 5.1.3]. For 2) see [3, Lemma 3.2.(ii) and Lemma 3.3]. $\square$

Given two Young functions $\Phi_0, \Phi_1 : [0, \infty) \to [0, \infty)$ and a parameter $0 < \theta < 1$, consider the function Φ_θ whose inverse is given by the relationship

$$\Phi_\theta^{-1} := \left[\Phi_0^{-1}\right]^{1-\theta} \left[\Phi_1^{-1}\right]^{\theta}. \tag{4}$$

It is well known (see [16, Lemma 6.3.2]) that Φ_θ is a Young function, and if $\Phi_0, \Phi_1 \in \mathcal{N}$, then so is $\Phi_\theta \in \mathcal{N}$. Moreover, if $\Phi_0, \Phi_1 \in \mathcal{N}_2$, then $\Phi_\theta \in \mathcal{N}_2$. However, these results don't cover the case of $\Phi_1(x) = x$. Next we are going to verify that the function Φ_θ defined by (4) also inherits good properties from the extreme functions Φ_0 and Φ_1 when $\Phi_0 \in \mathcal{N}$ and $\Phi_1(x) = x$. Recall that $\Phi_1 \in \Delta_2$ but $\Phi_1 \notin \mathcal{N}$. Namely, from Φ_0 we will deduce that $\Phi_\theta \in \mathcal{N}$, and from Φ_1 we will get that $\Phi_\theta \in \Delta_2$.

Proposition 3.3. *Suppose that $\Phi_0 \in \mathcal{N}$ and let $\Phi_1(x) = x$. Then*

$$\lim_{x\to 0} \frac{\Phi_\theta(x)}{x} = 0 \text{ and } \lim_{x\to\infty} \frac{\Phi_\theta(x)}{x} = \infty.$$

In particular, $\Phi_\theta \in \mathcal{N}$.

Proof. The function Φ_θ is given by $\Phi_\theta^{-1}(x) = \left[\Phi_0^{-1}(x)\right]^{1-\theta} x^\theta$, for all $x \geq 0$. Then, for all $x > 0$, we have

$$\frac{\Phi_\theta(x)}{x} = \frac{\Phi_\theta(x)}{\Phi_\theta^{-1}\left(\Phi_\theta(x)\right)} = \frac{\Phi_\theta(x)}{\left[\Phi_0^{-1}\left(\Phi_\theta(x)\right)\right]^{1-\theta}\left[\Phi_\theta(x)\right]^\theta} = \left[\frac{\Phi_\theta(x)}{\Phi_0^{-1}\left(\Phi_\theta(x)\right)}\right]^{1-\theta}$$
$$= \left[\frac{\Phi_0\left(\Phi_0^{-1}\left(\Phi_\theta(x)\right)\right)}{\Phi_0^{-1}\left(\Phi_\theta(x)\right)}\right]^{1-\theta}.$$

Then $\lim_{x\to\infty} \frac{\Phi_\theta(x)}{x} = \infty$ follows from $\lim_{x\to\infty} \Phi_\theta(x) = \infty$, $\lim_{y\to\infty} \Phi_0^{-1}(y) = \infty$ and $\lim_{z\to\infty} \frac{\Phi_0(z)}{z} = \infty$. On the other hand, $\lim_{x\to 0} \frac{\Phi_\theta(x)}{x} = 0$ follows from $\lim_{x\to 0} \Phi_\theta(x) = 0$, $\lim_{y\to 0} \Phi_0^{-1}(y) = 0$ and $\lim_{z\to 0} \frac{\Phi_0(z)}{z} = 0$. $\square$

Proposition 3.4. *Suppose that $\Phi_0 \in \mathcal{N}$ and let $\Phi_1(x) = x$. Then*

$$\Phi_\theta(2x) \leq 2^{\frac{1}{\theta}} \Phi_\theta(x), \quad x \geq 0,\ 0 < \theta < 1.$$

In particular, $\Phi_\theta \in \Delta_2$ for all $0 < \theta < 1$.

Proof. By definition $\Phi_\theta(2x) = \inf\left\{t > 0 : \left[\Phi_0^{-1}(t)\right]^{1-\theta} t^\theta > 2x\right\}$. Since Φ_0^{-1} is increasing,

$$\left[\Phi_0^{-1}\left(t\,2^{-\frac{1}{\theta}}\right)\right]^{1-\theta}\left[t\,2^{-\frac{1}{\theta}}\right]^{\theta} \le \left[\Phi_0^{-1}(t)\right]^{1-\theta}\left[t\,2^{-\frac{1}{\theta}}\right]^{\theta}.$$

Then we have $\left\{t > 0 : \Phi_\theta^{-1}\left(t\,2^{-\frac{1}{\theta}}\right) > x\right\} \subseteq \left\{t > 0 : \Phi_\theta^{-1}(t) > 2x\right\}$. Thus

$$\begin{aligned}\Phi_\theta(2x) &= \inf\left\{t > 0 : \Phi_\theta^{-1}(t) > 2x\right\} \le \inf\left\{t > 0 : \Phi_\theta^{-1}\left(t\,2^{-\frac{1}{\theta}}\right) > x\right\} \\ &= \inf\left\{2^{\frac{1}{\theta}} s > 0 : \Phi_\theta^{-1}(s) > x\right\} = 2^{\frac{1}{\theta}}\Phi_\theta(x). \quad \square\end{aligned}$$

Remark 3.5. We just checked that $\Phi_\theta \in \mathcal{N}_2$ for all $0 < \theta < 1$, where $\Phi_0 \in \mathcal{N}$ (recall that we don't assume that $\Phi_0 \in \Delta_2$), and $\Phi_1(x) = x$ (recall that $\Phi_1 \notin \mathcal{N}$). However, we do not know if it is true that $\Phi_\theta \in \mathcal{N}_2$ for all $0 < \theta < 1$, for general Young functions $\Phi_0 \in \mathcal{N}$ and $\Phi_1 \in \Delta_2$.

We finish this section with some others inclusion results between Orlicz spaces. Let us mention that next Proposition 3.7 will be the key to some of the results of the following section.

Proposition 3.6. *Suppose that $\Phi_0 \in \mathcal{N}$ and let $\Phi_1(x) = x$. Then, for each $0 < \theta < 1$:*

1) $\Phi_\theta(x) \le x^{\frac{1}{\theta}}\Phi_\theta(1)$, *for all $x \ge 1$.*
2) $\hat{\Phi}_\theta(y) \ge \left[\dfrac{y}{1+\Phi_\theta(1)}\right]^{\frac{1}{1-\theta}}$, *for all $y \ge y_0 := \hat{\Phi}_\theta^{-1}(1)$.*

In particular, this implies that $L^{\frac{1}{\theta}}(m) \subseteq L^{\Phi_\theta}(m)$ and $L^{\hat{\Phi}_\theta}(m) \subseteq L^{\frac{1}{1-\theta}}(m)$, for every vector measure m.

Proof. 1) By definition $\Phi_\theta(x) = \inf\left\{t > 0 : \left[\Phi_0^{-1}(t)\right]^{1-\theta} t^\theta > x\right\}$. Since Φ_0^{-1} is increasing, for all $x \ge 1$,

$$\left[\Phi_0^{-1}\left(t\,x^{-\frac{1}{\theta}}\right)\right]^{1-\theta}\left[t\,x^{-\frac{1}{\theta}}\right]^{\theta} \le \left[\Phi_0^{-1}(t)\right]^{1-\theta}\left[t\,x^{-\frac{1}{\theta}}\right]^{\theta}.$$

Then we have $\left\{t > 0 : \Phi_\theta^{-1}\left(t\,x^{-\frac{1}{\theta}}\right) > 1\right\} \subseteq \left\{t > 0 : \Phi_\theta^{-1}(t) > x\right\}$. Thus, for all $x \ge 1$,

$$\begin{aligned}\Phi_\theta(x) &= \inf\left\{t > 0 : \Phi_\theta^{-1}(t) > x\right\} \le \inf\left\{t > 0 : \Phi_\theta^{-1}\left(t\,x^{-\frac{1}{\theta}}\right) > 1\right\} \\ &= \inf\left\{x^{\frac{1}{\theta}} s > 0 : \Phi_\theta^{-1}(s) > 1\right\} = x^{\frac{1}{\theta}}\Phi_\theta(1).\end{aligned}$$

2) From (1) and using the inequality 1) that we have just proved we get

$$xy \le \Phi_\theta(x) + \hat{\Phi}_\theta(y) \le x^{\frac{1}{\theta}}\Phi_\theta(1) + \hat{\Phi}_\theta(y), \quad x \ge 1,\ y \ge 0. \tag{5}$$

Given $y \ge y_0 := \hat{\Phi}_\theta^{-1}(1)$, take $x = \left[\hat{\Phi}_\theta(y)\right]^\theta \ge 1$ and put it into (5) to obtain

$$y\left[\hat{\Phi}_\theta(y)\right]^\theta \le \hat{\Phi}_\theta(y)\Phi_\theta(1) + \hat{\Phi}_\theta(y),$$

for all $y \ge y_0$. Now an easy computation shows that $\hat{\Phi}_\theta(y) \ge \left[\dfrac{y}{1+\Phi_\theta(1)}\right]^{\frac{1}{1-\theta}}$, for all $y \ge y_0$. $\square$

Proposition 3.7. *Suppose that $\Phi_0 \in \mathcal{N}$ and let $\Phi_1(x) = x$. Then*

$$\Phi_1 \prec\prec \Phi_\theta \prec\prec \Phi_\alpha \prec\prec \Phi_0, \quad 0 < \alpha < \theta < 1$$

In particular, $L_w^{\Phi_0}(m) \subseteq L^{\Phi_\theta}(m)$ for all $0 < \theta < 1$, and this inclusion is L-weakly compact.

Proof. i) First we are going to see that $\Phi_\alpha \prec\prec \Phi_0$ for all $0 < \alpha < 1$. For a given $\varepsilon > 0$, since $\lim_{x\to\infty} \dfrac{\Phi_0(x)}{x} = \infty$, there is $x_\varepsilon > 0$ such that $\left[\dfrac{\Phi_0(\varepsilon x)}{\varepsilon x}\right]^\alpha \geq \dfrac{1}{\varepsilon}$, for all $x > x_\varepsilon$. Then, for $x > x_\varepsilon$, we have $x^\alpha \leq \varepsilon^{1-\alpha} [\Phi_0(\varepsilon x)]^\alpha$ and so, $x \leq [\varepsilon x]^{1-\alpha} [\Phi_0(\varepsilon x)]^\alpha$. Thus, $x \leq \left[\Phi_0^{-1}(\Phi_0(\varepsilon x))\right]^{1-\alpha} \left[\Phi_1^{-1}(\Phi_0(\varepsilon x))\right]^\alpha$. Consequently we get $x \leq \Phi_\alpha^{-1}(\Phi_0(\varepsilon x))$, and $\Phi_\alpha(x) \leq \Phi_0(\varepsilon x)$, for all $x > x_\varepsilon$, which means that $\Phi_\alpha \prec\prec \Phi_0$.

ii) Next we will check that $\Phi_\theta \prec\prec \Phi_\alpha$ for all $0 < \alpha < \theta < 1$. Recall that $\Phi_\alpha^{-1} := \left[\Phi_0^{-1}\right]^{1-\alpha} \left[\Phi_1^{-1}\right]^\alpha$ and $\Phi_\theta^{-1} := \left[\Phi_0^{-1}\right]^{1-\theta} \left[\Phi_1^{-1}\right]^\theta$. Take $\beta := \dfrac{\theta - \alpha}{1 - \alpha} \in (0, 1)$. Then

$$\left[\Phi_\alpha^{-1}\right]^{1-\beta} \left[\Phi_1^{-1}\right]^\beta = \left[\left[\Phi_0^{-1}\right]^{1-\alpha} \left[\Phi_1^{-1}\right]^\alpha\right]^{1-\beta} \left[\Phi_1^{-1}\right]^\beta = \left[\Phi_0^{-1}\right]^{1-\theta} \left[\Phi_1^{-1}\right]^\theta = \Phi_\theta^{-1}.$$

Now it is enough to apply exactly what we just seen in i) for the couple (Φ_0, Φ_1), with parameter $0 < \alpha < 1$, but now applied to the couple (Φ_α, Φ_1), with parameter $0 < \beta < 1$, to deduce that $\Phi_\theta \prec\prec \Phi_\alpha$.

iii) Once we know that Φ_θ is a N-function, we have $\lim_{x\to+\infty} \dfrac{\Phi_\theta(x)}{x} = +\infty$. Therefore, for each $\varepsilon > 0$ we have $\lim_{x\to+\infty} \dfrac{\Phi_\theta(\varepsilon x)}{\varepsilon x} = +\infty$. From here we have that there exists $x_\varepsilon > 0$ such that $\dfrac{\Phi_\theta(\varepsilon x)}{\varepsilon x} \geq \dfrac{1}{\varepsilon}$ for all $x \geq x_\varepsilon$. Then $\Phi_1(x) := x \leq \Phi_\theta(\varepsilon x)$ for all $x \geq x_\varepsilon$. That is, $\Phi_1 \prec\prec \Phi_\theta$.

The last assertion follows from Proposition 3.2. Indeed, given $0 < \theta < 1$, take $0 < \alpha < \theta$. Then we know that $L_w^{\Phi_0}(m) \subseteq L_w^{\Phi_\alpha}(m)$ since Φ_0 and Φ_α are Young functions. Now taking into account that $\Phi_\theta, \Phi_\alpha \in \mathcal{N}_2$ and $\Phi_\theta \prec\prec \Phi_\alpha$ it follows that the inclusion $L_w^{\Phi_\alpha}(m) \subseteq L^{\Phi_\theta}(m)$ is L-weakly compact, and so it is also the inclusion $L_w^{\Phi_0}(m) \subseteq L^{\Phi_\theta}(m)$, as we wanted to see. $\square$

4. Some consequences of the de la Vallée-Poussin theorem for vector measures

In this section we present the de la Vallée-Poussin theorem in the context of spaces $L^1(m)$ of scalar integrable functions with respect to a vector measure m and then, using this result, we find some consequences related to the compactness in these spaces in the style of Dunford–Pettis theorem which will allow us to locate each compact subset of $L^1(m)$ as a compact subset of a smaller Orlicz space $L^\Phi(m)$ associated to the measure m. Of course, the de la Vallée-Poussin theorem characterizes the uniformly integrable subsets of $L_w^1(m)$ as bounded subsets of some Orlicz space $L_w^\Phi(m)$. A subset $H \subseteq L_w^1(m)$ is said to be *uniformly integrable* if $\lim_{c\to\infty} \int_{[|f|>c]} |f|\, d|\langle m, x^*\rangle| = 0$, uniformly in $\|x^*\| \leq 1$ and $f \in H$, or equivalently,

$$\lim_{c\to\infty} \sup_{f\in H} \left\| f\chi_{[|f|>c]} \right\|_{L_w^1(m)} = 0. \tag{6}$$

Note that any uniformly integrable subset $H \subseteq L_w^1(m)$ is in fact a bounded subset of $L^1(m)$. See the proof of the implication 1) ⇒ 2) in Proposition 4.4.

Theorem 4.1 *(de la Vallée-Poussin). For a subset $H \subseteq L^0(m)$, the following conditions are equivalent:*

a) *H is uniformly integrable.*

b) *There exists a convex function* $\Phi \in \mathcal{D}$ *such that* $\{\Phi(|f|) : f \in H\}$ *is a bounded subset of* $L^1_w(m)$.
c) *There exists* $\Psi \in \mathcal{N}$ *such that* H *is a bounded subset of* $L^\Psi_w(m)$.

Proof. *b)* $\Rightarrow$ *a)* Let $M := \sup\left\{\|\Phi(|f|)\|_{L^1_w(m)} : f \in H\right\} < \infty$. For a given $\varepsilon > 0$ let $c \geq 0$ be such that $\Phi(x) \geq \dfrac{M}{\varepsilon} x$ for $x > c$. Then

$$\Phi(|f|\chi_{[|f|>c]}) \geq \Phi(|f|)\chi_{[|f|>c]} \geq \frac{M}{\varepsilon}|f|\chi_{[|f|>c]},$$

and taking norm, we get $\left\|f\chi_{[|f|>c]}\right\|_{L^1_w(m)} \leq \dfrac{\varepsilon}{M}\left\|\Phi(|f|\chi_{[|f|>c]})\right\|_{L^1_w(m)} \leq \varepsilon$, for every $f \in H$. This also implies that $H \subset L^1(m)$.

a) $\Rightarrow$ *b)* From the hypothesis (6) we can select an increasing sequence $0 < c_1 < c_2 < \cdots \uparrow \infty$ such that $\left\|f\chi_{[|f|>c_n]}\right\|_{L^1_w(m)} \leq \dfrac{1}{2^n}$, for all $n \geq 1$, and all $f \in H$. Let us define $\Phi(x) := \sum_{k=1}^{\infty}(x - c_k)^+$ for $x \geq 0$. Then $\Phi : [0,\infty) \to [0,\infty)$ is convex, increasing and for $x \geq 2c_n$ we have

$$\frac{\Phi(x)}{x} \geq \sum_{k=1}^{n}\left(1 - \frac{c_k}{x}\right)^+ \geq \frac{n}{2}.$$

Obviously for each $f \in H$ and $\|x^*\| \leq 1$, applying the monotone convergence theorem for each positive finite measure $|\langle m, x^*\rangle|$ we get

$$\int_\Omega \Phi(|f|)\, d|\langle m, x^*\rangle| = \sum_{k=1}^{\infty}\int_\Omega (|f| - c_k)^+\, d|\langle m, x^*\rangle| \leq 1,$$

which means that $\{\Phi(|f|) : f \in H\}$ is a bounded subset of $L^1_w(m)$.

b) $\Rightarrow$ *c)* We know by applying [9, Theorem 3.3] that Φ is the *principal part* of some $\Psi \in \mathcal{N}$, which means that there exists $x_0 \geq 0$ such that $\Phi(x) = \Psi(x)$, for all $x \geq x_0$. To prove that H is a bounded subset of $L^\Psi_w(m)$ it is enough to check (see Lemma 2.1) that $\{\Psi(|f|) : f \in H\}$ is a bounded subset of $L^1_w(m)$. Note that $\Psi(|f|) \leq \Phi(|f|)\chi_{[|f|\geq x_0]} + \Psi(x_0)\chi_{[|f|<x_0]}$, and then

$$\begin{aligned}\|\Psi(|f|)\|_{L^1_w(m)} &\leq \left\|\Phi(|f|)\chi_{[|f|\geq x_0]}\right\|_{L^1_w(m)} + \left\|\Psi(x_0)\chi_{[|f|<x_0]}\right\|_{L^1_w(m)} \\ &\leq \|\Phi(|f|)\|_{L^1_w(m)} + \Psi(x_0)\,\|m\|\,(\Omega).\end{aligned}$$

Thus, $\{\Psi(|f|) : f \in H\}$ is a bounded subset of $L^1_w(m)$ because $\{\Phi(|f|) : f \in H\}$ is a bounded subset of $L^1_w(m)$ by the hypothesis. The implication *c)* $\Rightarrow$ *b)* is trivial. $\square$

We present now a first consequence of the above result. Recall that $\mathcal{N}_2$ denotes the set of all N-functions with the Δ_2-property.

Corollary 4.2. *For any vector measure* m *we have*

$$L^1(m) = \bigcup_{\Phi\in\mathcal{N}_2} L^\Phi(m) = \bigcup_{\Phi\in\mathcal{N}_2} L^\Phi_w(m). \tag{7}$$

Proof. Clearly $\bigcup_{\Phi\in\mathcal{N}_2} L^\Phi(m) \subseteq \bigcup_{\Phi\in\mathcal{N}_2} L^\Phi_w(m)$. For any function $\Phi \in \mathcal{N}$ (in particular if $\Phi \in \mathcal{N}_2$) Propositions 3.3, 3.7 and 3.4 together assure that there exists another $\Psi \in \mathcal{N}_2$ such that $L^\Phi_w(m) \subseteq L^\Psi(m)$. That means the equality $\bigcup_{\Phi\in\mathcal{N}_2} L^{\Phi}(m) = \bigcup_{\Phi\in\mathcal{N}_2} L^{\Phi}_w(m)$ holds. To finish the proof take a function $f \in L^1(m)$. Since $\{f\}$ is a uniformly integrable subset, the de la Vallée-Poussin theorem provides a function $\Phi_0 \in \mathcal{N}$ such that $f \in L^{\Phi_0}_w(m)$. Once again Propositions 3.3, 3.7 and 3.4 together assure that there exists another $\Phi \in \mathcal{N}_2$ such that $f \in L^{\Phi}(m)$, and the proof is over. □

Remark 4.3. Let us make some observations on the equality (7).

1) For a finite measure μ and $\Phi \in \mathcal{N}$ it is known (see [19, Corollary 15.4.2] or [16, Theorem VII.3.2]) that $L^{\Phi}(\mu)$ is reflexive if and only if both $\Phi, \hat{\Phi} \in \mathcal{N}_2$. Denote by $\mathcal{N}_{\mathrm{R}}$ the set of all $\Phi \in \mathcal{N}$ such that $L^{\Phi}(\mu)$ is reflexive. Note that $\{x \mapsto x^p,\ p > 1\} \subsetneq \mathcal{N}_{\mathrm{R}} \subsetneq \mathcal{N}_2$. On the other hand (see the proof of [9, Theorem 4.1] and the comments therein), if $\Phi \in \mathcal{N}_2$ then $L^{p_1}(\mu) \subseteq L^{\Phi}(\mu)$ for some $p_1 > 1$. Using the same argument for $\hat{\Phi}$ we conclude that $\Phi \in \mathcal{N}_{\mathrm{R}}$ if and only if $L^{p_1}(\mu) \subseteq L^{\Phi}(\mu) \subseteq L^{p_2}(\mu)$ for some $p_1 > p_2 > 1$. Then

$$\bigcup_{\Phi\in\mathcal{N}_{\mathrm{R}}} L^{\Phi}(\mu) = \bigcup_{p>1} L^{p}(\mu) \subsetneq L^1(\mu).$$

2) Another natural question is to ask whether equality $L^1_w(m) = \bigcup_{\Phi\in\mathcal{N}_2} L^{\Phi}_w(m)$ holds. In general the answer is negative because there exist vector measures m such that $L^1(m) \subsetneq L^1_w(m)$. A such vector measure m can be constructed with values in the space c_0 of all null sequences. For the equality between the spaces $L^1(m)$ and $L^1_w(m)$ see [10].

Clearly, if H is relatively norm compact in $L^{\Phi}(m)$ for some $\Phi \in \mathcal{N}$, then H is relatively norm compact in $L^1(m)$. Now we are going to prove that, reciprocally, every relatively norm compact subset of $L^1(m)$ is located within some $L^{\Phi}(m)$. Relatively norm compactness in $L^1(m)$ is connected with *L-weakly compactness* and *sequentially compactness in measure*. The connection between L-weakly compactness and uniform integrability is given by the following

Proposition 4.4. *For a subset $H \subseteq L^1_w(m)$ the following conditions are equivalent:*

1) *H is uniformly integrable.*
2) *$H \subset L^1(m)$ is bounded and* $\lim\limits_{\|m\|(A)\to 0} \sup\limits_{f\in H} \|f\chi_A\|_{L^1_w(m)} = 0.$
3) *H is L-weakly compact in $L^1(m)$.*

Proof. 1) ⇒ 2) Let's fix a function $f \in H$, and consider the bounded functions $f_n := f\chi_{[|f|\le n]} \in L^1(m)$ for all $n = 1, 2, \ldots$ From (6) we have that $\lim\limits_{n\to\infty} \|f - f_n\|_{L^1_w(m)} = 0$. Then $f \in L^1(m)$ because this space is closed in $L^1_w(m)$, and therefore $H \subseteq L^1(m)$. By using (6) again there exists a constant $c > 0$ such that $\|f\chi_{[|f|>c]}\|_{L^1_w(m)} \le 1$ for all $f \in H$. Then

$$\|f\|_{L^1_w(m)} \le \|f\chi_{[|f|\le c]}\|_{L^1_w(m)} + \|f\chi_{[|f|>c]}\|_{L^1_w(m)} \le c\|m\|(\Omega) + 1$$

for all $f \in H$, and thus H is bounded in $L^1(m)$. Finally, note that

$$|f|\chi_A = |f|\chi_{A\cap[|f|>c]} + |f|\chi_{A\cap[|f|\le c]} \le |f|\chi_{[|f|>c]} + c\chi_A$$

for all $f \in L^1_w(m)$ and all $A \in \Sigma$. Then

$$\|f\chi_A\|_{L^1_w(m)} \le \|f\chi_{[|f|>c]}\|_{L^1_w(m)} + c\|m\|(A)$$

for all $f \in L^1_w(m)$ and all $A \in \Sigma$, and the conclusion follows from (6).

The reverse implication 2) ⇒ 1) follows from the Markov–Chebyshev's inequality. For a direct proof see [6, Theorem 1]. For the equivalence of 2) and 3) see [15, Lemma 2.37(iii)] since $L^1(m)$ is a σ-order continuous Banach function space with respect every Rybakov control measure for m. □

Proposition 4.5. *Let X be a σ-order continuous Banach function space with respect to μ. Let $(f_n)_n \subseteq X$, and $f \in L^0(\mu)$. The following conditions are equivalent:*

1) *$f \in X$, and $\|f_n - f\|_X \to 0$.*
2) *$(f_n)_n$ is L-weakly compact, and $f_n \to f$ in measure.*

Proof. 1) ⇒ 2) It is well-known that every relatively compact subset of a σ-order continuous Banach function space is L-weakly compact, and also that norm convergence implies convergence in measure.

2) ⇒ 1) It is enough to prove that $(f_n)_n$ is norm Cauchy. Given $\varepsilon > 0$, by the L-weak compactness (see [15, Lemma 2.37(iii)]), there exists $\delta_1 > 0$ such that $\|f_n \chi_A\|_X < \frac{\varepsilon}{3}$ for all $n \geq 1$ and all measurable subsets with $\mu(A) < \delta_1$. Now, taking into account that $(f_n)_n$ is Cauchy in measure, given $\delta_1 > 0$ and $\varepsilon_1 := \frac{\varepsilon}{3\|\chi_\Omega\|_X}$, the exists $n_0 \geq 1$ such that $\mu([|f_n - f_k| > \varepsilon_1]) < \delta_1$ for all $n, k \geq n_0$. Put $A_{n,k} := [|f_n - f_k| > \varepsilon_1]$ and note that $\mu(A_{n,k}) < \delta_1$ for all $n, k \geq n_0$. For those n, k we have

$$\begin{aligned} \|f_n - f_k\|_X &\leq \left\|(f_n - f_k)\,\chi_{A_{n,k}}\right\|_X + \left\|(f_n - f_k)\,\chi_{\Omega \smallsetminus A_{n,k}}\right\|_X \\ &\leq \left\|f_n \chi_{A_{n,k}}\right\|_X + \left\|f_k \chi_{A_{n,k}}\right\|_X + \varepsilon_1 \left\|\chi_{\Omega \smallsetminus A_{n,k}}\right\|_X \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \varepsilon_1 \|\chi_\Omega\|_X = \varepsilon. \end{aligned}$$

We just checked that $(f_n)_n$ is norm Cauchy, and thus convergent. Necessarily, its limit must be the function f. □

Corollary 4.6. *Let X be a σ-order continuous Banach function space with respect to μ, and let $H \subseteq X$. The following conditions are equivalent:*

1) *H is relatively norm compact.*
2) *H is L-weakly compact and relatively (sequentially) compact in measure.*

Corollary 4.7. *A subset $H \subseteq L^1(m)$ is relatively norm compact if and only if there exists $\Phi \in \mathcal{N}_2$ such that $H \subseteq L^\Phi(m)$ is relatively norm compact.*

Proof. One implication is trivial because the inclusion $L^\Phi(m) \subseteq L^1(m)$ is continuous. On the other hand, if $H \subseteq L^1(m)$ is relatively norm compact, then H is L-weakly compact and relatively (sequentially) compact in measure by Corollary 4.6. Moreover, by the de la Vallée-Poussin Theorem 4.1, there exists $\Phi_0 \in \mathcal{N}$ such that $H \subseteq L_w^{\Phi_0}(m)$ is bounded. We know (see Proposition 3.7) that the inclusion $L_w^{\Phi_0}(m) \subseteq L^{\Phi_\theta}(m)$ is L-weakly compact for $0 < \theta < 1$. Moreover, $\Phi_\theta \in \mathcal{N}_2$ by Proposition 3.4. Thus $H \subseteq L^{\Phi_\theta}(m)$ is L-weakly compact, and relatively (sequentially) compact in measure. Since $L^{\Phi_\theta}(m)$ has σ-order continuous norm, Corollary 4.6 tell us that H is relatively norm compact in $L^{\Phi_\theta}(m)$ as we want to see. □

Remark 4.8. The above result is essentially the generalization of [1, Theorem 2.2] for vector measures. Alexopoulos also proved in the same article that a similar result (see [1, Theorem 2.5]) is true for the weak topology of $L^1(\mu)$, where μ is a positive finite measure defined on a σ-algebra Σ. Nevertheless, we can not expect a similar result to Corollary 4.7 for the weak topology of $L^1(m)$. In order to check this take $p > 1$ and the vector measure

$$m : A \in \Sigma \to m(A) := \chi_A \in L^p(\mu), \tag{8}$$

in which case it is well-known that $L^1(m) = L^p(\mu)$. Therefore, $L^1(m)$ is reflexive and a subset $H \subseteq L^1(m)$ is relatively weakly compact if and only if it is bounded. If H were contained in some $L^{\Phi_0}(m)$, being bounded, we know that there exists $\Phi_\theta \in \mathcal{N}_2$ such that $H \subseteq L^{\Phi_\theta}(m)$ is L-weakly compact. Then H would be L-weakly compact in $L^1(m) = L^p(\mu)$, but not every bounded subset of $L^p(\mu)$ is L-weakly compact.

Remark 4.9. In Corollary 4.6 or Proposition 4.5 we can not weaken hypothesis 2) replacing the L-weak compactness with weak compactness of the subset H. Consider a finite measure μ and let $1 < p < \infty$. Take a sequence of measurable sets $(A_n)_n$ such that $A_n \downarrow \varnothing$, and construct the sequence of functions $f_n := \frac{1}{\mu(A_n)^{\frac{1}{p}}} \chi_{A_n}$, for all $n \geq 1$. For every measurable set B, note that

$$\int_B f_n d\mu = \frac{1}{\mu(A_n)^{\frac{1}{p}}} \mu(A_n \cap B) \leq \mu(A_n)^{1-\frac{1}{p}} \to 0.$$

Thus, $f_n \to 0$ in the weak topology of $L^p(\mu)$. On the other hand, its easy to see that $f_n \to 0$ in measure, but $\|f_n\|_{L^p(\mu)} = 1$, for all $n \geq 1$.

However, the case $p = 1$ is particularly interesting because it points out another difference between the Lebesgue space of a positive scalar measure $L^1(\mu)$ and the Lebesgue space of a vector measure $L^1(m)$. For a bounded sequence $(f_n)_n \subseteq L^1(\mu)$ and a function $f \in L^0(\mu)$, the Lebesgue–Vitali and Dunford–Pettis theorems assert that $f_n \to f$ in $L^1(\mu)$ if and only if $f_n \to f$ in the weak topology of $L^1(\mu)$ and $f_n \to f$ in measure. This equivalence fails for some Lebesgue spaces $L^1(m)$ as the measure of (8) points out.

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