INTERPOLATION WITH A PARAMETER FUNCTION OF L^p-SPACES WITH RESPECT TO A VECTOR MEASURE ON A δ-RING

R. DEL CAMPO,[1*] A. FERNÁNDEZ,[2] A. MANZANO,[3]
F. MAYORAL,[2] and F. NARANJO[2]

Abstract. Let ν be a σ-finite Banach-space-valued measure defined on a δ-ring. We find a wide class of measures ν for which interpolation with a parameter function of couples of Banach lattices of p-integrable and weakly p-integrable functions with respect to ν produces a Lorentz-type space. Moreover, we prove that if we interpolate between sums and intersections of them, then they still yield another Lorentz-type space closely related with the first one.

1. Introduction

Let m be a vector measure defined on a σ-algebra Σ of Ω with values in a Banach space X, let ρ be a parameter function in the class $Q(0,1)$ of Persson, let $0 < q \leq \infty$, and let $1 < p_0 \neq p_1 < \infty$. We proved in [5, Corollary 4] that

$$\big(L^{p_0}(m), L^{p_1}(m)\big)_{\rho,q} = \big(L^{p_0}_w(m), L^{p_1}_w(m)\big)_{\rho,q} = \Lambda^q_\varphi(\|m\|), \tag{1.1}$$

where $\varphi(t) = \frac{t^{\frac{1}{p_0}}}{\rho(t^{\frac{1}{p_0}-\frac{1}{p_1}})}$. In particular, for the classical real interpolation method, which is obtained for the parameter function $\rho(t) = t^\theta$ with $0 < \theta < 1$, we have

$$\big(L^{p_0}(m), L^{p_1}(m)\big)_{\theta,q} = \big(L^{p_0}_w(m), L^{p_1}_w(m)\big)_{\theta,q} = L^{p,q}(\|m\|), \tag{1.2}$$

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*Corresponding author.

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where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. This particular situation (1.2) was generalized in [6, Corollary 3.11], replacing m by a σ-finite, locally strongly additive vector measure ν defined on a weaker structure than a σ-algebra, namely, on a δ-ring $\mathcal{R}$ of Ω. Therefore, a natural question is to find out if (1.1) keeps on verifying with m replaced by ν. The answer lies in the affirmative (even for $1 \leq p_0 \neq p_1 \leq \infty$), and Section 3 is devoted to sketch the reasons why that works (see Corollary 3.5).

Moreover, in the setting of vector measures on δ-rings the L^p-spaces are no longer ordered by inclusion as it occurs in the case of measures on σ-algebras, and so it becomes interesting to investigate what happens when we interpolate between sums and intersections of them. Recall that integration with respect to vector measures defined on δ-rings is the natural vector-valued generalization of the case of integration with respect to positive σ-finite measures μ, which does not fit into the frame of vector measures on σ-algebras if μ is nonfinite. When μ is a σ-finite measure, it is known that

$$\big(L^p(\mu) + L^\infty(\mu), L^p(\mu) \cap L^\infty(\mu)\big)_{\rho,q} = \Lambda^q_{\widetilde{\varphi}}(\|\mu\|) \tag{1.3}$$

with $\widetilde{\varphi}(t) = \frac{t^{\frac{1}{p}}}{\widetilde{\rho}(t^{\frac{1}{p}})}$ and $\widetilde{\rho}(t) = \rho(t)\chi_{(0,1]}(t) + t\rho(t^{-1})\chi_{(1,\infty)}(t)$ (see [17, Example 7.1]). Therefore, in light of (1.1) and (1.3), one can expect that

$$\big(L^{p_0}(\nu) + L^{p_1}(\nu), L^{p_0}(\nu) \cap L^{p_1}(\nu)\big)_{\rho,q} = \Lambda^q_{\widetilde{\varphi}}(\|\nu\|) \tag{1.4}$$

with $\widetilde{\varphi}(t) = \frac{t^{\frac{1}{p_0}}}{\widetilde{\rho}(t^{\frac{1}{p_0}-\frac{1}{p_1}})}$ (and $\widetilde{\rho}$ as above) for any σ-finite locally strongly additive vector measure ν defined on a δ-ring and $1 \leq p_0 \neq p_1 \leq \infty$.

Given an interpolation couple $\bar{A} = (A_0, A_1)$, it has been studied that both the relationship between its interpolation spaces and the interpolation spaces of the couple $(\Sigma(\bar{A}), \Delta(\bar{A}))$ are obtained by the interpolation method with a parameter function (see [12, Proposition 3] or [17, Proposition 7.2]). Applying this to a couple of L^p-spaces with respect to ν and using Corollary 3.5, we can obtain (1.4) under the hypothesis that $\rho \in Q(0, \frac{1}{2}] \cup Q[\frac{1}{2}, 1)$. However, with the more general and natural hypothesis $\rho \in Q(0,1)$, it cannot be deduced in such a way. Therefore, a deeper insight into the involved K-functionals is needed in order to see that (1.4) can be achieved for any $\rho \in Q(0,1)$ (see Corollary 5.3). The cases $p_1 = \infty$ or $p_1 \neq \infty$ in (1.4) must be treated separately. The former is done in Section 4 and the latter in Section 5.

2. Preliminaries

Let X be a real Banach space with dual X' and unit ball $B(X)$, and let $\nu : \mathcal{R} \to X$ be a (countably additive) vector measure defined on a δ-ring $\mathcal{R}$ of subsets of some nonempty set Ω. We denote by $\mathcal{R}^{\mathrm{loc}}$ the σ-algebra of subsets $A \subseteq \Omega$ such that $A \cap B \in \mathcal{R}$ for each $B \in \mathcal{R}$. Measurability of functions $f : \Omega \to \mathbb{R}$ will be considered with respect to the measurable space $(\Omega, \mathcal{R}^{\mathrm{loc}})$. The *semivariation* of ν is the set function $\|\nu\| : \mathcal{R}^{\mathrm{loc}} \to [0, \infty]$ defined by

$$\|\nu\|(A) := \sup\big\{|\langle \nu, x'\rangle|(A) : x' \in B(X')\big\}, \quad A \in \mathcal{R}^{\mathrm{loc}},$$

where $|\langle \nu, x' \rangle|$ is the variation of the scalar measure $\langle \nu, x' \rangle : \mathcal{R} \to \mathbb{R}$ given by $\langle \nu, x' \rangle(A) := \langle \nu(A), x' \rangle$ for all $A \in \mathcal{R}$. The measure ν is said to be *locally strongly additive* if, for every disjoint sequence $(A_n)_n \subseteq \mathcal{R}$ with $\|\nu\|(\bigcup_{n\geq 1} A_n) < \infty$, we have $\|\nu(A_n)\|_X \to 0$.

A set $N \in \mathcal{R}^{\mathrm{loc}}$ is called *ν-null* if $\|\nu\|(N) = 0$, and a property holds *ν-almost everywhere* (ν-a.e.) if it holds except on a ν-null set. In what follows we will always consider vector measures ν which are *σ-finite*; that is, there exist a pairwise disjoint sequence $(\Omega_k)_k$ in $\mathcal{R}$ and a ν-null set N such that $\Omega = (\bigcup_{k\geq 1} \Omega_k) \cup N$.

Let $L^0(\nu)$ denote the space of all measurable functions $f : \Omega \to \mathbb{R}$. Two functions $f, g \in L^0(\nu)$ will be identified if they are equal ν-a.e. A measurable function $f \in L^0(\nu)$ is said to be *weakly integrable* (with respect to ν) if $f \in L^1(|\langle \nu, x' \rangle|)$ for all $x' \in X'$. In this case, for each $A \in \mathcal{R}^{\mathrm{loc}}$, there exists an element $\int_A f \, d\nu \in X''$ (called the *weak integral* of f over A) such that $\langle \int_A f \, d\nu, x' \rangle = \int_A f \, d\langle \nu, x' \rangle$ for all $x' \in X'$. The space $L^1_w(\nu)$ of all (ν-a.e. equivalence classes of) weakly integrable functions becomes a Banach lattice when it is endowed with the natural order ν-a.e. and the norm

$$\|f\|_1 := \sup\Big\{ \int_\Omega |f| \, d|\langle \nu, x' \rangle| : x' \in B(X') \Big\}, \quad f \in L^1_w(\nu).$$

A weakly integrable function f is called *integrable* (with respect to ν) if the vector $\int_A f \, d\nu \in X$ for all $A \in \mathcal{R}^{\mathrm{loc}}$. The space $L^1(\nu)$ of all (ν-a.e. equivalence classes of) integrable functions becomes an order-continuous closed ideal of $L^1_w(\nu)$, and in general $L^1(\nu) \subsetneq L^1_w(\nu)$.

If $1 < p < \infty$, then a function $f \in L^0(\nu)$ is said to be *weakly p-integrable* (with respect to ν) if $|f|^p \in L^1_w(\nu)$, and it is said to be *p-integrable* (with respect to ν) if $|f|^p \in L^1(\nu)$. We denote by $L^p_w(\nu)$ the space of (ν-a.e. equivalence classes of) weakly p-integrable functions and by $L^p(\nu)$ the space of (ν-a.e. equivalence classes of) p-integrable functions. Obviously, we have that $L^p(\nu) \subseteq L^p_w(\nu)$. The natural norm for both spaces is given by

$$\|f\|_p := \sup\Big\{ \Big(\int_\Omega |f|^p \, d|\langle \nu, x' \rangle| \Big)^{\frac{1}{p}} : x' \in B(X') \Big\}, \quad f \in L^p_w(\nu).$$

The Banach lattices $L^p(\nu)$ and $L^p_w(\nu)$ were initially studied in [8] for vector measures on a σ-algebra (see [15]), and its basic properties can be extended and remain true for vector measures on δ-rings (see [3], [4]). The space $L^\infty(\nu)$ consists of all (ν-a.e. equivalence classes of) essentially bounded functions equipped with the essential supremum norm $\|\cdot\|_\infty$.

Given $f \in L^0(\nu)$, we shall consider its distribution function (with respect to the semivariation $\|\nu\|$) $\|\nu\|_f : [0, \infty) \to [0, \infty]$ defined by

$$\|\nu\|_f(s) := \|\nu\|\big(\{ w \in \Omega : |f(w)| > s \}\big), \quad s \geq 0.$$

This distribution function has similar properties as in the scalar case (see [7]). For instance, $\|\nu\|_f$ is nonincreasing and right-continuous. The decreasing rearrangement of f (with respect to the semivariation $\|\nu\|$) is the function $f_* : (0, \infty) \to [0, \infty)$ given by $f_*(t) := \inf\{ s > 0 : \|\nu\|_f(s) \leq t \}$ for all $t > 0$. In particular, f_* is nonincreasing and right-continuous.

For $0 < q \leq \infty$ and a nonnegative measurable function φ defined on $(0, \infty)$, we denote by $\Lambda^q_\varphi(\|\nu\|)$ the set of all $f \in L^0(\nu)$ such that the quantity

$$\|f\|_{\Lambda^q_\varphi(\|\nu\|)} := \begin{cases} (\int_0^\infty (\varphi(t) f_*(t))^q \frac{dt}{t})^{\frac{1}{q}}, & \text{if } 0 < q < \infty, \\ \sup_{t>0} \varphi(t) f_*(t), & \text{if } q = \infty, \end{cases}$$

is finite.

When $\varphi(t) = t^{\frac{1}{p}}$ with $1 \leq p < \infty$, we obtain the Lorentz space $L^{p,q}(\|\nu\|)$ introduced in [7] for vector measures on σ-algebras. We also note that $L^{p,q}(\|\nu\|)$ is a quasi-Banach lattice with the Fatou property. For the special case $p = q$, we denote the space $L^{p,p}(\|\nu\|)$ simply by $L^p(\|\nu\|)$. As was pointed out in [7], in general, the spaces $L^p(\|\nu\|)$ and $L^p(\nu)$ do not coincide if $1 \leq p < \infty$. If the measure ν is defined on a σ-algebra, then it holds that

$$L^{p,1}(\|\nu\|) \subseteq L^p(\|\nu\|) \subseteq L^p(\nu) \subseteq L^p_w(\nu) \subseteq L^{p,\infty}(\|\nu\|), \tag{2.1}$$

and all these inclusions are continuous (see [7, Proposition 7]). If the vector measure ν is defined on a δ-ring, then the (continuous) inclusions that remain true are

$$L^{p,1}(\|\nu\|) \subseteq L^p(\|\nu\|) \subseteq L^p_w(\nu) \subseteq L^{p,\infty}(\|\nu\|). \tag{2.2}$$

However, if ν is locally strongly additive, then we recover the chain of inclusions (2.1) (see [6, Proposition 2.2, Remark 3.3] for the details).

Throughout the paper, we will use parameter functions that belong to the class $Q(0,1)$ considered by Persson [17]. Let us review the definition of the class $Q(0,1)$ and some other related classes. Given two real numbers $a_0 < a_1$, the class $Q[a_0, a_1]$ denotes all nonnegative functions ρ on $(0, \infty)$ such that $\rho(t)t^{-a_0}$ is nondecreasing and $\rho(t)t^{-a_1}$ is nonincreasing. We write $\rho \in Q(a_0, a_1)$ if $\rho \in Q[a_0 + \varepsilon, a_1 - \varepsilon]$ for some $\varepsilon > 0$. Moreover, $\rho \in Q(a_0, -)$ (resp., $\rho \in Q(-, a_1)$) means that $\rho \in Q(a_0, b)$ (resp., $\rho \in Q(b, a_1)$) for a certain real number b. Observe that $\rho \in Q(0, 1)$ if and only if $\rho(t)t^{-\alpha}$ is nondecreasing and $\rho(t)t^{-\beta}$ is nonincreasing for some $0 < \alpha < \beta < 1$.

Let us recall briefly the construction of the real interpolation method with a parameter function. Let $\bar{A} := (A_0, A_1)$ be a quasi-Banach couple, that is, two quasi-Banach spaces A_0, A_1 which are continuously embedded in some Hausdorff topological vector space. The Peetre's K-functional is defined for $f \in A_0 + A_1$ and $t > 0$ by

$$K(t, f) = K(t, f; A_0, A_1) = \inf\{\|f_0\|_{A_0} + t\|f_1\|_{A_1} : f = f_0 + f_1, f_i \in A_i\}.$$

For $\rho \in Q(0,1)$ and $0 < q \leq \infty$, the space $(A_0, A_1)_{\rho,q}$ is formed by all those elements $f \in A_0 + A_1$ such that the quasinorm

$$\|f\|_{\rho,q} := \begin{cases} (\int_0^\infty (\frac{K(t,f;A_0,A_1)}{\rho(t)})^q \frac{dt}{t})^{\frac{1}{q}}, & \text{if } 0 < q < \infty, \\ \sup_{t>0} \frac{K(t,f;A_0,A_1)}{\rho(t)}, & \text{if } q = \infty, \end{cases}$$

is finite. In the particular case when $\rho(t) = t^\theta, 0 < \theta < 1$, the space $(A_0, A_1)_{\rho,q}$ coincides with the interpolation space $(A_0, A_1)_{\theta,q}$ obtained by the classical real method (see [2]).

The interpolation space $(A_0, A_1)_{\rho,q}$ can be also defined by using a parameter function ρ belonging to other similar function classes such as the class $\mathcal{P}^{+-}$ or B_ψ (see [10], [9], [17]). We refer to [16], [10], [9], [11], [14], and [17], among others, for complete information about the real interpolation method with a parameter function.

Given a quasinormed function space A in $L^0(\nu)$, the r-convexification of A is the space $A^{(r)}$ defined by $A^{(r)} := \{f \in L^0(\nu) : |f|^r \in A\}$ and equipped with the quasinorm $\|f\|_{A^{(r)}} := \||f|^r\|_A^{\frac{1}{r}}$. It is not difficult to check the following result using the definitions of the function spaces that we have introduced.

Proposition 2.1. *Let* $1 \le r < \infty$*, and let* $0 < q \le \infty$*. Then*

(i) $(\Lambda^q_\varphi(\|\nu\|))^{(r)} = \Lambda^{rq}_{\varphi^{\frac{1}{r}}}(\|\nu\|)$.

In particular, for $\varphi(t) = t$*, we have*

(ii) $(L^1(\|\nu\|))^{(r)} = L^r(\|\nu\|)$ *for* $q = 1$.

(iii) $(L^{1,\infty}(\|\nu\|))^{(r)} = L^{r,\infty}(\|\nu\|)$ *for* $q = \infty$.

As usual, the equivalence $a \approx b$ (resp., $a \preccurlyeq b$) means that $\frac{1}{c}a \le b \le ca$ (resp., $a \le cb$) for some positive constant c independent of the appropriate parameters. Two quasinormed spaces, A and B, are considered as equal, and we write $A = B$ whenever they coincide as sets and their quasinorms are equivalent.

3. Interpolation of couples of L^p-spaces

In this section, we provide a description of the interpolation spaces for couples of L^p-spaces associated to a σ-finite vector measure ν. We start studying when $\Lambda^q_\varphi(\|\nu\|)$ is intermediate for the couples $(L^1(\|\nu\|), L^\infty(\nu))$ and $(L^{1,\infty}(\|\nu\|), L^\infty(\nu))$.

Lemma 3.1. *Let* $0 < q \le \infty$*, let* $\rho \in Q(0,1)$*, and let* $\varphi(t) = \frac{t}{\rho(t)}$*. Then*

$$L^{1,\infty}(\|\nu\|) \cap L^\infty(\nu) \subseteq \Lambda^q_\varphi(\|\nu\|) \subseteq L^1(\|\nu\|) + L^\infty(\nu).$$

Proof. Assume that $q < \infty$ (the case $q = \infty$ is similar). Given $f \in \Lambda^q_\varphi(\|\nu\|)$, $f \ge 0$, let $M := 1 + f_*(t_0)$ for some $t_0 > 0$, $g := f\chi_{[f>M]}$, $h := f\chi_{[f\le M]}$, and $W(t) = \frac{t^{q-1}}{\rho(t)^q}$, and take $0 < \alpha < 1$ such that $\rho(t)t^{-\alpha}$ is nondecreasing. It is not difficult to check that

$$\int_r^\infty \frac{W(t)}{t^q}\,dt \le \frac{1-\alpha}{\alpha r^q}\int_0^r W(t)\,dt, \quad r > 0.$$

Since $g_*(t) \le f_*(t)$, for all $t > 0$, the weighted Hardy inequality for the nonincreasing function (see [1, Theorem 1.7], and see also [18, Theorem 3] for the case $0 < q < 1$) gives

$$\begin{aligned}\Big(\int_0^\infty \Big[\frac{1}{t}\int_0^t g_*(u)\,du\Big]^q W(t)\,dt\Big)^{\frac{1}{q}} &\le \Big(\int_0^\infty \Big[\frac{1}{t}\int_0^t f_*(u)\,du\Big]^q W(t)\,dt\Big)^{\frac{1}{q}} \\ &\preccurlyeq \Big(\int_0^\infty f_*(t)^q W(t)\,dt\Big)^{\frac{1}{q}}\end{aligned}$$

$$= \Big(\int_0^\infty \Big[\frac{t}{\rho(t)} f_*(t)\Big]^q \frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$= \|f\|_{\Lambda^q_\varphi(\|\nu\|)} < \infty.$$

In particular, the function $\frac{1}{t}\int_0^t g_*(u)\,du$ is finite almost everywhere. Moreover, $\|\nu\|([f > M]) = \|\nu\|_f(M) \le t_0$, and we can assume that $\|\nu\|(\Omega) = \infty$ (the case $\|\nu\|(\Omega) < \infty$ is evident since $L^\infty(\nu) \subseteq L^1(\|\nu\|)$); thus, $\|\nu\|([f \le M]) = \infty$ and $g = 0$ in $[f \le M]$, which implies that $g_*(t) = 0$ for all $t \ge t_0$. Hence $\int_0^\infty g_*(u)\,du < \infty$; that is, $g \in L^1(\|\nu\|)$. This proves that $f = g+h$ with $g \in L^1(\|\nu\|)$ and $h \in L^\infty(\nu)$, and so $f \in L^1(\|\nu\|) + L^\infty(\nu)$.

Let $f \in L^{1,\infty}(\|\nu\|) \cap L^\infty(\nu)$, let $K_1 := \|f\|_{L^\infty(\nu)} = f_*(0)$, let $K_2 := \|f\|_{L^{1,\infty}(\|\nu\|)}$, and let $M := \rho(1)^{-1}$, and take $0 < \alpha < \beta < 1$ such that $\rho(t)t^{-\alpha}$ is nondecreasing and $\rho(t)t^{-\beta}$ is nonincreasing. Thus $t^\beta\rho(t)^{-1} \le M$ for all $0 < t \le 1$ and $t^\alpha\rho(t)^{-1} \le M$ for all $t \ge 1$ and so

$$\|f\|^q_{\Lambda^q_\varphi(\|\nu\|)} = \int_0^1 \Big[\frac{t}{\rho(t)} f_*(t)\Big]^q \frac{dt}{t} + \int_1^\infty \Big[\frac{t}{\rho(t)} f_*(t)\Big]^q \frac{dt}{t}$$
$$\le (MK_1)^q \int_0^1 t^{q(1-\beta)-1}\,dt + (MK_2)^q \int_1^\infty t^{-q\alpha-1}\,dt < \infty. \qquad \square$$

The following result can be obtained using the estimates of [6, Proposition 3.5] and following the lines of the proof of [5, Theorem 3] (with Lemma 3.1 in mind).

Theorem 3.2. *Let $0 < q \le \infty$, let $\rho \in Q(0,1)$, and let $\varphi(t) = \frac{t}{\rho(t)}$. It holds that*

$$\big(L^1(\|\nu\|), L^\infty(\nu)\big)_{\rho,q} = \big(L^{1,\infty}(\|\nu\|), L^\infty(\nu)\big)_{\rho,q} = \Lambda^q_\varphi(\|\nu\|).$$

The reiteration theorem [17, Proposition 4.3] allows us to calculate the interpolation spaces for different couples of L^p-spaces from Theorem 3.2. We need first this technical lemma, which can be easily deduced from [17, Lemma 1.1].

Lemma 3.3. *Let $\rho \in Q(0,1)$, let $1 < p_0 < p_1 < \infty$, let $\rho_0(t) := t^{1-\frac{1}{p_0}}$, let $\rho_1(t) := t^{1-\frac{1}{p_1}}$ $\rho_2(t) := \rho_0(t)\rho(\frac{\rho_1(t)}{\rho_0(t)})$, let $\rho_3(t) := \rho_0(t)\rho(\frac{t}{\rho_0(t)})$, and let $\rho_4(t) := \rho(\rho_1(t))$. It holds that*

(i) $\rho_2(t) \in Q(1-\frac{1}{p_0}, 1-\frac{1}{p_1})$,
(ii) $\rho_3(t) \in Q(1-\frac{1}{p_0}, 1)$,
(iii) $\rho_4(t) \in Q(0, 1-\frac{1}{p_1})$.

In particular, we have that $\rho_2, \rho_3, \rho_4 \in Q(0,1)$.

Corollary 3.4. *Let $0 < q \le \infty$, let $\rho \in Q(0,1)$, let $1 \le p_0 < p_1 \le \infty$, and let $\varphi(t) = \frac{t^{\frac{1}{p_0}}}{\rho(t^{\frac{1}{p_0}-\frac{1}{p_1}})}$. It holds that*

$$\big(L^{p_0}(\|\nu\|), L^{p_1}(\|\nu\|)\big)_{\rho,q} = \big(L^{p_0,\infty}(\|\nu\|), L^{p_1,\infty}(\|\nu\|)\big)_{\rho,q} = \Lambda^q_\varphi(\|\nu\|).$$

Proof. Let $\rho_0, \rho_1, \rho_2, \rho_3$, and ρ_4 be as in Lemma 3.3. Observe that the extreme case $p_0 = 1$ and $p_1 = \infty$ is precisely Theorem 3.2. Otherwise, since $\frac{\rho_1}{\rho_0} \in Q(0,-)$, we have by [17, Corollary 4.4] that

$$\big(L^{p_0}(\|\nu\|), L^{p_1}(\|\nu\|)\big)_{\rho,q} = \big(L^1(\|\nu\|), L^\infty(\nu)\big)_{\rho_2,q}, \tag{3.1}$$

$$\big(L^{p_0}(\|\nu\|), L^{\infty}(\nu)\big)_{\rho,q} = \big(L^1(\|\nu\|), L^\infty(\nu)\big)_{\rho_3,q}, \tag{3.2}$$

$$\big(L^{1}(\|\nu\|), L^{p_1}(\|\nu\|)\big)_{\rho,q} = \big(L^1(\|\nu\|), L^\infty(\nu)\big)_{\rho_4,q}. \tag{3.3}$$

If $1 < p_0 < p_1 < \infty$, then Lemma 3.3 guarantees that $\rho_2 \in Q(0,1)$. Therefore, it follows from (3.1) and Theorem 3.2 that $(L^{p_0}(\|\nu\|), L^{p_1}(\|\nu\|))_{\rho,q} = \Lambda^q_{\varphi_2}(\|\nu\|)$, where $\varphi_2(t) = \frac{t}{\rho_2(t)} = \frac{t^{\frac{1}{p_0}}}{\rho(t^{\frac{1}{p_0}-\frac{1}{p_1}})} = \varphi(t)$.

If $1 < p_0 < \infty$ and $p_1 = \infty$, then Lemma 3.3 implies that $\rho_3 \in Q(0,1)$. Hence (3.2) and Theorem 3.2 give $(L^{p_0}(\|\nu\|), L^{p_1}(\|\nu\|))_{\rho,q} = \Lambda^q_{\varphi_3}(\|\nu\|)$, where $\varphi_3(t) = \frac{t}{\rho_3(t)} = \frac{t}{\rho_0(t)\rho(\frac{t}{\rho_0(t)})} = \frac{t^{\frac{1}{p_0}}}{\rho(t^{\frac{1}{p_0}})} = \varphi(t)$.

If $p_0 = 1$ and $1 < p_1 < \infty$, then Lemma 3.3 ensures that $\rho_4 \in Q(0,1)$. Thus, it follows from (3.3) and Theorem 3.2 that $(L^{p_0}(\|\nu\|), L^{p_1}(\|\nu\|))_{\rho,q} = \Lambda^q_{\varphi_4}(\|\nu\|)$, where $\varphi_4(t) = \frac{t}{\rho_4(t)} = \frac{t}{\rho(t^{1-\frac{1}{p_1}})} = \varphi(t)$.

The result for the couple $(L^{p_0,\infty}(\|\nu\|), L^{p_1,\infty}(\|\nu\|))$ is obtained with the same reasoning but using the other equality of Theorem 3.2. □

Corollary 3.5. *Let $0 < q \le \infty$, let $\rho \in Q(0,1)$, let $1 \le p_0 < p_1 \le \infty$, and let $\varphi(t) = \frac{t^{\frac{1}{p_0}}}{\rho(t^{\frac{1}{p_0}-\frac{1}{p_1}})}$. It holds that $(L^{p_0}_w(\nu), L^{p_1}_w(\nu))_{\rho,q} = \Lambda^q_\varphi(\|\nu\|)$.*

If in addition ν is locally strongly additive, then

$$\big(L^{p_0}(\nu), L^{p_1}(\nu)\big)_{\rho,q} = \big(L^{p_0}_w(\nu), L^{p_1}(\nu)\big)_{\rho,q} = \big(L^{p_0}(\nu), L^{p_1}_w(\nu)\big)_{\rho,q} = \Lambda^q_\varphi(\|\nu\|).$$

Proof. For general ν, it holds that $L^p(\|\nu\|) \subseteq L^p_w(\nu) \subseteq L^{p,\infty}(\|\nu\|)$ (see (2.2)), and if in addition ν is locally strongly additive, then it also holds that $L^p(\|\nu\|) \subseteq L^p(\nu) \subseteq L^{p,\infty}(\|\nu\|)$ (see (2.1) and the later comments). Therefore, the result directly follows from Corollary 3.4. □

Note that if ν is a σ-finite scalar measure, then this result recovers [17, Lemma 6.1].

4. Interpolation between sum and intersection of L^p and L^∞

Let $\rho \in Q(0,1)$, and let $0 < q \le \infty$. From now on $\rho^*(t) := t\rho(\frac{1}{t})$ and $\widetilde{\rho}(t) = \rho(t)\chi_{(0,1]}(t) + \rho^*(t)\chi_{(1,\infty)}(t)$. Note that $\rho^* \in Q(0,1)$ (see [17, Example 1.2]), and so $\widetilde{\rho} \in Q(0,1)$. The next general estimate of the norm of an element $a \in (\Sigma(\bar{A}), \Delta(\bar{A}))_{\rho,q}$ (see [17, (7.3)]) will be the key for obtaining our interpolation formulas:

$$\|a\|_{(\Sigma(\bar{A}),\Delta(\bar{A}))_{\rho,q}} \approx \Big(\int_0^1 \Big(\frac{K(t,a;\bar{A})}{\rho(t)}\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} + \Big(\int_1^\infty \Big(\frac{K(t,a;\bar{A})}{\rho^*(t)}\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \tag{4.1}$$

(for $q = \infty$, integrals are replaced by suitable suprema as usual).

Using the fact that $a^r + b^r \approx (a+b)^r$, for all $a, b \geq 0$ and $0 < r < \infty$, we can reformulate (4.1) in this way:

$$\|a\|_{(\Sigma(\bar{A}),\Delta(\bar{A}))_{\rho,q}} \approx \Big(\int_0^\infty \Big(\frac{K(t,a;\bar{A})}{\widetilde{\rho}(t)}\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}}. \tag{4.2}$$

Moreover, we will use the following estimates for the K-functional of the couples $(L^p(\|\nu\|), L^\infty(\nu))$ and $(L^{p,\infty}(\|\nu\|), L^\infty(\nu))$, which can be deduced from the ones in [6, Proposition 3.5] using Proposition 2.1.

Proposition 4.1. *Let $p \geq 1$.*

(i) *If $f \in L^p(\|\nu\|) + L^\infty(\nu)$, then $K(t, f; L^p(\|\nu\|), L^\infty(\nu)) \preccurlyeq (\int_0^{t^p} f_*(s)^p\, ds)^{\frac{1}{p}}$.*
(ii) *If $f \in L^{p,\infty}(\|\nu\|) + L^\infty(\nu)$, then $K(t, f; L^{p,\infty}(\|\nu\|), L^\infty(\nu)) \succcurlyeq t f_*(t^p)$.*

Proof. We can assume that $f \geq 0$ without lost of generality. Given a couple (A_0, A_1) of quasinormed function spaces, it is known (see [13]) that $A_0^{(p)} + A_1^{(p)} = (A_0 + A_1)^{(p)}$ and that

$$K(t, f; A_0^{(p)}, A_1^{(p)}) \approx K(t^p, f^p; A_0, A_1)^{\frac{1}{p}}. \tag{4.3}$$

Applying (4.3) to the couple $(A_0, A_1) = (L^1(\|\nu\|), L^\infty(\nu))$ and using Proposition 2.1 and [6, Proposition 3.5], we have

$$K\big(t, f; L^p(\|\nu\|), L^\infty(\nu)\big) \approx K\big(t^p, f^p; L^1(\|\nu\|), L^\infty(\nu)\big)^{\frac{1}{p}} \preccurlyeq \Big(\int_0^{t^p} f_*(s)^p\, ds\Big)^{\frac{1}{p}}.$$

Doing the same with the couple $(A_0, A_1) = (L^{1,\infty}(\|\nu\|), L^\infty(\nu))$, it follows that

$$\begin{aligned} K\big(t, f; L^{p,\infty}(\|\nu\|), L^\infty(\nu)\big) &\approx K\big(t^p, f^p; L^{1,\infty}(\|\nu\|), L^\infty(\nu)\big)^{\frac{1}{p}} \succcurlyeq \big(t^p f_*^p(t^p)\big)^{\frac{1}{p}} \\ &= t f_*(t^p). \end{aligned}$$

□

The equivalence (4.2) and the estimates in Proposition 4.1 yield the following.

Theorem 4.2. *Let $1 \leq p < \infty$, let $\rho \in Q(0,1)$, let $0 < q \leq \infty$, and let $\widetilde{\varphi}(t) = \frac{t^{\frac{1}{p}}}{\widetilde{\rho}(t^{\frac{1}{p}})}$. Then*

$$\begin{aligned} \Lambda^q_{\widetilde{\varphi}}(\|\nu\|) &= \big(L^p(\|\nu\|) + L^\infty(\nu), L^p(\|\nu\|) \cap L^\infty(\nu)\big)_{\rho,q} \\ &= \big(L^{p,\infty}(\|\nu\|) + L^\infty(\nu), L^{p,\infty}(\|\nu\|) \cap L^\infty(\nu)\big)_{\rho,q}. \end{aligned}$$

Proof. We assume $0 < q < \infty$ (the case $q = \infty$ is similar). Let us first prove that $\Lambda^q_{\widetilde{\varphi}}(\|\nu\|) \subseteq (L^p(\|\nu\|) + L^\infty(\nu), L^p(\|\nu\|) \cap L^\infty(\nu))_{\rho,q}$. First, observe that Corollary 3.4 guarantees that $\Lambda^q_{\widetilde{\varphi}}(\|\nu\|) = (L^p(\|\nu\|), L^\infty(\nu))_{\widetilde{\rho},q}$ since $\widetilde{\rho} \in Q(0,1)$. Thus, given $f \in \Lambda^q_{\widetilde{\varphi}}(\|\nu\|) \subseteq L^p(\|\nu\|) + L^\infty(\nu)$, from (4.2) and Proposition 4.1(i), we deduce that

$$\begin{aligned} \|f\|_{\rho,q} &\approx \Big(\int_0^\infty \Big(\frac{K(s, f; L^p(\|\nu\|), L^\infty(\nu))}{\widetilde{\rho}(s)}\Big)^q \frac{ds}{s}\Big)^{\frac{1}{q}} \\ &\preccurlyeq \Big(\int_0^\infty \Big(\frac{1}{\widetilde{\rho}(s)}\Big[\int_0^{s^p} \big(f_*(u)\big)^p\, du\Big]^{\frac{1}{p}}\Big)^q \frac{ds}{s}\Big)^{\frac{1}{q}} \end{aligned}$$

$$\approx \Big(\int_0^\infty \Big(\frac{1}{\widetilde{\rho}(t^{\frac{1}{p}})}\Big)^q \Big[\int_0^t \big(f_*(u)\big)^p \, du\Big]^{\frac{q}{p}} \frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$= \Big(\int_0^\infty \big(\varphi(t)\big)^q \Big[\int_0^t \big(f_*(u)\big)^p \, du\Big]^{\frac{q}{p}} \frac{dt}{t}\Big)^{\frac{1}{q}},$$
where $\varphi(t) := \frac{1}{\widetilde{\rho}(t^{\frac{1}{p}})}$.

Moreover, $\varphi \in Q(-\frac{1}{p}, 0)$ since $\rho \in Q(0,1)$ (see [17, Lemma 1.1]). Therefore, applying [17, Lemma 3.2(a)] (with $h(t) = f_*(t)$ and $\psi(t) = t^{\frac{1}{p}}$), it follows that
$$\|f\|_{\rho,q} \preccurlyeq \Big(\int_0^\infty \big(\varphi(t)\big)^q \Big[\int_0^t \big(u^{\frac{1}{p}} f_*(u)\big)^p \frac{du}{u}\Big]^{\frac{q}{p}} \frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\preccurlyeq \Big(\int_0^\infty \big(\varphi(t) t^{\frac{1}{p}} f_*(t)\big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} = \|f\|_{\Lambda^q_{\widetilde{\varphi}}(\|\nu\|)}.$$

Now, we will check that $(L^{p,\infty}(\|\nu\|) + L^\infty(\nu), L^{p,\infty}(\|\nu\|) \cap L^\infty(\nu))_{\rho,q} \subseteq \Lambda^q_{\widetilde{\varphi}}(\|\nu\|)$. Let $f \in (L^{p,\infty}(\|\nu\|) + L^\infty(\nu), L^{p,\infty}(\|\nu\|) \cap L^\infty(\nu))_{\rho,q}$. Using Proposition 4.1(ii) and (4.2), we obtain
$$\|f\|_{\Lambda^q_{\widetilde{\varphi}}(\|\nu\|)} = \Big(\int_0^\infty \Big(\frac{t^{\frac{1}{p}}}{\widetilde{\rho}(t^{\frac{1}{p}})} f_*(t)\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \approx \Big(\int_0^\infty \Big(\frac{s}{\widetilde{\rho}(s)} f_*(s^p)\Big)^q \frac{ds}{s}\Big)^{\frac{1}{q}}$$
$$\preccurlyeq \Big(\int_0^\infty \Big(\frac{K(s, f; L^{p,\infty}(\|\nu\|), L^\infty(\nu))}{\widetilde{\rho}(s)}\Big)^q \frac{ds}{s}\Big)^{\frac{1}{q}} \approx \|f\|_{\rho,q}.$$

Finally, observe that $(L^p(\|\nu\|) + L^\infty(\nu), L^p(\|\nu\|) \cap L^\infty(\nu))_{\rho,q}$ is contained in $(L^{p,\infty}(\|\nu\|) + L^\infty(\nu), L^{p,\infty}(\|\nu\|) \cap L^\infty(\nu))_{\rho,q}$ since $L^p(\|\nu\|) \subseteq L^{p,\infty}(\|\nu\|)$. ☐

Corollary 4.3. *Let $0 < q \le \infty$, let $\rho \in Q(0,1)$, let $1 \le p < \infty$, and let $\widetilde{\varphi}(t) = \frac{t^{\frac{1}{p}}}{\widetilde{\rho}(t^{\frac{1}{p}})}$. Then*
$$\big(L^p_w(\nu) + L^\infty(\nu), L^p_w(\nu) \cap L^\infty(\nu)\big)_{\rho,q} = \Lambda^q_{\widetilde{\varphi}}(\|\nu\|).$$
If in addition ν is locally strongly additive, then
$$\big(L^p(\nu) + L^\infty(\nu), L^p(\nu) \cap L^\infty(\nu)\big)_{\rho,q} = \Lambda^q_{\widetilde{\varphi}}(\|\nu\|).$$

Proof. Use the argument of the proof of Corollary 3.5 but replace Corollary 3.4 by Theorem 4.2. ☐

Observe that if ν is a σ-finite scalar measure, then this result includes [17, Example 7.1].

5. Interpolation between sum and intersection of L^p-spaces

In order to obtain a similar result to Corollary 4.3 for couples $(L^{p_0}(\nu), L^{p_1}(\nu))$ instead of couples $(L^p(\nu), L^\infty(\nu))$, we need to establish some new estimates for the K-functional of the couples $(L^{p_0}(\|\nu\|), L^{p_1}(\|\nu\|))$ and $(L^{p_0,\infty}(\|\nu\|), L^{p_1,\infty}(\|\nu\|))$ that replace the ones in Proposition 4.1. This can be done with the aid of Holmstedt's formula (see [17, Remark 4.4]), as the next result shows.

Proposition 5.1. *Let* $1 \le p_0 < p_1 < \infty$.

(i) *If* $f \in L^{p_0}(\|\nu\|) + L^{p_1}(\|\nu\|)$ *and we denote* $F(u) := \left(\frac{1}{u}\int_0^u f_*(v)^{p_0}\,dv\right)^{\frac{1}{p_0}}$, *then*

$$K\big(t, f; L^{p_0}(\|\nu\|), L^{p_1}(\|\nu\|)\big) \preccurlyeq t\Big(\int_{t^{\frac{p_0 p_1}{p_1-p_0}}}^{\infty} F(u)^{p_1}\,du\Big)^{\frac{1}{p_1}}.$$

(ii) *If* $f \in L^{p_0,\infty}(\|\nu\|) + L^{p_1,\infty}(\|\nu\|)$, *then*

$$K\big(t, f; L^{p_0,\infty}(\|\nu\|), L^{p_1,\infty}(\|\nu\|)\big) \succcurlyeq t^{\frac{p_1}{p_1-p_0}} f_*(t^{\frac{p_0 p_1}{p_1-p_0}}).$$

Proof. (i) Since [5, Corollary 1] is also valid for vector measures defined on a δ-ring (see [6, Theorem 3.6]), we have $L^{p_1}(\|\nu\|) = (L^{p_0}(\|\nu\|), L^\infty(\nu))_{\frac{p_1-p_0}{p_1},p_1}$. Therefore, applying [17, Remark 4.4], it follows that

$$K\big(t, f; L^{p_0}(\|\nu\|), L^{p_1}(\|\nu\|)\big) \approx t\Big(\int_{t^{\frac{p_1}{p_1-p_0}}}^{\infty} \Big(\frac{K(s, f; L^{p_0}(\|\nu\|), L^\infty(\nu))}{s^{\frac{p_1-p_0}{p_1}}}\Big)^{p_1} \frac{ds}{s}\Big)^{\frac{1}{p_1}},$$

and, using Proposition 4.1(i), we obtain

$$\begin{aligned}
K\big(t, f; L^{p_0}(\|\nu\|), L^{p_1}(\|\nu\|)\big) &\preccurlyeq t\Big(\int_{t^{\frac{p_1}{p_1-p_0}}}^{\infty} \Big(\frac{(\int_0^{s^{p_0}} f_*(v)^{p_0}\,dv)^{\frac{1}{p_0}}}{s^{\frac{p_1-p_0}{p_1}}}\Big)^{p_1} \frac{ds}{s}\Big)^{\frac{1}{p_1}} \\
&\approx t\Big(\int_{t^{\frac{p_0p_1}{p_1-p_0}}}^{\infty} \frac{(\int_0^{u} f_*(v)^{p_0}\,dv)^{\frac{p_1}{p_0}}}{u^{\frac{p_1}{p_0}}}\,du\Big)^{\frac{1}{p_1}} \\
&= t\Big(\int_{t^{\frac{p_0p_1}{p_1-p_0}}}^{\infty} \Big(\frac{1}{u}\int_0^{u} f_*(v)^{p_0}\,dv\Big)^{\frac{p_1}{p_0}}\,du\Big)^{\frac{1}{p_1}} \\
&= t\Big(\int_{t^{\frac{p_0p_1}{p_1-p_0}}}^{\infty} F(u)^{p_1}\,du\Big)^{\frac{1}{p_1}}.
\end{aligned}$$

(ii) We also have $L^{p_1,\infty}(\|\nu\|) = (L^{p_0,\infty}(\|\nu\|), L^\infty(\nu))_{\frac{p_1-p_0}{p_1},\infty}$ by [5, Corollary 1]. Thus, applying again [17, Remark 4.4], we deduce that

$$\begin{aligned}
K\big(t, f; L^{p_0,\infty}(\|\nu\|), L^{p_1,\infty}(\|\nu\|)\big) &\approx t \sup_{s \ge t^{\frac{p_1}{p_1-p_0}}} \frac{K(s, f; L^{p_0,\infty}(\|\nu\|), L^\infty(\nu))}{s^{\frac{p_1-p_0}{p_1}}} \\
&\succcurlyeq t \sup_{s \ge t^{\frac{p_1}{p_1-p_0}}} \frac{s f_*(s^{p_0})}{s^{\frac{p_1-p_0}{p_1}}} = t \sup_{s \ge t^{\frac{p_1}{p_1-p_0}}} \big(s^{\frac{p_0}{p_1}} f_*(s^{p_0})\big) \\
&\ge t t^{\frac{p_0}{p_1-p_0}} f_*(t^{\frac{p_0p_1}{p_1-p_0}}) = t^{\frac{p_1}{p_1-p_0}} f_*(t^{\frac{p_0p_1}{p_1-p_0}}).
\end{aligned}$$

□

Now, the equivalence (4.2) and Proposition 5.1 give the following result.

Theorem 5.2. *Let* $1 \le p_0 < p_1 \le \infty, \rho \in Q(0,1)$, *let* $0 < q \le \infty$, *and let* $\widetilde{\varphi}(t) = \frac{t^{\frac{1}{p_0}}}{\widetilde{\rho}(t^{\frac{1}{p_0}-\frac{1}{p_1}})}$. *It holds that*

$$\begin{aligned}
\Lambda^q_{\widetilde{\varphi}}(\|\nu\|) &= \big(L^{p_0}(\|\nu\|) + L^{p_1}(\|\nu\|), L^{p_0}(\|\nu\|) \cap L^{p_1}(\|\nu\|)\big)_{\rho,q} \\
&= \big(L^{p_0,\infty}(\|\nu\|) + L^{p_1,\infty}(\|\nu\|), L^{p_0,\infty}(\|\nu\|) \cap L^{p_1,\infty}(\|\nu\|)\big)_{\rho,q}.
\end{aligned}$$

Proof. The case $p_1 = \infty$ is precisely Theorem 4.2, and so we can assume that $p_1 < \infty$. Suppose that $0 < q < \infty$ (the case $q = \infty$ is similar). Let us first prove that

$$\Lambda^q_{\widetilde{\varphi}}(\|\nu\|) \subseteq \big(L^{p_0}(\|\nu\|) + L^{p_1}(\|\nu\|), L^{p_0}(\|\nu\|) \cap L^{p_1}(\|\nu\|)\big)_{\rho,q}.$$

First, note that Corollary 3.4 ensures that $\Lambda^q_{\widetilde{\varphi}}(\|\nu\|) = (L^{p_0}(\|\nu\|), L^{p_1}(\|\nu\|))_{\widetilde{\rho},q}$ since $\widetilde{\rho} \in Q(0,1)$. Thus, given $f \in \Lambda^q_{\widetilde{\varphi}}(\|\nu\|) \subseteq L^{p_0}(\|\nu\|) + L^{p_1}(\|\nu\|)$, from (4.2) and Proposition 5.1 we deduce that

$$\begin{aligned}
\|f\|_{\rho,q} &\approx \Big(\int_0^\infty \Big(\frac{K(s,f;L^{p_0}(\|\nu\|),L^{p_1}(\|\nu\|))}{\widetilde{\rho}(s)}\Big)^q \frac{ds}{s}\Big)^{\frac{1}{q}} \\
&\preccurlyeq \Big(\int_0^\infty \Big(\frac{s}{\widetilde{\rho}(s)}\Big[\int_{s^{\frac{p_0p_1}{p_1-p_0}}}^\infty F(u)^{p_1}\,du\Big]^{\frac{1}{p_1}}\Big)^q \frac{ds}{s}\Big)^{\frac{1}{q}} \\
&\preccurlyeq \Big(\int_0^\infty \Big(\frac{t^{\frac{p_1-p_0}{p_0p_1}}}{\widetilde{\rho}(t^{\frac{p_1-p_0}{p_0p_1}})}\Big)^q \Big[\int_t^\infty F(u)^{p_1}\,du\Big]^{\frac{q}{p_1}} \frac{dt}{t}\Big)^{\frac{1}{q}} \\
&= \Big(\int_0^\infty (\varphi(t))^q \Big[\int_t^\infty F(u)^{p_1}\,du\Big]^{\frac{q}{p_1}} \frac{dt}{t}\Big)^{\frac{1}{q}},
\end{aligned}$$

where $\varphi(t) := \dfrac{t^{\frac{p_1-p_0}{p_0p_1}}}{\widetilde{\rho}(t^{\frac{p_1-p_0}{p_0p_1}})}$.

Note that $\varphi \in Q(0, \frac{p_1-p_0}{p_0p_1})$ since $\rho \in Q(0,1)$ (see [17, Lemma 1.1]). Therefore, applying [17, Lemma 3.2(b)] (with $\psi(t) = t^{\frac{1}{p_1}}$ and $h(t) = F(t)$, which is nonincreasing), it follows that

$$\begin{aligned}
\|f\|_{\rho,q} &\simeq \Big(\int_0^\infty (\varphi(t))^q \Big[\int_t^\infty (u^{\frac{1}{p_1}}F(u))^{p_1}\frac{du}{u}\Big]^{\frac{q}{p_1}} \frac{dt}{t}\Big)^{\frac{1}{q}} \\
&\preccurlyeq \Big(\int_0^\infty (\varphi(t)t^{\frac{1}{p_1}}F(t))^q \frac{dt}{t}\Big)^{\frac{1}{q}} = \Big(\int_0^\infty (\widetilde{\varphi}(t)F(t))^q\frac{dt}{t}\Big)^{\frac{1}{q}} \\
&= \Big(\int_0^\infty \Big(\frac{\widetilde{\varphi}(t)}{t^{\frac{1}{p_0}}}\Big)^q \Big(\int_0^t f_*(v)^{p_0}\,dv\Big)^{\frac{q}{p_0}} \frac{dt}{t}\Big)^{\frac{1}{q}}.
\end{aligned}$$

Observe that $\frac{\widetilde{\varphi}(t)}{t^{\frac{1}{p_0}}} \in Q(-,0)$, and so applying [17, Lemma 3.2(a)] (now with $\psi(t) = t^{\frac{1}{p_0}}$ and $h(t) = f_*(t)$), it follows that

$$\begin{aligned}
\|f\|_{\rho,q} &\preccurlyeq \Big(\int_0^\infty \Big(\frac{\widetilde{\varphi}(t)}{t^{\frac{1}{p_0}}}\Big)^q \Big(\int_0^t (v^{\frac{1}{p_0}}f_*(v))^{p_0}\frac{dv}{v}\Big)^{\frac{q}{p_0}} \frac{dt}{t}\Big)^{\frac{1}{q}} \\
&\simeq \Big(\int_0^\infty (\widetilde{\varphi}(t)f_*(t))^q\frac{dt}{t}\Big)^{\frac{1}{q}} = \|f\|_{\Lambda^q_{\widetilde{\varphi}}(\|\nu\|)}.
\end{aligned}$$

Now, we will check that

$$\big(L^{p_0,\infty}(\|\nu\|) + L^{p_1,\infty}(\|\nu\|), L^{p_0,\infty}(\|\nu\|) \cap L^{p_1,\infty}(\|\nu\|)\big)_{\rho,q} \subseteq \Lambda^q_{\widetilde{\varphi}}(\|\nu\|).$$

Let $f \in (L^{p_0,\infty}(\|\nu\|) + L^{p_1,\infty}(\|\nu\|), L^{p_0,\infty}(\|\nu\|) \cap L^{p_1\infty}(\|\nu\|))_{\rho,q}$. By Proposition 5.1(ii) and (4.2) we obtain

$$\begin{aligned}\|f\|_{\Lambda^q_{\widetilde{\varphi}}(\|\nu\|)} &= \Big(\int_0^\infty \Big(\frac{t^{\frac{1}{p_0}}}{\widetilde{\rho}(t^{\frac{1}{p_0}-\frac{1}{p_1}})} f_*(t)\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \\ &\approx \Big(\int_0^\infty \Big(\frac{s^{\frac{p_1}{p_1-p_0}}}{\widetilde{\rho}(s)} f_*(s^{\frac{p_0 p_1}{p_1-p_0}})\Big)^q \frac{ds}{s}\Big)^{\frac{1}{q}} \\ &\preccurlyeq \Big(\int_0^\infty \Big(\frac{K(s,f;L^{p_0,\infty}(\|\nu\|),L^{p_1\infty}(\|\nu\|))}{\widetilde{\rho}(s)}\Big)^q \frac{ds}{s}\Big)^{\frac{1}{q}} \approx \|f\|_{\rho,q}. \qquad \square\end{aligned}$$

Corollary 5.3. *Let $0 < q \le \infty$, let $\rho \in Q(0,1)$, let $1 \le p_0 < p_1 \le \infty$, and let $\widetilde{\varphi}(t) = \frac{t^{\frac{1}{p_0}}}{\widetilde{\rho}(t^{\frac{1}{p_0}-\frac{1}{p_1}})}$. It holds that $(L^{p_0}_w(\nu) + L^{p_1}_w(\nu), L^{p_0}_w(\nu) \cap L^{p_1}_w(\nu))_{\rho,q} = \Lambda^q_{\widetilde{\varphi}}(\|\nu\|)$.*

If in addition ν is locally strongly additive, then

$$\begin{aligned}\big(L^{p_0}(\nu) + L^{p_1}(\nu), L^{p_0}(\nu) \cap L^{p_1}(\nu)\big)_{\rho,q} &= \big(L^{p_0}_w(\nu) + L^{p_1}(\nu), L^{p_0}_w(\nu) \cap L^{p_1}(\nu)\big)_{\rho,q} \\ &= \big(L^{p_0}(\nu) + L^{p_1}_w(\nu), L^{p_0}(\nu) \cap L^{p_1}_w(\nu)\big)_{\rho,q} \\ &= \Lambda^q_{\widetilde{\varphi}}(\|\nu\|).\end{aligned}$$

Proof. Use the argument of the proof of Corollary 3.5, but replace Corollary 3.4 by Theorem 5.2. □

Note that if ν is a vector measure on a σ-algebra, then this result recovers [5, Corollary 4].

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[1]Departamento de Matemática Aplicada I, Universidad de Sevilla, ETSIA, Carretera de Utrera, km 1, 41013, Sevilla, Spain.

E-mail address: rcampo@us.es

[2]Departamento de Matemática Aplicada II, Universidad de Sevilla, ETSI, Camino de los Descubrimientos, s/n, 41092, Sevilla, Spain.

E-mail address: afcarrion@us.es; mayoral@us.es; naranjo@us.es

[3]Departamento Matemáticas y Computación, Universidad de Burgos, Escuela Politécnica Superior, Calle Villadiego, s/n, 09001, Burgos, Spain.

E-mail address: amanzano@ubu.es