Complex interpolation of L^p-spaces of vector measures on δ-rings[☆]

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A B S T R A C T

We apply the Calderón interpolation methods to Banach lattices of p-integrable and weakly p-integrable functions with respect to a Banach-space-valued measure defined on a δ-ring. In general, the results we obtain are quite different from those in the case of vector measures on σ-algebras. However, we find a wide class of vector measures on δ-rings for which the results on σ-algebras hold true.

1. Introduction

For a Banach-space-valued measure m defined on a σ-algebra, we obtained in [8] the Calderón interpolation spaces $[X_0, X_1]_{[\theta]}$ and $[X_0, X_1]^{[\theta]}$ of the couples (X_0, X_1), where X_0 and X_1 are the Banach lattices $L^p(m)$ or $L^p_w(\nu)$ of equivalence classes of scalar p-integrable or, respectively, weakly p-integrable functions with respect to the measure m. In such a case, the first method always gives another $L^p(m)$-space and the second one yields an $L^p_w(m)$-space. More precisely, we obtained (see [8, Theorem 3.4]) for $1 \le p_0 \ne p_1 \le \infty,\ 0 < \theta < 1$, and $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ the following equalities:

$$\left[L^{p_0}(m), L^{p_1}(m)\right]_{[\theta]} \overset{(\star)}{=} L^p(m),$$

$$\left[L^{p_0}_w(m), L^{p_1}(m)\right]_{[\theta]} \overset{(\star)}{=} \left[L^{p_0}(m), L^{p_1}_w(m)\right]_{[\theta]} \overset{(\star)}{=} L^p(m),$$

$$\left[L^{p_0}_w(m), L^{p_1}_w(m)\right]_{[\theta]} \overset{(\diamond)}{=} L^p(m).$$

$$\left[L^{p_0}(m), L^{p_1}(m)\right]^{[\theta]} \overset{(\diamond)}{=} L^p_w(m),$$

$$\left[L^{p_0}_w(m), L^{p_1}(m)\right]^{[\theta]} \overset{(\diamond)}{=} \left[L^{p_0}(m), L^{p_1}_w(m)\right]^{[\theta]} \overset{(\diamond)}{=} L^p_w(m),$$

$$\left[L^{p_0}_w(m), L^{p_1}_w(m)\right]^{[\theta]} \overset{(\star)}{=} L^p_w(m).$$

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In particular, if the vector measure m is a (real) positive finite measure μ all the previous equalities collapse into the well-known interpolation formulas $[L^{p_0}(\mu), L^{p_1}(\mu)]_{[\theta]} = [L^{p_0}(\mu), L^{p_1}(\mu)]^{[\theta]} = L^p(\mu)$. Nevertheless the situation considered in [8] does not include the case

$$\left[L^{p_0}(\mathbb{R}), L^{p_1}(\mathbb{R})\right]_{[\theta]} = \left[L^{p_0}(\mathbb{R}), L^{p_1}(\mathbb{R})\right]^{[\theta]} = L^p(\mathbb{R}),$$

where the Lebesgue measure in the real line $\mathbb{R}$ is considered. In order to fill this gap we need to consider a more general structure than a σ-algebra: we must consider vector measures defined on a δ-ring. That is the motivation to study the Calderón interpolation methods of Banach lattices of p-integrable and weakly p-integrable functions with respect to a Banach-space-valued measure defined on a δ-ring. We will see that interpolation results for vector measures on δ-rings can be very different from those ones on the context of σ-algebras. Roughly speaking we can say that equalities ($\star$) for vector measures on σ-algebras remain true for vector measures on δ-rings (see Corollary 3.7), but equalities ($\diamond$) for vector measures on σ-algebras cease to be true for vector measures on δ-rings (see Example 3.10). However, we will identify a certain type of vector measures on δ-rings (called *locally strongly additive* measures) which keep completely the same behavior as in the σ-algebra case for all the different combinations of couples (see Corollaries 4.7 and 4.9).

2. Preliminaries

In this section we establish the preliminaries necessary for integration of scalar functions with respect to vector measures on δ-rings, in order to make the paper more self-contained and readable. The basic references about integration for us will be [6,11–13]. Throughout this paper $\nu : \mathcal{R} \to X$ will be a vector measure defined on a δ-ring $\mathcal{R}$ of subsets of some nonempty set Ω with values in a real Banach space X. We denote by $\mathcal{R}^{\text{loc}}$ the σ-algebra of subsets $A \subseteq \Omega$ such that $A \cap B \in \mathcal{R}$ for each $B \in \mathcal{R}$. The measurability of functions $f : \Omega \longrightarrow \mathbb{R}$ will be considered with respect to the measurable space $(\Omega, \mathcal{R}^{\text{loc}})$. The *semivariation* of ν is the set function $\|\nu\| : \mathcal{R}^{\text{loc}} \to [0, \infty]$ defined by $\|\nu\|(A) := \sup\{|\langle \nu, x^*\rangle|(A) : x^* \in B(X^*)\}$, where $|\langle \nu, x^*\rangle|$ is the variation of the scalar measure

$$\left\langle \nu, x^*\right\rangle : A \in \mathcal{R} \longrightarrow \left\langle \nu, x^*\right\rangle(A) := \left\langle \nu(A), x^*\right\rangle \in \mathbb{R},$$

and $B(X^*)$ is the unit ball of X^*, the dual space of X. A set $N \in \mathcal{R}^{\text{loc}}$ is called ν-*null* if $\|\nu\|(N) = 0$. A property holds ν-*almost everywhere* (ν-a.e.) if it holds except on a ν-null set.

A measurable function $f : \Omega \longrightarrow \mathbb{R}$ is called *weakly integrable* (with respect to ν) if $f \in L^1(\langle \nu, x^*\rangle)$ for all $x^* \in X^*$. A weakly integrable function f is said to be *integrable* (with respect to ν) if, for each $A \in \mathcal{R}^{\text{loc}}$ there exists an element (necessarily unique) $\int_A f d\nu \in X$, satisfying

$$\left\langle \int_A f d\nu, x^*\right\rangle = \int_A f d\left\langle \nu, x^*\right\rangle, \quad x^* \in X^*.$$

If $1 \le p < \infty$, a measurable function $f : \Omega \longrightarrow \mathbb{R}$ is called *weakly p-integrable* with respect to ν if $|f|^p$ is weakly integrable (with respect to ν) and *p-integrable* (with respect to ν) if $|f|^p$ is integrable with respect to ν. The space $L^p_w(\nu)$ of all (ν-a.e. equivalence classes of) weakly p-integrable functions becomes a Banach lattice when endowed with the usual order ν-a.e. and the norm

$$\|f\|_{L^p_w(\nu)} := \sup\left\{\left(\int_\Omega |f|^p \, d\left|\left\langle \nu, x^*\right\rangle\right|\right)^{\frac{1}{p}} : x^* \in B(X^*)\right\}.$$

The *Fatou property* holds in $L^p_w(\nu)$, meaning that if $(f_n)_n$ is a positive increasing sequence in $L^p_w(\nu)$ converging pointwise ν-a.e. to a function f and $\sup_n \|f_n\|_{L^p_w(\nu)} < \infty$, then $f \in L^p_w(\nu)$, and $\|f\|_{L^p_w(\nu)} = \sup_n \|f_n\|_{L^p_w(\nu)}$. Moreover, the space $L^p(\nu)$ of all (ν-a.e. equivalence classes of) p-integrable functions is a closed *order continuous* ideal of $L^p_w(\nu)$. In fact, it is the closure of $\mathcal{S}(\mathcal{R})$, the space of simple functions supported on $\mathcal{R}$. Recall that order continuity means that if $(f_n)_n$ is a positive increasing sequence in $L^p(\nu)$ converging pointwise ν-a.e. to a function $f \in L^p(\nu)$, then $\|f - f_n\|_{L^p_w(\nu)} \to 0$. The Banach lattices $L^p(\nu)$ and $L^p_w(\nu)$ of equivalence classes of scalar p-integrable and weakly p-integrable functions were initially studied in [7] for vector measures ν on a σ-algebra and its basic properties can be extended and remain true for vector measures on δ-rings. Also we can find in [14, Chapter 3] a very good material about spaces of integrable functions with respect to a vector measure on a σ-algebra. Finally, let us consider two more spaces strongly related with the spaces of p-integrable functions with respect to a vector measure. Denote by $L^\infty(\nu)$ the space of classes of essentially bounded measurable functions $f : \Omega \longrightarrow \mathbb{R}$ with the essential supremum norm. Consider also the vector space $L^0(\nu)$ of all classes of measurable functions $f : \Omega \longrightarrow \mathbb{R}$. If the vector measure ν is defined on a σ-algebra it is well-known (see [7, Corollary 3.2]) that the following inclusions hold for all $p > 1$

$$L^\infty(\nu) \subseteq L^p(\nu) \subseteq L^p_w(\nu) \subseteq L^1(\nu) \subseteq L^1_w(\nu) \subseteq L^0(\nu), \tag{1}$$

and all of them are continuous inclusions, where the topology of *convergence in measure* is considered on $L^0(\nu)$. As it is well-known, this topology is generated by the complete F-norm $\left\|\frac{|f|}{1+|f|}\right\|_{L^1_w(\nu)}$, where $f \in L^0(\nu)$. When the vector measure ν is defined on a δ-ring instead of a σ-algebra, the inclusions (1) are in general false, but we can save something. For each $A \in \mathcal{R}$ consider the σ-algebra $\Sigma_A := \{E \in \mathcal{R} : E \subseteq A\}$ of subsets of A and the vector measure $m_A : E \in \Sigma_A \longrightarrow m_A(E) = \nu(E) \in X$, that is, the restriction of ν to A. Note that $|\langle m_A, x^*\rangle|(E) = |\langle \nu, x^*\rangle|(E)$ for all $x^* \in X^*$ and $E \in \Sigma_A$. In particular, $\|m_A\|(E) = \|\nu\|(E)$ for all $E \in \Sigma_A$. Moreover, if $1 \le p < \infty$ and $f \in L^p_w(\nu)$ it is not difficult to check that

$$\int_A |f\chi_A|^p \, d\left|\langle m_A, x^*\rangle\right| = \int_\Omega |f\chi_A|^p \, d\left|\langle \nu, x^*\rangle\right|, \quad x^* \in X^*,$$

and so $f\chi_A \in L^p_w(m_A)$ and $\|f\chi_A\|_{L^p_w(m_A)} = \|f\chi_A\|_{L^p_w(\nu)}$. Moreover, if $f \in L^1(\nu)$, in which case $f\chi_A \in L^1(\nu)$, and $E \in \Sigma_A$, then

$$\left\langle \int_E f\chi_A dm_A, x^* \right\rangle = \int_E f\chi_A d\langle \nu, x^*\rangle = \int_E f\chi_A d\langle m_A, x^*\rangle, \quad x^* \in X^*,$$

and $f\chi_A \in L^1(m_A)$.

Lemma 2.1. *If $f \in L^p_w(\nu)$, with $p > 1$, and $A \in \mathcal{R}$, then $f\chi_A \in L^1(\nu)$. Moreover, $\|f\chi_A\|_{L^1_w(\nu)} \le (\|\nu\|(A))^{\frac{1}{q}} \|f\|_{L^p_w(\nu)}$, where q is the conjugate exponent of p.*

Proof. Given $A \in \mathcal{R}$, and $f \in L^p_w(\nu)$, with $p > 1$ we know that $f\chi_A \in L^p_w(m_A)$. Now by applying (1) we obtain $f\chi_A \in L^p_w(m_A) \subseteq L^1(m_A)$. Now the Hölder inequality gives $\|f\chi_A\|_{L^1_w(m_A)} \le \|f\chi_A\|_{L^p_w(m_A)} (\|m_A\|(A))^{\frac{1}{q}}$, that is, $\|f\chi_A\|_{L^1_w(\nu)} \le \|f\chi_A\|_{L^p_w(\nu)} (\|\nu\|(A))^{\frac{1}{q}} \le \|f\|_{L^p_w(\nu)} (\|\nu\|(A))^{\frac{1}{q}}$. $\square$

3. Interpolation for general vector measures

We wish to apply Calderón's two methods of complex interpolation to couples of Banach lattices $L^p(\nu)$ and $L^p_w(\nu)$. Since these methods are defined only for Banach spaces over the complex field $\mathbb{C}$ we must in fact apply them to the couple of *complexifications* of those spaces concerning complex valued functions $f : \Omega \longrightarrow \mathbb{C}$ and vector measures with values in complex Banach spaces. If $\nu : \mathcal{R} \to X$ is a vector measure defined on a δ-ring $\mathcal{R}$ of subsets of Ω with values into a complex Banach space X we can define the spaces $L^p_w(\nu)$ and $L^p(\nu)$, with $1 \le p < \infty$, analogously as we did in the previous section for the case of a real Banach space. Moreover, following a standard argument (see [8, Section 2]) we can see that $L^p_w(\nu)$ and $L^p(\nu)$ are *complex Banach lattices*, that is,

$$L^p_w(\nu) = \left\{f : \Omega \longrightarrow \mathbb{C} : \mathrm{Re}(f), \mathrm{Im}(f) \in L^p_w(\nu_\mathbb{R})\right\},$$
$$L^p(\nu) = \left\{f : \Omega \longrightarrow \mathbb{C} : \mathrm{Re}(f), \mathrm{Im}(f) \in L^p(\nu_\mathbb{R})\right\},$$

where $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are, respectively, the real and imaginary parts of f, and $\nu_\mathbb{R} : A \in \mathcal{R} \longrightarrow \nu_\mathbb{R}(A) := \nu(A) \in X_\mathbb{R}$, where $X_\mathbb{R}$ denotes X considered as a vector space over $\mathbb{R}$, in which case $X_\mathbb{R}$ is a real Banach space. Then a function $f \in L^p_w(\nu)$ (respectively $L^p(\nu)$) if and only if its modulus $|f| \in L^p_w(\nu)$ (respectively $L^p(\nu)$). This means that in the proofs of the interpolation results of this paper it suffices to consider only nonnegative real functions. We refer to [2,4,5] for general results concerning interpolation.

In what follows we will always consider vector measures $\nu : \mathcal{R} \to X$ which are σ-finite, that is, there exist a pairwise disjoint sequence $(\Omega_k)_k$ in $\mathcal{R}$ and a ν-null set $N \in \mathcal{R}^{\mathrm{loc}}$, such that $\Omega = \left(\cup_{k\ge 1} \Omega_k\right) \cup N$. There is some connection between σ-finite vector measures defined on δ-rings and vector measures defined on σ-algebras. The first ones always appear as densities ν_g of a certain measurable function g with respect to a measure ν defined on a σ-algebra, as we describe in the following example. In fact, the example is a natural procedure to construct vector measures on δ-rings coming from vector measures on σ-algebras. See [6, Theorem 3.3].

Example 3.1. Let (Ω, Σ) be a measurable space, and $m : \Sigma \to X$ a vector measure with values in a Banach space X. For a strictly positive measurable function $g : \Omega \longrightarrow \mathbb{R}$ consider the δ-ring $\mathcal{R}_g := \left\{A \in \Sigma : g \cdot \chi_A \in L^1(m)\right\}$, where $L^1(m)$ is the space of integrable functions with respect to the measure m. We shall denote by ν_g the measure with density g with respect to m, that is, the vector measure defined by $\nu_g : A \in \mathcal{R}_g \longrightarrow \nu_g(A) := \int_A g \, dm \in X$. Note that $\mathcal{R}_g^{\mathrm{loc}} = \Sigma$, and so $\mathcal{R}_g$ coincides with (not only a δ-ring) the σ-algebra Σ, if and only if $g \in L^1(m)$. Therefore, measurability and (since g is strictly positive) equality ν_g-a.e. and m-a.e. coincide. Moreover, the L^p-spaces ($1 \le p < \infty$) associated with this measure ν_g can be easily described in terms of the ones of the measure m. Namely,

$$L^p_w(\nu_g) = \left\{f \in L^0(m) : |f|^p \cdot g \in L^1_w(m)\right\},$$
$$L^p(\nu_g) = \left\{f \in L^0(m) : |f|^p \cdot g \in L^1(m)\right\},$$

with the norm $\|f\|_{L^p_w(\nu_g)} = \||f|^p \cdot g\|^{\frac{1}{p}}_{L^1_w(m)}$, for all $f \in L^p_w(\nu_g)$. Furthermore,

(A) If g is bounded from above, then $L^p_w(m) \subseteq L^p_w(\nu_g)$, and similarly $L^p(m) \subseteq L^p(\nu_g)$, both with continuous inclusions.
(B) If g is bounded from below, then $L^p_w(\nu_g) \subseteq L^p_w(m)$, and analogously $L^p(\nu_g) \subseteq L^p(m)$, both with continuous inclusions.

The first step for interpolation is to check that each pair of spaces $L^p_w(\nu)$ or $L^p(\nu)$, where $\nu : \mathcal{R} \to X$ is a σ-finite vector measure, forms a *compatible couple of Banach spaces*, that is, they are imbedded continuously in the same topological vector space. In our case the environment space will be the linear space $L^0(\nu)$ of all (ν-a.e. equivalence classes of) real measurable functions f defined on Ω, endowed with the topology generated by certain F-norm $\|\cdot\|_{L^0(\nu)}$ which we shall now describe. Consider the decomposition $\Omega = \left(\cup_{k\geq 1} \Omega_k\right) \cup N$, where $(\Omega_k)_k$ is a pairwise disjoint sequence in $\mathcal{R}$, and N is a ν-null set in $\mathcal{R}^{\mathrm{loc}}$. For each $k = 1, 2, \ldots$ consider the σ-algebra $\Sigma_k := \{A \in \mathcal{R} : A \subseteq \Omega_k\}$ of subsets of Ω_k and the vector measure

$$m_k : A \in \Sigma_k \longrightarrow m_k(A) = \nu(A) \in X,$$

that is, the restriction of ν to Ω_k. Note that $\|m_k\|(A) = \|\nu\|(A)$ for all $A \in \Sigma_k$, and consequently a set $B \in \mathcal{R}^{\mathrm{loc}}$ is ν-null if and only if $B \cap \Omega_k$ is m_k-null for all $k = 1, 2, \ldots$. Now define

$$\|f\|_{L^0(\nu)} := \sum_{k=1}^{\infty} \frac{1}{2^k\,(1 + \|\nu\|\,(\Omega_k))} \left\| \frac{|f|}{1+|f|} \chi_{\Omega_k} \right\|_{L^1_w(m_k)}, \quad f \in L^0(\nu).$$

Note that $\frac{|f|}{1+|f|}\chi_{\Omega_k} \in L^\infty(m_k) \subseteq L^1(m_k)$ for all $k = 1, 2, \ldots$.

Lemma 3.2. *Let $(f_n)_n$ be a sequence in $L^0(\nu)$. The following assertions are equivalent:*

(1) $\|f_n\|_{L^0(\nu)} \to 0$ *as* $n \to \infty$.
(2) $f_n \to 0$, *as* $n \to \infty$, *in* m_k-*measure on* Ω_k *for all* $k = 1, 2, \ldots$.

Proof. The implication (1) ⇒ (2) follows from the inequality

$$\left\| \frac{|f|}{1+|f|} \chi_{\Omega_k} \right\|_{L^1_w(m_k)} \leq 2^k\,(1 + \|\nu\|\,(\Omega_k))\,\|f\|_{L^0(\nu)}, \quad f \in L^0(\nu),\ k = 1, 2, \ldots.$$

For the converse implication (2) ⇒ (1) take an arbitrary $\varepsilon > 0$ and let k_0 such that $\sum_{k>k_0} 2^{-k} < \frac{\varepsilon}{2}$. Now using the hypothesis (2) choose n_0 such that $\left\| \frac{|f_n|}{1+|f_n|}\chi_{\Omega_k} \right\|_{L^1_w(m_k)} \leq \frac{2^{k-1}(1+\|\nu\|(\Omega_k))\varepsilon}{k_0}$ for all $n \geq n_0$ and $k = 1, 2, \ldots, k_0$. Then $\|f_n\|_{L^0(\nu)} < \varepsilon$ for all $n \geq n_0$. $\square$

Lemma 3.3. $\|\cdot\|_{L^0(\nu)}$ *is an* F-*norm, and* $\left(L^0(\nu), \|\cdot\|_{L^0(\nu)}\right)$ *is a complete metric linear space.*

Proof. (i) $\|f\|_{L^0(\nu)} = 0$ if and only if $f = 0$ ν-a.e. If $\|f\|_{L^0(\nu)} = 0$, then $\left\| \frac{|f|}{1+|f|}\chi_{\Omega_k} \right\|_{L^1_w(m_k)} = 0$ for all $k = 1, 2, \ldots$ and hence $f = 0$ on Ω_k m_k-a.e. for all $k = 1, 2, \ldots$. Now take into account the comment above the definition of $\|\cdot\|_{L^0(\nu)}$ to conclude that $f = 0$ ν-a.e.
Since the function $t \in [0, \infty) \to \frac{t}{1+t} \in [0, \infty)$ is increasing the next properties follow:

(ii) $\|\alpha f\|_{L^0(\nu)} \leq \|f\|_{L^0(\nu)}$ if $|\alpha| \leq 1$ and $f \in L^0(\nu)$, and
(iii) $\|f + g\|_{L^0(\nu)} \leq \|f\|_{L^0(\nu)} + \|g\|_{L^0(\nu)}$ for all $f, g \in L^0(\nu)$.

Next let us see that

(iv) $\|\alpha_n f\|_{L^0(\nu)} \to 0$ if $f \in L^0(\nu)$ and $\alpha_n \to 0$.

Indeed, if $(\alpha_n)_n$ is a sequence of scalars with $\alpha_n \to 0$, then $\frac{|\alpha_n f|}{1+|\alpha_n f|}\chi_{\Omega_k}$ converges pointwise to 0 on Ω_k m_k-a.e. for all $k = 1, 2, \ldots$. The order continuity of the space $L^1(\nu_k)$ means that $\left\| \frac{|\alpha_n f|}{1+|\alpha_n f|}\chi_{\Omega_k} \right\|_{L^1_w(m_k)} \to 0$ for all $k = 1, 2, \ldots$. Thus Lemma 3.2 assures that $\|\alpha_n f\|_{L^0(\nu)} \to 0$ as $n \to \infty$.
Finally, from the inequality $\|\alpha f\|_{L^0(\nu)} \leq \max\{1, \alpha\}\,\|f\|_{L^0(\nu)}$, where $f \in L^0(\nu)$ and $\alpha \in \mathbb{R}$, it follows that

(v) $\|\alpha f_n\|_{L^0(\nu)} \to 0$ if $\alpha \in \mathbb{R}$ and $\|f_n\|_{L^0(\nu)} \to 0$.

Properties (i)–(v) mean that $\|\cdot\|_{L^0(\nu)}$ is an F-norm on $L^0(\nu)$. Finally we are going to check that $\left(L^0(\nu), \|\cdot\|_{L^0(\nu)}\right)$ is complete. Take a Cauchy sequence $(f_n)_n$ in $L^0(\nu)$. Then, for every $k = 1, 2, \ldots$, $\left(\frac{|f_n|}{1+|f_n|}\chi_{\Omega_k}\right)_n$ is a Cauchy sequence in the Banach space $L^1(m_k)$. Thus for every $k = 1, 2, \ldots$ there exists $g_k \in L^1(m_k)$ such that $\left\| g_k - \frac{|f_n|}{1+|f_n|}\chi_{\Omega_k} \right\|_{L^1_w(m_k)} \to 0$ as $n \to \infty$. Define pointwise the function $g := \sum_{k\geq 1} g_k \in L^0(\nu)$ and conclude that $\|g - f_n\|_{L^0(\nu)} \to 0$, as $n \to \infty$, with the same argument as in Lemma 3.2. $\square$

Lemma 3.4. *For all $p \geq 1$, the space $L^p_w(\nu)$ is continuously included into $L^0(\nu)$.*

Proof. If $f \in L^1_w(\nu)$, then $\left\| \frac{|f|}{1+|f|} \chi_{\Omega_k} \right\|_{L^1_w(m_k)} \le \left\| f \chi_{\Omega_k} \right\|_{L^1_w(m_k)} \le \|f\|_{L^1_w(\nu)}$ for all $k = 1, 2, \ldots$ and $\|f\|_{L^0(\nu)} \le \|f\|_{L^1_w(\nu)}$. If $f \in L^p_w(\nu)$, with $p > 1$, Lemma 2.1 assures that $f \chi_{\Omega_k} \in L^1(\nu)$ for all $k = 1, 2, \ldots$ and then $\left\| \frac{|f|}{1+|f|} \chi_{\Omega_k} \right\|_{L^1_w(m_k)} \le \left\| f \chi_{\Omega_k} \right\|_{L^1_w(\nu)} \le \|f\|_{L^p_w(\nu)} \left(\|\nu\| (\Omega_k) \right)^{\frac{1}{q}}$, where $q > 1$ is the conjugate exponent of p. Thus

$$\begin{aligned} \|f\|_{L^0(\nu)} &:= \sum_{k=1}^{\infty} \frac{1}{2^k \left(1 + \|\nu\| (\Omega_k)\right)} \left\| \frac{|f|}{1+|f|} \chi_{\Omega_k} \right\|_{L^1_w(m_k)} \\ &\le \sum_{k=1}^{\infty} \frac{\left(\|\nu\| (\Omega_k)\right)^{\frac{1}{q}}}{2^k \left(1 + \|\nu\| (\Omega_k)\right)} \|f\|_{L^p_w(\nu)} \le \|f\|_{L^p_w(\nu)} . \quad \square \end{aligned}$$

The key to obtaining the $(\star)$-formulas for the interpolated spaces is the Calderón–Lozanovskii's product space. Let us now recall the basic properties of this space that we can see in [4]. Let $\nu : \mathcal{R} \to X$ be a σ-finite vector measure. For a given couple (X_0, X_1) of Banach lattice ideals of $L^0(\nu)$ and $0 \le \theta \le 1$, the *Calderón–Lozanovskii's product space* $X_0^{1-\theta} X_1^{\theta}$ is the Banach space of all (ν-a.e. equivalence classes of) scalar measurable functions $f \in L^0(\nu)$ such that there exist $f_0 \in B_1(X_0)$, $f_1 \in B_1(X_1)$ and $\lambda > 0$ for which

$$|f(w)| \le \lambda \, |f_0(w)|^{1-\theta} \, |f_1(w)|^{\theta}, \quad w \in \Omega \ (\nu\text{-a.e.}) \tag{2}$$

endowed with the norm $\|f\|_{X_0^{1-\theta} X_1^{\theta}} = \inf \lambda$, where the infimum is taken over those λ satisfying (2). The Calderón–Lozanovskii's product space has the following relationships to the Calderón interpolation spaces.

(CL1) $X_0 \cap X_1 \subseteq [X_0, X_1]_{[\theta]} \subseteq X_0^{1-\theta} X_1^{\theta} \subseteq [X_0, X_1]^{[\theta]} \subseteq X_0 + X_1$. Moreover we have equality of norms (see [1, Theorem]), that is,

$$\|x\|_{[X_0, X_1]_{[\theta]}} = \|x\|_{X_0^{1-\theta} X_1^{\theta}} = \|x\|_{[X_0, X_1]^{[\theta]}}, \quad x \in [X_0, X_1]_{[\theta]} . \tag{3}$$

(CL2) If X_0 or X_1 is order continuous, then $[X_0, X_1]_{[\theta]} = X_0^{1-\theta} X_1^{\theta}$.
(CL3) If X_0 and X_1 have the Fatou property then $[X_0, X_1]^{[\theta]} = X_0^{1-\theta} X_1^{\theta}$.

Let us compute the Calderón–Lozanovskii's products of spaces of p-integrable functions. The key is the following result.

Proposition 3.5. *Let $1 < p, q < \infty$ be conjugate exponents. Then*

(i) $\mathcal{S}(\mathcal{R}) \cdot L^p_w(\nu) \subseteq L^1(\nu)$.
(ii) $L^p_w(\nu) \cdot L^q_w(\nu) = L^1_w(\nu)$, *with* $\|fg\|_{L^1_w(\nu)} \le \|f\|_{L^p_w(\nu)} \|g\|_{L^q_w(\nu)}$.
(iii) $L^p(\nu) \cdot L^q(\nu) = L^p(\nu) \cdot L^q_w(\nu) = L^1(\nu)$.

Proof. (i) This inclusion follows from Lemma 2.1, because functions in $\mathcal{S}(\mathcal{R})$ are linear combinations of characteristic functions of subsets in $\mathcal{R}$.

(ii) Let $f \in L^p_w(\nu)$, $g \in L^q_w(\nu)$. The Hölder inequality gives $fg \in L^1(|\langle \nu, x^* \rangle|)$, for all $x^* \in X^*$, and moreover, if $x^* \in B(X^*)$, then

$$\int_{\Omega} |fg| \, d|\langle \nu, x^* \rangle| \le \|f\|_{L^p(|\langle \nu, x^* \rangle|)} \|g\|_{L^q(|\langle \nu, x^* \rangle|)} \le \|f\|_{L^p_w(\nu)} \|g\|_{L^q_w(\nu)} .$$

Therefore, $fg \in L^1_w(\nu)$ with $\|fg\|_{L^1_w(\nu)} \le \|f\|_{L^p_w(\nu)} \|g\|_{L^q_w(\nu)}$. Conversely, if $0 \le h \in L^1_w(\nu)$ then $h = h^{\frac{1}{p}} h^{\frac{1}{q}}$, with $h^{\frac{1}{p}} \in L^p_w(\nu)$ and $h^{\frac{1}{q}} \in L^q_w(\nu)$.

(iii) Clearly, $L^1(\nu) \subseteq L^p(\nu) \cdot L^q(\nu) \subseteq L^p(\nu) \cdot L^q_w(\nu)$. Let $f \in L^p(\nu)$, and $g \in L^q_w(\nu)$. There exists $(s_n)_n \subseteq \mathcal{S}(\mathcal{R})$ such that $s_n \to f$ in $L^p(\nu)$. From (i), it follows that $(s_n g)_n \subseteq L^1(\nu)$. Moreover,

$$\|fg - s_n g\|_{L^1_w(\nu)} = \|(f - s_n) g\|_{L^1_w(\nu)} \le \|f - s_n\|_{L^p_w(\nu)} \|g\|_{L^q_w(\nu)} \to 0,$$

which yields $(s_n g)_n \to fg$ in $L^1_w(\nu)$. Since $L^1(\nu)$ is closed in $L^1_w(\nu)$ we conclude that $fg \in L^1(\nu)$. $\square$

As we mentioned above, Proposition 3.5 allows us to compute the Calderón–Lozanovskii's product spaces of several couples of L^p and L^p_w-spaces.

Corollary 3.6. *Let $1 \le p_0 < p_1 < \infty$, $0 < \theta < 1$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then*

(i) $L^{p_0}(\nu)^{1-\theta} L^{p_1}(\nu)^{\theta} = L^p(\nu)$.
(ii) $L^{p_0}_w(\nu)^{1-\theta} L^{p_1}(\nu)^{\theta} = L^{p_0}(\nu)^{1-\theta} L^{p_1}_w(\nu)^{\theta} = L^p(\nu)$.
(iii) $L^{p_0}_w(\nu)^{1-\theta} L^{p_1}_w(\nu)^{\theta} = L^p_w(\nu)$.

Proof. It is enough to observe that $\frac{p_0}{(1-\theta)p}$ and $\frac{p_1}{\theta p}$ are conjugate exponents. Now, apply Proposition 3.5. $\square$

From Corollary 3.6, and equalities described in (CL2) and (CL3), it follows that

Corollary 3.7. *If* $1 \le p_0 < p_1 < \infty$, $0 < \theta < 1$, *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, *then*

$$\left[L^{p_0}(\nu), L^{p_1}(\nu)\right]_{[\theta]} = \left[L_w^{p_0}(\nu), L^{p_1}(\nu)\right]_{[\theta]} = \left[L^{p_0}(\nu), L_w^{p_1}(\nu)\right]_{[\theta]} = L^p(\nu),$$
$$\left[L_w^{p_0}(\nu), L_w^{p_1}(\nu)\right]^{[\theta]} = L_w^p(\nu).$$

The simplest example of a σ-finite vector measure on a δ-ring is given by a σ-finite measure space (Ω, Σ, μ) if we consider the measure μ defined on the δ-ring of measurable subsets of finite measure. For example, consider the Lebesgue measure λ on the σ-algebra $\mathcal{M}$ of Lebesgue measurable subsets of the real line $\mathbb{R}$. Let $\mathcal{R} := \{A \in \mathcal{M} : \lambda(A) < \infty\}$ and define the vector measure $\nu : A \in \mathcal{R} \longrightarrow \nu(A) = \lambda(A) \in \mathbb{R}$. Then $L_w^p(\nu) = L^p(\nu) = L^p(\mathbb{R})$ for all $p \ge 1$ and Corollary 3.7 assures that

$$\left[L^{p_0}(\mathbb{R}), L^{p_1}(\mathbb{R})\right]_{[\theta]} = \left[L^{p_0}(\mathbb{R}), L^{p_1}(\mathbb{R})\right]^{[\theta]} = L^p(\mathbb{R}),$$

as we have mentioned in the introduction.

Remark 3.8. Let $1 \le p_0 < p < p_1 < \infty$. From (CL1) and the above corollary we obtain the following inclusions:

(i) $L_w^{p_0}(\nu) \cap L_w^{p_1}(\nu) \subseteq L_w^p(\nu) \subseteq L_w^{p_0}(\nu) + L_w^{p_1}(\nu)$.
(ii) $L^{p_0}(\nu) \cap L^{p_1}(\nu) \subseteq L^p(\nu) \subseteq L^{p_0}(\nu) + L^{p_1}(\nu)$.
(iii) $L^{p_0}(\nu) \cap L_w^{p_1}(\nu) \subseteq L^p(\nu) \subseteq L^{p_0}(\nu) + L_w^{p_1}(\nu)$.
(iv) $L_w^{p_0}(\nu) \cap L^{p_1}(\nu) \subseteq L^p(\nu) \subseteq L_w^{p_0}(\nu) + L^{p_1}(\nu)$.

Each of them assures that the corresponding space that is in the middle of the inclusions is an *intermediate space*. Nevertheless, for a general vector measure ν on a δ-ring and $p_0 < p < p_1$, the space $L^p(\nu)$ does not need to be an intermediate space of the couple $\left(L_w^{p_0}(\nu), L_w^{p_1}(\nu)\right)$ because in some cases $L_w^{p_0}(\nu) \cap L_w^{p_1}(\nu) \not\subseteq L^p(\nu)$. Analogously the space $L_w^p(\nu)$ does not need to be an intermediate space of the couple $(L^{p_0}(\nu), L^{p_1}(\nu))$ because in such cases $L_w^p(\nu) \not\subseteq L^{p_0}(\nu) + L^{p_1}(\nu)$. The following example illustrates the above statements.

Example 3.9. Let $\mathcal{R}$ be the δ-ring of finite subsets of natural numbers $\mathbb{N}$, and consider the σ-finite vector measure $\nu : A \in \mathcal{R} \longrightarrow \nu(A) := \chi_A \in c_0(\mathbb{N})$, where $c_0(\mathbb{N})$ is the space of null sequences. For every $1 \le p < \infty$, it is easy to check that $L_w^p(m) = \ell^\infty(\mathbb{N})$, the space of bounded sequences, and $L^p(m) = c_0(\mathbb{N})$. In what follows it will be interesting to know that $\|\nu\|(A) = 1$, for every nonempty $A \subseteq \mathbb{N}$, and $\|\nu\|(\varnothing) = 0$.

As we noted in the introduction, if ν is a vector measure over a σ-algebra, then it is known that, in addition to the equalities established in the above Corollary 3.7, the following equalities hold $[L_w^{p_0}(\nu), L_w^{p_1}(\nu)]_{[\theta]} = L^p(\nu)$ and $[L^{p_0}(\nu), L^{p_1}(\nu)]^{[\theta]} = [L_w^{p_0}(\nu), L^{p_1}(\nu)]^{[\theta]} = [L^{p_0}(\nu), L_w^{p_1}(\nu)]^{[\theta]} = L_w^p(\nu)$. Nevertheless, the situation can be completely different in δ-rings as the next example shows.

Example 3.10. Consider the vector measure ν of Example 3.9. For every $1 \le p < \infty$, we know that $L_w^p(\nu) = \ell^\infty(\mathbb{N})$, and also $L^p(\nu) = c_0(\mathbb{N})$. Thus, for all $1 \le p_0 < p < p_1 < \infty$, we have

$$\left[L_w^{p_0}(\nu), L_w^{p_1}(\nu)\right]_{[\theta]} = \left[\ell^\infty(\mathbb{N}), \ell^\infty(\mathbb{N})\right]_{[\theta]} = \ell^\infty(\mathbb{N}) = L_w^p(\nu),$$
$$\left[L^{p_0}(\nu), L^{p_1}(\nu)\right]^{[\theta]} = [c_0(\mathbb{N}), c_0(\mathbb{N})]^{[\theta]} = c_0(\mathbb{N}) = L^p(\nu).$$

But there are cases where the situation is similar to the case of σ-algebras even for measures genuinely defined on δ-rings.

Example 3.11. With the same notation of the previous examples, let us consider now the vector measure (defined on the same δ-ring $\mathcal{R}$)

$$\nu : A \in \mathcal{R} \longrightarrow \nu(A) := \alpha \cdot \chi_A \in c_0(\mathbb{N}),$$

where $\alpha = (\alpha_n)_n$ is the sequence given by $\alpha_n = n$, for all $n = 1, 2, \dots$. It is easy to check, for all $1 \le p < \infty$, that

$$L_w^p(\nu) = \ell^\infty\left(\alpha^{\frac{1}{p}}\right) := \left\{(a_n)_n : \left(n^{\frac{1}{p}} a_n\right)_n \in \ell^\infty(\mathbb{N})\right\},$$
$$L^p(\nu) = c_0\left(\alpha^{\frac{1}{p}}\right) := \left\{(a_n)_n : \left(n^{\frac{1}{p}} a_n\right)_n \in c_0(\mathbb{N})\right\}.$$

In this case, we get the $(\diamond)$-formulas, that is, for all $1 \le p_0 < p_1 < \infty$ and $0 < \theta < 1$, we have

$$\left[L_w^{p_0}(\nu), L_w^{p_1}(\nu)\right]_{[\theta]} = L^p(\nu), \tag{4}$$

and

$$\left[L^{p_0}(\nu), L^{p_1}(\nu)\right]^{[\theta]} = \left[L^{p_0}_w(\nu), L^{p_1}(\nu)\right]^{[\theta]} = \left[L^{p_0}(\nu), L^{p_1}_w(\nu)\right]^{[\theta]} = L^p_w(\nu), \tag{5}$$

where $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Let us see how to obtain equality (4). The proof of equalities (5) must be postponed until Corollary 4.9 because we do not know an easy computation to obtain them. To prove equality (4) it is enough to have in mind the following:

(A) $c_0(\alpha^{\frac{1}{p}}) \subseteq \ell^\infty(\alpha^{\frac{1}{p}}) \subseteq c_0(\alpha^{\frac{1}{q}}) \subseteq \ell^\infty(\alpha^{\frac{1}{q}}),\ 1 \le p < q < \infty$.
(B) $\overline{\ell^\infty(\alpha^{\frac{1}{p}})}^{\ell^\infty(\alpha^{\frac{1}{q}})} = c_0(\alpha^{\frac{1}{q}}),\ 1 \le p < q < \infty$.
(C) $\left(\ell^\infty(\alpha^{\frac{1}{p_0}})\right)^{1-\theta} \left(c_0(\alpha^{\frac{1}{p_1}})\right)^{\theta} = c_0(\alpha^{\frac{1}{p}})$ (cf. Corollary 3.6(ii)).

Then, taking into account [2, Theorem 4.2.2(b)],

$$\begin{aligned}\left[L^{p_0}_w(\nu), L^{p_1}_w(\nu)\right]_{[\theta]} &= \left[\ell^\infty\left(\alpha^{\frac{1}{p_0}}\right), \ell^\infty\left(\alpha^{\frac{1}{p_1}}\right)\right]_{[\theta]} = \left[\ell^\infty\left(\alpha^{\frac{1}{p_0}}\right), \overline{\ell^\infty\left(\alpha^{\frac{1}{p_0}}\right)}^{\ell^\infty\left(\alpha^{\frac{1}{p_1}}\right)}\right]_{[\theta]} \\ &= \left[\ell^\infty\left(\alpha^{\frac{1}{p_0}}\right), c_0\left(\alpha^{\frac{1}{p_1}}\right)\right]_{[\theta]} = \left(\ell^\infty\left(\alpha^{\frac{1}{p_0}}\right)\right)^{1-\theta}\left(c_0\left(\alpha^{\frac{1}{p_1}}\right)\right)^{\theta} \\ &= c_0\left(\alpha^{\frac{1}{p}}\right) = L^p(\nu).\end{aligned}$$

Let us mention for this measure that for every $A \subseteq \mathbb{N}$ we have $\|\nu\|(A) = \max A$ if A is finite, and $\|\nu\|(A) = \infty$ if A is infinite.

4. Interpolation for locally strongly additive measures

As we have seen in Example 3.9, for a σ-finite vector measure ν on a δ-ring and $p_0 < p < p_1$, the space $L^p(\nu)$ does not need to be an *intermediate* space of the couple $\left(L^{p_0}_w(\nu), L^{p_1}_w(\nu)\right)$. However, there is a broad class of vector measures for which this occurs: locally strongly additive vector measures (see Theorem 4.5). Recall that a vector measure $\nu : \mathcal{R} \to X$ is called *locally strongly additive* if $\lim_{n\to\infty} \|\nu(A_n)\|_X = 0$ for all disjoint sequences $(A_n)_n$ in $\mathcal{R}$ such that $\|\nu\|\left(\cup_{n\ge1} A_n\right) < \infty$. This concept of locally strongly additivity differs a bit from that of Brooks and Dinculeanu [3], where *locally* means that the property is satisfied inside a set of the δ-ring $\mathcal{R}$ instead of a measurable set of finite semivariation. Note that the vector measure we have considered in the previous Example 3.11 is locally strongly additive, but the vector measure we considered in Example 3.9 is not locally strongly additive. In what follows we continue with a σ-finite vector measure $\nu : \mathcal{R} \to X$.

Lemma 4.1. *Let $B \in \mathcal{R}^{\mathrm{loc}}$. Then*

(1) $\chi_B \in L^1_w(\nu)$ *if and only if* $\|\nu\|(B) < \infty$.
(2) $\chi_B \in L^1(\nu)$ *if and only if* $\lim_{n\to\infty} \|\nu(A_n)\| = 0$ *for all disjoint sequences $(A_n)_n$ in $\mathcal{R}$ such that $A_n \subseteq B$, for all $n = 1, 2, \ldots$.*

Moreover, the following conditions are equivalent:

(A) *ν is locally strongly additive.*
(B) *If $B \in \mathcal{R}^{\mathrm{loc}}$ and $\chi_B \in L^1_w(\nu)$, then $\chi_B \in L^1(\nu)$.*
(C) *There is no set $B \in \mathcal{R}^{\mathrm{loc}}$ such that $\chi_B \in L^1_w(\nu) \setminus L^1(\nu)$.*

Proof. (1) If $B \in \mathcal{R}^{\mathrm{loc}}$, it is enough to note that

$$\|\nu\|(B) = \sup\left\{\left|\langle \nu, x^*\rangle\right| : x^* \in B(X^*)\right\} = \|\chi_B\|_{L^1_w(\nu)}.$$

(2) Suppose $\chi_B \in L^1(\nu)$ and let $(A_n)_n \subseteq \mathcal{R}$ be a pairwise disjoint sequence such that $A_n \subseteq B$, for all $n = 1, 2, \ldots$. Denote by $A := \cup_{n\ge1} A_n$. Then $\chi_A \le \chi_B$, and so $\chi_A \in L^1(\nu)$. Moreover, the order continuity of $L^1(\nu)$ implies that $\sum_{n\ge1} \chi_{A_n} = \chi_A$ in $L^1(\nu)$, so $\|\nu(A_n)\| \le \left\|\chi_{A_n}\right\|_{L^1_w(\nu)} \to 0$, as $n \to \infty$. Reciprocally, suppose that $\lim_{n\to\infty} \|\nu(A_n)\| = 0$ for all pairwise disjoint sequences $(A_n)_n$ in $\mathcal{R}$ such that $A_n \subseteq B$, for all $n = 1, 2, \ldots$. This means that the vector measure $\nu_B : A \in \mathcal{R} \longrightarrow \nu_B(A) := \nu(B \cap A) \in X$ is strongly additive, which is equivalent to $\chi_\Omega \in L^1(\nu_B)$ (see [6, Corollary 3.2(b)]). Moreover, for a function $f \in L^0(\nu)$ it is not difficult to check that $f \in L^1(\nu_B)$ if and only if $f\chi_B \in L^1(\nu)$. Thus, $\chi_B \in L^1(\nu)$ and the equivalence is over.
Finally note that (C) is a reformulation of (B) and the equivalence between (A) and (B) follows by applying characterizations (1) and (2). □

Notation 4.2. In what follows it will be convenient to consider the following notation. For a nonnegative measurable function $f : \Omega \longrightarrow \mathbb{R}$, and two real numbers $0 < a < b$, consider the three disjoint measurable subsets of Ω

$$\begin{aligned}
&[f < a] := \{w \in \Omega : 0 \le f(w) < a\} \in \mathcal{R}^{\mathrm{loc}}, \\
&[a \le f \le b] := \{w \in \Omega : a \le f(w) \le b\} \in \mathcal{R}^{\mathrm{loc}}, \quad \text{and} \\
&[f > b] := \{w \in \Omega : f(w) > b\} \in \mathcal{R}^{\mathrm{loc}}.
\end{aligned}$$

The next two lemmas will be useful in what follows.

Lemma 4.3. *Let* $1 \le p_0 < p < p_1 < \infty$.

(1) *If* $0 \le f \in L^p_w(\nu)$, *then*
 (i) $f\chi_{[f>b]} \in L^{p_0}_w(\nu)$, *and* $\lim_{b\to\infty} \left\|f\chi_{[f>b]}\right\|_{L^{p_0}_w(\nu)} = 0$.
 (ii) $f\chi_{[f<a]} \in L^{p_1}_w(\nu)$, *and* $\lim_{a\to 0} \left\|f\chi_{[f<a]}\right\|_{L^{p_1}_w(\nu)} = 0$.

(2) *If* $0 \le f \in L^{p_0}_w(\nu) \cap L^{p_1}_w(\nu)$, *then*

$$\lim_{b\to\infty} \left\|f\chi_{[f>b]}\right\|_{L^p_w(\nu)} = \lim_{a\to 0} \left\|f\chi_{[f<a]}\right\|_{L^p_w(\nu)} = 0.$$

Proof. (1i) Note that $f^{p_0}\chi_{[f>b]} = f^p f^{p_0-p}\chi_{[f>b]} \le \frac{1}{b^{p-p_0}} f^p\chi_{[f>b]} \in L^1_w(\nu)$, which means that $f\chi_{[f>b]} \in L^{p_0}_w(\nu)$. Taking norm in the above inequalities we have

$$\left\|f^{p_0}\chi_{[f>b]}\right\|_{L^1_w(\nu)} \le b^{p_0-p} \left\|f^p\chi_{[f>b]}\right\|_{L^1_w(\nu)} \le b^{p_0-p} \left\|f^p\right\|_{L^1_w(\nu)} \to 0,$$

as $b \to \infty$, that is, $\lim_{b\to\infty} \left\|f\chi_{[f>b]}\right\|_{L^{p_0}_w(\nu)} = 0$.

(1ii) In that case $f^{p_1}\chi_{[f<a]} = f^p f^{p_1-p}\chi_{[f<a]} \le a^{p_1-p} f^p\chi_{[f<a]} \in L^1_w(\nu)$, so we have $f\chi_{[f<a]} \in L^{p_1}_w(\nu)$. Now, taking norm

$$\left\|f^{p_1}\chi_{[f<a]}\right\|_{L^1_w(\nu)} \le a^{p_1-p} \left\|f^p\chi_{[f<a]}\right\|_{L^1_w(\nu)} \to 0,$$

as $a \to 0$, that is, $\lim_{a\to 0} \left\|f\chi_{[f<a]}\right\|_{L^{p_1}_w(\nu)} = 0$.

(2) According to Remark 3.8 the function $f \in L^p_w(\nu)$ and so the functions $f^p\chi_{[f<a]}$ and $f^p\chi_{[f>b]}$ belong to $L^1_w(\nu)$ too. Moreover, using the above arguments we have

$$\begin{aligned}
\left\|f^p\chi_{[f<a]}\right\|_{L^1_w(\nu)} &\le a^{p-p_0} \left\|f^{p_0}\chi_{[f<a]}\right\|_{L^1_w(\nu)} \le a^{p-p_0} \left\|f^{p_0}\right\|_{L^1_w(\nu)}, \\
\left\|f^p\chi_{[f>b]}\right\|_{L^1_w(\nu)} &\le b^{p-p_1} \left\|f^{p_1}\chi_{[f>b]}\right\|_{L^1_w(\nu)} \le b^{p-p_1} \left\|f^{p_1}\right\|_{L^1_w(\nu)},
\end{aligned}$$

that is, $\lim_{b\to\infty} \left\|f\chi_{[f>b]}\right\|_{L^p_w(\nu)} = \lim_{a\to 0} \left\|f\chi_{[f<a]}\right\|_{L^p_w(\nu)} = 0$. □

Lemma 4.4. *Let* $0 \le f \in L^0(\nu)$, $1 \le p_0 < p_1 < \infty$, *and* $0 \le a < b$.

(A) *If* $f\chi_{[a\le f\le b]} \in L^{p_0}_w(\nu) + L^{p_1}_w(\nu)$, *then* $f\chi_{[a\le f\le b]} \in L^{p_0}_w(\nu) \cap L^{p_1}_w(\nu)$.
(B) *If* $f\chi_{[a\le f\le b]} \in L^{p_0}(\nu) + L^{p_1}(\nu)$, *then* $f\chi_{[a\le f\le b]} \in L^{p_0}(\nu) \cap L^{p_1}(\nu)$.

Proof. (A) Assume that $f\chi_{[a\le f\le b]}$ belongs to $L^{p_0}_w(\nu) + L^{p_1}_w(\nu)$, so there exist $0 \le f_0 \in L^{p_0}_w(\nu)$ and $0 \le f_1 \in L^{p_1}_w(\nu)$ such that $f\chi_{[a\le f\le b]} = f_0 + f_1$. On the one hand note that $f_0^{p_1} \le b^{p_1-p_0} f_0^{p_0}$ since $f_0 \le f\chi_{[a\le f\le b]} \le b$. Therefore,

$$f^{p_1}\chi_{[a\le f\le b]} = (f_0 + f_1)^{p_1} \le 2^{p_1}(f_0^{p_1} + f_1^{p_1}) \le 2^{p_1}(b^{p_1-p_0} f_0^{p_0} + f_1^{p_1}) \in L^1_w(\nu),$$

which proves that $f\chi_{[a\le f\le b]} \in L^{p_1}_w(\nu)$. In order to prove that $f\chi_{[a\le f\le b]}$ also belongs to $L^{p_0}_w(\nu)$, consider the disjoint sets of $\mathcal{R}^{\mathrm{loc}}$

$$\begin{aligned}
D &:= \{u \in [a \le f \le b] : f_0(u) \le f_1(u)\}, \\
E &:= \{u \in [a \le f \le b] : f_1(u) < f_0(u)\},
\end{aligned}$$

and observe that

$$f = f_0 + f_1 = (f_0 + f_1)\chi_D + (f_0 + f_1)\chi_E \le 2f_1\chi_D + 2f_0\chi_E,$$

and also that $f_1\chi_D \ge \frac{a}{2}$ since $a \le f\chi_D \le 2f_1\chi_D$. Thus,

$$f^{p_0}\chi_{[a\le f\le b]} \le 2^{p_0} f_1^{p_0}\chi_D + 2^{p_0} f_0^{p_0}\chi_E \le \frac{2^{p_1}}{a^{p_1-p_0}} f_1^{p_1}\chi_D + 2^{p_0} f_0^{p_0}\chi_E \in L^1_w(\nu),$$

which proves that $f\chi_{[a\le f\le b]} \in L^{p_0}(\nu)$.

(B) The proof is similar to (A). □

Theorem 4.5. *Let $1 \le p_0 < p_1 < \infty$. The following are equivalent:*

(i) *ν is locally strongly additive.*
(ii) *$L_w^{p_0}(\nu) \cap L_w^{p_1}(\nu) \subseteq L^p(\nu)$, for some/all $p_0 < p < p_1$.*
(iii) *$L_w^{p}(\nu) \subseteq L^{p_0}(\nu) + L^{p_1}(\nu)$, for some/all $p_0 < p < p_1$.*

Proof. (i) $\Rightarrow$ (ii) Let $p_0 < p < p_1$ and take $0 \le f \in L_w^{p_0}(\nu) \cap L_w^{p_1}(\nu)$. Let us consider the sets $\left[f < \frac{1}{n}\right]$, $\left[\frac{1}{n} \le f \le n\right]$, and $[f > n]$, for all $n = 1, 2, \dots$. As we know, all these sets are in $\mathcal{R}^{\mathrm{loc}}$ since f is measurable. According to Remark 3.8, $f \in L_w^p(\nu)$ and so the functions $f^p \chi_{\left[f<\frac{1}{n}\right]}$, $f^p \chi_{\left[\frac{1}{n} \le f \le n\right]}$, and $f^p \chi_{[f>n]}$ belong to $L_w^1(\nu)$, for all $n = 1, 2, \dots$. From the inequalities

$$\frac{1}{n^p} \chi_{\left[\frac{1}{n} \le f \le n\right]} \le f^p \chi_{\left[\frac{1}{n} \le f \le n\right]} \le n^p \chi_{\left[\frac{1}{n} \le f \le n\right]}, \quad n = 1, 2, \dots \tag{6}$$

we conclude that $\chi_{\left[\frac{1}{n} \le f \le n\right]} \in L_w^1(\nu)$, for all $n = 1, 2, \dots$. By the hypothesis and Lemma 4.1 we get $\chi_{\left[\frac{1}{n} \le f \le n\right]} \in L^1(\nu)$, for all $n = 1, 2, \dots$. But, applying again inequalities (6) we obtain that $f^p \chi_{\left[\frac{1}{n} \le f \le n\right]} \in L^1(\nu)$, for all $n \in \mathbb{N}$. On the other hand, Lemma 4.3 assures that

$$\lim_{n\to\infty} \left\| f^p \chi_{\left[f<\frac{1}{n}\right]} \right\|_{L_w^1(\nu)} = \lim_{n\to\infty} \left\| f^p \chi_{[f>n]} \right\|_{L_w^1(\nu)} = 0,$$

and therefore,

$$\left\| f^p - f^p \chi_{\left[\frac{1}{n} \le f \le n\right]} \right\|_{L_w^1(\nu)} \le \left\| f^p \chi_{\left[f<\frac{1}{n}\right]} \right\|_{L_w^1(\nu)} + \left\| f^p \chi_{[f>n]} \right\|_{L_w^1(\nu)} \to 0,$$

when $n \to \infty$, which says that $\left(f^p \chi_{\left[\frac{1}{n} \le f \le n\right]}\right)_n$ converges to f^p in $L_w^1(\nu)$. Hence, f^p must be in $L^1(\nu)$ (or equivalently $f \in L^p(\nu)$), since $L^1(\nu)$ is closed in $L_w^1(\nu)$.

(ii) $\Rightarrow$ (iii) Let $p_0 < p < p_1$ and assume that $L_w^{p_0}(\nu) \cap L_w^{p_1}(\nu) \subseteq L^p(\nu)$. Let us see that $L_w^p(\nu) \subseteq L^{p_0}(\nu) + L^{p_1}(\nu)$. Let $0 \le f \in L_w^p(\nu)$ and consider again the sets $\left[f < \frac{1}{n}\right]$, $\left[\frac{1}{n} \le f \le n\right]$, and $[f > n]$ for $n = 1, 2, \dots$. By applying Lemma 4.3 we obtain $f\chi_{\left[f<\frac{1}{n}\right]} \in L_w^{p_1}(\nu)$, $f\chi_{[f>n]} \in L_w^{p_0}(\nu)$, and moreover

$$\lim_{n\to\infty} \left\| f\chi_{\left[f<\frac{1}{n}\right]} \right\|_{L_w^{p_1}(\nu)} = \lim_{n\to\infty} \left\| f\chi_{[f>n]} \right\|_{L_w^{p_0}(\nu)} = 0. \tag{7}$$

As $L_w^p(\nu) \subseteq L_w^{p_0}(\nu) + L_w^{p_1}(\nu)$, Lemma 4.4 leads to

$$f\chi_{\left[\frac{1}{n} \le f \le n\right]} \in L_w^{p_0}(\nu) \cap L_w^{p_1}(\nu), \quad n = 1, 2, \dots. \tag{8}$$

From (7) and (8) we obtain also that

$$\left\| f - f\chi_{\left[\frac{1}{n} \le f \le n\right]} \right\|_{L_w^{p_0}(\nu)+L_w^{p_1}(\nu)} \le \left\| f\chi_{\left[f<\frac{1}{n}\right]} \right\|_{L_w^{p_1}(\nu)} + \left\| f\chi_{[f>n]} \right\|_{L_w^{p_0}(\nu)} \to 0,$$

when $n \to \infty$, which says that the sequence $\left(f\chi_{\left[\frac{1}{n} \le f \le n\right]}\right)_n$ converges to f in $L_w^{p_0}(\nu) + L_w^{p_1}(\nu)$. If $\left(f\chi_{\left[\frac{1}{n} \le f \le n\right]}\right)_n$ were a Cauchy sequence in $L^{p_0}(\nu) + L^{p_1}(\nu)$, then f would be in $L^{p_0}(\nu) + L^{p_1}(\nu)$ and this would finish the proof. First note that $f\chi_{\left[\frac{1}{n} \le f \le n\right]} \in L^{p_0}(\nu) + L^{p_1}(\nu)$ for $n = 1, 2, \dots$. This follows from (8), the hypothesis $L_w^{p_0}(\nu) \cap L_w^{p_1}(\nu) \subseteq L^p(\nu)$, and Remark 3.8(ii). Thus, we have to check for natural numbers $k < n$ that

$$\lim_{k\to\infty} \left\| f\chi_{\left[\frac{1}{n} \le f \le n\right]} - f\chi_{\left[\frac{1}{k} \le f \le k\right]} \right\|_{L_w^{p_0}(\nu)+L_w^{p_1}(\nu)} = 0.$$

Let $k, n \in \mathbb{N}$, with $k < n$. Since

$$\begin{aligned} f\chi_{\left[\frac{1}{n} \le f \le n\right]} - f\chi_{\left[\frac{1}{k} \le f \le k\right]} &= f\chi_{\left[\frac{1}{n} \le f \le n\right] \cap \left[f<\frac{1}{k}\right]} + f\chi_{\left[\frac{1}{n} \le f \le n\right] \cap [f>k]} \\ &= f\chi_{\left[\frac{1}{n} \le f < \frac{1}{k}\right]} + f\chi_{[k<f\le n]}, \end{aligned}$$

then, having in mind (7) we conclude that

$$\begin{aligned}\left\| f\chi_{\left[\frac{1}{n}\le f\le n\right]} - f\chi_{\left[\frac{1}{k}\le f\le k\right]}\right\|_{L^{p_0}_w(\nu)+L^{p_1}_w(\nu)} &\le \left\| f\chi_{\left[\frac{1}{n}\le f<\frac{1}{k}\right]}\right\|_{L^{p_0}_w(\nu)} + \left\| f\chi_{[k<f\le n]}\right\|_{L^{p_1}_w(\nu)}\\ &\le \left\| f\chi_{\left[f<\frac{1}{k}\right]}\right\|_{L^{p_0}_w(\nu)} + \left\| f\chi_{[k<f]}\right\|_{L^{p_1}_w(\nu)} \to 0,\end{aligned}$$

as $k\to\infty$.

(iii) ⇒ (i) Let $B\in\mathcal{R}^{\mathrm{loc}}$ such that $\chi_B\in L^1_w(\nu)$. Then $\chi_B\in L^p_w(\nu)$, and by the hypothesis $\chi_B\in L^{p_0}(\nu)+L^{p_1}(\nu)$, that is, $\chi_B=f_0+f_1$ for some $f_0\in L^{p_0}(\nu)$ and $f_1\in L^{p_1}(\nu)$. We can choose $f_0,f_1\ge 0$ and so $\sup\{f_0,f_1\}\le 1$. Since $f_0^{p_0}, f_1^{p_1}\in L^1(\nu)$ and $f_0^{p_1}\le f_0^{p_0}$ we have

$$\chi_B=(\chi_B)^{p_1}=(f_0+f_1)^{p_1}\le 2^{p_1}(f_0^{p_1}+f_1^{p_1})\le 2^{p_1}(f_0^{p_0}+f_1^{p_1})\in L^1(\nu).$$

Therefore $\chi_B\in L^1(\nu)$, and Lemma 4.1 ensures that ν is locally strongly additive. □

Remark 4.6. In relation to the proof of the above implication (ii) ⇒ (iii) let us mention the following comment. If Y_0 and Y_1 are Banach spaces and $X_0\subseteq Y_0$ and $X_1\subseteq Y_1$ are closed subspaces, in general $X_0+X_1\subseteq Y_0+Y_1$ is not a closed subspace of the sum. Even more, the sum of two closed subspaces of a Hilbert space need not be closed.

Let us see what happens when $L^p(\nu)$ is an intermediate space of the couple $\left(L^{p_0}_w(\nu), L^{p_1}_w(\nu)\right)$ as is described in Theorem 4.5.

Corollary 4.7. *Let $1\le p_0<p_1<\infty$. The following are equivalent:*

(1) *ν is locally strongly additive.*
(2) *$\left[L^{p_0}_w(\nu), L^{p_1}_w(\nu)\right]_{[\theta]}=L^p(\nu)$, where $0<\theta<1$, and $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$.*

Proof. (1) ⇒ (2) Applying Theorem 4.5 and Corollary 3.7, we have

$$\begin{aligned}L^{p_0}_w(\nu)\cap L^{p_1}_w(\nu) &\subseteq L^p(\nu)=\left[L^{p_0}(\nu), L^{p_1}(\nu)\right]_{[\theta]}\subseteq \left[L^{p_0}_w(\nu), L^{p_1}_w(\nu)\right]_{[\theta]}\\ &\subseteq [L^{p_0}_w(\nu), L^{p_1}_w(\nu)]^{[\theta]}=L^p_w(\nu).\end{aligned}$$

On the other hand, the norm in $L^p(\nu)$ is the restriction of the norm in $\left[L^{p_0}_w(\nu), L^{p_1}_w(\nu)\right]_{[\theta]}$, because $L^p(\nu)$ and $L^p_w(\nu)$ have the same norm, and as we know from (3) the norm in $\left[L^{p_0}_w(\nu), L^{p_1}_w(\nu)\right]_{[\theta]}$ is the restriction of the norm of $\left[L^{p_0}_w(\nu), L^{p_1}_w(\nu)\right]^{[\theta]}$. Being $L^p(\nu)$ a Banach space it is closed in $\left[L^{p_0}_w(\nu), L^{p_1}_w(\nu)\right]_{[\theta]}$, and we get the equality $L^p(\nu)=\left[L^{p_0}_w(\nu), L^{p_1}_w(\nu)\right]_{[\theta]}$ because $L^{p_0}_w(\nu)\cap L^{p_1}_w(\nu)$ is dense in $\left[L^{p_0}_w(\nu), L^{p_1}_w(\nu)\right]_{[\theta]}$ (see [2, Theorem 4.2.2]).

The implication (2) ⇒ (1) follows clearly from Theorem 4.5, because the inclusion $L^{p_0}_w(\nu)\cap L^{p_1}_w(\nu)\subseteq\left[L^{p_0}_w(\nu), L^{p_1}_w(\nu)\right]_{[\theta]}$ holds for all $0<\theta<1$. □

The key to obtaining the missing ($\diamond$)-formulas for the interpolated spaces is the Gustavsson–Peetre's method. Let us now recall briefly this method. Its detailed description appears in [9]. For a given couple (X_0,X_1) of Banach spaces and $0<\theta<1$, the *Gustavsson–Peetre space* $\langle X_0,X_1,\theta\rangle$ is the Banach space of those elements $x\in X_0+X_1$ for which there exists a sequence $(x_k)_{k\in\mathbb{Z}}$ of elements of $x_k\in X_0\cap X_1$ such that

(GP1) $x=\sum_{k\in\mathbb{Z}}x_k$, where the series converges in X_0+X_1, and
(GP2) there exists $C>0$ such that for every finite subset $F\subset\mathbb{Z}$ and every real sequence $(\varepsilon_k)_{k\in F}$ with $|\varepsilon_k|\le 1$ we have $\left\|\sum_{k\in F}\frac{\varepsilon_k}{2^{k\theta}}x_k\right\|_{X_0}\le C$, and $\left\|\sum_{k\in F}\frac{\varepsilon_k}{2^{k(\theta-1)}}x_k\right\|_{X_1}\le C$.

We equip $\langle X_0,X_1,\theta\rangle$ with the norm $\|x\|_{\langle X_0,X_1,\theta\rangle}=\inf C$, where the inf is taken over all sequences $(x_k)_{k\in\mathbb{Z}}$ permissible in (GP1) and (GP2). The relation of the Gustavsson–Peetre's interpolation space and the Calderón interpolation spaces is given (see [10, Theorem 5 and Section 7]) by the continuous inclusion

$$\langle X_0,X_1,\theta\rangle\subseteq[X_0,X_1]^{[\theta]}\,. \tag{GP3}$$

Corollary 4.8. *Let $1\le p_0<p_1<\infty$. The following are equivalent:*

(1) *ν is locally strongly additive.*
(2) *$L^p_w(\nu)\subseteq\langle L^{p_0}(\nu), L^{p_1}(\nu),\theta\rangle$, where $0<\theta<1$, and $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$.*

Proof. (1) ⇒ (2) Let $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and put

$$c := 2^{\frac{-(1-\theta)p_1}{p_1 - p}} = 2^{\frac{-\theta p_0}{p - p_0}} < 1.$$

Take an arbitrary function $0 \le f \in L^p_w(\nu)$ and for all $k \in \mathbb{Z}$, define $f_k := f\chi_{[c^k \le f < c^{k-1}]}$, which belongs to $L^p_w(\nu)$. Since ν is locally strongly additive, by Theorem 4.5 we have that $f, f_k \in L^{p_0}(\nu) + L^{p_1}(\nu)$, and applying Lemma 4.4 it follows that $f_k \in L^{p_0}(\nu) \cap L^{p_1}(\nu)$. We are going to check conditions (GP1) and (GP2) for the function f and the sequence $(f_k)_{k\in\mathbb{Z}}$.

(GP1) First note that $f = \sum_{k\in\mathbb{Z}} f_k$ pointwise. Then, given $i < j \in \mathbb{Z}$, we have by applying Lemma 4.3

$$\begin{aligned}\left\| f - \sum_{k=i}^{j} f_k \right\|_{L^{p_0}(\nu)+L^{p_1}(\nu)} &= \left\| f\chi_{[f \ge c^{i-1}]} + f\chi_{[f \le c^j]} \right\|_{L^{p_0}(\nu)+L^{p_1}(\nu)} \\ &\le \left\| f\chi_{[f \ge c^{i-1}]} \right\|_{L^{p_0}_w(\nu)} + \left\| f\chi_{[f < c^j]} \right\|_{L^{p_1}_w(\nu)} \to 0,\end{aligned}$$

when $i \to -\infty$ and $j \to \infty$, that is, $f = \sum_{k\in\mathbb{Z}} f_k$ in $L^{p_0}(\nu) + L^{p_1}(\nu)$.

(GP2) Let $F \subseteq \mathbb{Z}$ be a finite set and $(\varepsilon_k)_{k\in F}$ with $|\varepsilon_k| \le 1$. Keeping in mind that $f_k^{p_0} \le c^{k(p_0-p)} f_k^p$ and also that $f_k^{p_1} \le c^{(k-1)(p_1-p)} f_k^p$, we obtain, on the one hand

$$\begin{aligned}\left\| \sum_{k\in F} \frac{\varepsilon_k}{2^{k\theta}} f_k \right\|^{p_0}_{L^{p_0}_w(\nu)} &\le \left\| \sum_{k\in F} \frac{1}{2^{k\theta p_0}} f_k^{p_0} \right\|_{L^1_w(\nu)} \le \left\| \sum_{k\in F} \frac{c^{k(p_0-p)}}{2^{k\theta p_0}} f_k^p \right\|_{L^1_w(\nu)} \\ &= \left\| \sum_{k\in F} f_k^p \right\|_{L^1_w(\nu)} \le \|f\|^p_{L^p_w(\nu)},\end{aligned}$$

and on the other hand

$$\begin{aligned}\left\| \sum_{k\in F} \frac{\varepsilon_k}{2^{k(\theta-1)}} f_k \right\|^{p_1}_{L^{p_1}_w(\nu)} &\le \left\| \sum_{k\in F} \frac{1}{2^{k(\theta-1)p_1}} f_k^{p_1} \right\|_{L^1_w(\nu)} \le \left\| \sum_{k\in F} \frac{c^{(k-1)(p_1-p)}}{2^{k(\theta-1)p_1}} f_k^p \right\|_{L^1_w(\nu)} \\ &= \left\| \sum_{k\in F} f_k^p \right\|_{L^1_w(\nu)} \le \|f\|^p_{L^p_w(\nu)}.\end{aligned}$$

Therefore, taking $C = \max\left\{ \|f\|^{\frac{p}{p_0}}_{L^p_w(\nu)}, \|f\|^{\frac{p}{p_1}}_{L^p_w(\nu)} \right\}$ the implication is over.

The implication (2) ⇒ (1) is clear from Theorem 4.5, because the inclusion $\langle L^{p_0}(\nu), L^{p_1}(\nu), \theta\rangle \subseteq L^{p_0}(\nu) + L^{p_1}(\nu)$ holds for all $0 < \theta < 1$. □

Corollary 4.9. *Let $1 \le p_0 < p_1 < \infty$. The following are equivalent:*

(1) *ν is locally strongly additive.*

(2) $[L^{p_0}(\nu), L^{p_1}(\nu)]^{[\theta]} = [L^{p_0}_w(\nu), L^{p_1}(\nu)]^{[\theta]} = [L^{p_0}(\nu), L^{p_1}_w(\nu)]^{[\theta]} = L^p_w(\nu)$, *where $0 < \theta < 1$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.*

Proof. (1) ⇒ (2) Let $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. By applying the property (GP3), Corollaries 4.8 and 3.7, we have

$$\begin{aligned}L^p_w(\nu) &\subseteq \left\langle L^{p_0}(\nu), L^{p_1}(\nu), \theta \right\rangle \subseteq \left[L^{p_0}(\nu), L^{p_1}(\nu)\right]^{[\theta]} \\ &\subseteq \left[L^{p_0}_w(\nu), L^{p_1}(\nu)\right]^{[\theta]} \subseteq \left[L^{p_0}_w(\nu), L^{p_1}_w(\nu)\right]^{[\theta]} = L^p_w(\nu).\end{aligned}$$

In the above chain of inclusions we can change the space $\left[L^{p_0}_w(\nu), L^{p_1}(\nu)\right]^{[\theta]}$ by the other one $\left[L^{p_0}(\nu), L^{p_1}_w(\nu)\right]^{[\theta]}$. This gives the desired equalities.

The implication (2) ⇒ (1) is clear from Theorem 4.5, because the inclusion $[L^{p_0}(\nu), L^{p_1}(\nu)]^{[\theta]} \subseteq L^{p_0}(\nu) + L^{p_1}(\nu)$ holds for all $0 < \theta < 1$. □

Remark 4.10. After Corollary 4.9 we can retrieve equalities (5) of Example 3.11 because the measure considered there was locally strongly additive. In particular, with the same notation as in Example 3.11, we obtain

$$\left\langle c_0\left(\alpha^{\frac{1}{p_0}}\right), c_0\left(\alpha^{\frac{1}{p_1}}\right), \theta \right\rangle = \left[c_0\left(\alpha^{\frac{1}{p_0}}\right), c_0\left(\alpha^{\frac{1}{p_1}}\right)\right]^{[\theta]} = \ell^\infty\left(\alpha^{\frac{1}{p}}\right). \tag{9}$$

Remark 4.11. Given $1 \le p_0 < p_1 < \infty$, $0 < \theta < 1$, and a σ-finite vector measure $\nu : \mathcal{R} \to X$, Corollary 3.7 tells us that the smallest space of the list of all possible Calderón interpolated spaces is $L^p(\nu) = [L^{p_0}(\nu), L^{p_1}(\nu),]_{[\theta]}$, and the biggest one is $\left[L_w^{p_0}(\nu), L_w^{p_1}(\nu), \right]^{[\theta]} = L_w^p(\nu)$, where $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then any other Calderón interpolated space must be laid between $L^p(\nu)$ and $L_w^p(\nu)$. We have seen that the method $[\cdot,\cdot]_{[\theta]}$ always produces an L^p-space whereas the method $[\cdot,\cdot]^{[\theta]}$ always produces an L_w^p-space, of course under the hypothesis that ν is locally strongly additive. Without this assumption the spaces $\left[L_w^{p_0}(\nu), L_w^{p_1}(\nu)\right]_{[\theta]}$ and $[L^{p_0}(\nu), L^{p_1}(\nu)]^{[\theta]}$ can be strictly located between $L^p(\nu)$ and $L_w^p(\nu)$. The illustration of this claim is the purpose of the following example which is a mixture of Examples 3.9 and 3.11.

Example 4.12. Let $\mathcal{R}$ be the δ-ring of finite subsets of natural numbers and consider the σ-finite vector measure

$$\nu : A \in \mathcal{R} \longrightarrow \nu(A) := \chi_{A\cap\mathbb{O}} + \alpha \cdot \chi_{A\cap\mathbb{E}} \in c_0(\mathbb{N}),$$

where $\alpha = (\alpha_n)_n$ is the sequence given by $\alpha_n = n$, for all $n = 1, 2, \ldots$, and $\mathbb{O}$ and $\mathbb{E}$ are, respectively, the subset of odd and even natural numbers. For every $1 \le p < \infty$, it is not difficult to convince yourself that

$$\begin{aligned} L_w^p(\nu) &= \left\{ f = (f_n)_n : f\chi_{\mathbb{O}} \in \ell^\infty(\mathbb{N}) \text{ and } f\alpha^{\frac{1}{p}}\chi_{\mathbb{E}} \in \ell^\infty(\mathbb{N}) \right\} \\ &:= \ell^\infty(\mathbb{O}) \oplus \ell^\infty\left(\alpha^{\frac{1}{p}}\mathbb{E}\right), \\ L^p(\nu) &= \left\{ f = (f_n)_n : f\chi_{\mathbb{O}} \in c_0(\mathbb{N}) \text{ and } f\alpha^{\frac{1}{p}}\chi_{\mathbb{E}} \in c_0(\mathbb{N}) \right\} \\ &:= c_0(\mathbb{O}) \oplus c_0\left(\alpha^{\frac{1}{p}}\mathbb{E}\right). \end{aligned}$$

Analogously we define the spaces $\ell^\infty(\mathbb{O}) \oplus c_0(\alpha^{\frac{1}{p}}\mathbb{E})$ and $c_0(\mathbb{O}) \oplus \ell^\infty(\alpha^{\frac{1}{p}}\mathbb{E})$. Let us consider $1 \le p_0 < p_1 < \infty$, $0 < \theta < 1$, and let $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. First we are going to see that $L^p(\nu) \subsetneq \left[L_w^{p_0}(\nu), L_w^{p_1}(\nu)\right]_{[\theta]} \subsetneq L_w^p(\nu)$. Clearly the sequence $f := (1, 0, 1, 0, \ldots)$ belongs to $L_w^{p_0}(\nu) \cap L_w^{p_1}(\nu) \subseteq \left[L_w^{p_0}(\nu), L_w^{p_1}(\nu)\right]_{[\theta]}$, but $f \notin L^p(\nu)$ because $f\chi_{\mathbb{O}} \notin c_0(\mathbb{N})$. Now recall that $L_w^p(\nu) = \left[L_w^{p_0}(\nu), L_w^{p_1}(\nu)\right]^{[\theta]}$, and therefore $\left[L_w^{p_0}(\nu), L_w^{p_1}(\nu)\right]_{[\theta]} = \overline{L_w^{p_0}(\nu) \cap L_w^{p_1}(\nu)}^{L_w^p(\nu)}$. But, taking into account inclusions (A) stated in Example 3.11 we can easily check that $L_w^{p_0}(\nu) \cap L_w^{p_1}(\nu) \subseteq \ell^\infty(\mathbb{O}) \oplus c_0(\alpha^{\frac{1}{p}}\mathbb{E})$. Thus

$$\overline{L_w^{p_0}(\nu) \cap L_w^{p_1}(\nu)}^{L_w^p(\nu)} \subseteq \ell^\infty(\mathbb{O}) \oplus c_0\left(\alpha^{\frac{1}{p}}\mathbb{E}\right) \subsetneq \ell^\infty(\mathbb{O}) \oplus \ell^\infty\left(\alpha^{\frac{1}{p}}\mathbb{E}\right) = L_w^p(\nu).$$

Second, we will see that $L^p(\nu) \subsetneq [L^{p_0}(\nu), L^{p_1}(\nu)]^{[\theta]} \subsetneq L_w^p(\nu)$. Observe that

$$\left[L^{p_0}(\nu), L^{p_1}(\nu)\right]^{[\theta]} \subseteq L^{p_0}(\nu) + L^{p_1}(\nu) = L^{p_1}(\nu) = c_0(\mathbb{O}) \oplus c_0\left(\alpha^{\frac{1}{p_1}}\mathbb{E}\right).$$

Clearly the sequence $f := (1, 0, 1, 0, \ldots) \in L_w^p(\nu)$, but $f \notin [L^{p_0}(\nu), L^{p_1}(\nu)]^{[\theta]}$. Finally note that

$$\begin{aligned} L^p(\nu) &= c_0(\mathbb{O}) \oplus c_0\left(\alpha^{\frac{1}{p}}\mathbb{E}\right) \subsetneq c_0(\mathbb{O}) \oplus \ell^\infty\left(\alpha^{\frac{1}{p}}\mathbb{E}\right) \overset{(*)}{=} \left\langle L^{p_0}(\nu), L^{p_1}(\nu), \theta \right\rangle \\ &\subseteq \left[L^{p_0}(\nu), L^{p_1}(\nu)\right]^{[\theta]}. \end{aligned}$$

The above equality $(*)$ follows by using similar arguments of those used to obtain (9).

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