Reflexivity of function spaces associated to a σ-finite vector measure ☆

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Keywords:

Reflexivity
Integrable function
Vector measure
δ-ring
Locally strongly additive measure

A B S T R A C T

For a vector measure ν defined on a δ-ring with values in a Banach space and $1 < p < \infty$, we characterize the *reflexivity* of the different spaces $L^p_w(\nu)$ (integrability in the weak sense), $L^p(\nu)$ (integrability in the strong sense), and $L^p(\|\nu\|)$ (integrability in the Choquet sense).

1. Introduction

From the point of view of functional analysis the second most desired property of infinite spaces is *reflexivity* (the first one is *completeness*) and probably it is the most used in applications due to the weak compactness of its unit ball. Typical undergraduate examples of reflexive Banach spaces are Lebesgue L^p-spaces ($1 < p < \infty$) of a positive σ-finite measure. The corresponding scalar function spaces associated to a *vector measure* ν with values into a Banach space have been long studied (see, for example [18] and most of the references in the present paper). In this new context the things are really different. There appear several L^p-spaces associated to the vector measure: in the weak sense $L^p_w(\nu)$, in the strong sense $L^p(\nu)$, and finally, integrability in the Choquet sense $L^p(\|\nu\|)$, of course for $1 \leq p < \infty$. These kind of spaces are, in general, different from each other and nonreflexive, even for $1 < p < \infty$. When the vector measure ν is

☆ This research has been partially supported by La Junta de Andalucía. The authors acknowledge the support of the Ministerio de Economía y Competitividad of Spain and FEDER, under the project MTM2012-36740-C02-01.

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defined on a σ-algebra the reflexivity of $L^p_w(\nu)$ and $L^p(\nu)$ has been studied in [12]. Roughly speaking, for $1 < p < \infty$, the space $L^p_w(\nu)$, or equivalently $L^p(\nu)$, is reflexive if and only if they coincide. Also in the same context of a vector measure defined on a σ-algebra, the reflexivity of $L^p(\|\nu\|)$ is obtained as a byproduct of a general result about interpolation from [10], namely, $L^p(\|\nu\|)$ is always reflexive for all $1 < p < \infty$. In the present paper we study the reflexivity of these spaces when the measure is defined on a δ-ring, a more general (but natural) structure than a σ-algebra. In this new context we can say that a similar result characterizing reflexivity of $L^p_w(\nu)$ and $L^p(\nu)$ holds (see Theorem 2.3). Nevertheless $L^p(\|\nu\|)$ is not always reflexive. We characterize those vector measures for which $L^p(\|\nu\|)$ is reflexive as the *locally strongly additive* vector measures (see Theorem 4.3). Much of this work deals with this kind of measures.

2. Reflexivity of L^p and L^p_w

The basic references for us about integration will be [7,13,16,17] and [18, Chapter 3]. Throughout this paper we will consider a vector measure $\nu : \mathcal{R} \to X$ defined on a δ-ring $\mathcal{R}$ of subsets of some nonempty set Ω with values in a real Banach space X, with dual X'. We denote by $\mathcal{R}^{\text{loc}}$ the σ-algebra of subsets $A \subseteq \Omega$ such that $A \cap B \in \mathcal{R}$ for each $B \in \mathcal{R}$. Measurability of functions $f : \Omega \longrightarrow \mathbb{R}$ will be considered with respect to the measurable space $(\Omega, \mathcal{R}^{\text{loc}})$. The *semivariation* of ν is the set function $\|\nu\| : \mathcal{R}^{\text{loc}} \to [0,\infty]$ defined by $\|\nu\|(A) := \sup\{|\langle \nu, x'\rangle|(A) : \|x'\|_{X'} \le 1\}$, where $|\langle \nu, x'\rangle|$ is the variation of the scalar measure

$$\langle \nu, x'\rangle : A \in \mathcal{R} \longrightarrow \langle \nu, x'\rangle(A) := \langle \nu(A), x'\rangle \in \mathbb{R}.$$

Recall that for every subset $A \in \mathcal{R}^{\text{loc}}$, we have the following inequalities

$$\frac{1}{2}\|\nu\|(A) \le \sup\{\|\nu(B)\| : B \in \mathcal{R}, B \subseteq A\} \le \|\nu\|(A).$$

The semivariation is a subadditive set function that may be nonadditive. A set $N \in \mathcal{R}^{\text{loc}}$ is called ν-*null* if $\|\nu\|(N) = 0$, and a property holds ν-*almost everywhere* (ν-a.e.) if it holds except on a ν-null set. In what follows we will always consider vector measures $\nu : \mathcal{R} \to X$ which are σ-finite, that is, there exist a pairwise disjoint sequence $(\Omega_k)_k$ in $\mathcal{R}$, and a ν-null set $N \in \mathcal{R}^{\text{loc}}$, such that $\Omega = (\cup_{k\ge1}\Omega_k) \cup N$. Simple examples of σ-finite vector measures defined on δ-rings are given by the *Lebesgue measure* λ defined on the δ-ring $\mathcal{R} := \{A \in \mathcal{M} : \lambda(A) < \infty\}$, where $\mathcal{M}$ is the σ-algebra of all Lebesgue measurable subsets of the real line $\mathbb{R}$, and the *counting measure* defined on the δ-ring $\mathcal{P}_f(\mathbb{N})$ of finite subsets of the natural numbers $\mathbb{N}$. Other examples of σ-finite vector measure will be considered in Examples 3.2 and 4.5 below. Moreover, σ-finite vector measures have special scalar control measures as we see in the following result (see [7, Theorem 3.3]).

Lemma 2.1. *Let ν be a σ-finite vector measure. Then there exists $x'_0 \in X'$, with $\|x'_0\|_{X'} \le 1$, such that $|\langle \nu, x'_0\rangle|(A) = 0$ if and only if $\|\nu\|(A) = 0$, with $A \in \mathcal{R}^{\text{loc}}$.*

Proof. If ν is σ-finite, then there exists $0 < f \in L^1(\nu)$. Consider the vector measure $\nu_f : \mathcal{R}^{\text{loc}} \to X$ defined by $\nu_f(A) := \int_A f d\nu \in X$. Note that ν_f is defined on a σ-algebra, and $\|\nu_f\|(A) = \|f\chi_A\|_{L^1(\nu)}$, for all $A \in \mathcal{R}^{\text{loc}}$ (see [13, Theorem 3.2]). Let $x'_0 \in X'$, with $\|x'_0\|_{X'} \le 1$, such that $|\langle \nu_f, x'_0\rangle|$ is a Rybakov control measure for ν_f (see [9, Theorem IX.1.2]). Then $|\langle \nu, x'_0\rangle|(A) = 0$ if and only if $\|\nu\|(A) = 0$, with $A \in \mathcal{R}^{\text{loc}}$, because we know that $|\langle \nu_f, x'_0\rangle|(A) = \int_A f d|\langle \nu, x'_0\rangle|$, for all $A \in \mathcal{R}^{\text{loc}}$. □

A measurable function $f : \Omega \longrightarrow \mathbb{R}$ is called *weakly integrable* (with respect to ν) if $f \in L^1(|\langle \nu, x'\rangle|)$ for all $x' \in X'$. A weakly integrable function f is said to be *integrable* (with respect to ν) if, for each $A \in \mathcal{R}^{\text{loc}}$ there exists an element (necessarily unique) $\int_A f d\nu \in X$, satisfying

$$\left\langle \int\limits_A f d\nu, x' \right\rangle = \int\limits_A f d\langle \nu, x'\rangle, \quad x' \in X'.$$

If $1 \leq p < \infty$, a measurable function $f : \Omega \longrightarrow \mathbb{R}$ is called *weakly p-integrable* (with respect to ν) if $|f|^p$ is weakly integrable and *p-integrable* (with respect to ν) if $|f|^p$ is integrable. The space $L^p_w(\nu)$ of all (ν-a.e. equivalence classes of) weakly p-integrable functions becomes a Banach lattice when endowed with the usual ν-a.e. pointwise order and the norm

$$\|f\|_{L^p_w(\nu)} := \sup \left\{ \left(\int\limits_\Omega |f|^p \, d|\langle \nu, x'\rangle| \right)^{\frac{1}{p}} : \|x'\|_{X'} \leq 1 \right\}.$$

Moreover, the space $L^p(\nu)$ of all (ν-a.e. equivalence classes of) p-integrable functions is a closed *order continuous* ideal of $L^p_w(\nu)$. In fact, it is the closure of $\mathcal{S}(\mathcal{R})$, the space of simple functions supported on $\mathcal{R}$ (see [13, Theorem 3.5]). Recall that order continuous means that $\|f - f_n\|_{L^p(\nu)} \to 0$ for every $0 \leq f_n \uparrow f \in L^p(\nu)$. For $p \geq 1$, note that

$$L^p_w(\nu) = \{f : \Omega \longrightarrow \mathbb{R} : |f|^p \in L^1_w(\nu)\}, \quad \|f\|_{L^p_w(\nu)} = \|\,|f|^p\|^{\frac{1}{p}}_{L^1_w(\nu)}.$$

These Banach lattices $L^p(\nu)$ and $L^p_w(\nu)$ were initially studied in [12] and [19] for vector measures ν defined on a σ-algebra and its basic properties can be extended and remain true for vector measures defined on δ-rings (see [4]). Let us mention, in particular, that $L^p_w(\nu)$ is p-convex, that is, there is a constant $K > 0$ such that

$$\left\| (|f_1|^p + \cdots + |f_n|^p)^{\frac{1}{p}} \right\|_{L^p_w(\nu)} \leq K \left(\|f_1\|^p_{L^p_w(\nu)} + \cdots + \|f_n\|^p_{L^p_w(\nu)} \right)^{\frac{1}{p}},$$

for every election of vectors $f_1, \ldots, f_n$ in $L^p_w(\nu)$, as we can see directly from the definition of the norm $\|\cdot\|_{L^p_w(\nu)}$.

The following result has been borrowed from [4, p. 75] (see also [2, Corollary 5.7]). We include here the proof for the sake of completeness.

Proposition 2.2. *Let $1 \leq p < \infty$ and let $0 \leq f_n \uparrow$ in $L^p_w(\nu)$ such that $\sup_n \|f_n\|_{L^p_w(\nu)} < \infty$. Then, there exists $\sup_n f_n \in L^p_w(\nu)$. Moreover $\sup_n \|f_n\|_{L^p_w(\nu)} = \|\sup_n f_n\|_{L^p_w(\nu)}$. That is, $L^p_w(\nu)$ has the sequential Fatou property.*

Proof. There exists a ν-null set $N \in \mathcal{R}^{\text{loc}}$ such that $0 \leq f_n(w) \uparrow$ for all $w \in \Omega \setminus N$. Consider the function $g : \Omega \longrightarrow [0, \infty]$ defined by $g(w) := \sup_n f_n(w)$, if $w \in \Omega \setminus N$ and $g(w) = 0$, if $w \in N$. Then we have $0 \leq f_n^p \chi_{\Omega \setminus N} \uparrow g^p$ pointwise, and the Lebesgue monotone convergence theorem assures that

$$\int\limits_\Omega g^p \, d|\langle \nu, x'\rangle| = \lim_n \int\limits_\Omega f_n^p \chi_{\Omega \setminus N} \, d|\langle \nu, x'\rangle| \leq \|x'\| \sup_n \|f_n\|^p_{L^p_w(\nu)} < \infty,$$

for all $x' \in X'$. In this way $g \in L^p(|\langle \nu, x'\rangle|)$ for all $x' \in X'$, and

$$\sup\left\{\int_\Omega g^p\, d|\langle \nu, x'\rangle| : \|x'\| \le 1\right\} \le \sup_n \|f_n\|^p_{L^p_w(\nu)} < \infty.$$

In particular, by applying the above for the vector x_0' of Lemma 2.1, we deduce that g is finite ν-a.e. and, in fact, it equals with $\sup_n f_n$. Thus $g = \sup_n f_n \in L^p_w(\nu)$, and moreover

$$\left\|\sup_n f_n\right\|_{L^p_w(\nu)} = \|g\|_{L^p_w(\nu)} \le \sup_n \|f_n\|_{L^p_w(\nu)} \le \left\|\sup_n f_n\right\|_{L^p_w(\nu)}. \qquad \square$$

Recall that a Banach lattice is a KB-space whenever every norm bounded, positive, increasing sequence is norm convergent [1, Definition 14.10]. Thus every reflexive space is a KB-space (see the comments to the aforementioned definition), and it is clear that every KB-space has order continuous norm. Moreover every KB-space has the sequential Fatou property because every convergent (in norm) increasing sequence, necessarily converges to its supremum. The next result is the analogue to [12, Corollary 3.10] for vector measures defined on δ-rings. Its proof is a small modification of that, but we include it here for the sake of completeness. The equivalence of *d*) and *h*) has been proved independently by Avalos-Ramos and Galaz-Fontes in [2, Corollary 5.20].

Theorem 2.3. *For every $p > 1$, the following conditions are equivalent:*

a) $L^p_w(\nu)$ *has order continuous norm.*
b) $L^p_w(\nu)$ *is a KB-space.*
c) $L^p_w(\nu)$ *is reflexive.*
d) $L^p(\nu)$ *is reflexive.*
e) $L^p(\nu)$ *is a KB-space.*
f) $L^p(\nu)$ *has the sequential Fatou property.*
g) $L^p_w(\nu) = L^p(\nu)$ *as Banach lattices.*
h) $L^1_w(\nu) = L^1(\nu)$ *as Banach lattices.*

All eight assertions are true whenever the Banach space X is weakly sequentially complete.

Proof. *a*) $\Longrightarrow$ *b*) Let $(f_n)_n$ be a norm bounded, positive, increasing sequence in $L^p_w(\nu)$. By applying Proposition 2.2, there exists f in $L^p_w(\nu)$ such that $f_n \uparrow f$. Then, from order continuity of the norm, we have that $(f_n)_n$ converges to f in $L^p_w(m)$.
b) $\Longrightarrow$ *c*) Since $L^p_w(\nu)$ is a p-convex (with $p > 1$) Banach lattice, the space of summable sequences ℓ_1 is not lattice embeddable in $L^p_w(\nu)$ (see [14, p. 51]). Moreover, $L^p_w(\nu)$ does not contain a lattice copy of the space of null sequences c_0 since it is a KB-space by hypothesis (see [1, Theorem 14.12]). The result then follows from Lozanovskii's result (see [1, Theorem 14.23]).
c) $\Longrightarrow$ *d*) $L^p(\nu)$ is a closed subspace of $L^p_w(\nu)$.
d) $\Longrightarrow$ *e*) It is well known that reflexive spaces are KB-spaces.
e) $\Longrightarrow$ *f*) Every KB-space has the sequential Fatou property.
f) $\Longrightarrow$ *g*) See [4, Proposition 5.4].
g) $\Longleftrightarrow$ *h*) It is enough to observe that $f \in L^1_w(\nu)$ if and only if $|f|^{\frac{1}{p}} \in L^p_w(\nu)$.
g) $\Longrightarrow$ *a*) Note that $L^p(\nu)$ has always order continuous norm. See [19, Proposition 6] or [13, Theorem 3.3].
For the last claim in the statement of the theorem, recall that $L^1_w(\nu) = L^1(\nu)$ whenever the Banach space X is weakly sequentially complete. See [13, Theorem 5.1]. $\square$

3. Fatou property and order continuity of L^p of the semivariation

Now we are going to consider, for $1 \leq p < \infty$, the spaces denoted by $L^p(\|\nu\|)$. These spaces appear in a natural way, as *Lorentz spaces* with respect to the semivariation $\|\nu\|$, when we describe the *interpolation spaces* obtained by applying the *real interpolation method* to couples of L^p-spaces of a vector measure $\nu : \mathcal{R} \to X$ (see [6] and [10]). Let us introduce it briefly and describe some basic properties of them.

Given a measurable function $f : \Omega \longrightarrow \mathbb{R}$, we shall consider its *distribution function* (with respect to the semivariation of the vector measure ν) $\|\nu\|_f : t \in [0, \infty) \longrightarrow \|\nu\|_f(t) \in [0, \infty]$, defined by

$$\|\nu\|_f(t) := \|\nu\| \left(\{w \in \Omega : |f(w)| > t\}\right), \quad t \geq 0.$$

This distribution function has similar properties as in the scalar case (see [10]). For instance, $\|\nu\|_f$ is non-increasing and right-continuous. Recall that $L^1(\|\nu\|)$ is the space of (ν-a.e. equivalence classes of) measurable functions $f : \Omega \longrightarrow \mathbb{R}$ such that the integral $\int_0^\infty \|\nu\|_f(t)dt < \infty$. Then $L^1(\|\nu\|)$, with the *quasi-norm* $\|f\|_{L^1(\|\nu\|)} := \int_0^\infty \|\nu\|_f(t)dt$ and the usual ν-a.e. pointwise order, becomes a quasi-Banach lattice. For $1 < p < \infty$, we also consider the space

$$L^p(\|\nu\|) := \left\{f : \Omega \longrightarrow \mathbb{R} : |f|^p \in L^1(\|\nu\|)\right\},$$

with the quasi-norm $\|f\|_{L^p(\|\nu\|)} := \| \, |f|^p\|_{L^1(\|\nu\|)}^{\frac{1}{p}}$. We would need to mention that a consequence of [6, Remark 3.8.1] is that $L^p(\|\nu\|)$ is *normable* for every $1 < p < \infty$. This means that there is a lattice norm $\| \cdot \|_p$ equivalent to the quasi-norm $\| \cdot \|_{L^p(\|\nu\|)}$. The case $p = 1$ is something special because we don't know if $L^1(\|\nu\|)$ is normable (see [11] for details).

The following result is the analogue to Proposition 2.2.

Proposition 3.1. *Let $1 \leq p < \infty$ and let $0 \leq f_n \uparrow$ in $L^p(\|\nu\|)$ such that $\sup_n \|f_n\|_{L^p(\|\nu\|)} < \infty$. Then, there exists $\sup_n f_n \in L^p(\|\nu\|)$. Moreover $\sup_n \|f_n\|_{L^p(\|\nu\|)} = \|\sup_n f_n\|_{L^p(\|\nu\|)}$. That is, $L^p(\|\nu\|)$ has the sequential Fatou property.*

Proof. There exists a subset $N \in \mathcal{R}^{\text{loc}}$, with $\|\nu\|(N) = 0$, such that $0 \leq f_n(w) \uparrow$ for all $w \in \Omega \setminus N$. Consider the function $g : \Omega \longrightarrow [0, \infty]$ defined by $g(w) := \sup_n f_n(w)$, if $w \in \Omega \setminus N$ and $g(w) = 0$, if $w \in N$. Then we have $0 \leq f_n^p \chi_{\Omega \setminus N} \uparrow g^p$ pointwise, and $\|\nu\|_{f_n^p \chi_{\Omega \setminus N}}(t) \uparrow \|\nu\|_{g^p}(t)$ for all $t \geq 0$. By applying the Lebesgue monotone convergence theorem we obtain

$$\begin{aligned}\int_0^\infty \|\nu\|_{g^p}(t)dt &= \lim_n \int_0^\infty \|\nu\|_{f_n^p \chi_{\Omega \setminus N}}(t)dt = \lim_n \int_0^\infty \|\nu\|_{f_n^p}(t)dt \\ &= \sup_n \|f_n\|^p_{L^p(\|\nu\|)} < \infty.\end{aligned}$$

Then $\|\nu\|_{g^p}(t) < \infty$ for all $t > 0$ and g is finite ν-a.e. We conclude that $\sup_n f_n \in L^p(\|\nu\|)$ and moreover $\sup_n \|f_n\|_{L^p(\|\nu\|)} = \|\sup_n f_n\|_{L^p(\|\nu\|)}$. $\square$

As it has been pointed out in [10], in general, the spaces $L^p(\|\nu\|)$, $L^p(\nu)$ and $L^p_w(\nu)$ do not coincide, and the three spaces can be different. If the measure ν is defined on a σ-algebra, we have the following inclusions $L^\infty(\nu) \subseteq L^p(\|\nu\|) \subseteq L^p(\nu) \subseteq L^p_w(\nu)$, and all these inclusions are continuous for all $1 \leq p < \infty$ (see [10, Proposition 7]). Here $L^\infty(\nu)$ denotes the space of (classes ν-a.e. of) essentially bounded measurable functions $f : \Omega \longrightarrow \mathbb{R}$ with the essential supremum norm. However, if the vector measure ν is defined on a δ-ring instead of a σ-algebra, the inclusion $L^p(\|\nu\|) \subseteq L^p(\nu)$ is in general false as the following example points out.

Example 3.2. (See [6, Example 2.1].) Consider the σ-finite vector measure

$$\nu : A \in \mathcal{P}_f(\mathbb{N}) \to \nu(A) := \chi_A \in c_0.$$

For every $1 \le p < \infty$, it is easy to check that $L^p_w(\nu) = \ell^\infty$, the space of bounded sequences, and $L^p(\nu) = c_0$. In what follows it will be interesting to note that $\|\nu\|(A) = 1$, for every nonempty $A \subseteq \mathbb{N}$, and $\|\nu\|(\varnothing) = 0$. This means, in particular, that $\|\nu\|_f = \chi_{[0,\infty)}$ if f is an unbounded sequence, but $\|\nu\|_f = \chi_{[0,\|f\|_\infty)}$ if $f \in \ell^\infty$. Consequently, $L^1(\|\nu\|) = \ell^\infty = L^1_w(\nu)$, and $L^1(\|\nu\|) \nsubseteq L^1(\nu)$.

Nevertheless, the inclusion $L^1(\|\nu\|) \subseteq L^1_w(\nu)$ remains and it is continuous for every vector measure ν defined on a δ-ring. And, moreover, the inclusion $L^1(\|\nu\|) \subseteq L^1(\nu)$ holds if and only if the measure ν is *locally strongly additive* (see [6, Proposition 3.2]). In particular, if $L^1(\nu) = L^1_w(\nu)$, then the measure ν is locally strongly additive. Recall that a vector measure ν is locally strongly additive if for every disjoint sequence $(A_n)_n \subseteq \mathcal{R}$, with $\|\nu\|\left(\cup_{n\ge1} A_n\right) < \infty$, we have $\|\nu(A_n)\|_X \to 0$. See [5] and [6], where these measures were introduced in connection with real and complex interpolation methods and function spaces associated to a vector measure.

Note that Example 3.2 tells us that $\mathcal{S}(\mathcal{R})$, the set of simple functions supported on subsets of the δ-ring $\mathcal{R}$, is not always a dense subset of $L^1(\|\nu\|)$. The things are different if the measure is locally strongly additive. The following technical results will be used to prove that $\mathcal{S}(\mathcal{R})$ is dense in $L^1(\|\nu\|)$ when the vector measure is locally strongly additive. In what follows it will be convenient to consider the following notation. For a measurable function $f : \Omega \longrightarrow \mathbb{R}$ and a real number M, consider the measurable subset

$$[f > M] := \{w \in \Omega : f(w) > M\}.$$

Similar meaning have $[f \le M]$ or $[f \ne 0]$.

Lemma 3.3. *Let $\nu : \mathcal{R} \to X$ be a vector measure and let $0 \le f \in L^1(\|\nu\|)$. Then $\|\nu\|\left([f > M]\right) < \infty$ for each $M > 0$, and $\lim\limits_{M\to0} \left\|f\chi_{[f\le M]}\right\|_{L^1(\|\nu\|)} = 0$.*

Proof. Note that $f \ge M\,\chi_{[f>M]}$, for each $M > 0$, and so

$$\|f\|_{L^1(\|\nu\|)} \ge M\,\|\chi_{[f>M]}\|_{L^1(\|\nu\|)} = M\|\nu\|\left([f > M]\right).$$

Thus, $\|\nu\|\left([f > M]\right) \le \frac{1}{M}\|f\|_{L^1(\|\nu\|)} < \infty$. For the second assertion note that $[f\chi_{[f\le M]} > t] = \varnothing$, if $t \ge M > 0$, and so $\|\nu\|\left([f\chi_{[f\le M]} > t]\right) = 0$ for those t. On the other hand, if $0 \le t < M$, then $[f\chi_{[f\le M]} > t] = [t < f \le M]$ and, in this case, $\|\nu\|\left([f\chi_{[f\le M]} > t]\right) = \|\nu\|\left([t < f \le M]\right)$. Thus

$$\begin{aligned}
\lim_{M\to0} \left\|f\chi_{[f\le M]}\right\|_{L^1(\|\nu\|)} &= \lim_{M\to0} \int_0^\infty \|\nu\|\left([f\chi_{[f\le M]} > t]\right) dt \\
&= \lim_{M\to0} \int_0^M \|\nu\|\left([t < f \le M]\right) dt \\
&\le \lim_{M\to0} \int_0^M \|\nu\|\left([f > t]\right) dt = 0,
\end{aligned}$$

since $f \in L^1(\|\nu\|)$. $\square$

Lemma 3.4. *Let $\nu : \mathcal{R} \to X$ be a vector measure. The following conditions are equivalent:*

1) ν is locally strongly additive.
2) $\|\nu\|(E_n) \to 0$ for each sequence $(E_n)_n \subseteq \mathcal{R}^{\mathrm{loc}}$, such that $E_n \downarrow \varnothing$ and $\|\nu\|(E_1) < \infty$.

In particular, if ν is locally strongly additive, then for every $A \in \mathcal{R}^{\mathrm{loc}}$, with $\|\nu\|(A) < \infty$, and every $\varepsilon > 0$ there exists $B_\varepsilon \in \mathcal{R}$, with $B_\varepsilon \subseteq A$, such that $\|\nu\|(A \setminus B_\varepsilon) = \|\chi_A - \chi_{B_\varepsilon}\|_{L^1(\|\nu\|)} < \varepsilon$.

Proof. 1) $\Longrightarrow$ 2) Suppose that $(E_n)_n \subseteq \mathcal{R}^{\mathrm{loc}}$, with $E_n \downarrow \varnothing$ and $\|\nu\|(E_1) < \infty$. Then $\chi_{E_n} \in L^1_w(\nu)$ for all $n \geq 1$ because the sequence $(E_n)_n$ is decreasing and $\|\nu\|(E_1) < \infty$. Now, locally strongly additivity of ν implies that $\chi_{E_n} \in L^1(\nu)$ for all $n \geq 1$ (see [6, Lemma 3.1]), and moreover $\chi_{E_n} \downarrow 0$ pointwise in $L^1(\nu)$. The order continuity of the norm implies that $\|\nu\|(E_n) = \|\chi_{E_n}\|_{L^1(\nu)} \to 0$ as we want to see.
2) $\Longrightarrow$ 1) Let $(A_n)_n \subseteq \mathcal{R}$ be a disjoint sequence with $\|\nu\|\left(\cup_{n\geq 1} A_n\right) < \infty$. Put $E_1 := \cup_{n\geq 1} A_n$ and $E_n := E_1 \setminus (A_1 \cup \cdots \cup A_{n-1})$ for each $n \geq 2$. Then it is clear that $(E_n)_n \subseteq \mathcal{R}^{\mathrm{loc}}$, $E_n \downarrow \varnothing$ and $\|\nu\|(E_1) < \infty$. Moreover $A_n \subseteq E_n$ for all $n \geq 1$. Thus $\|\nu(A_n)\| \leq \|\nu\|(E_n) \to 0$ and ν is locally strongly additive.

For the last assertion take $A \in \mathcal{R}^{\mathrm{loc}}$, with $\|\nu\|(A) < \infty$, and recall that ν is σ-finite. This allows us to choose a sequence $(\Omega_n)_n \subseteq \mathcal{R}$, with $\Omega_n \uparrow \Omega$. Then $A \setminus A \cap \Omega_n \downarrow \varnothing$ and $\|\nu\|(A \setminus A \cap \Omega_1) \leq \|\nu\|(A) < \infty$. Now the equivalence 2) assures that $\|\nu\|(A \setminus A \cap \Omega_n) \to 0$, but $\|\nu\|(A \setminus A \cap \Omega_n) = \|\chi_A - \chi_{A\cap\Omega_n}\|_{L^1(\|\nu\|)}$. □

Here is the result about density of simple functions.

Proposition 3.5. *Let $\nu : \mathcal{R} \to X$ be a locally strongly additive vector measure. Then $\mathcal{S}(\mathcal{R})$ is dense in $L^1(\|\nu\|)$.*

Proof. Decomposing functions into positive and negative parts, it is enough to consider only nonnegative functions. Note that $f = f\chi_{[f>M]} + f\chi_{[f\leq M]}$ for each $0 \leq f \in L^1(\|\nu\|)$ and $M > 0$. Then Lemma 3.3 assures that the set

$$L^1_{\mathrm{fs}}(\|\nu\|) := \left\{ g \in L^1(\|\nu\|) : \|\nu\| \left([g \neq 0] \right) < \infty \right\}$$

is dense in $L^1(\|\nu\|)$. Now we are going to prove that $\mathcal{S}(\mathcal{R}^{\mathrm{loc}}) \cap L^1_{\mathrm{fs}}(\|\nu\|)$ is dense in $L^1_{\mathrm{fs}}(\|\nu\|)$. Take $0 \leq g \in L^1_{\mathrm{fs}}(\|\nu\|)$ and $\varepsilon > 0$. Consider the sequence $g_n := \inf\{g, n\}$ for all $n \geq 1$. Then $0 \leq g_n \uparrow g$ and $[g_n \neq 0] \subseteq [g \neq 0]$ for all $n \geq 1$. Then

$$\begin{aligned} \lim_{n\to\infty} \|g - g_n\|_{L^1(\|\nu\|)} &= \lim_{n\to\infty} \int_0^\infty \|\nu\| \left([g - g_n > t] \right) dt \\ &= \lim_{n\to\infty} \int_0^\infty \|\nu\| \left([g > n + t] \right) dt \\ &= \lim_{n\to\infty} \int_n^\infty \|\nu\| \left([g > s] \right) ds = 0. \end{aligned}$$

This means that the there exists $m \geq 1$ such that $\|g - g_m\|_{L^1(\|\nu\|)} < \frac{\varepsilon}{4}$. Since g_m is bounded and $[g_m \neq 0] \subseteq [g \neq 0]$ there exists a simple function $\varphi := \sum_{k=1}^N \alpha_k \chi_{A_k}$, with $A_k \in \mathcal{R}^{\mathrm{loc}}$, $A_k \subseteq [g \neq 0]$, $\alpha_k > 0$, for all $1 \leq k \leq N$ and $0 \leq \varphi \leq g_m$ such that $\|g_m - \varphi\|_{L^\infty(\nu)} < \frac{\varepsilon}{4\|\nu\|([g\neq 0])}$. Thus, having in mind that $[g_m - \varphi \neq 0] \subseteq [g \neq 0]$, we obtain

$$\|g_m - \varphi\|_{L^1(\|\nu\|)} = \int_0^\infty \|\nu\|([g_m - \varphi > t])dt$$

$$= \int_0^{\frac{\varepsilon}{4\|\nu\|([g\neq 0])}} \|\nu\|([g_m - \varphi > t])dt$$

$$< \frac{\varepsilon}{4\|\nu\|([g\neq 0])} \|\nu\|([g\neq 0]) = \frac{\varepsilon}{4}$$

and, consequently, $\|g-\varphi\|_{L^1(\|\nu\|)} \leq 2\|g-g_m\|_{L^1(\|\nu\|)} + 2\|g_m-\varphi\|_{L^1(\|\nu\|)} < \varepsilon$.

Finally, note that Lemma 3.4 assures that $\mathcal{S}(\mathcal{R})$ is dense in $\mathcal{S}(\mathcal{R}^{\text{loc}}) \cap L^1_{\text{fs}}(\|\nu\|)$. Indeed, given $0 \leq \varphi := \sum_{k=1}^n \alpha_k \chi_{A_k} \in \mathcal{S}(\mathcal{R}^{\text{loc}}) \cap L^1_{\text{fs}}(\|\nu\|)$ and $\varepsilon > 0$ there exists $B_k \in \mathcal{R}$ such that $\|\chi_{A_k} - \chi_{B_k}\|_{L^1(\|\nu\|)} < \frac{\varepsilon}{n2^n \sum_{k=1}^n \alpha_k}$, for all $k = 1, \dots, n$. Now taking $\phi := \sum_{k=1}^n \alpha_k \chi_{B_k} \in \mathcal{S}(\mathcal{R})$, we obtain that $\|\varphi - \phi\|_{L^1(\|\nu\|)} < \varepsilon$, and the proof is over. □

Proposition 3.6. *Let $\nu : \mathcal{R} \to X$ be a vector measure. The following conditions are equivalent:*

1) ν is locally strongly additive.
2) $\|f\chi_{E_n}\|_{L^1(\|\nu\|)} \to 0$ for every $f \in L^1(\|\nu\|)$ and every sequence $(E_n)_n \subseteq \mathcal{R}^{\text{loc}}$, with $E_n \downarrow \varnothing$.
3) $\|f - f_n\|_{L^1(\|\nu\|)} \to 0$ for every sequence $(f_n)_n$ and f of $L^1(\|\nu\|)$ such that $0 \leq f_n \uparrow f$. That is, $L^1(\|\nu\|)$ is order continuous.
4) $L^p(\|\nu\|)$ is order continuous for every (some) $1 \leq p < \infty$.

Proof. 1) $\implies$ 2) Note that Lemma 3.4 assures that every simple function $\varphi \in \mathcal{S}(\mathcal{R})$ satisfies the above condition 2). Given the function $f \in L^1(\|\nu\|)$, the sequence $(E_n)_n \subseteq \mathcal{R}^{\text{loc}}$, with $E_n \downarrow \varnothing$ and $\varepsilon > 0$, from Proposition 3.5, we know that there exists $\varphi \in \mathcal{S}(\mathcal{R})$ such that $\|f - \varphi\|_{L^1(\|\nu\|)} < \frac{\varepsilon}{4}$. Then we have

$$\begin{aligned}\|f\chi_{E_n}\|_{L^1(\|\nu\|)} &\leq 2\,\|f\chi_{E_n} - \varphi\chi_{E_n}\|_{L^1(\|\nu\|)} + 2\,\|\varphi\chi_{E_n}\|_{L^1(\|\nu\|)} \\ &\leq 2\,\|f-\varphi\|_{L^1(\|\nu\|)} + 2\,\|\varphi\chi_{E_n}\|_{L^1(\|\nu\|)} < \frac{\varepsilon}{2} + 2\,\|\varphi\chi_{E_n}\|_{L^1(\|\nu\|)}\end{aligned}$$

and knowing that $\|\varphi\chi_{E_n}\|_{L^1(\|\nu\|)} \to 0$, it follows that $\|f\chi_{E_n}\|_{L^1(\|\nu\|)} \to 0$.
2) $\implies$ 3) Let $0 \leq f_n \uparrow f \in L^1(\|\nu\|)$ and let $\varepsilon > 0$. The Lemma 3.3 assures that there exists $B \in \mathcal{R}^{\text{loc}}$, with $0 < \|\nu\|(B) < \infty$ (we assume that f is not the null function), such that $\|f\chi_{\Omega\smallsetminus B}\|_{L^1(\|\nu\|)} < \frac{\varepsilon}{24}$. For every $n \geq 1$ consider the measurable subsets $E_n := \left[f - f_n > \frac{\varepsilon}{12\|\nu\|(B)}\right] \in \mathcal{R}^{\text{loc}}$. Note that $E_n \downarrow \varnothing$. By the hypothesis $\|f\chi_{E_n}\|_{L^1(\|\nu\|)} < \frac{\varepsilon}{24}$ for large enough n. Then for those n we have that

$$\begin{aligned}\|f - f_n\|_{L^1(\|\nu\|)} &\leq 2\,\|(f-f_n)\chi_{\Omega\smallsetminus B}\|_{L^1(\|\nu\|)} + 2\,\|(f-f_n)\chi_B\|_{L^1(\|\nu\|)} \\ &\leq 4\,\|f\chi_{\Omega\smallsetminus B}\|_{L^1(\|\nu\|)} + 4\,\|f_n\chi_{\Omega\smallsetminus B}\|_{L^1(\|\nu\|)} \\ &\quad + 4\,\|(f-f_n)\chi_{E_n}\|_{L^1(\|\nu\|)} + 4\,\|(f-f_n)\chi_{B\smallsetminus E_n}\|_{L^1(\|\nu\|)} \\ &\leq 8\,\|f\chi_{\Omega\smallsetminus B}\|_{L^1(\|\nu\|)} + 8\,\|f\chi_{E_n}\|_{L^1(\|\nu\|)} \\ &\quad + \frac{4\varepsilon}{12\|\nu\|(B)}\|\nu\|(B\smallsetminus E_n) < \frac{8\varepsilon}{24} + \frac{8\varepsilon}{24} + \frac{4\varepsilon}{12} = \varepsilon,\end{aligned}$$

and $\|f - f_n\|_{L^1(\|\nu\|)} \to 0$.
3) $\implies$ 1) Let $(A_n)_n \subseteq \mathcal{R}$ be a disjoint sequence with $\|\nu\|(\cup_{n\geq 1} A_n) < \infty$. Put $B_n := A_1 \cup \cdots \cup A_n$ for every $n \geq 1$. Then $0 \leq \chi_{B_n} \uparrow \chi_A$, where $A := \cup_{n\geq 1} A_n$, since the sequence $(A_n)_n$ is pairwise disjoint. Moreover $\chi_A \in L^1(\|\nu\|)$, as $\|\nu\|(A) < \infty$. By the hypothesis it follows that $\|\chi_A - \chi_{B_n}\|_{L^1(\|\nu\|)} \to 0$, but

$$\|\nu(A_{n+1})\|_X \leq \|\nu\|(A_{n+1}) \leq \|\nu\|(B_{n+1}) = \|\chi_{B_{n+1}}\|_{L^1(\|\nu\|)} \leq \|\chi_A - \chi_{B_n}\|_{L^1(\|\nu\|)}.$$

3) $\iff$ 4) This equivalence follows from the definition of the space $L^p(\|\nu\|)$ and the fact that it is normable as we have commented previously. $\square$

Remark 3.7. Now, knowing that $L^p(\|\nu\|)$ has order continuous norm if the measure ν is locally strongly additive, it is not difficult to see that $\mathcal{S}(\mathcal{R})$ is dense in $L^p(\|\nu\|)$ for every $1 \leq p < \infty$.

4. Reflexivity of L^p of the semivariation

Example 3.2 tells us that not always $L^p(\|\nu\|)$ is a reflexive space even for $p > 1$. In this section we characterize those vector measures $\nu : \mathcal{R} \to X$ such that $L^p(\|\nu\|)$ is reflexive. First we need the following technical results which are interesting in themselves.

Proposition 4.1. *For every $p > 1$, the space $L^p(\|\nu\|)$ is a r-convex Banach lattice for every $1 \leq r < p$.*

Proof. As commented above, we know that $L^s(\|\nu\|)$ is a Banach lattice for the equivalent lattice norm $\|\cdot\|_s$ whenever $s > 1$. In order to prove that $L^p(\|\nu\|)$ is r-convex it is enough to show that there exists $K > 0$ such that

$$\left\|(|f_1|^r + \cdots + |f_n|^r)^{\frac{1}{r}}\right\|_{L^p(\|\nu\|)} \leq K\left(\|f_1\|^r_{L^p(\|\nu\|)} + \cdots + \|f_n\|^r_{L^p(\|\nu\|)}\right)^{\frac{1}{r}},$$

for every election of vectors $f_1, \ldots, f_n$ in $L^p(\|\nu\|)$. Take into account that $s := \frac{p}{r} > 1$, and so there exist two constants $C_1, C_2 > 0$ such that

$$C_1\|h\|_{L^s(\|\nu\|)} \leq \|h\|_s \leq C_2\|h\|_{L^s(\|\nu\|)}, \quad h \in L^s(\|\nu\|).$$

Recall also that $\|f\|_{L^p(\|\nu\|)} = \|\,|f|^r\|^{\frac{1}{r}}_{L^s(\|\nu\|)}$ for all $f \in L^p(\|\nu\|)$ or, equivalently, $\left\|\,|h|^{\frac{1}{r}}\right\|_{L^p(\|\nu\|)} = \|h\|^{\frac{1}{r}}_{L^s(\|\nu\|)}$ for all $h \in L^s(\|\nu\|)$. Then, for every election of vectors $f_1, \ldots, f_n$ in $L^p(\|\nu\|)$, we have

$$\begin{aligned}
\left\|\left(\sum_{k=1}^n |f_k|^r\right)^{\frac{1}{r}}\right\|_{L^p(\|\nu\|)} &= \left\|\sum_{k=1}^n |f_k|^r\right\|^{\frac{1}{r}}_{L^s(\|\nu\|)} \leq \frac{1}{C_1}\left\|\sum_{k=1}^n |f_k|^r\right\|^{\frac{1}{r}}_s \\
&\leq \frac{1}{C_1}\left(\sum_{k=1}^n \||f_k|^r\|_s\right)^{\frac{1}{r}} \leq \frac{C_2^{\frac{1}{r}}}{C_1}\left(\sum_{k=1}^n \||f_k|^r\|_{L^s(\|\nu\|)}\right)^{\frac{1}{r}} \\
&\leq \frac{C_2^{\frac{1}{r}}}{C_1}\left(\sum_{k=1}^n \|f_k\|^r_{L^p(\|\nu\|)}\right)^{\frac{1}{r}}
\end{aligned}$$

as we want to prove. $\square$

Proposition 4.2. *Let $\nu : \mathcal{R} \to X$ be a vector measure. For every $1 < p < \infty$, the inclusions $L^1_w(\nu) \cap L^\infty(\nu) \subseteq L^p(\|\nu\|) \subseteq L^1_w(\nu) + L^\infty(\nu)$ hold.*

Proof. For the second inclusion note that $L^p(\|\nu\|) \subseteq L^p_w(\nu)$. Now, if $f \in L^p_w(\nu)$ decompose it as $f = f\chi_{[|f|>1]} + f\chi_{[|f|\leq 1]}$. It is clear that $f\chi_{[|f|\leq 1]} \in L^\infty(\nu)$. On the other hand, for $p \geq 1$, we have

$$|f|\chi_{[|f|>1]} \leq |f|^p\chi_{[|f|>1]} \leq |f|^p \in L^1_w(\nu),$$

and $|f|\chi_{[|f|>1]} \in L^1_w(\nu)$. Consequently $f \in L^1_w(\nu) + L^\infty(\nu)$.

To prove the first inclusion take $f \in L^1_w(\nu) \cap L^\infty(\nu)$. Given $p > 1$, we can choose $\alpha < 1$, with $\alpha p > 1$. Then, for this α we have

$$t\chi_{[|f|^{\alpha p}>t]} \leq |f|^{\alpha p} = |f|^{\alpha p-1}|f| \leq \|f\|^{\alpha p-1}_{L^\infty(\nu)}|f| \in L^1_w(\nu),$$

and so $t\|\nu\|\left([|f|^{\alpha p} > t]\right) \leq \|f\|^{\alpha p-1}_{L^\infty(\nu)}\|f\|_{L^1_w(\nu)}$. In this way we get

$$t\|\nu\|\left(\left[|f|^p > t^{\frac{1}{\alpha}}\right]\right) \leq \|f\|^{\alpha p-1}_{L^\infty(\nu)}\|f\|_{L^1_w(\nu)},$$

or what is the same, $s^\alpha\|\nu\|\left([|f|^p > s]\right) \leq \|f\|^{\alpha p-1}_{L^\infty(\nu)}\|f\|_{L^1_w(\nu)}$, for all $s > 0$. Thus, we have the inequality $\|\nu\|\left([|f|^p > s]\right) \leq \frac{1}{s^\alpha}\|f\|^{\alpha p-1}_{L^\infty(\nu)}\|f\|_{L^1_w(\nu)}$, for all $s > 0$. Since $f \in L^\infty(\nu)$, there exists $M > 0$ such that

$$\int_0^\infty \|\nu\|\left([|f|^p > s]\right) ds = \int_0^M \|\nu\|\left([|f|^p > s]\right) ds.$$

Then, the integral $\int_0^\infty \|\nu\|\left([|f|^p > s]\right) ds < \infty$, because we have chosen $\alpha < 1$, and finally $f \in L^p(\|\nu\|)$ as we want to see. □

Theorem 4.3. *Let $\nu : \mathfrak{R} \to X$ be a vector measure. The following conditions are equivalent:*

a) ν is locally strongly additive.
b) $L^p(\|\nu\|)$ is a KB-space, for every (some) $1 < p < \infty$.
c) $L^p(\|\nu\|)$ is reflexive, for every (some) $1 < p < \infty$.
d) The inclusion $L^1_w(\nu) \cap L^\infty(\nu) \subseteq L^1_w(\nu) + L^\infty(\nu)$ is weakly compact.

Proof. *a)* $\Longrightarrow$ *b)* For every $1 < p < \infty$, the space $L^p(\|\nu\|)$ has the sequential Fatou property. From Proposition 3.6 we know that it has order continuous norm. Then it is a KB-space.
b) $\Longrightarrow$ *c)* Let $1 < p < \infty$. Since $L^p(\|\nu\|)$ is a r-convex Banach lattice for every $1 \leq r < p$ (see Proposition 4.1), the space ℓ_1 (recall that $p > 1$) is not lattice embeddable in $L^p(\|\nu\|)$ (see [14, p. 51]). Moreover, $L^p(\|\nu\|)$ does not contain a lattice copy of c_0 since it is a KB-space by hypothesis (see [1, Theorem 14.12]). The result then follows from Lozanovskii's result (see [1, Theorem 14.23]).
c) $\Longrightarrow$ *d)* We have seen in Proposition 4.2 that the inclusion $L^1_w(\nu) \cap L^\infty(\nu) \subseteq L^1_w(\nu) + L^\infty(\nu)$ always factorizes continuously through $L^p(\|\nu\|)$, with $1 < p < \infty$, and consequently it will be weakly compact if the space $L^p(\|\nu\|)$ is reflexive.
d) $\Longrightarrow$ *a)* Proceed by contradiction. Suppose that $(A_n)_n \subseteq \mathfrak{R}$ is a disjoint sequence such that $\|\nu\|\left(\cup_{n\geq 1}A_n\right) < \infty$, but $\|\nu(A_n)\|_X \not\to 0$. Then (by passing to a subsequence if necessary) there exists $\varepsilon > 0$ such that $\|\nu(A_n)\|_X > \varepsilon$ for all $n = 1, 2, \ldots$ Now consider the sets $B_n := A_1 \cup \cdots \cup A_n$ for all $n = 1, 2, \ldots$ Then $\|\chi_{B_n}\|_{L^1_w(\nu)\cap L^\infty(\nu)} \leq \max\left\{\|\nu\|\left(\cup_{n\geq 1}A_n\right), 1\right\}$, and $\{\chi_{B_n} : n \geq 1\}$ is a bounded set in $L^1_w(\nu) \cap L^\infty(\nu)$. By the hypothesis, it is then a relatively weakly compact set in $L^1_w(\nu) + L^\infty(\nu)$. By applying [8, Corollary 2.2] there exists a convex combination $g_n \in \operatorname{co}\left\{\chi_{B_n}, \chi_{B_{n+1}}, \ldots\right\}$ such that $(g_n)_n$ is norm convergent in $L^1_w(\nu) + L^\infty(\nu)$. Since $g_1 \in \operatorname{co}\{\chi_{B_1}, \chi_{B_2}, \ldots\}$, there exist a finite set $F_1 \subseteq \mathbb{N}$ and scalars $\{\alpha_n \geq 0, n \in F_1\}$, with $\sum_{n\in F_1}\alpha_n = 1$, such that $g_1 = \sum_{n\in F_1}\alpha_n\chi_{B_n}$. Note that $g_1 = 1$ on $B_{\min F_1}$ and $g_1 = 0$ outside $B_{\max F_1}$. Take $n_2 > \max F_1$. Since $g_{n_2} \in \operatorname{co}\left\{\chi_{B_{n_2}}, \chi_{B_{n_2+1}}, \ldots\right\}$, there exist a finite set $F_2 \subseteq \mathbb{N}$ and scalars

$\{\alpha_n \geq 0, n \in F_2\}$, with $\sum_{n\in F_2} \alpha_n = 1$, such that $g_{n_2} = \sum_{n\in F_2} \alpha_n \chi_{B_n}$. Now note that $g_{n_2} = 1$ on $B_{\min F_2}$ and $g_{n_2} = 0$ outside of $B_{\max F_2}$. Thus we have that $g_{n_2} - g_1 \geq \chi_{A_k}$, for all $k \in (\max F_1, \min F_2]$, and we get

$$\|g_{n_2} - g_1\|_{L^1_w(\nu)+L^\infty(\nu)} \geq \|\chi_{A_k}\|_{L^1_w(\nu)+L^\infty(\nu)} = \min\{\|\nu\|(A_k), 1\} > \min\{\varepsilon, 1\},$$

for some $k \in (\max F_1, \min F_2]$. For the next step take $n_3 > \max F_2$. Since $g_{n_3} \in \mathrm{co}\left\{\chi_{B_{n_3}}, \chi_{B_{n_3+1}}, \ldots\right\}$, there exist a finite set $F_3 \subseteq \mathbb{N}$ and scalars $\{\alpha_n \geq 0, n \in F_3\}$, with $\sum_{n\in F_3} \alpha_n = 1$, such that $g_{n_3} = \sum_{n\in F_3} \alpha_n \chi_{B_n}$. Now note that $g_{n_3} = 1$ on $B_{\min F_3}$ and $g_{n_3} = 0$ outside of $B_{\max F_3}$. Thus we have that $g_{n_3} - g_{n_2} \geq \chi_{A_k}$, for all $k \in (\max F_2, \min F_3]$, and we get now

$$\|g_{n_3} - g_{n_2}\|_{L^1_w(\nu)+L^\infty(\nu)} \geq \|\chi_{A_k}\|_{L^1_w(\nu)+L^\infty(\nu)} = \min\{\|\nu\|(A_k), 1\} > \min\{\varepsilon, 1\},$$

for some $k \in (\max F_2, \min F_3]$. Following this inductive process we construct a subsequence $(g_{n_k})_k \subseteq (g_n)_n$ such that

$$\|g_{n_{k+1}} - g_{n_k}\|_{L^1_w(\nu)+L^\infty(\nu)} > \min\{\varepsilon, 1\}, \quad k = 1, 2, \ldots$$

But the above inequality is in contradiction with the fact that the sequence $(g_n)_n$ converges in the norm of $L^1_w(\nu) + L^\infty(\nu)$. □

As we have seen with the equivalence *c)*–*d)* of the previous theorem the reflexivity of $L^p(\|\nu\|)$, with $1 < p < \infty$, is strongly connected with the weak compactness of the inclusion $L^1_w(\nu) \cap L^\infty(\nu) \subseteq L^1_w(\nu) + L^\infty(\nu)$. This equivalence can be deduced from a general and deep result on interpolation due to Maligranda and Quevedo [15, Theorem 1] (see also Beauzamy's results in [3]). The basic reason for that equivalence is the equality $L^p(\|\nu\|) = \left(L^1_w(\nu), L^\infty(\nu)\right)_{1-\frac{1}{p}, p}$, that is, $L^p(\|\nu\|)$ coincides with the interpolated space (by the real method) of the couple of Banach spaces $L^1_w(\nu)$ and $L^\infty(\nu)$ (see [6, Corollary 3.7]). However, we have chosen to present a direct proof of this equivalence by using the well-known and interesting result about weak compactness (without duality) due to Diestel, Ruess and Schachermayer [8, Corollary 2.2].

Remark 4.4. If the measure ν is defined on a σ-algebra, then $L^\infty(\nu) \subseteq L^1_w(\nu)$ and we know in that case that this inclusion is more than weakly compact, in fact, it is L-weakly compact (see [12, Proposition 3.3]). In particular, $\|f_n\|_{L^1_w(\nu)} \to 0$ for every disjoint bounded sequence $(f_n)_n \subseteq L^\infty(\nu)$. This is far from being true for measures ν defined on δ-rings, even being locally strongly additive. In the general case, a disjoint bounded sequence $(f_n)_n \subseteq L^1_w(\nu) \cap L^\infty(\nu)$ does not converge to the null function in the norm of $L^1_w(\nu) + L^\infty(\nu)$, as we can see easily by considering the Lebesgue measure λ and the sequence of characteristic functions $\chi_{[n,n+1)} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, for which we have that

$$\|\chi_{[n,n+1)}\|_{L^1(\mathbb{R})+L^\infty(\mathbb{R})} = 1, \quad n = 1, 2, \ldots$$

Moreover the sequence $(\chi_{[n,n+1)})_n$ neither converges to the null function in the weak topology of $L^1(\mathbb{R})$ since $\int_{\mathbb{R}} \chi_{[n,n+1)} d\lambda = 1$, for all $n = 1, 2, \ldots$ The latter indicates us that, in general, inclusion $L^1_w(\nu) \cap L^\infty(\nu) \subseteq L^1_w(\nu)$ is not weakly compact.

Finally let us point out the following fact. If $L^p_w(\nu)$ (or equivalently $L^p(\nu)$) is reflexive for some $1 < p < \infty$, in which case $L^1_w(\nu) = L^1(\nu)$ as we showed in Theorem 2.3, then the measure ν is necessarily locally strongly additive and, consequently, $L^p(\|\nu\|)$ is reflexive. However, this last space can be reflexive even if the spaces $L^1_w(\nu)$ and $L^1(\nu)$ do not coincide as we can see with next example.

Example 4.5. Consider the δ-ring $\mathcal{R} := \mathcal{P}_f(\mathbb{N})$ of the finite subsets of $\mathbb{N}$ and the vector measure $\nu : A \in \mathcal{P}_f(\mathbb{N}) \longrightarrow \nu(A) := \sum_{n\in A} ne_n \in c_0$. In this case $\mathcal{R}^{\text{loc}} = \mathcal{P}(\mathbb{N})$ and the semivariation is given by $\|\nu\|(A) =$

max A, if $A \subseteq \mathbb{N}$ is finite and $\|\nu\|(A) = \infty$, if $A \subseteq \mathbb{N}$ is infinite. Then ν is a locally strongly additive measure. Moreover it is not difficult to see that

$$L^1_w(\nu) = \{f = (f_n)_n : (nf_n)_n \in \ell_\infty\},$$
$$L^1(\nu) = \{f = (f_n)_n : (nf_n)_n \in c_0\}$$

with equality of norms, that is, $\|f\|_{L^1_w(\nu)} = \sup_n n|f_n|$. It is also true that $\ell_1(n) \subsetneq L^1(\|\nu\|) \subsetneq \ell_1$, and both inclusions are continuous, where $\ell_1(n)$ is the Banach space of all sequences $f = (f_n)_n$ such that $(nf_n)_n \in \ell_1$, with the norm $\|f\|_{\ell_1(n)} := \sum_{n=1}^\infty n|f_n|$. Then we have that $L^1_w(\nu) \cap L^\infty(\nu) = L^1_w(\nu)$ and $L^1_w(\nu) + L^\infty(\nu) = \ell^\infty$, being the inclusion $L^1_w(\nu) \subseteq \ell^\infty$ weakly compact. Indeed it is compact, as we can see easily by considering the sequence $(T_N)_N$ the finite range operators

$$T_N : f \in L^1_w(\nu) \longrightarrow T_N(f) := (f_1, \ldots, f_N, 0, \ldots) \in \ell^\infty,$$

which converges in norm to the inclusion operator from $L^1_w(\nu)$ into ℓ^∞. Thus we conclude that $L^p(\|\nu\|)$ is reflexive for all $1 < p < \infty$.

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