Weighted Hardy inequalities, real interpolation methods and vector measures

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Abstract We analyze the relationship of the Ariño–Muckenhoupt weights with the K -spaces obtained when the real interpolation method defined by a parameter function is applied to the pairs $(L^1,\ L^\infty)$ and $(L^{1,\infty},\ L^\infty)$ of function spaces associated to the semivariation of a vector measure.

Keywords Weighted Hardy inequality · Real interpolation · Vector measure · Semivariation · Lorentz space

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1 Introduction

If the classical Lions–Peetre real interpolation method $(\cdot,\cdot)_{\theta,q}$ is applied to the pair (L^1, L^∞) of Lebesgue spaces with respect to a positive scalar measure the result is a Lorentz space $L^{p,q}$. Namely, we have

$$(L^1, L^\infty)_{\theta,q} = L^{p,q}, \quad 0 < \theta < 1 \le q \le \infty,\ p = \frac{1}{1-\theta}, \tag{1}$$

as we can see in [3, Theorem 5.2.1]. The same Lorentz space $L^{p,q}$ is obtained by replacing the Banach space L^1 by the quasi-Banach weak-L^1 space $L^{1,\infty}$ (see [3, Theorem 5.3.1]). Then $(X, L^\infty)_{\theta,q} = L^{p,q}$ for every quasi-Banach space X such that $L^1 \subseteq X \subseteq L^{1,\infty}$. A more general formula than (1) holds when the real interpolation method $(\cdot,\cdot)_{\rho,q}$ with a parameter function ρ is considered. The construction of this interpolation method consists in replacing the function t^θ associated to the real method by a more general function ρ that satisfies certain suitable conditions so that the main theorems of interpolation theory to be still valid. This is the case, for instance, when ρ is in the class $Q(0,1)$ introduced by Persson in [23]. It holds that

$$(L^1, L^\infty)_{\rho,q} = (L^{1,\infty}, L^\infty)_{\rho,q} = \Lambda^q_{\frac{t}{\rho(t)}},\ 1 \le q \le \infty,\ \rho \in Q(0,1) \tag{2}$$

(see [23, Proposition 6.2] and also [18, Lemma 3.1]). For the precise definition of the Lorentz space Λ^q_v see Sect. 4.

The equality (1) for spaces of scalar integrable functions with respect to a vector measure has been considered in [14] by several of the present authors. On the other hand, in [5] we extend the results given in [14] establishing interpolation formulae for different pairs of spaces associated to a vector measure and a parameter function ρ that belongs to the class $Q(0,1)$, providing in particular the corresponding version of (2) for the case of vector measures. We would like to mention that (1) has also been considered in [10] for spaces associated to a capacity. In this paper we continue the research started in [14] and [5], obtaining results that complement those ones. Now we are interested in analyzing the relationship between some conditions on the pair (ρ, q) and the K-spaces obtained by applying $(\cdot,\cdot)_{\rho,q,K}$ to the pairs $\big(L^1(\|m\|), L^\infty(m)\big)$ and $\big(L^{1,\infty}(\|m\|), L^\infty(m)\big)$, when ρ is merely a positive measurable function defined on $(0,\infty)$ (see definitions in Sects. 2 and 5). We note that for a such kind of functions the equivalence theorem may fail, unlike it happens when $\rho \in Q(0,1)$. Our approach is based on the relationship of the pair (ρ, q) with a weighted Hardy type inequality for non-increasing functions and, therefore, with the Ariño–Muckenhoupt weights (see [1] and [24]).

The paper is organized as follows. In Sect. 2 we consider the basic terminology and results on vector measures and related spaces of integrable functions. In Sect. 3 we estimate the K-functionals of the interpolation pairs $\big(L^1(\|m\|), L^\infty(m)\big)$ and $\big(L^{1,\infty}(\|m\|), L^\infty(m)\big)$. These estimates will be used later in Sect. 5. Section 4 is devoted to define and study the Lorentz type spaces with respect to the semivariation of a vector measure. Namely, we establish some conditions that ensure the quasinormality of this kind of spaces. In Sect. 5 we consider the main results of the paper concerning interpolation with a parameter function and Ariño–Muckenhoupt weights. Finally, in Sect. 6 we relate our results with those obtained previously by Gustavsson [18] and Persson [23], for a scalar measure and parameter functions belonging to special classes.

For non explicit results and terminology on interpolation see [3] and [4]. For quasi-Banach spaces and quasi-Banach lattices of measurable functions we refer to [19] and [22]. Detailed information about Lorentz type spaces defined over positive scalar measures can be found in [8].

2 Vector measures and related spaces

Let $m : \Sigma \to Y$ be a countably additive vector measure defined on a σ-algebra Σ of subsets of a nonempty set Ω with values in a Banach space Y. Denote by Y' the dual space of Y and by $B(Y)$ its unit ball. The semivariation of m is the finite set function $\|m\| : A \in \Sigma \to \|m\|(A) \in [0, \infty)$ given by

$$\|m\| (A) := \sup \left\{ \left|\langle m, y'\rangle\right| (A) : y' \in B(Y') \right\}.$$

Here $\left|\langle m, y'\rangle\right|$ denotes the variation measure of the scalar measure $\langle m, y'\rangle$ defined by $\langle m, y'\rangle (A) := \langle m(A), y'\rangle$. The semivariation is a subadditive set function that may be non additive. However, every vector measure m has a Rybakov control measure (see [12, Theorem IX.1.2]), which means that there exists $y' \in B(Y')$ such that m is absolutely continuous with respect to $\left|\langle m, y'\rangle\right|$, or equivalently, $\|m\|$ and $\left|\langle m, y'\rangle\right|$ have the same null sets (see [12, Theorem I.2.1]).

Let $L^0(m)$ be the space of all scalar measurable functions defined on Ω. As usual, two functions $f, g \in L^0(m)$ will be identified if they are equal m-a.e., that is, if $\{w \in \Omega : f(w) \neq g(w)\}$ is an $\|m\|$-null set. A function $f \in L^0(m)$ is called *weakly integrable* (with respect to m) if $f \in L^1\left(\left|\langle m, y'\rangle\right|\right)$ for all $y' \in Y'$. The space $L^1_w(m)$ of all (equivalence classes of) weakly integrable functions becomes a Banach lattice when it is endowed with the natural order m-a.e., and the norm

$$\|f\|_{L^1_w(m)} := \sup \left\{ \int_\Omega |f|\, d\left|\langle m, y'\rangle\right| : y' \in B(Y') \right\}, \qquad f \in L^1_w(m).$$

A function $f \in L^1_w(m)$ is said to be *integrable* (with respect to m) if for every A in Σ there exists an element $\int_A f dm \in Y$ (called the *integral* of f over A) such that $\left\langle \int_A f dm, y' \right\rangle = \int_A f d\langle m, y'\rangle$ for all $y' \in Y'$. The space $L^1(m)$ of all (equivalence classes of) integrable functions becomes an order continuous closed lattice ideal of $L^1_w(m)$. In general, the inclusion $L^1(m) \subseteq L^1_w(m)$ may be proper.

For $1 < p < \infty$, we consider the spaces $L^p(m)$ ($L^p_w(m)$) of power p-integrable (weakly p-integrable) functions defined as those scalar measurable functions f defined on Ω such that $|f|^p \in L^1(m)$ ($|f|^p \in L^1_w(m)$). The space $L^\infty(m)$ of all (equivalence classes of) essentially bounded functions is equipped with the essential supremum norm $\|\cdot\|_{L^\infty(m)}$. The inclusion $L^\infty(m) \subseteq L^1(m)$ holds and $\|f\|_{L^1(m)} \leq \|f\|_{L^\infty(m)} \|m\| (\Omega)$ for all $f \in L^\infty(m)$. See [16] for a detailed study of Banach space properties of these spaces. On the other hand, complex and real interpolation methods have been considered for these classes of spaces in [6,7,14,17] and more recently in [5]. As a tool for describing real interpolation spaces of spaces of power integrable functions with respect to a vector measure several of the present authors introduced in [14] the *Lorentz spaces with respect to the semivariation* that we are going to recall briefly.

For a function $f \in L^0(m)$ we consider its *distribution function* with respect to the semivariation $\|m\|$ defined by $\|m\|_f(t) := \|m\| ([|f| > t])$ for all $t \geq 0$. Here $[|f| > t]$ denotes the measurable set $\{\omega \in \Omega : |f(\omega)| > t\}$. The distribution function $\|m\|_f$ has similar properties to the distribution of a function with respect to a scalar positive measure. The *decreasing* *rearrangement* f_* of the function f with respect to the semivariation $\|m\|$ is defined for all $s > 0$ as

$$f_*(s) := \inf\left\{t > 0 : \|m\|_f(t) \le s\right\}. \tag{3}$$

Thus $f_* : s \in (0, \infty) \longrightarrow f_*(s) \in [0, \infty)$ is a non-increasing right-continuous function such that $f_*(s) = 0$ for all $s \ge \|m\|(\Omega)$, and so, we may regard f_* as a function defined only on the interval $(0, \|m\|(\Omega))$.

The Lorentz spaces with respect to the semivariation are denoted by $L^{p,q}(\|m\|)$ for all $1 \le p < \infty$ and $1 \le q \le \infty$, and consists of all $f \in L^0(m)$ for which the quantity

$$\|f\|_{L^{p,q}(\|m\|)} := \begin{cases} \left(\int_0^\infty \left[s^{\frac{1}{p}} f_*(s)\right]^q \frac{ds}{s}\right)^{\frac{1}{q}}, & \text{for } 1 \le q < \infty, \\ \sup\left\{s^{\frac{1}{p}} f_*(s), s > 0\right\}, & \text{for } q = \infty, \end{cases}$$

is finite. This expression defines a lattice quasinorm for which $L^{p,q}(\|m\|)$ is a quasi-Banach lattice with the Fatou property. Obviously, if m is a scalar positive measure these spaces are the classical Lorentz spaces $L^{p,q}$. However, for a general vector measure m these spaces may be quite different from the classical Lorentz spaces. In particular, we do not have the equality of the Lorentz space $L^p(\|m\|) := L^{p,p}(\|m\|)$ with the space $L^p(m)$ of p-integrable functions with respect to m (see [14, Example 6]). For every $1 \le p < \infty$ we have the continuous inclusions

$$L^\infty(m) \subseteq L^p(\|m\|) \subseteq L^p(m) \subseteq L^p_w(m) \subseteq L^{p,\infty}(\|m\|) \subseteq L^{1,\infty}(\|m\|).$$

All these inclusions can be proper (see [14, Example 6]). The space $L^{p,q}(\|m\|)$ is normable for every $p > 1$ and $1 \le q \le \infty$ (see [14, Corollary 14]). The space $L^1(\|m\|)$ is normed with $\|\cdot\|_{L^1(\|m\|)}$ if and only if m is a *strongly subadditive measure*, but we do not know if it is normable for every vector measure (see [15]).

Finally, note that every pair of spaces $L^p(m)$, $L^p_w(m)$ or $L^{p,q}(\|m\|)$, is an interpolation pair since all of them are continuously embedded into $L^0(m)$.

3 Estimating the *K*-functionals

In this section we obtain some estimates that relate the distribution function and the decreasing rearrangement function (with respect to $\|m\|$) and the K-functionals of the interpolation pairs $\left(L^1(\|m\|), L^\infty(m)\right)$ and $\left(L^{1,\infty}(\|m\|), L^\infty(m)\right)$. Recall that the K-functional associated to a compatible pair (X_0, X_1) of quasi-Banach spaces is defined for $x \in X_0 + X_1$ and $t > 0$ by

$$K(t, x) := K(t, x; X_0, X_1) = \inf\left\{\|x_0\|_{X_0} + t\,\|x_1\|_{X_1} : x = x_0 + x_1, x_0 \in X_1, x_1 \in X_1\right\}.$$

Without explicit mention we shall use the following facts about the K-functional associated to these interpolation pairs:

(i) $K(t, |f|) = K(t, f)$ for f in $X_0 + X_1$ and $t > 0$, and
(ii) there exists $C > 0$ such that for each $f \in X_0 + X_1$ and $t > 0$ we can obtain $A \in \Sigma$ that satisfies $\|f\chi_A\|_{X_0} + t\,\left\|f\chi_{\Omega\setminus A}\right\|_{X_1} \le CK(t, f)$. The set $A := [|f_0| \ge |f_1|]$ gives this inequality with $C = 2$ if $f_0 \in X_0$ and $f_1 \in X_1$ satisfy that $f = f_0 + f_1$ and $\|f_0\|_{X_0} + \|f_1\|_{X_1} \le 2K(t, f)$.

In what follows to estimate the K-functional $K(t, f)$ we assume without loss of generality that $f \ge 0$.

As usual, for two non-negative functions F and G, $F \preceq G$ means that there exists $c > 0$ such that $F \leq c\, G$; $F \succeq G$ means that $G \preceq F$ and $F \approx G$ means simultaneously that $F \preceq G$ and $F \succeq G$.

Proposition 1 *Let f be a function in $L^{1,\infty}(\|m\|)$. Then*

$$\sup\left\{s \min\{t, \|m\|_f(s)\}, s > 0\right\} \preceq K\left(t, f; L^{1,\infty}(\|m\|), L^\infty(m)\right)$$

for every $t > 0$. In particular, $t f_*(t) \preceq K\left(t, f; L^{1,\infty}(\|m\|), L^\infty(m)\right)$ for every $t > 0$.

Proof The same proof as in [14, Proposition 8] works here because $\|\chi_A\|_{L^{1,\infty}(\|m\|)} = \|\chi_A\|_{L^1(m)} = \|m\|(A)$ for every A in Σ. □

Proposition 2 *Let f be a function in $L^1(\|m\|)$. Then*

$$K\left(t, f; L^1(\|m\|), L^\infty(m)\right) = \int_0^\infty \min\{t, \|m\|_f(s)\}\, ds = \int_0^t f_*(s)\, ds$$

for every $t > 0$.

Proof See [14, Lemma 3] for the second equality. For the first one, let us prove first that $K\left(t, f; L^1(\|m\|), L^\infty(m)\right) \leq \int_0^\infty \min\{t, \|m\|_f(s)\}\, ds$. Put $s_0 := f_*(t) \geq 0$. If $s_0 = 0$, then $\|m\|_f(s) < t$ for every $s > 0$, and consequently

$$\int_0^\infty \min\{t, \|m\|_f(s)\}\, ds = \int_0^\infty \|m\|_f(s)\, ds = \|f\|_{L^1(\|m\|)} \geq K\left(t, f; L^1(\|m\|), L^\infty(m)\right).$$

Now assume that $s_0 > 0$. Then

$$\int_0^\infty \min\{t, \|m\|_f(s)\}\, ds = \int_0^{s_0} t\, ds + \int_{s_0}^\infty \|m\|_f(s)\, ds = t\, s_0 + \int_0^\infty \|m\|_f(s_0 + u)\, du$$

since $\|m\|_f(s) \geq t$ for every $s < s_0$, and $\|m\|_f(s) \leq \|m\|_f(s_0) \leq t$ for every $s \geq s_0$. Let us consider $B := [f > s_0]$ and decompose $f = f_0 + f_1$ where $f_0 := (f - s_0)\chi_B$ and $f_1 := f - f_0 = s_0\, \chi_B + f\, \chi_{\Omega \setminus B}$. Then $\|f_1\|_{L^\infty(m)} \leq s_0$ and

$$\|f_0\|_{L^1(\|m\|)} = \int_0^\infty \|m\|_{f_0}(u)\, du \leq \int_0^\infty \|m\|_f(s_0 + u)\, du.$$

This gives the desired inequality

$$K\left(t, f; L^1(\|m\|), L^\infty(m)\right) \leq \|f_0\|_{L^1(\|m\|)} + t\, \|f_1\|_{L^\infty(m)} \leq \int_0^\infty \min\{t, \|m\|_f(s)\}\, ds.$$

For the reverse inequality $\int_0^\infty \min\{t, \|m\|_f(s)\}\, ds \leq K\left(t, f; L^1(\|m\|), L^\infty(m)\right)$ let us consider an arbitrary decomposition $f = f_0 + f_1$, with f_0 in $L^1(\|m\|)$ and f_1 in $L^\infty(m)$. Let us denote $s_1 := \|f_1\|_{L^\infty(m)} \geq 0$. For every $s > s_1$ we have $\|m\|_f(s) \leq \|m\|_{f_0}(s - s_1)$ since $[f = f_0 + f_1 > s] \subseteq \left[f_0 > s - \|f_1\|_{L^\infty(m)}\right]$. Then

$$\begin{aligned}\int_0^\infty \min\{t, \|m\|_f(s)\}\, ds &\leq \int_0^{s_1} t\, ds + \int_{s_1}^\infty \|m\|_f(s)\, ds \\ &\leq t\, \|f_1\|_{L^\infty(m)} + \int_{s_1}^\infty \|m\|_{f_0}(s - s_1)\, ds\end{aligned}$$

$$= t\,\|f_1\|_{L^\infty(m)} + \int_0^\infty \|m\|_{f_0}(u)\,du$$
$$= \|f_0\|_{L^1(\|m\|)} + t\,\|f_1\|_{L^\infty(m)}.$$

Taking the infimum over all possible representations $f = f_0 + f_1$ we obtain the desired inequality. □

Corollary 1 *Let X be a quasi-Banach space with* $L^1(\|m\|) \subseteq X \subseteq L^{1,\infty}(\|m\|)$. *Then*

(a) $K(t, f; X, L^\infty(m)) \succeq t\, f_*(t)$, *for every* $f \in X$ *and* $t > 0$.

(b) $K(t, f; X, L^\infty(m)) \preceq \int_0^t f_*(s)\,ds$, *for every* $f \in L^1(\|m\|)$ *and* $t > 0$.

4 Lorentz type spaces associated to a vector measure

In a similar way to the scalar measure case, we introduce the Lorentz spaces associated to (the semivariation of) a vector measure m by using the decreasing rearrangement defined in (3).

Definition 1 For an index $1 \le q \le \infty$ and a *weight* v [a non-negative measurable function on $(0, \infty)$ that is not identically zero] denote by $\Lambda_v^q(\|m\|)$ the set of functions f in $L^0(m)$ such that

$$\|f\|_{\Lambda_v^q(\|m\|)} := \begin{cases} \left(\int_0^\infty (v(t) f_*(t))^q \frac{dt}{t}\right)^{\frac{1}{q}}, & \text{for } 1 \le q < \infty, \\ \operatorname{ess\,sup}\{v(t) f_*(t), t > 0\}, & \text{for } q = \infty, \end{cases}$$

is finite.

For $v(t) = t^{\frac{1}{p}}$ we obtain the Lorentz spaces $L^{p,q}(\|m\|)$ previously considered, and $\|f\|_{\Lambda_v^q(\|m\|)} = \|f\|_{L^{p,q}(\|m\|)}$.

Now we are going to study the *quasinormability* of $\Lambda_v^q(\|m\|)$. It is obvious from the definition that $\|\cdot\|_{\Lambda_v^q(\|m\|)}$ is homogeneous. Then it is equivalent to a quasinorm if and only if itself is a quasinorm and, in that case, $\Lambda_v^q(\|m\|)$ will be necessarily a linear space. For a characterization of weights v for which $\Lambda_v^q(dt)$ is a linear space see [11].

Our first step is to describe $\|f\|_{\Lambda_v^q(\|m\|)}$ for a function $f \in \Lambda_v^q(\|m\|)$. In order to do that we denote by v_q the function defined by $v_q(t) := \frac{v(t)^q}{t}$ for a weight v and $1 \le q < \infty$. Also we will denote by $V_q(t) := \int_0^t v_q(s)\,ds$, for all $t > 0$. Note that a function f is in $\Lambda_v^q(\|m\|)$ if and only if its decreasing rearrangement f_* is in the weighted Lebesgue L^q-space $L^q\left(v_q(t)dt\right)$ and $\|f\|_{\Lambda_v^q(\|m\|)} = \|f_*\|_{L^q(v_q(t)dt)}$. Analogously f is in $\Lambda_v^\infty(\|m\|)$ if and only if f_* belongs to the weighted L^∞-space $L^\infty\,(v(t)dt)$ also with the equality $\|f\|_{\Lambda_v^\infty(\|m\|)} = \|f_*\|_{L^\infty(v(t)dt)}$.

The following lemma is the analogous of [8, Lemma 2.2.4] and [9, Theorem 2.1] for vector measures. It will be useful to characterize the quasinormability of the Lorentz type spaces.

Lemma 1 *Let* $1 \le q < \infty$ *and* $v \ge 0$ *a measurable function on* $(0, \infty)$. *Then*

$$\int_0^\infty f_*(t)^q\, v(t)\,dt = \int_0^\infty q\, t^{q-1} \left[\int_0^{\|m\|_f(t)} v(s)\,ds\right] dt \qquad (4)$$

for every $f \in L^0(m)$. In particular, we have the following equality

$$\|f\|_{\Lambda_v^q(\|m\|)} = \left(q \int_0^\infty t^{q-1} V_q \left(\|m\|_f(t) \right) dt \right)^{\frac{1}{q}}.$$

Proof Recall the every non-negative measurable function can be obtained as the m-a.e. pointwise limit of an increasing sequence of non-negative measurable simple functions. Thus, by applying the Monotone Convergence Theorem it is enough to obtain the desired equality (4) only for a non-negative measurable simple function. Then, let φ be a such function on Ω and write it as $\varphi := \sum_{k=1}^{N+1} a_k \chi_{A_k}$, where $a_1 > \cdots > a_N > a_{N+1} := 0$ and $\{A_1, \ldots, A_N, A_{N+1}\}$ is a measurable partition of Ω with $\|m\|(A_k) > 0$ for $1 \leq k \leq N$. A direct computation gives that $\|m\|_\varphi = \sum_{j=1}^{N} \alpha_j \chi_{[a_{j+1}, a_j)}$ and $\varphi_* = \sum_{k=1}^{N} a_k \chi_{[\alpha_{k-1}, \alpha_k)}$, where $\alpha_0 := 0$ and $\alpha_j := \|m\| \left(\bigcup_{i=1}^{j} A_i \right)$, for $j = 1, \ldots, N$. Then, we have the equality

$$\begin{aligned}
\int_0^\infty q\, t^{q-1} \left[\int_0^{\|m\|_\varphi(t)} v(s)\, ds \right] dt &= \int_0^{a_1} q\, t^{q-1} \left[\int_0^{\|m\|_\varphi(t)} v(s)\, ds \right] dt \\
&= \sum_{j=1}^{N} \int_{a_{j+1}}^{a_j} q\, t^{q-1} \left[\int_0^{\alpha_j} v(s)\, ds \right] dt \\
&= \sum_{j=1}^{N} \left[a_j^q - a_{j+1}^q \right] \left[\int_0^{\alpha_j} v(s)\, ds \right] \\
&= \sum_{k=1}^{N} a_k^q \left[\int_{\alpha_{k-1}}^{\alpha_k} v(s)\, ds \right] \\
&= \int_0^{\alpha_N} \varphi_*(t)^q v(t)\, dt = \int_0^\infty \varphi_*(t)^q v(t)\, dt.
\end{aligned}$$

□

To get a result like Lemma 1 but for $q = \infty$, we introduce the function V_∞ defined by $V_\infty(r) := \operatorname{ess\,sup} \{v(t), 0 < t \leq r\}$ for $r > 0$, and $V_\infty(0) := 0$. Note that V_q is a non-decreasing function for every $1 \leq q \leq \infty$, and therefore it has at most countably many discontinuities. Moreover the following property will be useful in the proof of the next result.

Lemma 2 *V_∞ is a left-continuous function.*

Proof Since V_∞ is a non-decreasing function, for every $r > 0$ there exists the limit $L := \lim_{\varepsilon \to 0^+} V_\infty(r - \varepsilon) \leq V_\infty(r)$. Then, it is enough to prove that $\{t \in (0, r) : v(t) > L\}$ is a null set. But for a certain $k \in \mathbb{N}$ we have the equality

$$\{t \in (0, r) : v(t) > L\} = \bigcup_{n \geq k} \left\{ t \in \left(0, r - \frac{1}{n}\right] : v(t) > L \right\}$$

and the Lebesgue measure of each set $\left\{ t \in \left(0, r - \frac{1}{n}\right] : v(t) > L \right\}$, where $n \geq k$, is zero since $V_\infty \left(r - \frac{1}{n} \right) \leq L$.

□

Lemma 3 *Let f be a measurable function on Ω. Then*

(a) $\|f\|_{\Lambda^{\infty}_{V_\infty}(\|m\|)} = \|f\|_{\Lambda^{\infty}_{v}(\|m\|)}$.
(b) $\sup\left\{s\, V_\infty\left(\|m\|_f(s)\right), s > 0\right\} = \sup\left\{V_\infty(t)\, f_*(t), t > 0\right\}$.

Proof (a) Obviously $\|f\|_{\Lambda^{\infty}_{v}(\|m\|)} \le \|f\|_{\Lambda^{\infty}_{V_\infty}(\|m\|)}$ since $v \le V_\infty$. Reciprocally, if f belongs to $\Lambda^{\infty}_{v}(\|m\|)$, then $V_\infty(t) f_*(t) \le \operatorname{ess\,sup}\{v(s) f_*(s), 0 < s \le t\} \le \|f\|_{\Lambda^{\infty}_{v}(\|m\|)} < \infty$, for every $t > 0$, and therefore $\|f\|_{\Lambda^{\infty}_{v}(\|m\|)} \ge \|f\|_{\Lambda^{\infty}_{V_\infty}(\|m\|)}$.

(b) To obtain $s\, V_\infty\left(\|m\|_f(s)\right) \le \sup\{V_\infty(t)\, f_*(t), t > 0\}$ for every $s > 0$ it is enough to consider the points $s > 0$ such that $\|m\|_f(s) > 0$. In this case, for every positive $\varepsilon < \|m\|_f(s)$ we have $0 < \|m\|_f(s) - \varepsilon < \|m\|_f(s)$ and the definition of f_* gives us $f_*\left(\|m\|_f(s) - \varepsilon\right) \ge s$. Then

$$\sup\{V_\infty(t)\, f_*(t), t > 0\} \ge V_\infty(\|m\|_f(s) - \varepsilon)\, f_*(\|m\|_f(s) - \varepsilon) \ge V_\infty(\|m\|_f(s) - \varepsilon)s.$$

Taking limit as $\varepsilon \to 0^+$ and using Lemma 2, we have the claimed inequality. To establish the reverse inequality, $\sup\left\{s\, V_\infty\left(\|m\|_f(s)\right), s > 0\right\} \ge V_\infty(t)\, f_*(t)$ for every $t > 0$, it suffices to consider the points $t > 0$ for which $f_*(t) > 0$. In this case, if $0 < \varepsilon < f_*(t)$, we have $0 < f_*(t) - \varepsilon < f_*(t)$ and therefore $t < \|m\|_f\left(f_*(t) - \varepsilon\right)$. Then, since V_∞ is non-decreasing, we obtain

$$\sup\left\{s\, V_\infty\left(\|m\|_f(s)\right), s > 0\right\} \ge (f_*(t) - \varepsilon)\, V_\infty\left(\|m\|_f\left(f_*(t) - \varepsilon\right)\right) \ge (f_*(t) - \varepsilon)\, V_\infty(t).$$

Taking limit as $\varepsilon \to 0^+$ we get the claimed inequality.

□

The following result characterizes the quasinormability of $\Lambda^q_v(\|m\|)$ by means of the behavior of V_q on the range of $\|m\|$. See [8, Lemma 2.2.10] for the scalar measure case with $1 \le q < \infty$.

Proposition 3 *Suppose $1 \le q \le \infty$. Then $\Lambda^q_v(\|m\|)$ is a quasinormed space if and only if there exists $C > 0$ such that*

$$0 < V_q\left(\|m\|(A \cup B)\right) \le C\left[V_q(\|m\|(A)) + V_q(\|m\|(B))\right] \tag{5}$$

for every pair $A, B \in \Sigma$ such that $\|m\|(A \cup B) > 0$.

Proof Let us assume first that $\|\cdot\|_{\Lambda^q_v(\|m\|)}$ is a quasinorm. Then, for some $D > 0$, we have $\|f + g\|_{\Lambda^q_v(\|m\|)} \le D\left(\|f\|_{\Lambda^q_v(\|m\|)} + \|g\|_{\Lambda^q_v(\|m\|)}\right)$ for every $f, g \in \Lambda^q_v(\|m\|)$. Applying this inequality to the characteristic functions of two measurable subsets A and B of Ω such that $\|m\|(A \cup B) > 0$, we obtain

$$\|\chi_{A\cup B}\|_{\Lambda^q_v(\|m\|)} \le \|\chi_A + \chi_B\|_{\Lambda^q_v(\|m\|)} \le D\left(\|\chi_A\|_{\Lambda^q_v(\|m\|)} + \|\chi_B\|_{\Lambda^q_v(\|m\|)}\right).$$

Then, condition (5) follows from the fact that for every E in Σ,

$$\|\chi_E\|_{\Lambda^q_v(\|m\|)} = \begin{cases} V_q(\|m\|(E))^{\frac{1}{q}} & \text{for } 1 \le q < \infty, \\ V_\infty(\|m\|(E)) & \text{for } q = \infty. \end{cases}$$

For the converse, we only need to prove that

$$\|f + g\|_{\Lambda^q_v(\|m\|)} \le D\left(\|f\|_{\Lambda^q_v(\|m\|)} + \|g\|_{\Lambda^q_v(\|m\|)}\right),$$

for some constant $D > 0$ and every $f, g \in \Lambda_v^q(\|m\|)$. To do this we will apply Lemmas 1 and 3 together with the well-known inequality $\|m\|_{f+g}(t) \leq \|m\|_f\left(\frac{t}{2}\right) + \|m\|_g\left(\frac{t}{2}\right)$ for every $t > 0$. Let us denote $h := f + g$. Now we consider two cases:

(i) For $1 \leq q < \infty$, condition (5) and Lemma 1 give

$$\begin{aligned}
\|h\|^q_{\Lambda_v^q(\|m\|)} &= \int_0^\infty h_*(t)^q v_q(t)\, dt = \int_0^\infty q\, t^{q-1}\left[\int_0^{\|m\|_h(t)} v_q(s)\, ds\right] dt \\
&\leq \int_0^\infty q\, t^{q-1} V_q\left(\|m\|_f\left(\frac{t}{2}\right) + \|m\|_g\left(\frac{t}{2}\right)\right) dt \\
&\leq C\int_0^\infty q\, t^{q-1}\left[V_q\left(\|m\|_f\left(\frac{t}{2}\right)\right) + V_q\left(\|m\|_g\left(\frac{t}{2}\right)\right)\right] dt \\
&= C\, 2^q \int_0^\infty q\, s^{q-1}\left[V_q\left(\|m\|_f(s)\right) + V_q\left(\|m\|_g(s)\right)\right] dt \\
&= 2^q C\left(\|f\|^q_{\Lambda_v^q(\|m\|)} + \|g\|^q_{\Lambda_v^q(\|m\|)}\right).
\end{aligned}$$

Hence, $\|f + g\|_{\Lambda_v^q(\|m\|)} \leq 2\, C^{\frac{1}{q}}\left(\|f\|_{\Lambda_v^q(\|m\|)} + \|g\|_{\Lambda_v^q(\|m\|)}\right)$.

(ii) For $q = \infty$, condition (5) and Lemma 3 give

$$\begin{aligned}
\|h\|_{\Lambda_v^\infty(\|m\|)} = \|h\|_{\Lambda^\infty_{V_\infty}(\|m\|)} &\leq \operatorname*{ess\,sup}_{s>0} s\ V_\infty\left(\|m\|_f\left(\frac{s}{2}\right) + \|m\|_g\left(\frac{s}{2}\right)\right) \\
&\leq C \operatorname*{ess\,sup}_{s>0} s\left[V_\infty\left(\|m\|_f\left(\frac{s}{2}\right)\right) + V_\infty\left(\|m\|_f\left(\frac{s}{2}\right)\right)\right] \\
&\leq 2\, C\left(\|f\|_{\Lambda^\infty_{V_\infty}(\|m\|)} + \|g\|_{\Lambda^\infty_{V_\infty}(\|m\|)}\right).
\end{aligned}$$

□

In order to give another sufficient condition for the quasinormability of $\Lambda_v^q(\|m\|)$ we consider the following definition.

Definition 2 It is said that a function $V \geq 0$ satisfies the Δ_2-condition on the interval $[0, L]$ if there exits $C > 0$ such that $V(2t) \leq C V(t)$ for every $0 < t \leq \frac{L}{2}$.

Remark 1 A non-negative non-decreasing function V satisfies the Δ_2-condition on the interval $[0, L]$ if and only if there exists $C > 0$ such that

$$V(s + t) \leq C\,(V(s) + V(t)), \tag{6}$$

for every $s > 0$ and $t > 0$ with $s + t \leq L$. Indeed, If V satisfies the Δ_2-condition on $[0, L]$ with constant C and s and t are given in $(0, L]$ such that $\max\{s, t\} \leq \frac{L}{2}$ then $V(s + t) \leq V(2\max\{s, t\}) \leq C\, V(\max\{s, t\}) \leq C\,(V(s) + V(t))$. If $s + t \leq L$, but now $\max\{s, t\} > \frac{L}{2}$, we also have

$$V(s + t) \leq V(L) \leq C\, V\left(\frac{L}{2}\right) \leq C\, V(\max\{s, t\}) \leq C\,(V(s) + V(t)).$$

For the reverse implication it is enough to take $s = t$ in (6).

Corollary 2 *Let* $1 \leq q \leq \infty$. *If* V_q *satisfies the* Δ_2-*condition on* $[0, \|m\|(\Omega)]$, *then* $\Lambda_v^q(\|m\|)$ *is a quasinormed space.*

Proof Let us assume that $V_q(s+t) \le C\left(V_q(s)+V_q(t)\right)$ for every s, t in $(0, \|m\|(\Omega)]$ with $s+t \le \|m\|(\Omega)$. For $A, B \in \Sigma$ arbitrary, we distinguish two cases.

If $\|m\|(A)+\|m\|(B) \le \|m\|(\Omega)$ we obtain

$$V_q\left(\|m\|(A\cup B)\right) \le V_q\left(\|m\|(A)+\|m\|(B)\right) \le C\left[V_q\left(\|m\|(A)\right)+V_q\left(\|m\|(B)\right)\right].$$

If $\|m\|(A)+\|m\|(B) > \|m\|(\Omega)$, we may assume that $v_q(t)=0$ for every $t \ge \|m\|(\Omega)$ and therefore $V_q\left(\|m\|(A)+\|m\|(B)\right) = V_q\left(\|m\|(\Omega)\right)$. Moreover, we may assume, in this case, that $2\|m\|(A) > \|m\|(\Omega)$. Then

$$\begin{aligned} V_q\left(\|m\|(A\cup B)\right) &\le V_q\left(\|m\|(\Omega)\right) \le C\, V_q\left(\frac{1}{2}\|m\|(\Omega)\right) \le C\, V_q\left(\|m\|(A)\right) \\ &\le C\left[V_q\left(\|m\|(A)\right)+V_q\left(\|m\|(B)\right)\right]. \end{aligned}$$

□

Remark 2 In the context of a positive non-atomic scalar measure μ it is easy to prove that V_q satisfies the Δ_2-condition on $[0, \mu(\Omega)]$ if and only if $\Lambda_v^q(\mu)$ is a quasinormed space. We do not know if this result remains true for a non-atomic vector measure.

5 Interpolation with a parameter function and Ariño–Muckenhoupt weights

From now on we mean by a *parameter function* ρ a positive measurable function defined on the interval $(0, \infty)$. For a parameter function ρ we denote by v the weight $v(t) := \dfrac{t}{\rho(t)}$ defined on $(0, \infty)$. We are going to study the pairs (ρ, q) for which $\left(L^1(\|m\|), L^\infty(m)\right)_{\rho,q,K} = \left(L^{1,\infty}(\|m\|), L^\infty(m)\right)_{\rho,q,K}$ holds. In these cases, we characterize such space as a Lorentz type space $\Lambda_v^q(\|m\|)$. To do it, no special assumptions is done on ρ, so our results also include some trivial cases. Let us recall briefly the construction of the space $(\cdot,\cdot)_{\rho,q,K}$ associated with the function lattice $\Phi(\rho, q)$. This function lattice consists of all measurable functions g on $(0, \infty)$ such that

$$\|g\|_{\Phi(\rho,q)} := \begin{cases} \left(\displaystyle\int_0^\infty \left[\frac{|g(t)|}{\rho(t)}\right]^q \frac{dt}{t}\right)^{\frac{1}{q}}, & \text{for } 1 \le q < \infty, \\ \operatorname*{ess\,sup}_{t>0} \dfrac{|g(t)|}{\rho(t)}, & \text{for } q = \infty, \end{cases}$$

is finite. For a given compatible pair (X_0, X_1) of quasi-Banach spaces $(X_0, X_1)_{\rho,q,K}$ denotes the set of all elements $x \in X_0 + X_1$ such that $K(\cdot, x) \in \Phi(\rho, q)$. In that case, we put $\|x\|_{\rho,q,K} := \|K(\cdot, x)\|_{\Phi(\rho,q)}$. The classical Lions–Peetre real interpolation space $(X_0, X_1)_{\theta,q}$ is obtained for the particular parameter function $\rho(t) := t^\theta$, with $0 < \theta < 1$.

We note that $(X_0, X_1)_{\rho,q,K}$ is an intermediate space with respect to (X_0, X_1) if and only if $\min\{1, t\}$ belongs to $\Phi(\rho, q)$. In fact, if we assume that $\min\{1, t\}$ belongs to $\Phi(\rho, q)$ it follows that $(X_0, X_1)_{\rho,q,K} \subseteq X_0 + X_1$ since $\min\{1, t\}K(1, x) \le K(t, x)$ for $t > 0$ (see (1) in [3, page 38]). Moreover, $K(t, x) \le \min\{1, t\}\|x\|_{X_0\cap X_1}$ for $t > 0$ (see (2) in [3, page 42]), and so $X_0 \cap X_1 \subseteq (X_0, X_1)_{\rho,q,K}$. On the other hand, if $(X_0, X_1)_{\rho,q,K}$ is an intermediate space, in particular, $X_0 \cap X_1 \subseteq (X_0, X_1)_{\rho,q,K}$. Thus taking into account that, for any $0 \ne x \in (X_0, X_1)_{\rho,q,K} \subseteq X_0 + X_1$, it holds that $\min\{1, t\}K(1, x) \le$

$K(t,x)$, it follows that $\min\{1,t\} \in \Phi(\rho,q)$. Moreover, $(X_0,X_1)_{\rho,q,K}$ is an interpolation space of the pair (X_0,X_1) when the condition $\min\{1,t\} \in \Phi(\rho,q)$ is fulfilled (see [4, Proposition 3.3.1]).

We start by proving the following inclusion.

Lemma 4 *It holds that* $\left(L^{1,\infty}(\|m\|), L^{\infty}(m)\right)_{\rho,q,K} \subseteq \Lambda_v^q(\|m\|)$ *for every* $1 \le q \le \infty$. *Moreover, if* $\Lambda_v^q(\|m\|)$ *is a quasinormed space the above inclusion is continuous.*

Proof From Proposition 1 we have

$$v(t) f_*(t) = \frac{t f_*(t)}{\rho(t)} \preceq \frac{K\left(t, f; L^{1,\infty}(\|m\|), L^{\infty}(m)\right)}{\rho(t)}.$$

Then for every f in $\left(L^{1,\infty}(\|m\|), L^{\infty}(m)\right)_{\rho,q,K}$ we obtain that f is in $\Lambda_v^q(\|m\|)$, and also $\|f\|_{\Lambda_v^q(\|m\|)} \preceq \|f\|_{\rho,q,K}$. □

The following theorem is the key result of this section. In the proof we make use of the weighted Hardy inequality (7) for non-increasing functions g and, in particular, the characterization of those weights w allowed in the inequality

$$\int_0^{\infty} \left[\frac{1}{t}\int_0^t g(s)\,ds\right]^q w(t)\,dt \le C\int_0^{\infty} g(t)^q w(t)\,dt. \tag{7}$$

If $1 \le q < \infty$, the weights w for which (7) is true for all non-increasing functions g are exactly those weights in the Ariño–Muckenhoupt class B_q, that is, weights w on $(0,\infty)$ such that there exists $C > 0$ for which

$$\int_r^{\infty} \frac{w(t)}{t^q}\,dt \le \frac{C}{r^q}\int_0^r w(t)\,dt, \tag{8}$$

for all $r > 0$ (see [1, Theorem 1.7] and [24]).

We continue with the notation of Sect. 4 and consider in what follows the functions v_q for $1 \le q < \infty$, and V_q for $1 \le q \le \infty$, associated to $v(t) := \dfrac{t}{\rho(t)}$.

Theorem 1 *If* $v_q \in B_q$, *then* $\left(L^{1}(\|m\|), L^{\infty}(m)\right)_{\rho,q,K} = \Lambda_v^q(\|m\|)$ *for all* $1 \le q < \infty$.

Proof First note that $\Lambda_v^q(\|m\|) \subseteq L^1(\|m\|)$ if $v_q \in B_q$. Indeed, if f is in $\Lambda_v^q(\|m\|)$ we have

$$\int_0^{\infty} \left[\frac{1}{t}\int_0^t f_*(s)\,ds\right]^q v_q(t)\,dt \le C\int_0^{\infty} f_*(t)^q v_q(t)\,dt = C\,\|f\|^q_{\Lambda_v^q(\|m\|)} < \infty.$$

Thus, the integrand of the left hand side $\left[\frac{1}{t}\int_0^t f_*(s)\,ds\right]^q v_q(t)$ is finite $a.e.$ and

$$\|f\|_{L^1(\|m\|)} = \int_0^{\infty} f_*(s)\,ds = \int_0^{\|m\|(\Omega)} f_*(s)\,ds < \infty.$$

We only need to prove that $\Lambda_v^q(\|m\|) \subseteq \left(L^{1}(\|m\|), L^{\infty}(m)\right)_{\rho,q,K}$. Let $f \ge 0$ be in $\Lambda_v^q(\|m\|)$ and denote by $K(t,f)$ the K-functional associated to the pair $\left(L^{1}(\|m\|), L^{\infty}(m)\right)$. Taking into account that $f_* \ge 0$ is a non-increasing function and that $v_q \in B_q$, the quasinorm $\|f\|_{\rho,q,K}$ of f in $(L^1(\|m\|), L^{\infty}(m))_{\rho,q,K}$ can be estimated as follows, having in mind Proposition 2 and the Hardy inequality (7)

$$\begin{aligned}\|f\|^q_{\rho,q,K} &= \int_0^\infty \left[\frac{K(t,f)}{\rho(t)}\right]^q \frac{dt}{t} = \int_0^\infty \left[\frac{1}{\rho(t)}\int_0^t f_*(s)\,ds\right]^q \frac{dt}{t}\\ &= \int_0^\infty \left[\frac{1}{t}\int_0^t f_*(s)\,ds\right]^q \frac{t^{q-1}}{\rho(t)^q}\,dt \preceq \int_0^\infty f_*(t)^q \frac{t^{q-1}}{\rho(t)^q}\,dt\\ &= \|f\|^q_{\Lambda^q_v(\|m\|)} < \infty.\end{aligned}$$

Therefore $\Lambda^q_v(\|m\|) \subseteq \left(L^1(\|m\|), L^\infty(m)\right)_{\rho,q,K}$. □

Remark 3 (a) Note that Theorem 1 holds, in particular, if $m = \mu$ is a finite positive scalar measure (in which case $\|m\| = \mu$). Our proof also works for the pair $\left(L^1(\mu), L^\infty(\mu)\right)$, even for $\left(L^{1,\infty}(\mu), L^\infty(\mu)\right)$, if μ is merely a σ-finite measure.

(b) If $v_q \in B_q$, then V_q satisfies the Δ_2-condition on the interval $[0, \|m\|(\Omega)]$, and therefore $\|\cdot\|_{\Lambda^q_v(\|m\|)}$ is a quasinorm as we have seen in Corollary 2. In addition, if $\min\{1,t\} \in \Phi(\rho,q)$ we have that $\Lambda^q_v(\|m\|)$ is an interpolation space, and otherwise we obtain the null space. Note that the function $\min\{1,t\}$ belongs to $\Phi(\rho,q)$ for $1 \le q < \infty$ if, and only if,

$$\begin{aligned}\int_0^\infty \left(\frac{\min\{1,t\}}{\rho(t)}\right)^q \frac{dt}{t} &= \int_0^1 \left[\frac{t}{\rho(t)}\right]^q \frac{dt}{t} + \int_1^\infty \left[\frac{1}{\rho(t)}\right]^q \frac{dt}{t}\\ &= \int_0^1 v_q(t)\,dt + \int_1^\infty \frac{v_q(t)}{t^q}\,dt < \infty. \qquad (9)\end{aligned}$$

Having in mind the above expression (9) for the condition $\min\{1,t\} \in \Phi(\rho,q)$, and Theorem 1 it follows directly the following corollary.

Corollary 3 *Let X be a quasi-Banach space with $L^1(\|m\|) \subseteq X \subseteq L^{1,\infty}(\|m\|)$ and let $1 \le q < \infty$. If $v_q \in B_q \cap L^1(0,1)$, then $(X, L^\infty(m))_{\rho,q,K} = \Lambda^q_v(\|m\|)$.*

Since $L^1(m)$ is an intermediate Banach space between $L^1(\|m\|)$ and $L^{1,\infty}(\|m\|)$, the above Corollary 3 says in particular that $\Lambda^q_v(\|m\|)$ is an interpolation space of the pair of Banach spaces $\left(L^1(m), L^\infty(m)\right)$ and therefore it is a normable space for the corresponding interpolation norm which is, in fact, equivalent to the functional $\|\cdot\|_{\Lambda^q_v(\|m\|)}$.

Now, we are going to consider a weak version of the reciprocal of Theorem 1. Note that in its proof we only need to consider non-increasing functions supported on the finite interval $(0, \|m\|(\Omega))$. Then, the condition $v_q \in B_q$ does not seem to be necessary. Nevertheless, we can say some thing about the function v_q when the measure m is *non-atomic*. Recall that a set $A \in \Sigma$ is called an *atom* of the vector measure m if $m(A) \ne 0$ and if $B \subseteq A$, $B \in \Sigma$ implies that either $m(B) = 0$ or $m(A \backslash B) = 0$, and a measure without atoms is called *non-atomic* (see [21, page 32]). By a standard set-theoretic argument it can be proved that m is non-atomic if and only if its semivarition $\|m\|$ is non-atomic. Then, we can see that the semivariation has the *Darboux property* (see [13, Theorem 10] or [20, Corollary 3]), which means that range of the semivariation is the closed interval $[0, \|m\|(\Omega)]$, that is,

$$\{\|m\|(A) : A \in \Sigma\} = [0, \|m\|(\Omega)]. \qquad (10)$$

Proposition 4 *Let m be a non-atomic vector measure and $1 \le q < \infty$. If $\Lambda^q_v(\|m\|)$ is a quasinormed space and $\left(L^1(\|m\|), L^\infty(m)\right)_{\rho,q,K} = \Lambda^q_v(\|m\|)$ algebraically, then $v_q \chi_{(0,T)} \in B_q$ for each $0 < T < \infty$.*

Proof It is standard to prove that $\Lambda_v^q(\|m\|)$ is complete if $\|\cdot\|_{\Lambda_v^q(\|m\|)}$ is a quasinorm (see [8, Theorem 2.3.1] for the scalar case), that is, under our assumption $\Lambda_v^q(\|m\|)$ is a quasi-Banach space. Then, by the hypothesis and Lemma 4 the open mapping theorem assures that $\|\cdot\|_{\Lambda_v^q(\|m\|)}$ and $\|\cdot\|_{\rho,q,K}$ are equivalent quasinorms. Therefore there exists $C > 0$ such that $\|f\|_{\rho,q,K} \leq C\|f\|_{\Lambda_v^q(\|m\|)}$ for every $f \in \Lambda_v^q(\|m\|)$. Equivalently,

$$\int_0^\infty \left[\frac{1}{\rho(t)}\int_0^t f_*(s)\,ds\right]^q \frac{dt}{t} \leq C^q \int_0^\infty \left[\frac{t}{\rho(t)} f_*(t)\right]^q \frac{dt}{t}.$$

Since m is non-atomic, by using (10), for every $r \in (0, \|m\|(\Omega)]$ there exists $A \in \Sigma$ such that $\|m\|(A) = r$. Having in mind that $(\chi_A)_* = \chi_{(0,\|m\|(A))}$ the above inequality read as

$$\int_0^\infty \left[\frac{1}{\rho(t)}\int_0^t \chi_{(0,r)}(s)\,ds\right]^q \frac{dt}{t} \leq C^q \int_0^\infty \left[\frac{t}{\rho(t)} \chi_{(0,r)}(t)\right]^q \frac{dt}{t},$$

for every $r \in (0, \|m\|(\Omega)]$. Then, we obtain for those r the following inequality

$$\begin{aligned}\int_0^\infty \left[\frac{1}{\rho(t)}\int_0^t \chi_{(0,r)}(s)\,ds\right]^q \frac{dt}{t} &= \int_0^r \left[\frac{t}{\rho(t)}\right]^q \frac{dt}{t} + \int_r^\infty \left[\frac{r}{\rho(t)}\right]^q \frac{dt}{t} \\ &\leq C^q \int_0^r \left[\frac{t}{\rho(t)}\right]^q \frac{dt}{t},\end{aligned}$$

and therefore

$$\int_r^\infty \frac{v_q(t)}{t^q}\,dt \leq \frac{C^q - 1}{r^q}\int_0^r v_q(t)\,dt \tag{11}$$

for every $r \in (0, \|m\|(\Omega)]$. Recall that in order to prove that $v_q\chi_{(0,T)}$ belongs to B_q for every $0 < T < \infty$ we have to check (8). If $T \leq \|m\|(\Omega)$ it follows directly from (11). Suppose now that $\|m\|(\Omega) < T$ and $\|m\|(\Omega) < r$. By using (11) again we obtain

$$\begin{aligned}\int_r^\infty \frac{v_q(t)}{t^q}\chi_{(0,T)}(t)dt &\leq \int_{\|m\|(\Omega)}^\infty \frac{v_q(t)}{t^q}\chi_{(0,T)}(t)dt \\ &\leq \frac{C^q - 1}{(\|m\|(\Omega))^q}\int_0^{\|m\|(\Omega)} v_q(t)\chi_{(0,T)}(t)dt \\ &\leq \frac{C^q - 1}{(\|m\|(\Omega))^q}\left[\frac{T}{r}\right]^q \int_0^r v_q(t)\chi_{(0,T)}(t)dt,\end{aligned}$$

and $v_q\chi_{(0,T)} \in B_q$. □

For the case $q = \infty$ we have the following result analogous to Theorem 1.

Theorem 2 *If* $\int_0^t \frac{\rho(s)}{s}\,ds \leq C\,\rho(t)$ *for some constant* $C > 0$ *and all* $t > 0$, *then*

$$\left(L^1(\|m\|), L^\infty(m)\right)_{\rho,\infty,K} = \Lambda_v^\infty(\|m\|). \tag{12}$$

Proof We only need to prove that $\Lambda_v^\infty(\|m\|) \subseteq \left(L^1(\|m\|), L^\infty(m)\right)_{\rho,\infty,K}$. The other inclusion is already proved in Lemma 4. Now, if $f \in \Lambda_v^\infty(\|m\|)$ we have $tf_*(t) \leq D\,\rho(t)$ for some constant $D > 0$ and all $t > 0$. Therefore

$$K\left(t, f; L^1(\|m\|), L^\infty(m)\right) = \int_0^t f_*(s)\,ds \leq D\int_0^t \frac{\rho(s)}{s}\,ds \leq D\,C\,\rho(t),$$

for all $t > 0$. Thus $f \in \left(L^1(\|m\|), L^\infty(m)\right)_{\rho,\infty,K}$. □

Note that this theorem includes some trivial cases. For example, it is easy to prove that $\Lambda_v^\infty(\|m\|)$ is the null space for $v(t) := t^{1-\alpha}$, with $\alpha > 1$, in which case, $\rho(t) = t^\alpha$. Recall that $(X_0, X_1)_{\rho,\infty,K}$ is an interpolation space for every pair (X_0, X_1) if $\min\{1, t\} \in \Phi(\rho, \infty)$. Otherwise we obtain the null space. If $\Lambda_v^\infty(\|m\|)$ is a non-trivial quasinormed space, then we have the equality (12) as quasi-Banach spaces.

Remark 4 Under the hypothesis of the above theorem, the following equality

$$\left(X, L^\infty(m)\right)_{\rho,\infty,K} = \Lambda_v^\infty(\|m\|)$$

holds for every quasi-Banach space X such that $L^1(\|m\|) \subseteq X \subseteq L^{1,\infty}(\|m\|)$. If moreover $t \le D\,\rho(t)$ for some constant $D > 0$ and all $t \in (0, 1)$, the function $\min\{1, t\} \in \Phi(\rho, \infty)$. Take into account that the hypothesis of Theorem 2 implies that $\int_0^1 \frac{\rho(s)}{s}\, ds \le C\,\rho(t)$ for all $t > 1$. Consequently, under this additional assumption, $\Lambda_v^\infty(\|m\|)$ is an interpolation space of each pair $(X, L^\infty(m))$. In particular, $\Lambda_v^\infty(\|m\|)$ is normable with the norm of the space $\left(L^1(m), L^\infty(m)\right)_{\rho,\infty,K}$.

6 Examples

Next let us relate our results with the corresponding results for the scalar measure case given by Gustavsson [18] and Persson [23]. These authors study the interpolation with a parameter function belonging to certain classes of functions, such as B_ψ (see [18, Definition 1.2]), $Q(0, 1)$ or $\mathcal{P}^{+-}$ (see [23, pages 201–202]). These classes can be considered (in some sense) the same class ([23, Proposition 1.3]). Next we recall the definition of the class $Q(0, 1)$.

Definition 3 A parameter function ρ is said to be in the class $Q(0, 1)$ if there exist $0 < \alpha < \beta < 1$ such that $\rho(t)\,t^{-\alpha}$ is non-decreasing and $\rho(t)\,t^{-\beta}$ is non-increasing.

Example 1 Note that the function $\rho(t) := t^\theta\,(1 + |\log t|)^\gamma$ belongs to the class B_ψ whenever $0 < \theta < 1$ and $|\gamma| < \min\{\theta, 1 - \theta\}$. On the other hand, recall that $B_\psi \subseteq Q(0, 1)$ (see [23, Proposition 1.3]). For this function ρ and a positive σ-finite scalar measure μ the Lorentz space $\Lambda_v^q(\mu)$, associated to $v(t) := \frac{t}{\rho(t)}$, is the Lorentz–Zygmund space $L^{p,q}(\log L)^\gamma(\mu)$ for $\frac{1}{p} := 1 - \theta$ (see [2] and [23, page 218]).

We are ready to analyze the role of each one of the conditions involved in the definition of the class $Q(0, 1)$ in connection with the Ariño–Muckenhoupt classes B_q considered above.

Proposition 5 *If $\rho(t)\,t^{-\alpha}$ is non-decreasing for some $\alpha > 0$, then:*

(a) $v_q \in B_q$, *for each* $1 \le q < \infty$.
(b) $\int_0^t \frac{\rho(s)}{s}\, ds \le \frac{1}{\alpha}\rho(t)$, *for every* $t > 0$.

Proof (a) For $0 < r \le t$, we have $\rho(r)\,r^{-\alpha} \le \rho(t)\,t^{-\alpha}$ by the hypothesis. Then $\frac{1}{\rho(t)} \le \frac{r^\alpha}{\rho(r)\,t^\alpha}$. Hence

$$\int_r^\infty \frac{v_q(t)}{t^q} dt = \int_r^\infty \frac{1}{t\,\rho(t)^q} dt \le \frac{r^{\alpha q}}{\rho(r)^q} \int_r^\infty \frac{1}{t^{1+\alpha q}} dt = \frac{1}{q\,\alpha\,\rho(r)^q}. \tag{13}$$

For $0 < t < r$, we have $\rho(t)\,t^{-\alpha} \le \rho(r)\,r^{-\alpha}$ and therefore $\dfrac{1}{\rho(t)} \ge \dfrac{r^{\alpha}}{\rho(r)\,t^{\alpha}}$. Hence

$$\int_0^r v_q(t)\,dt = \int_0^r \frac{t^{q-1}}{\rho(t)^q}dt \ge \frac{r^{\alpha q}}{\rho(r)^q}\int_0^r t^{q(1-\alpha)-1}dt = \frac{r^q}{q(1-\alpha)\rho(r)^q}. \tag{14}$$

Then from (13) and (14) we get $\displaystyle\int_r^{\infty} \frac{v_q(t)}{t^q}dt \le \frac{1-\alpha}{\alpha}\frac{1}{r^q}\int_0^r v_q(t)\,dt.$

(b) Using that $\rho(s)\,s^{-\alpha}$ is non-decreasing we have

$$\int_0^t \frac{\rho(s)}{s}\,ds = \int_0^t \frac{\rho(s)}{s}\,ds \le \int_0^t \frac{\rho(t)}{t^{\alpha}}\frac{s^{\alpha}}{s}\,ds = \frac{1}{\alpha}\,\rho(t),$$

for every $t > 0$.

□

Next corollary follows directly from Proposition 5 and Theorems 1 and 2.

Corollary 4 *If $\rho(t)\,t^{-\alpha}$ is non-decreasing for some $\alpha > 0$, then*

$$\left(L^1(\|m\|), L^{\infty}(m)\right)_{\rho,q,K} = \Lambda_v^q(\|m\|),$$

with equivalence of quasinorms, for all $1 \le q \le \infty$.

Proposition 6 *If $\rho(t)\,t^{-\beta}$ is non-increasing for some $\beta < 1$, then:*

(a) *$v_q \in L^1(0,T)$ for every $0 < T < \infty$ and $1 \le q < \infty$.*
(b) *$\rho(1)\,t \le \rho(t)$, for all $0 < t < 1$.*

Proof (a) Since $\rho(t)\,t^{-\beta}$ is non-increasing, we have $\dfrac{\rho(t)}{t^{\beta}} \ge \dfrac{\rho(T)}{T^{\beta}}$ for every $0 < t < T$, and therefore

$$\int_0^T v_q(t)\,dt = \int_0^T \frac{t^{q-1}}{\rho(t)^q}\,dt \le \frac{T^{q\beta}}{\rho(T)^q}\int_0^T \frac{t^{q-1}}{t^{q\beta}}\,dt = \frac{1}{q(1-\beta)}\frac{T^q}{\rho(T)^q} < \infty.$$

(b) Suppose $0 < t < 1$. Then $\rho(1)\,t \le \rho(t)\,t^{-\beta}t \le \rho(t)$.

□

The following result follows directly from Propositions 5 and 6 and condition (9).

Corollary 5 *If $\rho \in Q(0,1)$ then:*

(a) *$v_q \in B_q \cap L^1(0,1)$, for each $1 \le q < \infty$.*
(b) *There exits a constant $C > 0$ such that $t \le C\,\rho(t)$ for all $0 < t < 1$, and $\displaystyle\int_0^t \frac{\rho(s)}{s}\,ds \le C\,\rho(t)$ for all $t > 0$.*
(c) *$\min\{1,t\} \in \Phi(\rho,q)$, for each $1 \le q \le \infty$.*

Finally, we summarize the relationship between our Corollary 3 and Remark 4 with the results obtained by Gustavsson [18, Lemma 3.1] and Persson [23, Proposition 6.2]: If X is a quasi-Banach space with $L^1(\|m\|) \subseteq X \subseteq L^{1,\infty}(\|m\|)$ and $\rho \in Q(0,1)$, then the equality $(X, L^{\infty}(m))_{\rho,q,K} = \Lambda_v^q(\|m\|)$ holds for all $1 \le q \le \infty$, with equivalence of quasinorms. In particular, $\Lambda_v^q(\|m\|)$ is an interpolation space of the pair $(L^1(m), L^{\infty}(m))$ and so it is normable with the norm of the space $\left(L^1(m), L^{\infty}(m)\right)_{\rho,q,K}$.

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