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A note on real interpolation of L^p-spaces of vector measures on δ-rings ☆

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ABSTRACT

We describe the real interpolation spaces obtained when we apply the real K-method of Lions–Peetre to Banach lattices of p-integrable and weakly p-integrable functions with respect to a Banach-space-valued measure defined on a δ-ring. In general, the obtained results are quite different from those in the case of vector measures on σ-algebras described in [9]. However, we find a wide class of vector measures on δ-rings for which the results on σ-algebras hold true.

1. Introduction

A basic problem in interpolation theory is to describe the spaces obtained by applying an interpolation method to concrete compatible couples of spaces. For a Banach-space-valued measure m defined on a σ-algebra, we obtained in [11] the Calderón interpolation spaces $[X_0, X_1]_{[\theta]}$ and $[X_0, X_1]^{[\theta]}$, and in [9] the real interpolation spaces $(X_0, X_1)_{\theta,q}$ of the couples (X_0, X_1), where X_0 and X_1 are the Banach lattices $L^p(m)$ or $L^p_w(m)$ of equivalence classes of scalar p-integrable or, respectively, weakly p-integrable functions with respect to the measure m. Later we investigated in [5] the Calderón interpolation methods of the same spaces, but for measures defined on δ-rings. We showed in [5] that the interpolation results for vector measures on δ-rings can be very different from those on the context of σ-algebras. However, we identified

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a certain type of vector measures on δ-rings (called *locally strongly additive* measures) which keep completely the same behavior (for all the different combinations of couples) as measures defined on σ-algebras. In the present note we complete the picture with the study of real interpolation methods of Banach lattices of p-integrable and weakly p-integrable functions with respect to a Banach-space-valued measure defined on a δ-ring. As in the case of complex methods we can say that certain interpolation equalities for vector measures on σ-algebras described in [9] remain true for vector measures on δ-rings, but some others cease to be true for vector measures on δ-rings. Curiously, for the same type of measures (locally strongly additive measures) real interpolation equalities in the setting of measures defined on σ-algebra remain true for measures on δ-rings. However, the reasons why this happens are very different from those on the context of complex interpolation methods.

2. Preliminaries

In this section we establish the preliminaries necessaries about integration of scalar functions with respect to vector measures on δ-rings, in order to make the paper more self-contained and readable. The basic references about integration for us will be [7,12–14]. Throughout this paper we will consider a vector measure $\nu : \mathcal{R} \to X$ defined on a δ-ring $\mathcal{R}$ of subsets of some nonempty set Ω with values in a real Banach space X, with dual X'. We denote by $\mathcal{R}^{\mathrm{loc}}$ the σ-algebra of subsets $A \subseteq \Omega$ such that $A \cap B \in \mathcal{R}$ for each $B \in \mathcal{R}$. Measurability of functions $f : \Omega \to \mathbb{R}$ will be considered with respect to the measurable space $(\Omega, \mathcal{R}^{\mathrm{loc}})$. The *semivariation* of ν is the set function $\|\nu\| : \mathcal{R}^{\mathrm{loc}} \to [0,\infty]$ defined by $\|\nu\|(A) := \sup\{|\langle \nu, x'\rangle|(A) : \|x'\|_{X'} \le 1\}$, where $|\langle \nu, x'\rangle|$ is the variation of the scalar measure

$$\langle \nu, x'\rangle : A \in \mathcal{R} \to \langle \nu, x'\rangle(A) := \langle \nu(A), x'\rangle \in \mathbb{R}.$$

A set $N \in \mathcal{R}^{\mathrm{loc}}$ is called *ν-null* if $\|\nu\|(N) = 0$. A property holds *ν-almost everywhere* (ν-a.e.) if it holds except on a ν-null set.

A measurable function $f : \Omega \to \mathbb{R}$ is called *weakly integrable* (with respect to ν) if $f \in L^1(\langle \nu, x'\rangle)$ for all $x' \in X'$. A weakly integrable function f is said to be *integrable* (with respect to ν) if, for each $A \in \mathcal{R}^{\mathrm{loc}}$ there exists an element (necessarily unique) $\int_A f d\nu \in X$, satisfying

$$\left\langle \int_A f d\nu, x' \right\rangle = \int_A f d\langle \nu, x'\rangle, \quad x' \in X'.$$

If $1 \le p < \infty$, a measurable function $f : \Omega \to \mathbb{R}$ is called *weakly p-integrable* (with respect to ν) if $|f|^p$ is weakly integrable and *p-integrable* (with respect to ν) if $|f|^p$ is integrable. The space $L^p_w(\nu)$ of all (ν-a.e. equivalence classes of) weakly p-integrable functions becomes a Banach lattice with the *Fatou property* when endowed with the usual ν-a.e. pointwise order and the norm

$$\|f\|_{L^p_w(\nu)} := \sup\left\{ \left(\int_\Omega |f|^p \, d\left|\langle \nu, x'\rangle\right| \right)^{\frac{1}{p}} : \left\|x'\right\|_{X'} \le 1 \right\}.$$

Moreover, the space $L^p(\nu)$ of all (ν-a.e. equivalence classes of) p-integrable functions is a closed *order continuous* ideal of $L^p_w(\nu)$. In fact, it is the closure of $\mathcal{S}(\mathcal{R})$, the space of simple functions supported on $\mathcal{R}$. The Banach lattices $L^p(\nu)$ and $L^p_w(\nu)$ of equivalence classes of scalar p-integrable and weakly p-integrable functions were initially studied in [10] for vector measures ν on a σ-algebra and its basic properties can be extended and remain true for vector measures on δ-rings (see [4]). Also we can find in [15, Chapter 3] a very good material about spaces of integrable functions with respect to a vector measure on a σ-algebra. Finally, let us consider two more spaces strongly related with the spaces of p-integrable functions with respect to a vector measure. Denote by $L^\infty(\nu)$ the space of classes of essentially bounded measurable functions $f : \Omega \to \mathbb{R}$ with the essential supremum norm. Consider also the vector space $L^0(\nu)$ of all classes of measurable functions $f : \Omega \to \mathbb{R}$. If the vector measure ν is defined on a σ-algebra it is well-known (see [10, Corollary 3.2]) that the following inclusions hold for all $p > 1$

$$L^\infty(\nu) \subseteq L^p(\nu) \subseteq L^p_w(\nu) \subseteq L^1(\nu) \subseteq L^1_w(\nu) \subseteq L^0(\nu), \tag{1}$$

and all of them are continuous inclusions, where the topology of *convergence in measure* is considered on $L^0(\nu)$. When the vector measure ν is defined on a δ-ring instead of a σ-algebra, the inclusions (1) are in general false, but we can save something (see, for example, Proposition 2.2 and Remark 3.3).

In what follows we will always consider vector measures $\nu : \mathcal{R} \to X$ which are σ-finite, that is, there exist a pairwise disjoint sequence $(\Omega_k)_k$ in $\mathcal{R}$, and a ν-null set $N \in \mathcal{R}^{\mathrm{loc}}$, such that $\Omega = (\bigcup_{k\geq 1} \Omega_k) \cup N$. The simplest example of a σ-finite vector measure on a δ-ring is given by the Lebesgue measure λ defined on the δ-ring $\mathcal{R} := \{A \in \mathcal{M} : \lambda(A) < \infty\}$, where $\mathcal{M}$ is the σ-algebra of all Lebesgue measurable subsets of the real line $\mathbb{R}$. If we consider the vector measure $\nu : A \in \mathcal{R} \to \nu(A) = \lambda(A) \in \mathbb{R}$, then $L^p_w(\nu) = L^p(\nu) = L^p(\mathbb{R})$ for all $p \geq 1$.

In the context of interpolation it is well-known that we need a topological vector space as an environment space in order to consider *couples of Banach spaces*. In our case it is the linear space $L^0(\nu)$, endowed with the topology of convergence in measure on each subset Ω_k. This topology is generated by the F-norm $\|\cdot\|_{L^0(\nu)}$ that we shall now describe. For each $k = 1, 2, \dots$ consider the σ-algebra $\Sigma_k := \{A \in \mathcal{R} : A \subseteq \Omega_k\}$ of subsets of Ω_k and the vector measure $\nu_k : A \in \Sigma_k \to \nu_k(A) = \nu(A) \in X$, that is, the restriction of ν to Σ_k. Now define

$$\|f\|_{L^0(\nu)} := \sum_{k=1}^{\infty} \frac{1}{2^k(1 + \|\nu\|(\Omega_k))} \left\| \frac{|f|}{1+|f|} \chi_{\Omega_k} \right\|_{L^1_w(\nu_k)}, \quad f \in L^0(\nu).$$

For details see [5, Lemmas 3.2, 3.3 and 3.4]. In particular, let us mention that each pair of spaces $L^p_w(\nu)$ or $L^p(\nu)$ forms a *compatible couple of Banach spaces*, that is, they are imbedded continuously in the same topological vector space $L^0(\nu)$.

Given $f \in L^0(\nu)$, we shall consider its *distribution function* (with respect to the vector measure ν) defined by

$$\|\nu\|_f : s \in [0, \infty) \to \|\nu\|_f(s) := \|\nu\|\big(\{w \in \Omega : |f(w)| > s\}\big) \in [0, \infty],$$

where $\|\nu\|$ is the semivariation of the measure ν. This distribution function has similar properties as in the scalar case (see [9]). For instance, $\|\nu\|_f$ is non-increasing and right-continuous. The *decreasing rearrangement* of f (with respect to the measure ν) is given by

$$f_* : t \in (0, \infty) \to f_*(t) := \inf\{s > 0 : \|\nu\|_f(s) \leq t\} \in [0, \infty].$$

Some properties of f_* can be found in [9] when the measure is defined on a σ-algebra. Nevertheless, it is not difficult to see (even for measures on δ-rings) that f_* is a non-increasing, right-continuous function. Moreover the following two equalities hold

$$\int_0^\infty \|\nu\|_f(t)\,dt = \int_0^\infty f_*(t)\,dt \quad \text{and} \quad \sup_{t>0} t\|\nu\|_f(t) = \sup_{t>0} t f_*(t).$$

For $1 \leq p, q \leq \infty$ the Lorentz space $L^{p,q}(\|\nu\|)$ with respect to the vector measure ν consists of all functions $f \in L^0(\nu)$ for which the quantity

$$\|f\|_{L^{p,q}(\|\nu\|)} := \begin{cases} (\int_0^\infty (s^{\frac{1}{p}} f_*(s))^q \frac{ds}{s})^{\frac{1}{q}} & (1 \le q < \infty) \\ \sup_{s>0} s^{\frac{1}{p}} f_*(s) & (q = \infty) \end{cases}$$

is finite. The functional $f \mapsto \|f\|_{L^{p,q}(\|\nu\|)}$ is not always a norm, even when $p, q \ge 1$, because the triangle inequality fails. Nevertheless it is not difficult to prove that $\|f+g\|_{L^{p,q}(\|\nu\|)} \le C(\|f\|_{L^{p,q}(\|\nu\|)} + \|g\|_{L^{p,q}(\|\nu\|)})$, where $C \ge 1$ is a constant depending on p and q. Therefore $\|\cdot\|_{L^{p,q}(\|\nu\|)}$ is only a *quasi-norm*. We also note that $L^{p,q}(\|\nu\|)$ is a quasi-Banach lattice with the Fatou property. For the special case $p = q$, we denote the space $L^{p,p}(\|\nu\|)$ simply by $L^p(\|\nu\|)$. As it has been pointed out in [9], in general, the spaces $L^p(\|\nu\|)$ and $L^p(\nu)$ do not coincide. For $p > 1$ and $1 \le q \le \infty$ the Lorentz spaces $L^{p,q}(\|\nu\|)$ are intermediate spaces of the couple $(L^1(\|\nu\|), L^\infty(\nu))$, that is, $L^1(\|\nu\|) \cap L^\infty(\nu) \subseteq L^{p,q}(\|\nu\|) \subseteq L^1(\|\nu\|) + L^\infty(\nu)$. Moreover, if the measure ν is defined on a σ-algebra, it holds the following inclusions, for all $1 \le p < \infty$ (see [9, Proposition 7])

$$L^\infty(\nu) \subseteq L^{p,1}(\|\nu\|) \subseteq L^p(\|\nu\|) \subseteq L^p(\nu) \subseteq L^p_w(\nu) \subseteq L^{p,\infty}(\|\nu\|), \tag{2}$$

and all these inclusions are continuous. However, if the vector measure ν is defined on a δ-ring instead of a σ-algebra, the inclusion $L^p(\|\nu\|) \subseteq L^p(\nu)$ is in general false as Example 2.1 below points out. The inclusion $L^\infty(\nu) \subseteq L^{p,1}(\|\nu\|)$ is false even for a non-finite positive scalar measure. On the contrary the others inclusions of the chain (2) remain true as we shall see with the following Proposition 2.2 (see also Remark 3.3).

Example 2.1. Let $\mathcal{R}$ be the δ-ring of finite subsets of natural numbers $\mathbb{N}$, and consider the σ-finite vector measure $\nu : A \in \mathcal{R} \to \nu(A) := \chi_A \in c_0$, where c_0 is the space of null sequences. For every $1 \le p < \infty$, it is easy to check that $L^p_w(\nu) = \ell^\infty$, the space of bounded sequences, and $L^p(\nu) = c_0$. In what follows it will be interesting to note that $\|\nu\|(A) = 1$, for every nonempty $A \subseteq \mathbb{N}$, and $\|\nu\|(\varnothing) = 0$. This means, in particular, that $\|\nu\|_f = \chi_{[0,\infty)}$ if f is an unbounded sequence, but $\|\nu\|_f = \chi_{[0,\|f\|_\infty)}$ if $f \in \ell^\infty$. Consequently, we have for an unbounded sequence f that $f_*(t) = \infty$ if $t \in (0,1)$ and $f_*(t) = 0$ if $t \ge 1$. On the other hand, $f_* = \|f\|_\infty \chi_{(0,1)}$ if $f \in \ell^\infty$. Thus $L^1(\|\nu\|) = \ell^\infty = L^1_w(\nu)$, and $L^1(\|\nu\|) \nsubseteq L^1(\nu)$.

Proposition 2.2. *The following continuous inclusions hold*

$$L^1(\|\nu\|) \subseteq L^1_w(\nu) \subseteq L^{1,\infty}(\|\nu\|) \subseteq L^0(\nu). \tag{3}$$

Proof. First we check the inclusion $L^1(\|\nu\|) \subseteq L^1_w(\nu)$. Take $f \in L^1(\|\nu\|)$, and choose any $x' \in X'$, with $\|x'\| \le 1$. For the positive σ-additive measure $|\langle \nu, x' \rangle|$ we have (see [2, Proposition II.1.8] for the first equality)

$$\int_\Omega |f|\, d\big|\langle \nu, x' \rangle\big| = \int_0^\infty \big|\langle \nu, x' \rangle\big|_f(t)dt \le \int_0^\infty \|\nu\|_f(t)dt = \|f\|_{L^1(\|\nu\|)}. \tag{4}$$

Taking supremum in (4) when $\|x'\| \le 1$, we obtain $\|f\|_{L^1_w(\nu)} \le \|f\|_{L^1(\|\nu\|)}$.

Now we prove second inclusion $L^1_w(\nu) \subseteq L^{1,\infty}(\|\nu\|)$. Take $f \in L^1_w(\nu)$ and let $t > 0$. Then $t\chi_{\{w \in \Omega : |f(w)| > t\}} \le |f|$, and so $t\chi_{\{w \in \Omega : |f(w)| > t\}} \in L^1_w(\nu)$. Moreover

$$t\|\nu\|_f(t) = \|t\chi_{\{w \in \Omega : |f(w)| > t\}}\|_{L^1_w(\nu)} \le \|f\|_{L^1_w(\nu)}. \tag{5}$$

Taking supremum in (5) we obtain $\|f\|_{L^{1,\infty}(\nu)} := \sup_{t>0} t\|\nu\|_f(t) \le \|f\|_{L^1_w(\nu)}$.

The continuity of the last inclusion $L^{1,\infty}(\|\nu\|) \subseteq L^0(\nu)$ (and also all other inclusions in the paper) follows from [1, Theorem 16.6]. □

3. Real interpolation results for measures on δ-rings

Let us recall briefly the construction of the *real interpolation method of Lions–Peetre*. Let (A_0, A_1) be a quasi-Banach couple, that is, two quasi-Banach spaces A_0, A_1 which are continuously embedded in some Hausdorff topological vector space. The Peetre K-functional is defined, for $t > 0$ and $f \in A_0 + A_1$, by

$$K(t, f; A_0, A_1) := \inf\big\{\|f_0\|_{A_0} + t\|f_1\|_{A_1} : f = f_0 + f_1,\ f_0 \in A_0,\ f_1 \in A_1\big\}.$$

For $0 < \theta < 1$ and $1 \le q \le \infty$, the space $(A_0, A_1)_{\theta,q}$ is formed by all those elements $f \in A_0 + A_1$ such that the quasi-norm

$$\|f\|_{(A_0,A_1)_{\theta,q}} := \begin{cases} (\int_0^\infty (t^{-\theta} K(t, f; A_0, A_1))^q \frac{dt}{t})^{\frac{1}{q}}, & \text{if } 1 \le q < \infty, \\ \sup_{t>0} t^{-\theta} K(t, f; A_0, A_1), & \text{if } q = \infty, \end{cases}$$

is finite. One of the main results in [9] is Corollary 17 which assures, for $0 < \theta < 1 \le q \le \infty$, $1 \le p_0 \ne p_1 \le \infty$, and a vector measure ν defined on a σ-algebra that

$$\big(L^{p_0}(\nu), L^{p_1}(\nu)\big)_{\theta,q} = \big(L^{p_0}_w(\nu), L^{p_1}_w(\nu)\big)_{\theta,q} = L^{p,q}\big(\|\nu\|\big), \tag{6}$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. As Example 2.1 shows, the above equalities are not longer true if the measure ν is defined on a δ-ring. That is, for such a measure $(L^{p_0}(\nu), L^{p_1}(\nu))_{\theta,q} = c_0$ but $(L^{p_0}_w(\nu), L^{p_1}_w(\nu))_{\theta,q} = \ell^\infty$. Nevertheless, there are cases where the situation is similar to the case of σ-algebras, described in (6), even for measures genuinely defined on δ-rings. There is a broad class of vector measures for which this occurs: *locally strongly additive vector measures*. Recall that a vector measure $\nu : \mathcal{R} \to X$ is called locally strongly additive if $\lim_{n\to\infty} \|\nu(A_n)\|_X = 0$ for all disjoint sequences $(A_n)_n$ in $\mathcal{R}$ such that $\|\nu\|(\bigcup_{n\ge1} A_n) < \infty$. Note that the vector measure we have considered in Example 2.1 is not locally strongly additive. In what follows we continue with a σ-finite vector measure $\nu : \mathcal{R} \to X$. Locally strongly additive vector measures were characterized in [5] in the following form.

Lemma 3.1. *(See Lemma 4.1 in [5].) The following conditions are equivalent:*

A) The measure ν is locally strongly additive.
B) If $B \in \mathcal{R}^{\mathrm{loc}}$ and $\chi_B \in L^1_w(\nu)$, then $\chi_B \in L^1(\nu)$.

Now we add some more equivalent conditions to that characterization.

Proposition 3.2. *The following conditions are equivalent:*

A) The measure ν is locally strongly additive.
C) If $B \in \mathcal{R}^{\mathrm{loc}}$ and $\chi_B \in L^1(\|\nu\|)$, then $\chi_B \in L^1(\nu)$.
D) $L^\infty(\nu) \cap L^1(\|\nu\|) \subseteq L^\infty(\nu) \cap L^1(\nu)$.
E) $L^1(\|\nu\|) \subseteq L^1(\nu)$.

Proof. A) $\Leftrightarrow$ C). For a set $B \in \mathcal{R}^{\mathrm{loc}}$ note that $\chi_B \in L^1(\|\nu\|)$ if and only if $\chi_B \in L^1_w(\nu)$, because $\|\chi_B\|_{L^1(\|\nu\|)} = \|\chi_B\|_{L^1_w(\nu)} = \|\nu\|(B)$. Then, the equivalence A) $\Leftrightarrow$ C) follows from Lemma 3.1.

C) $\Rightarrow$ D). Take $f \in L^\infty(\nu) \cap L^1(\|\nu\|)$. For every $k \ge 1$ consider the subsets $B_k := \{w \in \Omega : \frac{1}{k} \le |f(w)|\} \in \mathcal{R}^{\mathrm{loc}}$. Note that $\frac{1}{k}\chi_{B_k} \le |f|$. Thus $\chi_{B_k} \in L^1(\|\nu\|)$, and by the hypothesis $\chi_{B_k} \in L^1(\nu)$. Consider for all $k \ge 1$ the functions $g_k := f\chi_{B_k}$, and note that $g_k \le \|f\|_{L^\infty(\nu)}\chi_{B_k}$. Thus $g_k \in L^1(\nu)$ for all $k \ge 1$ and moreover (taking into account Proposition 2.2) we get

$$\|f-g_k\|_{L^1_w(\nu)} \le \|f-g_k\|_{L^1(\|\nu\|)} = \int_0^\infty \|\nu\|_{f-g_k}(t)dt \le \int_0^{\frac{1}{k}} \|\nu\|_f(t)dt \to 0,$$

as $k \to \infty$ because $f \in L^1(\|\nu\|)$. Then $f \in L^1(\nu)$ since $L^1(\nu) \subseteq L^1_w(\nu)$ is closed.

D) $\Rightarrow$ E). Take $0 \le f \in L^1(\|\nu\|)$. For every $n \ge 1$ consider the functions $f_n := \min\{f, n\} \in L^\infty(\nu)$. Note that $f_n \in L^\infty(\nu) \cap L^1(\|\nu\|)$, and so $f_n \in L^1(\nu)$ for all $n \ge 1$. Again taking into account Proposition 2.2 we get

$$\begin{aligned}\|f-f_n\|_{L^1_w(\nu)} \le \|f-f_n\|_{L^1(\|\nu\|)} &= \int_0^\infty \|\nu\|_{f-f_n}(t)dt \\ &= \int_0^n \|\nu\|_{f-f_n}(t)dt + \int_n^\infty \|\nu\|_{f-f_n}(t)dt \\ &\le \int_0^n \|\nu\|_f(n)dt + \int_n^\infty \|\nu\|_f(t)dt \\ &= n\|\nu\|_f(n) + \int_n^\infty \|\nu\|_f(t)dt \to 0,\end{aligned}$$

as $n \to \infty$ because $f \in L^1(\|\nu\|)$. Then $f \in L^1(\nu)$ since $L^1(\nu) \subseteq L^1_w(\nu)$ is closed.

The implication E) $\Rightarrow$ C) is obvious. □

Remark 3.3. Let $\nu : \mathcal{R} \to X$ be a locally strongly additive σ-finite vector measure. Then, for all $1 \le p < \infty$, we have the following continuous inclusions

$$L^{p,1}(\|\nu\|) \subseteq L^p(\|\nu\|) \subseteq L^p(\nu) \subseteq L^p_w(\nu) \subseteq L^{p,\infty}(\|\nu\|). \tag{7}$$

The chain of inclusions $L^{p,1}(\|\nu\|) \subseteq L^p(\|\nu\|) \subseteq L^{p,\infty}(\|\nu\|)$ is similar to the case of a positive scalar measure (see [2, Proposition IV.4.2]). The rest of the inclusions in (7) follow from the equivalence E) of Proposition 3.2 and also the continuous inclusions (3) in Proposition 2.2, by noting that $L^p(\|\nu\|) = \{f \in L^0(\nu) : |f|^p \in L^1(\|\nu\|)\}$.

Remark 3.4. Lewis proved in [12, Theorem 5.1] the equivalence of the following assertions:

i) The Banach space X has no subspace isomorphic to c_0.
ii) $L^1(\nu) = L^1_w(\nu)$ for every X-valued vector measure ν defined on a δ-ring.

Thus Lemma 3.1 tells us that every σ-finite measure $\nu : \mathcal{R} \to X$ is locally strongly additive if the Banach space X has no subspace isomorphic to c_0. This result is a sort of Diestel–Faires theorem for measures on δ-rings (see [8, Theorem I.4.2]).

In what follows we need some estimates for the K-functional that will be useful to establish our interpolation results. These estimates can be obtained following the same techniques used in [9] with minor modifications (see [9, Lemma 3 and Propositions 8 and 10] for details). Let us also mention that similar estimates were obtained independently by Cerdà, Martín and Silvestre in [6] for capacities. As usual, in what follows $a \preccurlyeq b$ means that $a \le cb$ for some positive constant c independent of the quantities a and b.

Proposition 3.5. *For a σ-finite vector measure $\nu : \mathcal{R} \to X$ the following estimates for the K-functional hold:*

i) *If $f \in L^1(\|\nu\|) + L^\infty(\nu)$, then $K(t, f; L^1(\|\nu\|), L^\infty(\nu)) \preccurlyeq \int_0^t f_*(s)ds$.*
ii) *If $f \in L^{1,\infty}(\|\nu\|) + L^\infty(\nu)$, then $tf_*(t) \preccurlyeq K(t, f; L^{1,\infty}(\|\nu\|), L^\infty(\nu))$.*

Theorem 3.6. *Let $\nu : \mathcal{R} \to X$ be a σ-finite vector measure, and $0 < \theta < 1 \le q \le \infty$. Then $(L^1(\|\nu\|), L^\infty(\nu))_{\theta,q} = (L^{1,\infty}(\|\nu\|), L^\infty(\nu))_{\theta,q} = L^{\frac{1}{1-\theta},q}(\|\nu\|)$.*

Proof. The inclusion $(L^1(\|\nu\|), L^\infty(\nu))_{\theta,q} \subseteq (L^{1,\infty}(\|\nu\|), L^\infty(\nu))_{\theta,q}$ follows from the inclusion $L^1(\|\nu\|) \subseteq L^{1,\infty}(\|\nu\|)$, and since this last inclusion is continuous we have the inequality

$$\|f\|_{(L^{1,\infty}(\|\nu\|),L^\infty(\nu))_{\theta,q}} \preccurlyeq \|f\|_{(L^1(\|\nu\|),L^\infty(\nu))_{\theta,q}}, \quad f \in \big(L^1\big(\|\nu\|\big), L^\infty(\nu)\big)_{\theta,q}. \tag{8}$$

We have also the inclusion $(L^{1,\infty}(\|\nu\|), L^\infty(\nu))_{\theta,q} \subseteq L^{\frac{1}{1-\theta},q}(\|\nu\|)$ as a consequence of the inequality *ii)* in Proposition 3.5. In particular, we obtain

$$\|f\|_{L^{\frac{1}{1-\theta},q}(\|\nu\|)} \preccurlyeq \|f\|_{(L^{1,\infty}(\|\nu\|),L^\infty(\nu))_{\theta,q}}, \quad f \in \big(L^{1,\infty}\big(\|\nu\|\big), L^\infty(\nu)\big)_{\theta,q}. \tag{9}$$

In order to check that the inclusion $L^{\frac{1}{1-\theta},q}(\|\nu\|) \subseteq (L^1(\|\nu\|), L^\infty(\nu))_{\theta,q}$ holds, we assume first that $q < \infty$. Proposition 3.5.*i)* and the Hardy inequality (see [2, Lemma III.3.9]) give, for any $f \in L^{\frac{1}{1-\theta},q}(\|\nu\|)$,

$$\begin{aligned}
\|f\|_{(L^1(\|\nu\|),L^\infty(\nu))_{\theta,q}} &= \left(\int_0^\infty \big(t^{-\theta} K\big(t, f; L^1\big(\|\nu\|\big), L^\infty(\nu)\big)\big)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\preccurlyeq \left(\int_0^\infty \left[t^{-\theta} \int_0^t f_*(u)du\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty \left[t^{1-\theta} \frac{1}{t} \int_0^t f_*(u)du\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\preccurlyeq \left(\int_0^\infty \big[t^{1-\theta} f_*(t)\big]^q \frac{dt}{t}\right)^{\frac{1}{q}} = \|f\|_{L^{\frac{1}{1-\theta},q}(\|\nu\|)}.
\end{aligned} \tag{10}$$

This implies that $L^{\frac{1}{1-\theta},q}(\|\nu\|) \subseteq (L^1(\|\nu\|), L^\infty(\nu))_{\theta,q}$. For the case $q = \infty$, the inclusion $L^{\frac{1}{1-\theta},\infty}(\|\nu\|) \subseteq (L^1(\|\nu\|), L^\infty(\nu))_{\theta,\infty}$ can be obtained by using the estimate *i)* in Proposition 3.5 and noting that

$$\begin{aligned}
t^{-\theta} K\big(t, f; L^1\big(\|\nu\|\big), L^\infty(\nu)\big) &\preccurlyeq t^{-\theta} \int_0^t f_*(s)ds = t^{-\theta} \int_0^t s^{1-\theta} f_*(s) s^{\theta-1} ds \\
&\le \frac{1}{\theta} \|f\|_{L^{\frac{1}{1-\theta},\infty}(\|\nu\|)}.
\end{aligned}$$

Taking supremum, we obtain $L^{\frac{1}{1-\theta},\infty}(\|\nu\|) \subseteq (L^1(\|\nu\|), L^\infty(\nu))_{\theta,\infty}$, and

$$\|f\|_{(L^1(\|\nu\|),L^\infty(\nu))_{\theta,\infty}} \preccurlyeq \|f\|_{L^{\frac{1}{1-\theta},\infty}(\|\nu\|)}, \quad f \in L^{\frac{1}{1-\theta},\infty}\big(\|\nu\|\big). \tag{11}$$

Finally note that we get the equality between the three spaces (even for $q = \infty$) as metric spaces. The equivalence of their quasi-norms is given by (8), (9) and (10), or (11) for $q = \infty$. □

Corollary 3.7. *Let $\nu : \mathcal{R} \to X$ be a σ-finite locally strongly additive vector measure, and $0 < \theta < 1 \leq q \leq \infty$. Then*

$$\big(L^1(\nu), L^\infty(\nu)\big)_{\theta,q} = \big(L^1_w(\nu), L^\infty(\nu)\big)_{\theta,q} = L^{\frac{1}{1-\theta},q}\big(\|\nu\|\big).$$

Proof. For a σ-finite locally strongly additive vector measure $\nu : \mathcal{R} \to X$ take into account the equivalence E) of Proposition 3.2 and also the continuous inclusions (3) in Proposition 2.2, that is,

$$L^1\big(\|\nu\|\big) \subseteq L^1(\nu) \subseteq L^1_w(\nu) \subseteq L^{1,\infty}\big(\|\nu\|\big) \subseteq L^0(\nu),$$

and apply the above Theorem 3.6. □

Remark 3.8. 1) Note that the second equality in Corollary 3.7 holds even for a non-locally strongly additive σ-finite vector measure ν due to the inclusions (3) in Proposition 2.2. Then if ν is a vector measure for which $L^1(\nu) = L^1_w(\nu)$ (recall Remark 3.4) both equalities in Corollary 3.7 hold.

2) On the other hand, if ν is a *strongly additive*, that is, $\lim_{n\to\infty} \|\nu(A_n)\|_X = 0$ for all disjoint sequences $(A_n)_n$ in $\mathcal{R}$, then $L^1(\nu)$, $L^1_w(\nu)$ and also $L^{p,q}(\|\nu\|)$, $1 \leq p, q \leq \infty$ coincide with the corresponding spaces for certain vector measure defined on a σ-algebra, see [7, Corollary 3.2.a)]. In that case, Corollary 3.7 follows from [9, Corollary 13]. Nevertheless, there are locally strongly additive σ-finite vector measures ν which are not strongly additive and such that $L^1(\nu) \subsetneq L^1_w(\nu)$ as the following example shows.

Example 3.9. (See Example 3.11 in [5].) Let $\mathcal{R}$ be the δ-ring of finite subsets of natural numbers $\mathbb{N}$, and let $\alpha := (\alpha_n)_n$ be a sequence without any bounded subsequence. Consider the σ-finite vector measure

$$\nu : A \in \mathcal{R} \to \nu(A) := \alpha \cdot \chi_A \in c_0.$$

It is easy to check that

$$L^1_w(\nu) = \big\{(f_n)_n : (f_n\alpha_n)_n \in \ell^\infty\big\},$$
$$L^1(\nu) = \big\{(f_n)_n : (f_n\alpha_n)_n \in c_0\big\}.$$

On the other hand it is not difficult to see that $\|\nu\|(A) = \sup_{n\in A} |\alpha_n|$, for every nonempty $A \subseteq \mathbb{N}$. This means, in particular, that $\|\nu\|(A) < \infty$ if and only if $A \in \mathcal{R}$ since $(\alpha_n)_n$ has not bounded subsequences, and consequently ν is locally strongly additive. Finally note that ν is clearly not strongly additive and $L^1(\nu) \subsetneq L^1_w(\nu)$.

Remark 3.10. Let $1 < p < \infty$ and take $\theta = 1 - \frac{1}{p}$. Putting $q = 1$ in the above Corollary 3.7 we obtain, in particular, $(L^1(\nu), L^\infty(\nu))_{\theta,1} = L^{p,1}(\|\nu\|)$. Similarly, we have $(L^1(\nu), L^\infty(\nu))_{\theta,\infty} = L^{p,\infty}(\|\nu\|)$ if we take $q = \infty$ in Corollary 3.7. Now from (7) in Remark 3.3 we conclude that

$$\big(L^1(\nu), L^\infty(\nu)\big)_{\theta,1} \subseteq L^p(\nu) \subseteq L^p_w(\nu) \subseteq \big(L^1(\nu), L^\infty(\nu)\big)_{\theta,\infty}. \tag{12}$$

In the terminology of [3, Theorem 3.5.2] the inclusions above say that the spaces $L^p(\nu)$ and $L^p_w(\nu)$ belong both to the both classes $\mathcal{C}_J(\theta, L^1(\nu), L^\infty(\nu))$ and $\mathcal{C}_K(\theta, L^1(\nu), L^\infty(\nu))$. See [3, p. 49] just after Definition 3.5.1. Also note that for a general vector measure ν it follows that $L^p_w(\nu)$ belongs to the classes $\mathcal{C}_J(\theta, L^1_w(\nu), L^\infty(\nu))$ and $\mathcal{C}_K(\theta, L^1_w(\nu), L^\infty(\nu))$.

Corollary 3.11. *Let $\nu : \mathcal{R} \to X$ be a σ-finite locally strongly additive vector measure, $0 < \eta < 1 \leq q \leq \infty$, and $1 < p_0 \neq p_1 < \infty$. Then*

$$\big(L^{p_0}(\nu), L^{p_1}(\nu)\big)_{\eta,q} = \big(L^{p_0}_w(\nu), L^{p_1}_w(\nu)\big)_{\eta,q} = L^{p,q}\big(\|\nu\|\big),$$

where $\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}$.

Proof. Having in mind the inclusions (12) in the previous remark, we can apply the reiteration theorem [3, Theorem 3.5.3] with parameters $\theta_0 = 1 - \frac{1}{p_0}$ and $\theta_1 = 1 - \frac{1}{p_1}$. The reiteration theorem tells us that

$$\big(L^{p_0}(\nu), L^{p_1}(\nu)\big)_{\eta,q} = \big(L^{p_0}_w(\nu), L^{p_1}_w(\nu)\big)_{\eta,q} = \big(L^1(\nu), L^\infty(\nu)\big)_{\theta,q},$$

where $\theta = (1-\eta)\theta_0 + \eta\theta_1$, in which case $1-\theta = \frac{1}{p}$. Finally the above Corollary 3.7 gives $(L^1(\nu), L^\infty(\nu))_{\theta,q} = L^{\frac{1}{1-\theta},q}(\|\nu\|) = L^{p,q}(\|\nu\|)$, which is the last equality. □

Remark 3.12. Note that the second equality in Corollary 3.11 holds even for non-locally strongly additive vector measures.

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