INTERPOLATION WITH A PARAMETER FUNCTION AND INTEGRABLE FUNCTION SPACES WITH RESPECT TO VECTOR MEASURES

RICARDO DEL CAMPO [1], ANTONIO FERNÁNDEZ [1], ANTONIO MANZANO [2], FERNANDO MAYORAL [1] AND FRANCISCO NARANJO [1]

Abstract. We establish interpolation formulae for different compatible couples formed by spaces of scalar integrable functions with respect to a vector measure in connection with a parameter function.

1. Introduction

An intrinsic problem in interpolation theory is to describe the spaces obtained by applying an interpolation method to concrete compatible couples of spaces. For instance, it is well-known that interpolating (L^1, L^∞) by the classical real method we obtain the Lorentz spaces $L^{p,q}$, and the Lebesgue spaces L^p in particular. Namely, if (Ω, Σ) is a measurable space, μ is a positive σ-finite measure on (Ω, Σ), $1 \leqslant p_0 \neq p_1 \leqslant \infty$, $0 < \theta < 1$, $0 < q \leqslant \infty$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, then it holds with equivalence of quasi-norms (see [4, Theorem 5.3.1]) that $(L^{p_0}, L^{p_1})_{\theta,q} = L^{p,q}$, and, in particular,

$$\left(L^1, L^\infty\right)_{1-\frac{1}{p},p} = L^p, \quad 1 < p < \infty. \tag{1}$$

An extension of the classical real method is the so-called *real interpolation method with a parameter function*. Its construction consists in replacing the function t^θ by a more general function ρ, called *parameter function*, that satisfies certain suitable conditions for the main theorems from interpolation theory (equivalence, reiteration, duality, etc.) to be still valid. This is the case, for instance, when ρ belongs to the class $Q(0,1)$, introduced by Persson [27]. The origin of this method can be found in a paper by Peetre [26] and it has been considered by many different authors (see [17], [16], [19], [23], [27] and [8] among others). Let us just mention that an advantage of this interpolation method is that if it applies to the couple (L^1, L^∞) with the parameter function $\rho(t) = t^{1-\frac{1}{p}}(1+|\log t|)^{-\alpha}$, with $1 < p < \infty$ and $\alpha \in \mathbb{R}$, then one obtains the

Mathematics subject classification (2010): Primary 46E30, Secundary 46G10, 46B42.
Keywords and phrases: Real interpolation; parameter function; vector measures; integrable function spaces; Lorentz-Zygmund spaces.
1 Partially supported by Ministerio de Ciencia e Innovación MTM2012-36740-C02-01.
2 Partially suported by Ministerio de Ciencia e Innovación MTM2010-15814.

Lorentz-Zygmund spaces $L^{p,q}(\log L)^{\alpha}$, which generalize the classical Lebesgue spaces L^p, Lorentz spaces $L^{p,q}$ and Zygmund spaces $L^p(\log L)^{\alpha}$ (see [3]).

Several of the present authors have shown (see [13]) that a similar result to (1) does not hold in the case of a vector measure m. Indeed, if $1 < p < \infty$ and m is a vector measure, the space $\left(L^1(m), L^{\infty}(m)\right)_{1-\frac{1}{p},p}$ is reflexive because the natural embedding $L^{\infty}(m) \subseteq L^1(m)$ is a weakly compact operator (see [14, Proposition 3.3] and [2, Proposition II.2.3]), but $L^p(m)$ is non-reflexive whenever $L^1(m) \neq L^1_w(m)$ (see [14, Corollary 3.10]). However, as it is shown in [13, Theorem 12], it is possible to establish a similar formula to (1) in terms of the Lorentz spaces $L^{p,q}(\|m\|)$, defined by means of the *semivariation* $\|m\|$ of the vector measure m.

In this paper we continue the research started in [13]. On the one hand, we extend the results given in [13] when general parameter functions are considered, providing a description of the interpolated spaces for the couples $(L^{p_0}(m), L^{p_1}(m))$ and $\left(L^{p_0}_w(m), L^{p_1}_w(m)\right)$ consisting in spaces of p-integrable and weakly p-integrable functions with respect to a vector measure m. On the other hand, we derive interpolation formulae for couples $\left(\Lambda^{q_0}_{\varphi_0}(\|m\|), \Lambda^{q_1}_{\varphi_1}(\|m\|)\right)$ of Lorentz spaces defined by the semivariation $\|m\|$ of m, which generalize some classical results established by Gustavsson [16], Merucci [23] and Persson [27], when m is in particular a finite positive scalar measure.

2. Preliminaries

We start by introducing the spaces of scalar integrable functions with respect to a vector measure that we are going to use in the paper. Let X be a real Banach space and $m : \Sigma \to X$ be a countably additive vector measure, where Σ is a σ-algebra of subsets of some nonempty set Ω. Let X' and X'' denote the dual and bidual spaces of X, respectively, and let $B(X)$ be the unit ball of X. The *semivariation* of m is the set function $\|m\| : \Sigma \to [0,\infty)$ defined by

$$\|m\|(A) := \sup\{|\langle m, x'\rangle|(A) : x' \in B(X')\}, \quad A \in \Sigma,$$

where $|\langle m, x'\rangle|$ is the total variation measure of the scalar measure $\langle m, x'\rangle$, given by $\langle m, x'\rangle(A) := \langle m(A), x'\rangle$ for all $A \in \Sigma$. We note that the semivariation $\|m\|$ need not be an additive function, but $\|m\|$ and the measure m coincide if m is a finite positive scalar measure. Basic properties of the semivariation can be found in [9, Chapter IV, §10]. In particular, we would like to point out that $\|m\|(\Omega) < \infty$ for a (countably additive) vector measure m (see [9, Lemma IV.10.4]).

Let $L^0(m)$ denote the space of all measurable functions $f : \Omega \to \mathbb{R}$. Two functions $f, g \in L^0(m)$ will be identified if are equal m-a.e., that is, if $\|m\|(\{w \in \Omega : f(w) \neq g(w)\}) = 0$. We also recall that $f \in L^0(m)$ is said to be *weakly integrable* (with respect to m) if $f \in L^1(|\langle m, x'\rangle|)$ for all $x' \in X'$. In this case (see [28, Corollary 3]) for each $A \in \Sigma$ there exists an element $\int_A f dm \in X''$ (called the weak integral of f over A) such that $\left\langle \int_A f dm, x' \right\rangle = \int_A f d\langle m, x'\rangle$ for all $x' \in X'$. The space $L^1_w(m)$ of all (m-a.e.

equivalence classes of) weakly integrable functions becomes a Banach lattice when it is endowed with the natural order m-a.e., and the norm

$$\|f\|_1 := \sup\left\{ \int_\Omega |f|\, d|\langle m, x'\rangle| : x' \in B(X') \right\}, \quad f \in L^1_w(m).$$

A weakly integrable function f is called *integrable* (with respect to m) if the vector $\int_A f dm \in X$ for all $A \in \Sigma$ (see [20] or [25]). The set $L^1(m)$ of all (m-a.e. equivalence classes of) integrable functions becomes an order continuous closed ideal of $L^1_w(m)$, and in general $L^1(m) \subsetneq L^1_w(m)$.

If $1 < p < \infty$, a function $f \in L^0(m)$ is said to be *weakly p-integrable* (with respect to m) if $|f|^p \in L^1_w(m)$, and *p-integrable* (with respect to m) if $|f|^p \in L^1(m)$. We denote by $L^p_w(m)$ the space of (m-a.e. equivalence classes of) weakly p-integrable functions and by $L^p(m)$ the space of (m-a.e. equivalence classes of) p-integrable functions. Obviously we have that $L^p(m) \subseteq L^p_w(m)$. The natural norm for both spaces is given by

$$\|f\|_p := \sup\left\{ \left(\int_\Omega |f|^p\, d|\langle m, x'\rangle| \right)^{\frac{1}{p}} : x' \in B(X') \right\}, \quad f \in L^p_w(m).$$

The spaces $L^p(m)$ and $L^p_w(m)$ have been deeply studied in [14]. The space $L^\infty(m)$ consists in those (m-a.e. equivalence classes of) essentially bounded functions equipped with the supremum norm $\|\cdot\|_\infty$. It holds that $L^\infty(m) \subseteq L^1(m)$ with

$$\|f\|_1 \leqslant \|m\|(\Omega) \cdot \|f\|_\infty, \quad f \in L^\infty(m).$$

Given $f \in L^0(m)$, we shall consider its *distribution function* (with respect to the vector measure m) $\|m\|_f : [0,\infty) \to [0,\infty)$ defined by

$$\|m\|_f(s) := \|m\|(\{w \in \Omega : |f(w)| > s\}),$$

where $\|m\|$ is the semivariation of the measure m. This distribution function has similar properties that in the scalar case (see [13]). For instance, $\|m\|_f$ is bounded, non-increasing and right-continuous. The *decreasing rearrangement* of f (with respect to the measure m) $f_* : (0,\infty) \to [0,\infty)$ is given by

$$f_*(t) := \inf\left\{ s > 0 : \|m\|_f(s) \leqslant t \right\}.$$

Some properties of f_* can be found in [13]. In particular, the function f_* is non-increasing and right-continuous. It also verifies that $f_*(t) = 0$ for any $t \geqslant \|m\|(\Omega)$.

For $0 < q \leqslant \infty$ and a non-negative measurable function φ defined on $(0,\infty)$, we denote by $\Lambda^q_\varphi(\|m\|)$ the set of all $f \in L^0(m)$ such that the quantity

$$\|f\|_{\Lambda^q_\varphi(\|m\|)} := \begin{cases} \left(\int_0^\infty (\varphi(t) f_*(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}, & \text{if } 0 < q < \infty, \\ \sup_{t>0} \varphi(t) f_*(t), & \text{if } \quad q = \infty, \end{cases} \tag{2}$$

is finite. When $\varphi(t)=t^{\frac{1}{p}}(1+|\log t|)^{\alpha}$, where $1\leqslant p<\infty$ and $\alpha\in\mathbb{R}$, we obtain the space $L^{p,q}(\log L)^{\alpha}(\|m\|)$, which can be considered as a version of the Lorentz-Zygmund space in the vector measure setting (see the definition in [3] for a finite positive scalar measure). In particular, if $\alpha=0$ we recover the Lorentz space $L^{p,q}(\|m\|)$ introduced in [13]. For the special case $p=q$, we denote the space $L^{p,p}(\|m\|)$ simply by $L^p(\|m\|)$. As it has been pointed out in [13], in general, the spaces $L^p(\|m\|)$ and $L^p(m)$ do not coincide if $1\leqslant p<\infty$. However, it holds that (see [13, Proposition 7])

$$L^{\infty}(m)\subseteq L^{p,1}(\|m\|)\subseteq L^{p}(\|m\|)\subseteq L^{p}(m)\subseteq L^{p}_{w}(m)\subseteq L^{p,\infty}(\|m\|),\quad 1\leqslant p<\infty,$$

and all these inclusions are continuous. We also note that $L^{p,q}(\|m\|)$ is a quasi-Banach lattice with the Fatou property.

Next we review some basic notions and facts related to the definition of certain classes of parameter functions defined on $(0,\infty)$ that will be considered throughout the paper. We shall follow the notation used by Persson [27]. Namely, given two real numbers $a_0<a_1$, the class $Q[a_0,a_1]$ denotes all non-negative functions ρ on $(0,\infty)$ such that $\rho(t)t^{-a_0}$ is non-decreasing and $\rho(t)t^{-a_1}$ is non-increasing. We write $\rho\in Q(a_0,a_1)$ if $\rho\in Q[a_0+\varepsilon,a_1-\varepsilon]$ for some $\varepsilon>0$. Moreover, $\rho\in Q(a_0,-)$ (respectively, $\rho\in Q(-,a_1)$) means that $\rho\in Q(a_0,b)$ (respectively, $\rho\in Q(b,a_1)$) for certain real number b. Specially important for us will be the class $Q(0,1)$. Observe that $\rho\in Q(0,1)$ if and only if ρ is non-negative, $\rho(t)t^{-\alpha}$ is non-decreasing and $\rho(t)t^{-\beta}$ is non-increasing, for some $0<\alpha<\beta<1$.

Let us recall briefly the construction of the *real interpolation method with a parameter function*. Let (A_0,A_1) be a quasi-Banach couple, that is, two quasi-Banach spaces A_0, A_1 which are continuously embedded in some Hausdorff topological vector space. The Peetre's K-functional is defined, for $f\in A_0+A_1$ and $t>0$, by

$$K(t,f)=K(t,f;A_0,A_1):=\inf\left\{\|f_0\|_{A_0}+t\|f_1\|_{A_1}: f=f_0+f_1\,,\,f_0\in A_0, f_1\in A_1\right\}.$$

For $\rho\in Q(0,1)$ and $0<q\leqslant\infty$, the space $(A_0,A_1)_{\rho,q}$ is formed by all those elements $f\in A_0+A_1$ such that the quasi-norm

$$\|f\|_{(A_0,A_1)_{\rho,q}}:=\begin{cases}\left(\displaystyle\int_0^{\infty}\left(\frac{K(t,f;A_0,A_1)}{\rho(t)}\right)^{q}\frac{dt}{t}\right)^{\frac{1}{q}}, & \text{if } 0<q<\infty,\\ \displaystyle\sup_{t>0}\frac{K(t,f;A_0,A_1)}{\rho(t)}, & \text{if } \quad q=\infty,\end{cases}$$

is finite. In the particular case when $\rho(t)=t^{\theta}$, $0<\theta<1$, the space $(A_0,A_1)_{\rho,q}$ coincides with the interpolation space $(A_0,A_1)_{\theta,q}$ obtained by the classical real method (see [4]).

The interpolation space $(A_0,A_1)_{\rho,q}$ can be also defined by using a parameter function ρ belonging to other similar function classes, such as the class $\mathscr{P}^{+-}$ or B_{ψ} (see [17], [16] and [27]). In fact, these classes $Q(0,1)$, $\mathscr{P}^{+-}$ and B_{ψ} can be considered (in some sense) the same class (see [27, Proposition 1.3]), sometime called *quasi-power function class* (see [27, Proposition 1.2]). In this paper we focus on parameter functions in the class $Q(0,1)$, but it is possible to consider other classes of parameter functions as *quasi concave functions* (see [3, Definition II.5.6] and [4, Lemma 5.4.3]), *logarithmic type functions* (see [10], [11] and [12]) or *slowly varying functions* (see [15]). We refer to [26], [17], [16], [19], [23], and [27] for a complete information about the real interpolation method with a parameter function.

We shall use the following equality (3) that relates the interpolation space with respect a parameter function of quasi-normed ideal function spaces and their corresponding *r-convexifications*. With the above notation, if A_0 is one of the spaces $L^1(\|m\|)$ or $L^{1,\infty}(\|m\|)$ and A_1 is the space $L^\infty(m)$, it is hold that

$$\left(A_0^{(r)},A_1^{(r)}\right)_{\rho,q} = \left[(A_0,A_1)_{\rho_1,\frac{q}{r}}\right]^{(r)}, \tag{3}$$

where $\rho_1(t) := \left(\rho(t^{\frac{1}{r}})\right)^r$, for $1 \leqslant r < \infty$. For a quasi-normed function space $A \subseteq L^0(m)$, its r-convexification is defined by $A^{(r)} := \left\{f \in L^0(m) : |f|^r \in A\right\}$ and equipped with the quasi-norm $\|f\|_{A^{(r)}} := \||f|^r\|_A^{\frac{1}{r}}$. The above equality (3) follows by applying the estimates of the K-functional obtained in [22, Theorem 1] (see also [22, Remark 2]). Regarding the function spaces we have introduced by means of (2) it is not difficult to check by using their definitions the following result.

PROPOSITION 1. Let $1 \leqslant r < \infty$ and $0 < q \leqslant \infty$. Then
1) $\left(\Lambda_\varphi^q(\|m\|)\right)^{(r)} = \Lambda^{rq}_{\varphi^{\frac{1}{r}}}(\|m\|)$.
In particular, for $\varphi(t) = t$, we have
2) $\left(L^1(\|m\|)\right)^{(r)} = L^r(\|m\|)$, for $q = 1$, and
3) $\left(L^{1,\infty}(\|m\|)\right)^{(r)} = L^{r,\infty}(\|m\|)$, for $q = \infty$.

As usual, the equivalence $a \simeq b$ (respectively $a \preccurlyeq b$) means that $\frac{1}{c}a \leqslant b \leqslant ca$ (respectively $a \leqslant cb$) for some positive constant c independent of appropriate parameters. Two quasinormed spaces, A and B, are considered as equal and we write $A = B$ whenever their quasi-norms are equivalent.

We finish this section with some estimates for the K-functional that will be useful to establish our interpolation results in Section 3. These estimates can be obtained following the same techniques used in [13] with minor modifications (see [13, Lemma 3 and Propositions 8 and 10] for details). Let us also mention that similar estimates were obtained independently by Cerdà, Martín and Silvestre in [6] for capacities.

PROPOSITION 2. i) If $f \in L^1(\|m\|)$, then $K(t,f;L^1(\|m\|),L^\infty(m)) \preccurlyeq \int_0^t f_*(s)ds$.
ii) If $f \in L^{1,\infty}(\|m\|)$, then $tf_*(t) \preccurlyeq K(t,f;L^{1,\infty}(\|m\|),L^\infty(m))$.

3. Description of the interpolated spaces

In this section we provide a description of the interpolated spaces for the couples $(L^{p_0}(m), L^{p_1}(m))$ and $\left(L^{p_0}_w(m), L^{p_1}_w(m)\right)$ and also derive interpolation formulae for couples $\left(\Lambda^{q_0}_{\varphi_0}(\|m\|), \Lambda^{q_1}_{\varphi_1}(\|m\|)\right)$ of Lorentz spaces, generalizing some classical results established by Gustavsson [16], Merucci [23] and Persson [27]. We start with the following key result that we shall use throughout the section.

THEOREM 3. *Let* $0 < q \leqslant \infty$, $\rho \in Q(0,1)$, *and* $\varphi(t) = \dfrac{t}{\rho(t)}$. *Then,*

$$\left(L^1(\|m\|), L^\infty(m)\right)_{\rho,q} = \left(L^{1,\infty}(\|m\|), L^\infty(m)\right)_{\rho,q} = \Lambda^q_\varphi(\|m\|).$$

In particular, if $0 < \theta < 1$, *it holds that*

$$\left(L^1(\|m\|), L^\infty(m)\right)_{\theta,q} = \left(L^{1,\infty}(\|m\|), L^\infty(m)\right)_{\theta,q} = L^{\frac{1}{1-\theta},q}(\|m\|).$$

Proof. Since $L^1(\|m\|) \subseteq L^{1,\infty}(\|m\|)$, it is clear the inclusion $(L^1(\|m\|), L^\infty(m))_{\rho,q} \subseteq (L^{1,\infty}(\|m\|), L^\infty(m))_{\rho,q}$, and the inequality

$$\|f\|_{\left(L^{1,\infty}(\|m\|),L^\infty(m)\right)_{\rho,q}} \leqslant \|f\|_{\left(L^1(\|m\|),L^\infty(m)\right)_{\rho,q}}, \quad f \in \left(L^1(\|m\|), L^\infty(m)\right)_{\rho,q}. \tag{4}$$

We have also the inclusion $\left(L^{1,\infty}(\|m\|), L^\infty(m)\right)_{\rho,q} \subseteq \Lambda^q_\varphi(\|m\|)$ as a consequence of the inequality ii) in Proposition 2. In particular, we obtain

$$\|f\|_{\Lambda^q_\varphi(\|m\|)} \preccurlyeq \|f\|_{\left(L^{1,\infty}(\|m\|),L^\infty(m)\right)_{\rho,q}}, \quad f \in \left(L^{1,\infty}(\|m\|), L^\infty(m)\right)_{\rho,q}. \tag{5}$$

In order to check that the inclusion $\Lambda^q_\varphi(\|m\|) \subseteq \left(L^1(\|m\|), L^\infty(m)\right)_{\rho,q}$ holds, we assume first that $q < \infty$, and consider the function $W(t) = \dfrac{t^{q-1}}{\rho(t)^q}$. Since $\rho \in Q(0,1)$, in particular $\rho(t)t^{-\alpha}$ is non-decreasing for some $0 < \alpha < 1$, it is not difficult to check that

$$\int_r^\infty \frac{W(t)}{t^q} dt \leqslant \frac{1-\alpha}{\alpha r^q} \int_0^r W(t)dt, \quad r > 0.$$

The weighted Hardy inequality for non-increasing functions [1, Theorem 1.7] (see also [29, Theorem 3] or [5, Proposition 2.6] for the case $0 < q < 1$) gives, for any $f \in \Lambda^q_\varphi(\|m\|)$,

$$\begin{aligned}\left(\int_0^\infty \left[\frac{1}{t}\int_0^t f_*(u)du\right]^q W(t)dt\right)^{\frac{1}{q}} &\preccurlyeq \left(\int_0^\infty f_*(t)^q W(t)dt\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \left(\frac{t}{\rho(t)} f_*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} = \|f\|_{\Lambda^q_\varphi(\|m\|)} < \infty.\end{aligned}$$

In particular, the function $\frac{1}{t}\int_0^t f_*(u)du$ is finite a.e. and, since f_* is non-increasing and $f_*(t)=0$ for all $t \geqslant \|m\|(\Omega)$, this means that $\int_0^\infty f_*(u)du < \infty$, that is, the inclusion $\Lambda_\varphi^q(\|m\|) \subseteq L^1(\|m\|)$ holds. Then, for any $f \in \Lambda_\varphi^q(\|m\|)$, by applying the estimate i) in Proposition 2 and the weighted Hardy inequality again, we obtain

$$\begin{aligned}
\|f\|_{\left(L^1(\|m\|),L^\infty(m)\right)_{\rho,q}} &= \left(\int_0^\infty \left(\frac{K(t,f;L^1(\|m\|),L^\infty(m))}{\rho(t)}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\preccurlyeq \left(\int_0^\infty \left(\frac{1}{\rho(t)}\left[\int_0^t f_*(u)du\right]\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty \left[\frac{1}{t}\int_0^t f_*(u)du\right]^q \frac{t^{q-1}}{\rho(t)^q}dt\right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty \left[\frac{1}{t}\int_0^t f_*(u)du\right]^q W(t)dt\right)^{\frac{1}{q}} \preccurlyeq \|f\|_{\Lambda_\varphi^q(\|m\|)}. \qquad (6)
\end{aligned}$$

This implies that $\Lambda_\varphi^q(\|m\|) \subseteq \left(L^1(\|m\|),L^\infty(m)\right)_{\rho,q}$. For the case $q=\infty$, the inclusion $\Lambda_\varphi^\infty(\|m\|) \subseteq \left(L^1(\|m\|),L^\infty(m)\right)_{\rho,\infty}$ can be obtained in the following way. Recall that $\rho(s)s^{-\alpha}$ is non-decreasing for some $0<\alpha<1$. Then, for every $f \in \Lambda_\varphi^\infty(\|m\|)$, in which case $\|f\|_{\Lambda_\varphi^\infty(\|m\|)} := \sup_{s>0} \frac{sf_*(s)}{\rho(s)} < \infty$, we have

$$\begin{aligned}
\frac{1}{\rho(t)}\int_0^t f_*(s)ds &= \frac{1}{\rho(t)}\int_0^t \frac{sf_*(s)}{\rho(s)}\frac{\rho(s)}{s}ds \leqslant \|f\|_{\Lambda_\varphi^\infty(\|m\|)} \frac{1}{\rho(t)}\int_0^t \frac{\rho(s)}{s}ds \\
&= \|f\|_{\Lambda_\varphi^\infty(\|m\|)} \frac{1}{\rho(t)}\int_0^t \frac{\rho(s)}{s^\alpha}s^{\alpha-1}ds \leqslant \frac{1}{\alpha}\|f\|_{\Lambda_\varphi^\infty(\|m\|)} < \infty. \qquad (7)
\end{aligned}$$

This means that $\int_0^\infty f_*(s)ds < \infty$, that is, the inclusion $\Lambda_\varphi^\infty(\|m\|) \subseteq L^1(\|m\|)$ holds. Then, for any $f \in \Lambda_\varphi^\infty(\|m\|)$, by applying the estimate i) in Proposition 2 and the above inequality (7), we obtain

$$\frac{K(t,f;L^1(\|m\|),L^\infty(m))}{\rho(t)} \preccurlyeq \frac{1}{\rho(t)}\int_0^t f_*(s)ds \leqslant \frac{1}{\alpha}\|f\|_{\Lambda_\varphi^\infty(\|m\|)}.$$

Taking supremum, we obtain $\Lambda_\varphi^\infty(\|m\|) \subseteq \left(L^1(\|m\|),L^\infty(m)\right)_{\rho,\infty}$, and

$$\|f\|_{(L^1(\|m\|),L^\infty(m))_{\rho,\infty}} \preccurlyeq \|f\|_{\Lambda_\varphi^\infty(\|m\|)}, \quad f \in \Lambda_\varphi^\infty(\|m\|). \qquad (8)$$

Finally we get the equality between the three spaces (even for $q=\infty$)

$$\left(L^1(\|m\|),L^\infty(m)\right)_{\rho,q} = \left(L^{1,\infty}(\|m\|),L^\infty(m)\right)_{\rho,q} = \Lambda_\varphi^q(\|m\|)$$

as metric spaces (the equivalence of their quasi-norms is given by (4), (5) and (6), or (8) for $q=\infty$). For last part of the statement it is enough to take the function $\rho(t)=t^\theta$, $0<\theta<1$. $\square$

REMARK 1. Note that in the proof of Theorem 3 the assumption $\rho \in Q(0,1)$ is only required for using that $\rho(t)t^{-\alpha}$ is non-decreasing for some $0<\alpha<1$. Thus, Theorem 3 still continues being valid with the weaker assumption $\rho \in Q(0,-)$.

COROLLARY 1. *Let* $1 \leqslant r < \infty$, $\rho \in Q(0,1)$, $\varphi_r(t) = \dfrac{t^{\frac{1}{r}}}{\rho\left(t^{\frac{1}{r}}\right)}$ *and* $0 < q \leqslant \infty$. *Then* $(L^r(\|m\|), L^\infty(m))_{\rho,q} = (L^{r,\infty}(\|m\|), L^\infty(m))_{\rho,q} = \Lambda^q_{\varphi_r}(\|m\|)$. *In particular, if* $0 < \theta < 1$, *it holds that* $(L^r(\|m\|), L^\infty(m))_{\theta,q} = (L^{r,\infty}(\|m\|), L^\infty(m))_{\theta,q} = L^{\frac{r}{1-\theta},q}(\|m\|)$.

Proof. Taking into account the equality (3) and the previous Theorem 3, we have for $\tau(t) = \left(\rho(t^{\frac{1}{r}})\right)^r$ and $\phi(t) = \dfrac{t}{\tau(t)}$ the following equalities

$$\begin{aligned}(L^r(\|m\|), L^\infty(m))_{\rho,q} &= \left[\left(L^1(\|m\|), L^\infty(m)\right)_{\tau,\frac{q}{r}}\right]^{(r)} = \left[\Lambda^{\frac{q}{r}}_{\phi}(\|m\|)\right]^{(r)} \\ &= \Lambda^q_{\phi^{\frac{1}{r}}}(\|m\|) = \Lambda^q_{\varphi_r}(\|m\|).\end{aligned}$$

The above chain of equalities also works for the pair $(L^{r,\infty}(\|m\|), L^\infty(m))$. □

When, in particular, m is a finite positive scalar measure Corollary 1 turns out to be [27, Lemma 6.1] (see also [16, Lemma 3.1]). The following result extends [13, Theorem 12 and Corollary 13].

COROLLARY 2. *Let* $1 \leqslant r < \infty$, $\rho \in Q(0,1)$, $\varphi_r(t) = \dfrac{t^{\frac{1}{r}}}{\rho\left(t^{\frac{1}{r}}\right)}$ *and* $0 < q \leqslant \infty$. *Then* $(L^r(m), L^\infty(m))_{\rho,q} = (L^r_w(m), L^\infty(m))_{\rho,q} = \Lambda^q_{\varphi_r}(\|m\|)$. *In particular, if* $0 < \theta < 1$, *it holds that* $(L^r(m), L^\infty(m))_{\theta,q} = (L^r_w(m), L^\infty(m))_{\theta,q} = L^{\frac{r}{1-\theta},q}(\|m\|)$.

Proof. It is enough to use the following chain of continuous inclusions

$$L^r(\|m\|) \subseteq L^r(m) \subseteq L^r_w(m) \subseteq L^{r,\infty}(\|m\|), \quad 1 \leqslant r < \infty,$$

and Corollary 1. □

REMARK 2. Note that the function $\rho(t) = t^{1-\frac{1}{p}}(1+|\log t|)^{-\alpha}$ belongs to the class B_ψ whenever $1<p<\infty$ and $|\alpha| < \min\left\{\frac{1}{p}, 1-\frac{1}{p}\right\}$. On the other hand, recall that $B_\psi \subseteq Q(0,1)$ (see [27, Proposition 1.3]). As a consequence of this observation and the Cororally 2, the space $L^{p,q}(\log L)^\alpha(\|m\|)$ can be obtained as an interpolation space with respect to couple $\left(L^1(m), L^\infty(m)\right)$. Indeed $L^{p,q}(\log L)^\alpha(\|m\|) = (L^1(m), L^\infty(m))_{\rho,q}$ for all $0 < q \leqslant \infty$. However, the restriction $|\alpha| < \min\left\{\frac{1}{p}, 1-\frac{1}{p}\right\}$ can be removed as we shall see just now. For $\alpha \in \mathbb{R}$ (without any restriction), it is easy to check that the submultiplicative function $\bar{\rho}(t) := \sup_{u>0} \dfrac{\rho(ut)}{\rho(u)}$ is precisely the function $\bar{\rho}(t) = t^{1-\frac{1}{p}}(1+|\log t|)^{|\alpha|}$ (see, for instance, [23]), which is continuous and satisfies that $\bar{\rho}(t) = o(\max\{1,t\})$ as $t \to 0$ and $t \to \infty$. Hence, there exists a function $\eta \in Q[0,1]$ such that $\eta \simeq \rho$ (see Lemma 1.2 in [21, Chapter II, Section 1]). In fact $\eta \in \mathscr{P}^{+-}$, and then $\rho \simeq \tau$ for some $\tau \in Q(0,1)$ (see [27, Proposition 1.3]). Note that $\varphi \simeq \phi$, where $\varphi(t) = \dfrac{t}{\rho(t)}$ and $\phi(t) = \dfrac{t}{\tau(t)}$, and therefore $\Lambda^q_\varphi(\|m\|) = \Lambda^q_\phi(\|m\|)$. Then

$$L^{p,q}(\log L)^\alpha(\|m\|) = \Lambda^q_\varphi(\|m\|) = \Lambda^q_\phi(\|m\|) = (L^1(m), L^\infty(m))_{\tau,q} = (L^1(m), L^\infty(m))_{\rho,q}.$$

We summarize these comments in the following

COROLLARY 3. *Let* $1 < p < \infty$, $0 < q \leqslant \infty$, $\alpha \in \mathbb{R}$, *and* $\rho(t) = t^{1-\frac{1}{p}}(1+|\log t|)^{-\alpha}$. *Then* $\big(L^1(m), L^\infty(m)\big)_{\rho,q} = L^{p,q}(\log L)^\alpha(\|m\|)$.

Before stating the next result, we collect some useful facts we shall use in the rest of the paper.

REMARK 3. If $1 < p < \infty$ and $\rho(t) := t^{1-\frac{1}{p}}$, it holds by Corollary 2 that

$$\begin{aligned}\big(L^1(m), L^\infty(m)\big)_{\rho,1} &= L^{p,1}(\|m\|) \subseteq L^p(m) \subseteq L^p_w(m)\\ &\subseteq L^{p,\infty}(\|m\|) = \big(L^1(m), L^\infty(m)\big)_{\rho,\infty}.\end{aligned}$$

The above (continuous) inclusions show that the spaces $L^p(m)$ and $L^p_w(m)$ are of the class $C(\rho; L^1(m), L^\infty(m))$ (see [27, Examples 2.1 and 2.2] and [4, Theorem 3.11.4]).

LEMMA 1. *If* $\rho \in Q(0,1)$ *and* $\tau \in Q(0,-)$, *then* $\rho(\tau(t)) \in Q(0,-)$.

Proof. Since $\rho \in Q(0,1)$ there exists $0 < \varepsilon < \frac{1}{2}$ such that $\rho(t)t^{-\varepsilon}$ is non-decreasing and $\rho(t)t^{-(1-\varepsilon)}$ is non-increasing. Furthermore, by the assumption $\tau \in Q(0,-)$, there also exist $b \in \mathbb{R}$ and $0 < \delta < \frac{b}{2}$ so that $\tau(t)t^{-\delta}$ is non-decreasing and $\tau(t)t^{-(b-\delta)}$ is non-increasing. In particular, τ is non-decreasing. Thus, if $t \leqslant s$,

$$\rho(\tau(t))t^{-\varepsilon\delta} = \rho(\tau(t))\tau(t)^{-\varepsilon}\left(\tau(t)t^{-\delta}\right)^\varepsilon \leqslant \rho(\tau(s))\tau(s)^{-\varepsilon}\left(\tau(s)s^{-\delta}\right)^\varepsilon = \rho(\tau(s))s^{-\varepsilon\delta},$$

and so $\rho(\tau(t))t^{-\varepsilon\delta}$ is non-decreasing. A similar argument shows that the function $\rho(\tau(t))t^{-(1-\varepsilon)(b-\delta)}$ is non-increasing. In other words, $\rho(\tau(t)) \in Q[\varepsilon\delta, c-\varepsilon\delta]$ with $c = (1-\varepsilon)(b-\delta)+\varepsilon\delta$. $\square$

LEMMA 2. (see Lemma 3.3 in [27]) *Let* $\rho \in Q(0,1)$, $\rho_0, \rho_1 \in Q(0,-)$, *and put* $\tau(t) = \frac{\rho_1(t)}{\rho_0(t)}$. *If* $\tau \in Q(0,-)$ *or* $\tau \in Q(-,0)$, *then the function* $\rho_0(t)\rho(\tau(t)) \in Q(0,-)$. *When in addition* $\rho_0, \rho_1 \in Q(0,1)$, *then the function* $\rho_0(t)\rho(\tau(t)) \in Q(0,1)$.

Proof. For instance, assume that $\tau \in Q(0,-)$. Then, for $i=0,1$, there exist numbers $a_i \in \mathbb{R}$, $0<\delta_i<\frac{a_i}{2}$, such that $\rho_i(t)t^{-\delta_i}$ is non-decreasing and $\rho_i(t)t^{-(a_i-\delta_i)}$ is non-increasing. Choose any $0<\delta<\min\{\delta_0,\delta_1\}$. Since $\frac{\rho(\tau(t))}{\tau(t)}$ is non-increasing, we have whenever $t \leqslant s$ that

$$\begin{aligned}\rho_0(t)\rho(\tau(t))t^{-(a_1-\delta)} &= \rho_1(t)\frac{\rho(\tau(t))}{\tau(t)}t^{-(a_1-\delta)} = \rho_1(t)t^{-(a_1-\delta_1)}\frac{\rho(\tau(t))}{\tau(t)}t^{-\delta_1+\delta}\\ &\geqslant \rho_1(s)s^{-(a_1-\delta_1)}\frac{\rho(\tau(s))}{\tau(s)}s^{-\delta_1+\delta} = \rho_0(s)\rho(\tau(s))s^{-(a_1-\delta)}.\end{aligned}$$

Reasoning in a similar way we obtain that $\rho_0(t)\rho(\tau(t))t^{-\delta}$ is non-decreasing. Hence, the function $\rho_0(t)\rho(\tau(t)) \in Q[\delta, a_1-\delta]$.

Note that if $\tau \in Q(-,0)$, the proof works with slight changes, but now taking into account that $\rho(\tau(t))$ is non-increasing and $\frac{\rho(\tau(t))}{\tau(t)}$ is non-decreasing. $\square$

COROLLARY 4. *Let* $1<p_0 \neq p_1<\infty$, $\rho \in Q(0,1)$, $\varphi(t)=\dfrac{t^{\frac{1}{p_0}}}{\rho\left(t^{\frac{1}{p_0}-\frac{1}{p_1}}\right)}$, *and* $0<q\leqslant\infty$. *Then*

$$(L^{p_0}(m),L^{p_1}(m))_{\rho,q} = (L^{p_0}_w(m),L^{p_1}(m))_{\rho,q} = (L^{p_0}_w(m),L^{p_1}_w(m))_{\rho,q} = \Lambda^q_\varphi(\|m\|).$$

In particular, if $0<\theta<1$ *and* $\dfrac{1}{p}=\dfrac{1-\theta}{p_0}+\dfrac{\theta}{p_1}$, *it holds that*

$$(L^{p_0}(m),L^{p_1}(m))_{\theta,q} = (L^{p_0}_w(m),L^{p_1}(m))_{\theta,q} = (L^{p_0}_w(m),L^{p_1}_w(m))_{\theta,q} = L^{p,q}(\|m\|).$$

Proof. Without loss of generality we may suppose that $p_0<p_1$, because (see [16, Proposition 2.2] and also [27, Example 1.2]) $(A_0,A_1)_{\rho,q}=(A_1,A_0)_{\tilde{\rho},q}$, for the function $\tilde{\rho}(t):=t\rho\left(\frac{1}{t}\right)$. Let $\rho_i(t):=t^{1-\frac{1}{p_i}}$, for $i=0,1$. Since $\tau:=\frac{\rho_1}{\rho_0}\in Q(0,1)$, we have by Remark 3 and [27, Proposition 4.3] (see also [24]) that

$$\begin{aligned}(L^{p_0}(m),L^{p_1}(m))_{\rho,q} &= (L^{p_0}_w(m),L^{p_1}(m))_{\rho,q} = (L^{p_0}_w(m),L^{p_1}_w(m))_{\rho,q}\\ &= \left(L^1(m),L^\infty(m)\right)_{\eta,q}\end{aligned}$$

with $\eta(t):=\rho_0(t)\rho(\tau(t))$. Moreover, Lemma 2 implies that $\eta \in Q(0,1)$. It follows from Theorem 3 that

$$\begin{aligned}(L^{p_0}(m),L^{p_1}(m))_{\rho,q} &= (L^{p_0}_w(m),L^{p_1}(m))_{\rho,q} = (L^{p_0}_w(m),L^{p_1}_w(m))_{\rho,q}\\ &= \left(L^1(m),L^\infty(m)\right)_{\eta,q} = \Lambda^q_\varphi(\|m\|),\end{aligned}$$

where $\varphi(t)=\dfrac{t}{\eta(t)}=\dfrac{t^{\frac{1}{p_0}}}{\rho\left(t^{\frac{1}{p_0}-\frac{1}{p_1}}\right)}$. $\square$

Our Theorem 3, Corollary 1, and the reiteration theorem (see for example [27, Corollary 4.4] and [24]) allow us to establish the next result, which can be read as a version of [27, Proposition 6.2] for the case of a vector measure.

THEOREM 4. *Let* $\rho \in Q(0,1)$ *and* $0 < q_0, q, q_1 \leqslant \infty$.

a) If $\varphi_0 \in Q(0,1)$, *then* $\left(\Lambda^{q_0}_{\varphi_0}(\|m\|), L^\infty(m)\right)_{\rho,q} = \Lambda^q_\varphi(\|m\|)$, *where* $\varphi(t) = \dfrac{\varphi_0(t)}{\rho(\varphi_0(t))}$.

b) If $\varphi_1 \in Q(0,\frac{1}{p})$, $1 \leqslant p < \infty$, *then* $\left(L^p(\|m\|), \Lambda^{q_1}_{\varphi_1}(\|m\|)\right)_{\rho,q} = \Lambda^q_\varphi(\|m\|)$, *with* $\varphi(t) = \dfrac{t^{\frac{1}{p}}}{\rho\left(\frac{t^{\frac{1}{p}}}{\varphi_1(t)}\right)}$.

c) Let $\varphi_0, \varphi_1 \in Q(0,1)$ *and put* $\phi := \frac{\varphi_0}{\varphi_1}$. *If* $\phi \in Q(0,-)$ *or* $\phi \in Q(-,0)$, *then* $\left(\Lambda^{q_0}_{\varphi_0}(\|m\|), \Lambda^{q_1}_{\varphi_1}(\|m\|)\right)_{\rho,q} = \Lambda^q_\varphi(\|m\|)$, *where* $\varphi(t) = \dfrac{\varphi_0(t)}{\rho(\phi(t))}$.

Proof. a) Since $\varphi_0 \in Q(0,1)$, the function $\rho_0(t) := \dfrac{t}{\varphi_0(t)} \in Q(0,1)$. Applying Theorem 3 and reiteration, we get that

$$\left(\Lambda^{q_0}_{\varphi_0}(\|m\|), L^\infty(m)\right)_{\rho,q} = \left(\left(L^1(\|m\|), L^\infty(m)\right)_{\rho_0,q_0}, L^\infty(m)\right)_{\rho,q} = \left(L^1(\|m\|), L^\infty(m)\right)_{\eta,q},$$

where $\eta(t) := \rho_0(t)\rho\left(\frac{t}{\rho_0(t)}\right) = \rho_0(t)\rho\left(\varphi_0(t)\right)$. It follows from Lemma 1 that the function $\rho\left(\varphi_0(t)\right) \in Q(0,-)$. Therefore, it also holds that $\eta \in Q(0,-)$. Using again Theorem 3 (and Remark 1), we conclude that

$$\left(\Lambda^{q_0}_{\varphi_0}(\|m\|), L^\infty(m)\right)_{\rho,q} = \left(L^1(\|m\|), L^\infty(m)\right)_{\eta,q} = \Lambda^q_\varphi(\|m\|),$$

with $\varphi(t) = \dfrac{t}{\eta(t)} = \dfrac{\varphi_0(t)}{\rho(\varphi_0(t))}$.

b) From $\varphi_1 \in Q(0,\frac{1}{p})$, it follows that $\rho_1(t) := \dfrac{t}{\varphi_1(t^p)} \in Q(0,1)$. According to Corollary 1, and using reiteration, we have that

$$\begin{aligned}\left(L^p(\|m\|), \Lambda^{q_1}_{\varphi_1}(\|m\|)\right)_{\rho,q} &= \left(L^p(\|m\|), \left(L^p(\|m\|), L^\infty(m)\right)_{\rho_1,q_1}\right)_{\rho,q} \\ &= \left(L^p(\|m\|), L^\infty(m)\right)_{\eta,q},\end{aligned}$$

where $\eta(t) = \rho(\rho_1(t))$. By Lemma 1, we obtain that $\eta(t) \in Q(0,-)$. Applying Corollary 1 (see also Remark 1), it follows that

$$\left(L^p(\|m\|), \Lambda^{q_1}_{\varphi_1}(\|m\|)\right)_{\rho,q} = \left(L^p(\|m\|), L^\infty(m)\right)_{\eta,q} = \Lambda^q_\varphi(\|m\|),$$

where $\varphi(t) = \dfrac{t^{\frac{1}{p}}}{\eta\left(t^{\frac{1}{p}}\right)} = \dfrac{t^{\frac{1}{p}}}{\rho\left(\frac{t^{\frac{1}{p}}}{\varphi_1(t)}\right)}$.

c) For $i=0,1$ put $\rho_i(t) := \dfrac{t}{\varphi_i(t)}$. Note that $\frac{\rho_1}{\rho_0} = \phi := \frac{\varphi_0}{\varphi_1}$. Due to $\varphi_0, \varphi_1 \in Q(0,1)$, the functions $\rho_0, \rho_1 \in Q(0,1)$ too. By Theorem 3 and reiteration, it holds that

$$\begin{aligned}\left(\Lambda^{q_0}_{\varphi_0}(\|m\|), \Lambda^{q_1}_{\varphi_1}(\|m\|)\right)_{\rho,q} &= \left(\left(L^1(\|m\|), L^\infty(m)\right)_{\rho_0,q_0}, \left(L^1(\|m\|), L^\infty(m)\right)_{\rho_1,q_1}\right)_{\rho,q} \\ &= \left(L^1(\|m\|), L^\infty(m)\right)_{\eta,q},\end{aligned}$$

with $\eta(t) = \rho_0(t)\rho(\phi(t))$. On the other hand, since $\rho, \rho_0, \rho_1 \in Q(0,1)$, and $\phi \in Q(0,-)$ or $\phi \in Q(-,0)$, the function $\eta \in Q(0,1)$ by Lemma 2. Finally, it follows from Theorem 3 that $\left(\Lambda^{q_0}_{\varphi_0}(\|m\|), \Lambda^{q_1}_{\varphi_1}(\|m\|)\right)_{\rho,q} = \left(L^1(\|m\|), L^\infty(m)\right)_{\eta,q} = \Lambda^{q}_{\varphi}(\|m\|)$, where $\varphi(t) = \dfrac{t}{\eta(t)} = \dfrac{\varphi_0(t)}{\rho(\phi(t))}$. $\square$

Corollary 3 and Theorem 4 give the following result.

COROLLARY 5. *Assume that* $1 < p_0 \neq p_1 < \infty$, $0 < q_0, q, q_1 \leqslant \infty$, *and* $\alpha_0, \alpha_1 \in \mathbb{R}$. *If* $0 < \theta < 1$, $\dfrac{1}{p} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$, *and* $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, *it holds that*

$$\left(L^{p_0,q_0}(\log L)^{\alpha_0}(\|m\|), L^{p_1,q_1}(\log L)^{\alpha_1}(\|m\|)\right)_{\theta,q} = L^{p,q}(\log L)^{\alpha}(\|m\|).$$

In the particular case when m is a finite positive scalar measure, Corollary 5 provides an interpolation result in the same direction that [23, Corollaire 1] (see also [16, Proposition 3.3]).

As an application of our results, we finish the paper showing the reflexivity of the space $L^{p,q}(\log L)^{\alpha}(\|m\|)$, for $1 < p, q < \infty$, and $\alpha \in \mathbb{R}$. According to Corollary 3 such a space is a Banach space. We recall that $L^\infty(m) \subseteq L^1(m)$ is a weakly compact inclusion since $L^1(m)$ has order continuous norm (see [14, Proposition 3.3]). Following the ideas of Heinrich [18, Proposition 2.2] (see also [2, Proposition II.2.3] and [7, Corollary 4.4]) it can be proved that $(A_0, A_1)_{\rho,q}$, with $1 < q < \infty$, is reflexive if and only if the inclusion $A_0 \cap A_1 \subseteq A_0 + A_1$ is weakly compact. In particular, by applying Corollary 3 we derive the following result.

COROLLARY 6. *If* $1 < p, q < \infty$, *and* $\alpha \in \mathbb{R}$, *the space* $L^{p,q}(\log L)^{\alpha}(\|m\|)$ *is reflexive.*

Acknowledgements. The authors would like to thank the referee for his/her useful comments and, in particular, for pointing out the papers [10], [11], [12] and [15] to us.

REFERENCES

[1] M. A. ARIÑO AND B. MUCKENHOUPT, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans. Amer. Math. Soc. **320**, 2 (1990), 727–735.

[2] B. BEAUZAMY, *Espaces d'interpolation réels: topologie et géométrie*, Springer, Lecture Notes in Math. **666**, Berlin, (1978).

[3] C. BENNETT AND R. SHARPLEY, *Interpolation of operators*, Academic Press Inc. Pure and Applied Mathematics **129**, Boston, MA, (1988).

[4] J. BERGH AND J. LÖFSTRÖM, *Interpolation spaces (An Introduction)*, Springer–Verlag, Grundlehren der Mathematischen Wissenschaften **223**, Berlin, (1976).

[5] M. J. CARRO AND J. SORIA, *Boundedness of some integral operators*, Canad. J. Math. **45**, 6 (1993), 1155–1166.

[6] J. CERDÀ, J. MARTÍN AND P. SILVESTRE, *Capacitary function spaces*, Collect. Math. **62**, 1 (2011), 95–118.

[7] F. COBOS, L. M. FERNÁNDEZ-CABRERA, A. MANZANO AND A. MARTÍNEZ, *Real Interpolation and closed operator ideals*, J. Math. Pures Appl. **83**, 3 (2004), 417–432.

[8] F. COBOS, M. L. FERNÁNDEZ-CABRERA AND H. TRIEBEL, *Abstract and concrete logarithmic interpolation spaces*, J. London Math. Soc. **70**, 1 (2004), 231–243.

[9] N. DUNFORD AND J. T. SCHWARTZ, *Linear Operators. Part I*, Wiley Classics Lib., John Wiley and Sons Inc., Grundlehren der Mathematischen Wissenschaften, New York, (1988).

[10] W. D. EVANS AND B. OPIC, *Real interpolation with logarithmic functors and reiteration*, Canad. J. Math. **52**, 5 (2000), 920–960.

[11] W. D. EVANS, B. OPIC AND L. PICK, *Interpolation of operators on scales of generalized Lorentz-Zygmund spaces*, Math. Nachr. **182**, (1996), 127–181.

[12] W. D. EVANS, B. OPIC AND L. PICK, *Real interpolation with logarithmic functors*, J. Inequal. Appl. **7**, 2 (2002), 187–269.

[13] A. FERNÁNDEZ, F. MAYORAL AND F. NARANJO, *Real interpolation method on spaces of scalar integrable functions with respect to vector measures*, J. Math. Anal. Appl. **376**, 1 (2011), 203–211.

[14] A. FERNÁNDEZ, F. MAYORAL, F. NARANJO, C. SÁEZ AND E. A. SÁNCHEZ–PÉREZ, *Spaces of p-integrable functions with respect to a vector measure*, Positivity **10**, 1 (2006), 1–16.

[15] A. GOGATISHVILI, B. OPIC AND W. TREBELS, *Limiting reiteration for real interpolation with slowly varying functions*, Math. Nachr. **278**, 1–2 (2005), 86–107.

[16] J. GUSTAVSSON, *A function parameter in connection with interpolation of Banach spaces*, Math. Scand. **42**, 2 (1978), 289–305.

[17] J. GUSTAVSSON AND J. PEETRE, *Interpolation of Orlicz spaces*, Studia Math. **60**, 1 (1977), 33–59.

[18] S. HEINRICH, *Closed operator ideals and interpolation*, J. Funct. Anal. **35**, 3 (1980), 397–411.

[19] S. JANSON, *Minimal and maximal methods of interpolation*, J. Funct. Anal. **44**, 1 (1981), 50–73.

[20] I. KLUVÁNEK AND G. KNOWLES *Vector measures and Control Systems*, North-Holland, Notas Mat. **58**, Amsterdam, (1975).

[21] S. G. KREIN, JU. I. PETUNIN AND E. M. SEMENOV, *Interpolation of linear operators*, American Mathematical Society, Translations of Mathematical Monographs **54**, Providence R.I., (1982).

[22] L. MALIGRANDA, *The K-functional for p-convexifications*, Positivity **17**, 3 (2013), 707–710.

[23] C. MERUCCI, *Interpolation réele avec fonction paramètre: réitération et applications aux espaces* $\Lambda^p(\varphi)$ $(0 < p \leqslant +\infty)$, C. R. Acad. Sc. Paris **295**, 6 (1982), 427–430.

[24] P. NILSSON, *Reiteration theorems for real interpolation and approximation spaces*, Ann. Mat. Pura Appl. **32**, 4 (1982), 291–330.

[25] S. OKADA, W. J. RICKER AND E. A. SÁNCHEZ–PÉREZ, *Optimal Domain and Integral Extension of Operators Acting in Function Spaces*, Birkhäuser-Verlag, Oper. Theory Adv. Appl. **180**, Basel (2008).

[26] J. PEETRE, *A theory of interpolation of normed spaces*, Instituto de Matemática Pura e Aplicada, Conselho Nacional de Pesquisas, Notas de Matemática, **39**, Rio de Janeiro (1968).

[27] L. E. PERSSON, *Interpolation with a parameter function*, Math Scand. **59**, 2 (1986), 199–222.
[28] G. F. STEFANSSON, L_1 *of a vector measure*, Le Matematiche **48**, 2 (1993), 219–234.
[29] V. D. STEPANOV, *The weighted Hardy's inequality for nonincreasing functions*, Trans. Amer. Math. Soc. **338**, 1 (1993), 173–186.

Ricardo del Campo
Dpto. Matemática Aplicada I
Universidad de Sevilla, EUITA
Ctra. de Utrera Km. 1
41013–Sevilla, Spain
e-mail: `rcampo@us.es`

Antonio Fernández
Dpto. Matemática Aplicada II
Universidad de Sevilla
Escuela Técnica Superior de Ingenieros
Camino de los Descubrimientos, s/n
41092–Sevilla, Spain
e-mail: `afcarrion@etsi.us.es`

Antonio Manzano
Dpto. Matemáticas y Computación
Universidad de Burgos
Escuela Politécnica Superior
c/ Villadiego, s/n
09001–Burgos, Spain
e-mail: `amanzano@ubu.es`

Fernando Mayoral
Dpto. Matemática Aplicada II
Universidad de Sevilla
Escuela Técnica Superior de Ingenieros
Camino de los Descubrimientos, s/n
41092–Sevilla, Spain
e-mail: `mayoral@us.es`

Francisco Naranjo
Dpto. Matemática Aplicada II
Universidad de Sevilla
Escuela Técnica Superior de Ingenieros
Camino de los Descubrimientos, s/n
41092–Sevilla, Spain
e-mail: `naranjo@us.es`