Complex Interpolation of Operators and Optimal Domains

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Abstract. Let X_0 and X_1 be two order continuous Banach function spaces on a finite measure space, (E_0, E_1) a Banach space interpolation pair, and $T\colon X_0 + X_1 \to E_0 + E_1$ an admissible operator between the pairs (X_0, X_1) and (E_0, E_1). If $T_\theta\colon [X_0, X_1]_{[\theta]} \to [E_0, E_1]_{[\theta]}$ is the interpolated operator by the first complex method of Calderón and m_0, m_1 and m_θ are the vector measures coming from $T|_{X_0}$ and $T|_{X_1}$ and T_θ, respectively, then we study the relationship between the optimal domain $L^1(m_\theta)$ of T_θ and the complex interpolation space $[L^1(m_0), L^1(m_1)]_{[\theta]}$ of the optimal domains of $T|_{X_0}$ and $T|_{X_1}$. Then, we apply the obtained result to study interpolation of p-th power factorable and bidual (p, q)-power-concave operators.

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1. Introduction

Let $T\colon X \to E$ be a (continuous and linear) operator from a Banach function space X, over a measure space (Ω, Σ, μ), into a Banach space E. Under some mild conditions, a vector measure $m_T : \Sigma \to E$ is defined by $m_T(A) := T(\chi_A)$. The space $L^1(m_T)$ of scalar integrable functions with respect to m_T is the *optimal domain* of T in the sense that it is the largest space among all order continuous Banach function spaces (based on (Ω, Σ, μ)) into which X is continuously embedded and to which T admits an E-valued continuous linear extension (see [5] or [12, Theorem 4.14]).

Let (X_0, X_1) be an interpolation pair of Banach function spaces, (E_0, E_1) an interpolation pair of Banach spaces and T an admissible operator

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between the pairs (X_0, X_1) and (E_0, E_1), that is, an operator $T\colon X_0 + X_1 \to E_0 + E_1$ such that its restrictions $T_0 := T|_{X_0}\colon X_0 \to E_0$ and $T_1 := T|_{X_1}\colon X_1 \to E_1$ are continuous. Moreover, let $T_\theta\colon [X_0, X_1]_{[\theta]} \to [E_0, E_1]_{[\theta]}$ be the interpolated operator for $0 < \theta < 1$, where $[\cdot,\cdot]_{[\theta]}$ denotes Calderón's first complex interpolation method, and set $m_\theta := m_{T_\theta}$ for all $0 \le \theta \le 1$.

Mastylo asked one of the authors whether the equality between the interpolated space $[L^1(m_0), L^1(m_1)]_{[\theta]}$ of the optimal domains of T_0 and T_1, and the optimal domain $L^1(m_\theta)$ of the interpolated operator T_θ holds. In general, the answer to this question lies in the negative (see Remark 3.3). However, in the present paper we prove that the inclusion $[L^1(m_0), L^1(m_1)]_{[\theta]} \subseteq L^1(m_\theta)$ holds. This is carried out in Sect. 3 (Theorem 3.1), after having established the necessary preliminaries in Sect. 2. Finally, in Sect. 4, we apply the obtained results to study the complex interpolation of certain classes of operators characterized by some extension or Maurey–Rosenthal factorization property. In concrete, we study the complex interpolation of p-th power factorable operators (Theorem 4.4), bidual (p, q)-power-concave operators (Theorem 4.9) and q-concave operators (Theorem 4.12).

2. Preliminaries and Notation

Let (Ω, Σ, μ) be a finite measure space. A Banach function space X over μ (B.f.s. for short) is an ideal of the space of (equivalence classes of) measurable functions $L^0(\mu)$ endowed with a complete norm $\|\cdot\|_X$ that is compatible with the μ-a.e. order and such that $L^\infty(\mu) \subseteq X \subseteq L^1(\mu)$ (see [9, p. 28]). The unit ball of X will be denoted by $B(X)$. The Banach space of all integral functionals on X is the *Köthe dual space* and is denoted by X'. The topological dual is denoted by X^*.

A B.f.s. X is *order continuous* if for every sequence $(f_n)_n$ in X such that $0 \le f_n \downarrow 0$ pointwise we have that $\|f_n\|_X \downarrow 0$. In [9, p. 29-30] we can find the following characterization: X is order continuous if and only if $X' = X^*$. Given $0 < p < \infty$, the *p-th power space* of a B.f.s. X with norm $\|\cdot\|_X$ is the space $X_{[p]} := \left\{ f \in L^0(\mu) : |f|^{\frac{1}{p}} \in X \right\}$ which is a quasi-Banach function space with quasi-norm

$$\|f\|_{X_{[p]}} := \|\,|f|^{\frac{1}{p}}\|_X^p, \quad f \in X_{[p]}.$$

This quasi-norm is equivalent to a norm if and only if X is p-convex, and for $0 < p < 1$, it is in fact a norm (see [12, Proposition 2.23]). Therefore, $X_{[p]}$ is a B.f.s. for $0 < p < 1$. If $p \ge 1$ then $X \subseteq X_{[p]}$.

For a given pair (X_0, X_1) of B.f.s. over μ and $0 < \theta < 1$, the *Calderón–Lozanovskii's product space* $X_0^{1-\theta} X_1^\theta$ (see [3]) is the Banach space consisting of all $f \in L^0(\mu)$ such that there exist $f_0 \in B(X_0)$, $f_1 \in B(X_1)$ and $\lambda > 0$ for which

$$|f(w)| \le \lambda |f_0(w)|^{1-\theta} |f_1(w)|^\theta, \quad w \in \Omega \ \ (\mu\text{-a.e.}), \tag{2.1}$$

endowed with the norm $\|x\|_{X_0^{1-\theta} X_1^\theta} = \inf \lambda$, where the infimum is taken over all λ satisfying (2.1). The Calderón–Lozanovskii's product space has the following relationships with the first Calderón's complex interpolation method (see [1] or [2]): $X_0 \cap X_1 \subseteq [X_0, X_1]_{[\theta]} \subseteq X_0^{1-\theta} X_1^{\theta} \subseteq X_0 + X_1$ with $\|x\|_{[X_0,X_1]_{[\theta]}} = \|x\|_{X_0^{1-\theta}X_1^{\theta}}$ for all $x \in [X_0, X_1]_{[\theta]}$. Furthermore, if X_0 or X_1 is order continuous, then $[X_0, X_1]_{[\theta]} = X_0^{1-\theta} X_1^{\theta}$.

Now we present the essential definitions and results about integration with respect to vector measures. Let $m : \Sigma \to E$ be a vector measure defined on a σ–algebra Σ of subsets of a nonempty set Ω. This will always mean that m is countably additive on Σ with values in a real Banach space E. The semivariation of m is the subadditive set function $\|m\| : \Sigma \to [0, \infty)$ defined by

$$\|m\| (A) := \sup \{ |\langle m, x^* \rangle| (A) : x^* \in B(E^*) \}, \quad A \in \Sigma,$$

where $|\langle m, x^* \rangle|$ is the total variation measure of the scalar measure $\langle m, x^* \rangle$ given by $\langle m, x^* \rangle (A) := \langle m(A), x^* \rangle$, for all $A \in \Sigma$. Note that for a positive scalar (finite) measure m, the semivariation of m and the measure m coincide. A set $A \in \Sigma$ is called m–null if $\|m\| (A) = 0$. Let $L^0(m)$ be the space of all $\mathbb{R}$–valued Σ–measurable functions on Ω. Two functions $f, g \in L^0(m)$ are identified if they are equal m–a.e., that is, if $\{w \in \Omega : f(w) \neq g(w)\}$ is an m–null set.

A function $f \in L^0(m)$ is called *integrable* (with respect to m) if $f \in L^1(|\langle m, x^* \rangle|)$ for all $x^* \in E^*$ and for each $A \in \Sigma$ there exists an element $\int_A f dm \in E$ (called the *integral* of f over A) such that $\left\langle \int_A f dm, x^* \right\rangle = \int_A f d \langle m, x^* \rangle$ for all $x^* \in E^*$ (see [10], [8]). The space $L^1(m)$ of all (equivalence classes of) integrable functions becomes a Banach lattice when it is endowed with the natural order m–a.e., and the norm

$$\|f\|_{L^1(m)} := \sup \left\{ \int_{\Omega} |f| \, d |\langle m, x^* \rangle| : x^* \in B(E^*) \right\}, \quad f \in L^1(m).$$

The integration map $I_m : L^1(m) \to E$, given by $I_m(f) := \int_\Omega f \, dm$ for all $f \in L^1(m)$, is an operator.

Let $1 \leq p < \infty$. A measurable function $f : \Omega \to \mathbb{R}$ is called p*–integrable* (with respect to m) if $|f|^p \in L^1(m)$. We denote by $L^p(m)$ the space of (equivalence classes of) p–integrable functions. The natural norm for this space is given by

$$\|f\|_{L^p(m)} := \sup \left\{ \left(\int_{\Omega} |f|^p \, d |\langle m, x^* \rangle| \right)^{\frac{1}{p}} : x^* \in B(E^*) \right\}, \quad f \in L^p(m).$$

A measure $|\langle m, x^* \rangle|$ that is equivalent to m (in the sense that they have both the same null sets) is called a Rybakov control measure for m. Such a measure always exists. We refer to [6] for this notion and basic results on vector measures and to [7] for spaces $L^p(m)$. If μ is a Rybakov control measure for m then $L^p(m)$ is a B.f.s. on (Ω, Σ, μ) for all $p \geq 1$. Moreover, $L^p(m)$ is order continuous and it can be easily checked that $L^p(m) = (L^1(m))_{[\frac{1}{p}]}$.

3. Interpolation and Optimal Domains

Let $T\colon X \to E$ be a Banach-space valued operator on an order continuous B.f.s. X. Under these conditions the expression $m_T(A) := T(\chi_A)$ defines a vector measure $m_T\colon \Sigma \to E$ which is called the *vector measure associated to* T. The operator T is said to be μ-determined if the measures μ and m_T have exactly the same null sets. When T is μ-determined, the space $L^1(m_T)$ is an order continuous B.f.s. on (Ω, Σ, μ), X is continuously embedded into $L^1(m_T)$ via the natural inclusion $J_T\colon X \to L^1(m_T)$ and the integration operator $I_{m_T}\colon L^1(m_T) \to E$ is the unique continuous linear extension of T satisfying $T = I_{m_T} \circ J_T$ (see [5] or [12, Proposition 4.4]).

Therefore, if Y is another order continuous B.f.s such that $X \subseteq Y \subseteq L^0(\mu)$ and $T\colon Y \to E$ is a continuous linear extension of T then $Y \subseteq L^1(m_T)$. In this sense, it is said that $L^1(m_T)$ is the (order continuous) *optimal domain* for the operator T.

From now on, X_0 and X_1 will be order continuous B.f.s. on the same finite measure space (Ω, Σ, μ), (E_0, E_1) a Banach space pair and T an admissible μ-determined operator. In this situation we have the optimal domains $L^1(m_0)$, $L^1(m_1)$ and $L^1(m_\theta)$ corresponding to the restricted μ-determined operators $T_0\colon X_0 \to E_0$, $T_1\colon X_1 \to E_1$ and $T_\theta\colon [X_0, X_1]_{[\theta]} \to [E_0, E_1]_{[\theta]}$. The following result relates the interpolation space $[L^1(m_0), L^1(m_1)]_{[\theta]}$ of the optimal domains $L^1(m_0)$ and $L^1(m_1)$ with the optimal domain $L^1(m_\theta)$ of the interpolated operator T_θ.

Theorem 3.1. *$[L^1(m_0), L^1(m_1)]_{[\theta]}$ is continuously embedded into $L^1(m_\theta)$.*

Proof. For $i = 0, 1$, the space X_i is continuously embedded into the space $L^1(m_i)$ and there exists a unique extension I_{m_i} of T_i to $L^1(m_i)$ (see [5] or [12, Theorem 4.14]). By the order continuity of X_i and $L^1(m_i)$ we have the following chain of inclusions

$$[X_0, X_1]_{[\theta]} = X_0^{1-\theta} X_1^{\theta} \subseteq L^1(m_0)^{1-\theta} L^1(m_1)^{\theta} = [L^1(m_0), L^1(m_1)]_{[\theta]}.$$

Note that if $f \in L^1(m_0) \cap L^1(m_1)$ then $I_{m_0}(f) = I_{m_1}(f)$. In fact, if $0 \le f \in L^1(m_0) \cap L^1(m_1)$ then we can take a sequence of simple functions $0 \le \varphi_n \uparrow f$ pointwise. Since $L^1(m_i)$ has order continuous norm we have $\varphi_n \to f$ in $L^1(m_i)$. By the continuity of I_{m_i} we obtain $I_{m_i}(f) = \lim I_{m_i}(\varphi_n)$, but $\varphi_n \in X_i$, so $I_{m_i}(\varphi_n) = T_i(\varphi_n) = T(\varphi_n)$ and hence $I_{m_i}(f) = \lim T(\varphi_n)$.

Now we define the operator $\hat{T}\colon L^1(m_0) + L^1(m_1) \to E_0 + E_1$ by

$$\hat{T}(f_0 + f_1) := I_{m_0}(f_0) + I_{m_1}(f_1), \quad f_0 + f_1 \in L^1(m_0) + L^1(m_1).$$

$\hat{T}$ is well–defiend since $I_{m_0}(f) = I_{m_1}(f)$ for every $f \in L^1(m_0) \cap L^1(m_1)$. Moreover, $\hat{T}|_{L^1(m_0)} = I_{m_0}$, $\hat{T}|_{L^1(m_1)} = I_{m_1}$ and $\hat{T}|_{X_0+X_1} = T$. Therefore, we have that the interpolated operator $\hat{T}_\theta\colon [L^1(m_0), L^1(m_1)]_{[\theta]} \to [E_0, E_1]_{[\theta]}$ satisfies $\hat{T}_\theta|_{[X_0, X_1]_{[\theta]}} = T_\theta$.

Thus, $\hat{T}_\theta$ is a continuous linear extension of $T_\theta\colon [X_0, X_1]_{[\theta]} \to [E_0, E_1]_{[\theta]}$ to the order continuous Banach space $[L^1(m_0), L^1(m_1)]_{[\theta]}$. By the optimality of the domain $L^1(m_\theta)$ for the operator T_θ (see again [5] or [12, Theorem

4.14]) we conclude that $[L^1(m_0), L^1(m_1)]_{[\theta]}$ is continuously embedded into $L^1(m_\theta)$. $\square$

Remark 3.2. When (E_0, E_1) is a pair of B.f.s. on the same finite measure space and the operator T is positive, the vector measure m_θ coincides with the interpolated measure $[m_0, m_1]_\theta$ introduced and studied in [4] and then Theorem 3.1 is a generalization of [4, Corollary 2.7].

Remark 3.3. (see the interesting paper [11] for details). In general we can not expect the equality between the spaces involved in Theorem 3.1 as the Fourier transform map $\mathcal{F}\colon L^1(\mathbb{T}) \to \ell^\infty(\mathbb{Z})$ shows. Recall that $\mathcal{F}$ assigns to each $f \in L^1(\mathbb{T})$ its Fourier transform $\hat{f} : \mathbb{Z} \to \mathbb{C}$ given by

$$\hat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx, \quad n \in \mathbb{Z}.$$

It is known that $L^1(m_0) = L^1(\mathbb{T})$ and $L^1(m_1) = L^2(\mathbb{T})$ for the vector measures $m_0(A) = \mathcal{F}_0(\chi_A)$ and $m_1(A) = \mathcal{F}_1(\chi_A)$ associated to $\mathcal{F}_0\colon L^1(\mathbb{T}) \to \ell^\infty(\mathbb{Z})$ and $\mathcal{F}_1\colon L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$. Therefore, for $0 < \theta < 1$,

$$[L^1(m_0), L^1(m_1)]_{[\theta]} = [L^1(\mathbb{T}), L^2(\mathbb{T})]_{[\theta]} = L^p(\mathbb{T}),$$

where $\frac{1}{p} = 1 - \frac{\theta}{2}$. However, the optimal domain $L^1(m_\theta)$ for the interpolated operator $T_\theta\colon L^p(\mathbb{T}) \to \ell^{\frac{2}{\theta}}$ can be described in this way

$$L^1(m_\theta) = F^p(\mathbb{T}) := \left\{ f \in L^1(\mathbb{T}) : \mathcal{F}(f\chi_A) \in \ell^{\frac{2}{\theta}}(\mathbb{Z}), \forall A \in \mathcal{B}(\mathbb{T}) \right\},$$

and, what is more interesting and deeper, the inclusion $L^p(\mathbb{T}) \subseteq F^p(\mathbb{T})$ is proper (see [11, Theorem 1.4]).

4. Interpolation of p-th Power Factorable Operators and Bidual (p, q)-Power-Concave Operators

A classical question on interpolation theory is the following: if we have an admissible operator T between the pairs (X_0, X_1) and (E_0, E_1) such that both $T_0 : X_0 \to E_0$ and $T_1 : X_1 \to E_1$ have a certain property P then does the operator $T_\theta : [X_0, X_1]_{[\theta]} \to [E_0, E_1]_{[\theta]}$ between the interpolated spaces satisfy the same property P? We are going to apply our result about optimal domains to answer this question in the affirmative for the properties of being *p-th power factorable* operator and being *bidual* (p, q)*-power-concave* operator.

We begin by recalling the definition and properties of p-th power factorable operators introduced in [12].

Definition 4.1. Let $1 \leq p < \infty$, X an order continuous Banach function space, and E a Banach space. An operator $T\colon X \to E$ is said to be *p-th power factorable* if there exists an operator $T_{[p]}\colon X_{[p]} \to E$, which equals T over $X \subseteq X_{[p]}$.

When we consider a μ-determined operator $T\colon X \to E$ on an order continuous B.f.s. X it is easy to see that the restriction $I^p_{m_T}\colon L^p(m_T) \to E$ of the integration operator $I_{m_T}: L^1(m_T) \to E$ is μ-determined and p-th power factorable. The characterization of p-th power factorable operator is given by the following result (see [12, Theorem 5.7] for more equivalences).

Theorem 4.2. *Given $1 \le p < \infty$, an order continuous B.f.s. X, a Banach space E and a μ-determined operator $T\colon X \to E$, the following assertions are equivalent:*

(i) *T is p-th power factorable.*
(ii) *$X \subseteq L^p(m_T)$ with a continuous inclusion.*

Remark 4.3. If an operator T is p-th power factorable then it is q-th power factorable for every $1 \le q < p$. This is due to the facts that $L^p(m_T)$ is continuously embedded in $L^q(m_T)$ for $1 \le q < p$ and that the composition of a p-th power factorable operator with an operator is a p-th power factorable operator (see [12, Lemma 5.4]).

Now we are going to prove that the property of being p-th power factorable is preserved by the first Calderón's complex interpolation method. Recall that from now on (X_0, X_1) is a pair of order continuous B.f.s on the same measure space (Ω, Σ, μ), (E_0, E_1) is a Banach space interpolation pair and T is an admissible μ-determined operator with restrictions $T_0 := T|_{X_0}\colon X_0 \to E_0$ and $T_1 =: T|_{X_1}\colon X_1 \to E_1$.

Theorem 4.4. *If T_0 and T_1 are p-th power factorable then the interpolated operator T_θ is p-th power factorable.*

Proof. By Theorem 4.2 we have to prove that $[X_0, X_1]_{[\theta]} \subseteq L^p(m_\theta)$. Since X_0 and X_1 have order continuous norm and T_0 and T_1 are p-th factorable we have $[X_0, X_1]_{[\theta]} = X_0^{1-\theta} X_1^\theta \subseteq L^p(m_0)^{1-\theta} L^p(m_1)^\theta$. So, we only need to check that $L^p(m_0)^{1-\theta} L^p(m_1)^\theta \subseteq L^p(m_\theta)$, but this condition is equivalent to prove that $L^1(m_0)^{1-\theta} L^1(m_1)^\theta \subseteq L^1(m_\theta)$ and this follows from Theorem 3.1 since $L^1(m_0)$ (or $L^1(m_1)$) has order continuous norm. $\square$

What can we say if we consider an admissible operator T such that T_0 is p_0-th power factorable and T_1 is p_1-th power factorable? The next result sheds some light on this issue:

Theorem 4.5. *Let $0 < \theta < 1 \le p_0 \le p_1 < \infty$. If T_0 is p_0-th power factorable and T_1 is p_1-th power factorable then $T_{\theta,\alpha}\colon [X_0, X_1]_{[\theta]} \to [E_0, E_1]_{[\alpha]}$ is well-defined and it is a p-th power factorable restriction of T, where p is given by $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\alpha := \frac{\theta p}{p_1}$.*

Proof. First, observe that $0 < \alpha < 1$. During the proof we will check that $[X_0, X_1]_{[\theta]} \subseteq L^p(m_\alpha)$, where m_α is the measure associated to the operator $T_\alpha\colon [X_0, X_1]_{[\alpha]} \to [E_0, E_1]_{[\alpha]}$. Thus, in particular we have $[X_0, X_1]_{[\theta]} \subseteq L^1(m_\alpha)$ which allows us to consider the operator $T_{\theta,\alpha}\colon [X_0, X_1]_{[\theta]} \to [E_0, E_1]_{[\alpha]}$ given by $T_{\theta,\alpha} := I_{m_\alpha}|_{[X_0, X_1]_{[\theta]}}$. Note that $m_{T_{\theta,\alpha}} = m_\alpha$. Therefore, keeping in mind Theorem 4.2, we only have to prove that $[X_0, X_1]_{[\theta]} \subseteq L^p(m_\alpha)$.

Since X_0 is p_0-th factorable and X_1 is p_1-th factorable we have

$$[X_0, X_1]_{[\theta]} = X_0^{1-\theta} X_1^{\theta} \subseteq L^{p_0}(m_0)^{1-\theta} L^{p_1}(m_1)^{\theta}.$$

Thus, if we check that $L^{p_0}(m_0)^{1-\theta} L^{p_1}(m_1)^{\theta} \subseteq L^p(m_\alpha)$ then the proof will be completed. So, let $0 \leq f_0 \in L^{p_0}(m_0)$, $0 \leq f_1 \in L^{p_1}(m_1)$ and take $g_0 := f_0^{p_0} \in L^1(m_0)$ and $g_1 := f_1^{p_1} \in L^1(m_1)$. By Theorem 3.1 we have $L^1(m_0)^{1-\alpha} L^1(m_1)^{\alpha} \subseteq L^1(m_\alpha)$. Hence,

$$\left(f_0^{1-\theta} f_1^{\theta}\right)^p = f_0^{(1-\theta)p} f_1^{\theta p} = (f_0^{p_0})^{\frac{(1-\theta)p}{p_0}} (f_1^{p_1})^{\frac{\theta p}{p_1}} = g_0^{1-\alpha} g_1^{\alpha} \in L^1(m_\alpha),$$

that is, $f_0^{1-\theta} f_1^{\theta} \in L^p(m_\alpha)$. □

Observe that if $p_0 = p_1$ in Theorem 4.5 then $\alpha = \theta$ and we recover Theorem 4.4. Furthermore, if $p_0 < p_1$, T_0 is p_0-th power factorable and T_1 is p_1-th power factorable then T_1 is also p_0-th power factorable and so the operator $T_\theta \colon [X_0, X_1]_{[\theta]} \to [E_0, E_1]_{[\theta]}$ is p_0-th power factorable by Theorem 4.4. Theorem 4.5 provides a better result in the sense that the operator $T_{\theta,\alpha}$ is even p-th power factorable (with $p_0 < p$). However, the range space is not $[E_0, E_1]_{[\theta]}$ but $[E_0, E_1]_{[\alpha]}$. In particular when the pair of Banach spaces (E_0, E_1) satisfies that $E_0 \subseteq E_1$ we have the following result.

Corollary 4.6. *Let $1 \leq p_0 < p_1 < \infty$ and $E_0 \subseteq E_1$. If T_0 is p_0-th power factorable and T_1 is p_1-th power factorable then T_θ is p-th power factorable, where p is given by $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.*

Proof. If $p_0 < p_1$ then $p_0 < p < p_1$ and so $\alpha = \frac{\theta p}{p_1} < \theta$. From $E_0 \subseteq E_1$ and [2, Theorem 4.2.1] it follows that $[E_0, E_1]_{[\alpha]} \subseteq [E_0, E_1]_{[\theta]}$. By Theorem 4.5, the operator $T_{\theta,\alpha} \colon [X_0, X_1]_{[\theta]} \to [E_0, E_1]_{[\alpha]}$ is p-th power factorable and hence $T_\theta = T_{\theta,\theta} \colon [X_0, X_1]_{[\theta]} \to [E_0, E_1]_{[\theta]}$ is p-th power factorable. □

In connection with the Maurey–Rosenthal factorization theory, Okada, Ricker and Sánchez-Pérez introduced in [12] the concept of bidual (p, q)-power-concave operator.

Definition 4.7. Let $1 \leq p, q < \infty$, X an order continuous Banach function space, and E a Banach space. An operator $T \colon X \to E$ is said to be *bidual (p, q)-power-concave* if there exists a constant $C > 0$ such that

$$\sum_{j=1}^{n} \|T(f_j)\|_E^{\frac{q}{p}} \leq C \sup\left\{ \left| \left\langle \sum_{j=1}^{n} |f_j|^{\frac{q}{p}}, \xi \right\rangle \right| : \xi \in B(X_{[q]}^*) \right\}$$

for all $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in X$. A bidual $(1, q)$-power-concave operator is called simply *bidual q-concave.*

The following theorem characterizes bidual (p, q)-power-concave operators (see [12, Theorem 6.9] for other equivalences).

Theorem 4.8. *Let $T : X \to E$ a μ-determined operator. The following assertions are equivalent:*

(i) *T is bidual (p, q)-power-concave operator.*

(ii) *There exists* $0 < g \in L^0(\mu)$ *such that the inclusions*

$$X \subseteq L^q(g d\mu) \subseteq L^p(m_T)$$

hold and are continuous.

Now we can obtain a result about interpolation of bidual (p,q)-power-concave operators.

Theorem 4.9. *If* T_0 *and* T_1 *are bidual* (p,q)*-power-concave then* T_θ *is bidual* (p,q)*-power-concave.*

Proof. By Theorem 4.8 there exists $0 < g_i \in L^0(\mu)$ such that $X_i \subseteq L^q(g_i d\mu) \subseteq L^p(m_i)$ for $i = 0, 1$. Therefore,

$$[X_0, X_1]_{[\theta]} \subseteq [L^q(g_0 d\mu), L^q(g_1 d\mu)]_{[\theta]} \subseteq [L^p(m_0), L^p(m_1)]_{[\theta]}.$$

According to [2, Theorem 5.5.3] we have $[L^q(g_0 d\mu), L^q(g_1 d\mu)]_{[\theta]} = L^q(g d\mu)$, where $g := g_0^{1-\theta} g_1^{\theta} > 0$. On the other hand, by Theorem 3.1 it follows that $L^1(m_0)^{1-\theta} L^1(m_1)^{\theta} \subseteq L^1(m_\theta)$ and hence $L^p(m_0)^{1-\theta} L^p(m_1)^{\theta} \subseteq L^p(m_\theta)$. Thus, $[X_0, X_1]_{[\theta]} \subseteq L^q(g d\mu) \subseteq [L^p(m_0), L^p(m_1)]_{[\theta]} \subseteq L^p(m_\theta)$ and so the operator T_θ is bidual (p,q)-power-concave (again by Theorem 4.8). $\square$

We can obtain similar results to Theorem 4.5 and Proposition 4.6 for bidual (p,q)-power-concave operators.

Theorem 4.10. *Let* $1 \le p_0 \le p_1 < \infty$, $1 \le q_0 \le q_1 < \infty$, $0 < \theta < 1$. *If* T_0 *is bidual* (p_0, q_0)*-power-concave and* T_1 *is bidual* (p_1, q_1)*-power-concave then* $T_{\theta,\alpha} \colon [X_0, X_1]_{[\theta]} \to [E_0, E_1]_{[\alpha]}$ *is bidual* (p,q)*-power-concave, where* $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ *and* $\alpha := \frac{\theta p}{p_1}$.

Proof. Again there exists $0 < g_i \in L^0(\mu)$ such that $X_i \subseteq L^{q_i}(g_i d\mu) \subseteq L^{p_i}(m_i)$, for $i = 0, 1$ and hence

$$[X_0, X_1]_{[\theta]} \subseteq [L^q(g_0 d\mu), L^q(g_1 d\mu)]_{[\theta]} = L^q(g d\mu) \subseteq [L^p(m_0), L^p(m_1)]_{[\theta]},$$

with $g = g_0^{1-\theta} g_1^{\theta} > 0$. Now we can repeat the same argument as in Theorem 4.5 to prove that $L^{p_0}(m_0)^{1-\theta} L^{p_1}(m_1)^{\theta} \subseteq L^p(m_\alpha)$ and this completes the proof. $\square$

Observe that if $p_0 = p_1$ and $q_0 = q_1$ then Theorem 4.10 reduces to Theorem 4.9.

Corollary 4.11. *Let* $1 \le p_0 < p_1 < \infty$, $1 \le q < \infty$ *and* $E_0 \subseteq E_1$. *If* T_0 *is bidual* (p_0, q)*-power-concave and* T_1 *is bidual* (p_1, q)*-power-concave then* T_θ *is bidual* (p,q)*-power-concave, where* p *is given by* $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Proof. If $p_0 < p_1$ then $p_0 < p < p_1$ and so $\alpha = \frac{\theta p}{p_1} < \theta$. From $E_0 \subseteq E_1$ and [2, Theorem 4.2.1] we deduce that $[E_0, E_1]_{[\alpha]} \subseteq [E_0, E_1]_{[\theta]}$. By Theorem 4.10, the operator $T_{\theta,\alpha} \colon [X_0, X_1]_{[\theta]} \to [E_0, E_1]_{[\alpha]}$ is bidual (p,q)-power-concave and thus $T_\theta = T_{\theta,\theta} \colon [X_0, X_1]_{[\theta]} \to [E_0, E_1]_{[\theta]}$ is bidual (p,q)-power-concave (see [12, Proposition 6.2 (vi)]). $\square$

Finally we obtain a new result about interpolation of q-concave operators. Recall that a bidual q-concave operator is in particular q-concave (see [12, Proposition 6.2 (i)] with $p = 1$).

Corollary 4.12. *Let (X_0, X_1) be a pair of q-convex order continuous B.f.s and T an admissible operator. If T_0 and T_1 are q-concave then T_θ is q-concave.*

Proof. Since X_0 and X_1 are q-convex and T_0 and T_1 are q-concave, then applying [12, Proposition 6.2 (iv) and (6.6)] it follows that T_0 and T_1 are bidual q-concave operators and so is T_θ by Theorem 4.9 with $p = 1$. Then, [12, Proposition 6.2 (i)] guarantees that T_θ is q-concave. □

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