L^p-SPACES AND IDEAL PROPERTIES OF INTEGRATION OPERATORS FOR FRÉCHET-SPACE-VALUED MEASURES

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ABSTRACT. An investigation is made of L^p-spaces generated by Fréchet-space -valued measures, together with various ideal properties (compactness, weak compactness, complete continuity) of their associated integration map. Such ideal properties influence the nature of the L^p-spaces. Significant differences and new features occur which are not present in the Banach space setting.

KEYWORDS: *Fréchet space (lattice), vector measure, Fatou property, Lebesgue topology, integration map.*

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1. INTRODUCTION

Let X be a Banach space and ν a σ-additive X-valued measure. The Banach space $L^1(\nu)$ of all *ν-integrable* functions is due to Kluvánek and Knowles [19]. Later, Curbera stressed the role of order and obtained a better understanding of the nature of $L^1(\nu)$ [6]. The Banach space $L^1_{\mathrm{w}}(\nu)$ of all *scalarly ν-integrable* functions is due to Stefansson [30]. Spaces of the kind $L^1_{\mathrm{w}}(\nu)$ characterize a broad class of Banach lattices with the σ-Fatou property [8]. For $p > 1$ the Banach space $L^p(\nu)$ of p-integrable functions is due to Sánchez-Pérez [28]. The corresponding Banach spaces $L^p_{\mathrm{w}}(\nu)$ were treated by Fernández et al. in [15]; see also [10] and Chapter 3 of [25]. For further properties and applications of the spaces $L^p(\nu)$ and $L^p_{\mathrm{w}}(\nu)$, $1 \leqslant p < \infty$, see [9], [25] and the references therein.

For X a (locally convex) *Fréchet* space, the analogous spaces $L^1(\nu)$, $L^1_{\mathrm{w}}(\nu)$ are more recent. Again $L^1_{\mathrm{w}}(\nu)$ is a Fréchet lattice containing $L^1(\nu)$ as a closed subspace ([4] and [26], Section 4.4). One of our aims is to develop some of the main properties of $L^p(\nu)$ and $L^p_{\mathrm{w}}(\nu)$ for $p > 1$. Beyond the facts that both $L^p(\nu)$ and $L^p_{\mathrm{w}}(\nu)$ are Fréchet lattices, with $L^p(\nu)$ closed in $L^p_{\mathrm{w}}(\nu)$, that the simple functions are dense in $L^p(\nu)$, and that both $L^p(\nu) \subseteq L^r(\nu)$ and $L^p_{\mathrm{w}}(\nu) \subseteq L^r_{\mathrm{w}}(\nu)$ hold whenever $1 \leqslant r \leqslant p$ ([26], Chapter 4) not much more is known. Let us formulate some further facts concerning these spaces which are presented here. For X a Banach space and $p > 1$, we always have $L^p_{\mathrm{w}}(\nu) \subseteq L^1(\nu)$ with a continuous (natural) inclusion map ([15], Proposition 3.1) which is always weakly compact ([15], Proposition 3.3). For a *non-normable* Fréchet space X the situation can be different. The natural inclusion map $L^p_{\mathrm{w}}(\nu) \subseteq L^1(\nu)$ still exists and is continuous, but it may *fail* to be weakly compact. Indeed, we identify Fréchet spaces X for which there always exists an X-valued measure ν such that the inclusion $L^p_{\mathrm{w}}(\nu) \subseteq L^1(\nu)$ fails to be weakly compact for every $p > 1$. The underlying reason is, for X a Banach space, that the inclusion $L^p(\nu) \subseteq L^r(\nu)$ is *proper* for every non-trivial X-valued measure ν whenever $1 \leqslant r < p$ ([25], Proposition 3.28). An examination of the proof in [25] reveals it can be adapted to show that also $L^p_{\mathrm{w}}(\nu) \subsetneq L^r_{\mathrm{w}}(\nu)$ whenever $1 \leqslant r < p$. This *fails* for Fréchet spaces in general; it can even happen that $L^p_{\mathrm{w}}(\nu) = L^p(\nu) = L^1(\nu)$ for all $1 \leqslant p < \infty$. So, differences such as these (and others) appear in the non-normable setting.

On the positive side, we characterize weak compactness of the integration map $I_\nu : L^1(\nu) \to X$ (i.e., $I_\nu(f) := \int_\Omega f d\nu$). In this case we have $L^1_{\mathrm{w}}(\nu) = L^1(\nu)$, a known result in Banach spaces ([7], Corollary 2.3). It is also shown that $L^1_{\mathrm{w}}(\nu) = L^1(\nu)$ whenever I_ν is completely continuous (new even for Banach spaces). Concerning lattice properties, there is a good correspondence with Banach space results. For each $p \geqslant 1$, the Fréchet lattice $L^p(\nu)$ has a Lebesgue topology and $L^p_{\mathrm{w}}(\nu)$ has the Fatou property. More precisely, $L^p(\nu)$ is the order continuous part of $L^p_{\mathrm{w}}(\nu)$ and $L^p_{\mathrm{w}}(\nu)$ is the Fatou completion of $L^p(\nu)$. So, the containment $L^p(\nu) \subseteq L^p_{\mathrm{w}}(\nu)$ is proper precisely when either $L^p(\nu)$ or $L^p_{\mathrm{w}}(\nu)$ fails to be weakly sequentially complete, i.e., whenever either $L^p(\nu)$ or $L^p_{\mathrm{w}}(\nu)$ contains a copy of the Banach lattice c_0.

2. FRÉCHET SPACES OF p-INTEGRABLE FUNCTIONS

Let X be a (real) metrizable locally convex space (briefly, metrizable lcs) generated by an increasing fundamental sequence of seminorms $(\|\cdot\|^{(n)})_{n\in\mathbb{N}}$ and with continuous dual space X^*. Consider the neighbourhood base of $0 \in X$ generated by the sets $B_n := \{x \in X : \|x\|^{(n)} \leqslant 1\}$ and their polars $B_n^\circ := \{x^* \in X^* : |\langle x, x^*\rangle| \leqslant 1,\ \forall x \in B_n\}$, in which case $B_{n+1} \subseteq B_n$ and $B_n^\circ \subseteq B_{n+1}^\circ$, for each $n \in \mathbb{N}$.

Let (Ω, Σ) be a measurable space, $\nu : \Sigma \to X$ be a vector measure (i.e. σ-additive) and $f : \Omega \to \mathbb{R}$ be a Σ–measurable function. We call f *scalarly ν-integrable* if it is integrable for each $\mathbb{R}$-valued measure $\langle \nu, x^*\rangle : A \mapsto \langle \nu(A), x^*\rangle$, for $x^* \in X^*$. A scalarly ν-integrable function f is called *ν-integrable* if, for each $A \in \Sigma$, there exists an element $\int_A f d\nu \in X$ such that $\left\langle \int_A f d\nu, x^* \right\rangle = \int_A f d\langle \nu, x^*\rangle$, for $x^* \in X^*$. The total variation measure of $\langle \nu, x^*\rangle$ is denoted by $|\langle \nu, x^*\rangle|$.

Let $p \geqslant 1$. A Σ–measurable function $f : \Omega \to \mathbb{R}$ is called *scalarly ν-p-integrable* if $|f|^p$ is scalarly ν-integrable and *ν-p-integrable* if $|f|^p$ is ν-integrable. The linear space of all individual scalarly ν-p-integrable (respectively ν-p-integrable) functions on Ω is denoted by $\mathcal{L}^p_{\mathrm{w}}(\nu)$ (respectively $\mathcal{L}^p(\nu)$). For each $n \in \mathbb{N}$, define a $[0,\infty]$–valued "seminorm" $\|\cdot\|^{(n)}_{\nu,p}$ on the space $\mathcal{L}^0(\nu)$, consisting of all Σ-measurable functions, by

$$\|f\|^{(n)}_{\nu,p} := \sup_{x^*\in B_n^\circ} \Big(\int_\Omega |f|^p \mathrm{d}|\langle \nu, x^*\rangle| \Big)^{1/p}, \quad f \in \mathcal{L}^0(\nu), \tag{2.1}$$

and denote $\|\cdot\|^{(n)}_{\nu,1}$ simply by $\|\cdot\|^{(n)}_{\nu}$; see Section 4.3 of [26].

Observe that $f \in \mathcal{L}^0(\nu)$ belongs to $\mathcal{L}^p_{\mathrm{w}}(\nu)$ if and only if $\|f\|^{(n)}_{\nu,p} < \infty$ for all $n \in \mathbb{N}$; the argument for $p = 1$ given in Proposition 2.1 of [4] can be adapted to $p > 1$. That is, (2.1) is a $[0,\infty)$-valued seminorm, for each $n \in \mathbb{N}$, precisely on $\mathcal{L}^p_{\mathrm{w}}(\nu) \subseteq \mathcal{L}^0(\nu)$. The spaces $\mathcal{L}^p_{\mathrm{w}}(\nu)$ and $\mathcal{L}^p(\nu)$ are complete lcs' for the sequence of seminorms given by (2.1) (provided X is complete in the latter case) ([26], Theorems 4.4.2 and 4.4.8). Clearly $\mathcal{L}^p(\nu) \subseteq \mathcal{L}^p_{\mathrm{w}}(\nu)$ for all $p \geqslant 1$. By Lemma II.1.2 of [19], for each $n \in \mathbb{N}$,

$$\sup_{E\in\Sigma} \Big\| \int_E f \mathrm{d}\nu \Big\|^{(n)} \leqslant \|f\|^{(n)}_{\nu} \leqslant 2 \sup_{E\in\Sigma} \Big\| \int_E f \mathrm{d}\nu \Big\|^{(n)}, \quad f \in \mathcal{L}^1(\nu). \tag{2.2}$$

Fix $n \in \mathbb{N}$. Then X_n is the completion of the quotient normed space X/M_n, where $M_n := \{x \in X : \|x\|^{(n)} = 0\}$, and $\pi_n : X \to X_n$ is the corresponding quotient map. Hence, $\nu_n : \Sigma \to X_n$ defined by

$$\nu_n(A) := \pi_n(\nu(A)), \quad A \in \Sigma, \tag{2.3}$$

is a *Banach-space*-valued vector measure. Its variation measure $|\nu_n| : \Sigma \to [0,\infty]$ is defined analogous to that for scalar measures ([11], Chapter I, Definition 1.4).

LEMMA 2.1. *Let X be a Fréchet space and $\nu : \Sigma \to X$ be a vector measure. If $f \in \mathcal{L}^0(\nu)$, then $\|f\|^{(n)}_{\nu} = \|f\|_{\nu_n}$, for $n \in \mathbb{N}$. Moreover,*

(i) *f is scalarly ν-integrable if and only if f is scalarly ν_n-integrable for each $n \in \mathbb{N}$.*

(ii) *f is ν-integrable if and only if f is ν_n-integrable for each $n \in \mathbb{N}$.*

(iii) *For $n \in \mathbb{N}$, if f is scalarly ν_{n+1}-integrable, then f is also scalarly ν_n-integrable.*

(iv) *For $n \in \mathbb{N}$, if f is ν_{n+1}-integrable, then f is also ν_n-integrable.*

Proof. The first statement and (i) follow from π_n^* being a linear isometric bijection between the Banach spaces X_n^* and $\mathrm{Lin}(B_n^\circ)$ ([22], Remark 24.5(b)). For the details we refer to the proof of Proposition 2.1 in [4].

(ii) This is part of Lemma 2 in [24].

(iii) Let $f \in \mathcal{L}^1_{\mathrm{w}}(\nu_{n+1})$. Given $y^* \in X_n^*$ we have $\pi_n^*(y^*) \in \mathrm{Lin}(B_n^\circ) \subseteq \mathrm{Lin}(B_{n+1}^\circ)$. So, there is $x^* \in X_{n+1}^*$ with $\pi_n^*(y^*) = \pi_{n+1}^*(x^*)$ and $\int_\Omega |f| \mathrm{d}|\langle \nu_n, y^*\rangle|$

$= \int_\Omega |f| \mathrm{d}|\langle \nu, \pi_n^*(y^*)\rangle| = \int_\Omega |f| \mathrm{d}|\langle \nu, \pi_{n+1}^*(x^*)\rangle| = \int_\Omega |f| \mathrm{d}|\langle \nu_{n+1}, x^*\rangle| < \infty$. That is, $f \in \mathcal{L}^1_{\mathrm{w}}(\nu_n)$.

(iv) Let $f \in \mathcal{L}^1(\nu_{n+1})$. Since $\|\cdot\|^{(n)} \leqslant \|\cdot\|^{(n+1)}$, we have $M_{n+1} \subseteq M_n$ and hence, there is a continuous linear map $\pi_{n+1,n} : X_{n+1} \to X_n$ satisfying $\pi_n = \pi_{n+1,n} \circ \pi_{n+1}$. Given $A \in \Sigma$, let $x_A := \int_A f \mathrm{d}\nu_{n+1} \in X_{n+1}$ in which case $u_A := \pi_{n+1,n}(x_A) \in X_n$. For each $y^* \in X_n^*$ we have $\pi_n^*(y^*) = \pi_{n+1}^*(x^*)$ for some $x^* \in X_{n+1}^*$ (see the proof of (iii)) and so

$$\begin{aligned}\langle u_A, y^*\rangle &= \langle \pi_{n+1,n}(x_A), y^*\rangle = \langle x_A, \pi_{n+1,n}^*(y^*)\rangle \\ &= \langle x_A, (\pi_{n+1}^*)^{-1} \circ \pi_n^*(y^*)\rangle = \langle x_A, (\pi_{n+1}^*)^{-1} \circ \pi_{n+1}^*(x^*)\rangle \\ &= \langle x_A, x^*\rangle = \int_A f \mathrm{d}\langle \nu_{n+1}, x^*\rangle = \int_A f \mathrm{d}\langle \nu, \pi_{n+1}^*(x^*)\rangle \\ &= \int_A f \mathrm{d}\langle \nu_n, \pi_n^*(y^*)\rangle = \int_A f \mathrm{d}\langle \nu_n, y^*\rangle.\end{aligned}$$

It follows that $f \in \mathcal{L}^1(\nu_n)$ and

$$\int_A f \mathrm{d}\nu_n = u_A = \pi_{n+1,n}\Big(\int_A f \mathrm{d}\nu_{n+1}\Big), \quad A \in \Sigma. \tag{2.4}$$ ∎

For Banach spaces, the following version of Hölder's inequality occurs in Theorem 3.1.13 of [26] for (i) and in Theorem 3.5.1 of [26] for (ii).

PROPOSITION 2.2. *Let X be a Fréchet space, $\nu : \Sigma \to X$ be a vector measure and $1 < p, q < \infty$ satisfy $1/p + 1/q = 1$.*

(i) If $f \in \mathcal{L}^p_{\mathrm{w}}(\nu)$ and $g \in \mathcal{L}^q_{\mathrm{w}}(\nu)$, then $fg \in \mathcal{L}^1_{\mathrm{w}}(\nu)$.

(ii) If $f \in \mathcal{L}^p(\nu)$ and $g \in \mathcal{L}^q(\nu)$, then $fg \in \mathcal{L}^1(\nu)$.

In both cases, $\|fg\|_\nu^{(n)} \leqslant \|f\|_{\nu,p}^{(n)} \|g\|_{\nu,q}^{(n)}$, for $n \in \mathbb{N}$.

Proof. (i) By definition $|f|^p \in \mathcal{L}^1_{\mathrm{w}}(\nu)$, $|g|^q \in \mathcal{L}^1_{\mathrm{w}}(\nu)$. Fix $x^* \in X^*$. Then, both $|f|^p, |g|^q \in \mathcal{L}^1(|\langle \nu, x^*\rangle|)$. By the classical Hölder inequality for scalar measures, $fg \in \mathcal{L}^1(|\langle \nu, x^*\rangle|)$ and

$$\int_\Omega |fg| \mathrm{d}|\langle \nu, x^*\rangle| \leqslant \Big(\int_\Omega |f|^p \mathrm{d}|\langle \nu, x^*\rangle|\Big)^{1/p} \Big(\int_\Omega |g|^q \mathrm{d}|\langle \nu, x^*\rangle|\Big)^{1/q} < \infty.$$

This shows that $fg \in \mathcal{L}^1_{\mathrm{w}}(\nu)$.

Fix $n \in \mathbb{N}$. For $x^* \in B_n^\circ$, the previous inequality yields $\int_\Omega |fg| \mathrm{d}|\langle \nu, x^*\rangle| \leqslant \|f\|_{\nu,p}^{(n)} \|g\|_{\nu,q}^{(n)}$ and so

$$\|fg\|_\nu^{(n)} = \sup_{x^* \in B_n^\circ} \int_\Omega |fg| \mathrm{d}|\langle \nu, x^*\rangle| \leqslant \|f\|_{\nu,p}^{(n)} \, \|g\|_{\nu,q}^{(n)}.$$

(ii) Here $|f|^p \in \mathcal{L}^1(\nu), |g|^q \in \mathcal{L}^1(\nu)$. By Lemma 2.1, $|f|^p \in \mathcal{L}^1(\nu_n), |g|^q \in \mathcal{L}^1(\nu_n)$, that is, $f \in \mathcal{L}^p(\nu_n), g \in \mathcal{L}^q(\nu_n)$, for all $n \in \mathbb{N}$. Applying Theorem 3.5.1 of [26] to each complete seminormed space $\mathcal{L}^1(\nu_n)$ we deduce that $fg \in \mathcal{L}^1(\nu_n)$, for all $n \in \mathbb{N}$; see also the proof of Lemma 2.21(i) in [25]. Again Lemma 2.1 ensures that $fg \in \mathcal{L}^1(\nu)$. ∎

Since $\chi_\Omega \in \mathcal{L}^q(\nu) \subseteq \mathcal{L}^q_{\mathrm{w}}(\nu)$, the following result is clear from Proposition 2.2; see also Theorem 4.5.13(v)(a) of [26] for part of the conclusion.

COROLLARY 2.3. *Let ν be a Fréchet-space-valued measure and $1 < p, q < \infty$ satisfy $1/p + 1/q = 1$. Let $f \in \mathcal{L}^p_{\mathrm{w}}(\nu)$. Then*

$$\|f\|_\nu^{(n)} \leqslant \|f\|_{\nu,p}^{(n)} (\|\chi_\Omega\|_\nu^{(n)})^{1/q}, \quad n \in \mathbb{N}. \tag{2.5}$$

Also, $\mathcal{L}^p(\nu) \subseteq \mathcal{L}^1(\nu)$ and $\mathcal{L}^p_{\mathrm{w}}(\nu) \subseteq \mathcal{L}^1_{\mathrm{w}}(\nu)$ with continuous inclusions.

Scalarly ν-p-integrable functions are more than scalarly ν-integrable. For Banach spaces the next result is Proposition 3.1 of [15].

PROPOSITION 2.4. *Let X be a Fréchet space and $\nu : \Sigma \to X$ be a vector measure. If $p > 1$, then $\mathcal{L}^p_{\mathrm{w}}(\nu) \subseteq \mathcal{L}^1(\nu)$ with a continuous inclusion.*

Proof. Let $f \in \mathcal{L}^p_{\mathrm{w}}(\nu)$. For any $k \in \mathbb{N}$, consider the set $A_k := \{\omega \in \Omega : |f(w)| \leqslant k\}$ and the function $f_k := f\chi_{A_k}$. Each $f_k \in \mathcal{L}^1(\nu)$ since it is bounded ([19], p. 26). Moreover, (f_k) converges pointwise to f. To verify that $f \in \mathcal{L}^1(\nu)$ it suffices to show that $(\int_E f_k \mathrm{d}\nu)$ is Cauchy in X uniformly with respect to $E \in \Sigma$ ([21], Theorem 2.4). Fix $n \in \mathbb{N}$. For $i > j$, by Proposition 2.2 and (2.2) we have, for $E \in \Sigma$, that

$$\begin{aligned}\Big\| \int_E f_i \mathrm{d}\nu - \int_E f_j \mathrm{d}\nu \Big\|^{(n)} &\leqslant \sup_{F \in \Sigma} \Big\| \int_F (f_i - f_j) \mathrm{d}\nu \Big\|^{(n)} \leqslant \|f_i - f_j\|_\nu^{(n)} = \| |f| \chi_{(A_i \setminus A_j)} \|_\nu^{(n)} \\ &\leqslant \|f\|_{\nu,p}^{(n)} \|\chi_{(\Omega \setminus A_j)}\|_{\nu,q}^{(n)} = \|f\|_{\nu,p}^{(n)} (\|\chi_{(\Omega \setminus A_j)}\|_\nu^{(n)})^{1/q}.\end{aligned}$$

Since $\chi_{(\Omega \setminus A_j)} \downarrow 0$ pointwise, the dominated convergence theorem ([19], p. 30) implies that $\|\chi_{(\Omega \setminus A_j)}\|_\nu^{(n)} \to 0$ as $j \to \infty$. Thus $f \in \mathcal{L}^1(\nu)$.

Continuity of the inclusion $\mathcal{L}^p_{\mathrm{w}}(\nu) \subseteq \mathcal{L}^1(\nu)$ is clear from (2.5). ∎

A set $A \in \Sigma$ is called *ν-null* if $\nu(B) = 0$ for every $B \in \Sigma$ with $B \subseteq A$. Equivalently, A is ν_n-null for all $n \in \mathbb{N}$. Denote the σ-ideal of all ν-null sets by $\mathcal{N}_0(\nu)$. Let $f \in \mathcal{L}^0(\nu)$. Then f is called *ν-null* whenever $\|f\|_\nu^{(n)} = 0$ for all $n \in \mathbb{N}$. If $f^{-1}(\mathbb{R}\setminus\{0\}) \in \mathcal{N}_0(\nu)$, then it is also $|\langle \nu, x^* \rangle|$-null for all $x^* \in X^*$. Hence, f is a ν-null function because

$$\|f\|_\nu^{(n)} = \sup_{x^* \in B_n^\circ} \int_{f^{-1}(\mathbb{R}\setminus\{0\})} |f| \mathrm{d}|\langle \nu, x^* \rangle| = 0, \quad n \in \mathbb{N}.$$

Conversely, suppose f is a ν-null function. Fix $n \in \mathbb{N}$. For the Banach-space-valued measure ν_n, Rybakov's theorem states that there is a unit vector $\xi^* \in X_n^*$ (i.e., $\pi_n^*(\xi^*) \in B_n^\circ$) such that ν_n and $|\langle \nu_n, \xi^* \rangle|$ have the same null sets ([11], p. 268). Since $\int_{f^{-1}(\mathbb{R}\setminus\{0\})} |f| \mathrm{d} |\langle \nu, \xi^* \rangle| \leqslant \|f\|_\nu^{(n)} = 0$, the set $f^{-1}(\mathbb{R}\setminus\{0\})$ is $|\langle \nu_n, \xi^* \rangle|$-null and hence, also ν_n-null. But, $n \in \mathbb{N}$ is arbitrary and so $f^{-1}(\mathbb{R}\setminus\{0\}) \in \mathcal{N}_0(\nu)$. So, the subspace $\mathcal{N}(\nu)$ of $\mathcal{L}^0(\nu)$ consisting of all ν-null functions is the space of all $f \in \mathcal{L}^0(\nu)$ such that $f^{-1}(\mathbb{R}\setminus\{0\}) \in \mathcal{N}_0(\nu)$ (briefly, we say f is ν-null). Observe that $\mathcal{N}(\nu) \subseteq \mathcal{L}^1(\nu)$. Two functions from $\mathcal{L}^0(\nu)$ are ν-equivalent if their difference is a ν-null function. Noting that $\mathcal{N}(\nu)$ is a closed ideal in both $\mathcal{L}^p_{\mathrm{w}}(\nu)$ and $\mathcal{L}^p(\nu)$, the quotient spaces $L^p_{\mathrm{w}}(\nu) := \mathcal{L}^p_{\mathrm{w}}(\nu)/\mathcal{N}(\nu)$ and $L^p(\nu) := \mathcal{L}^p(\nu)/\mathcal{N}(\nu)$, for $1 \leqslant p < \infty$, are then complete Fréchet lattices for the (quotient) seminorms induced via (2.1) ([26], Theorem 4.5.11). So, Proposition 2.2, Corollary 2.3 and Proposition 2.4 also hold for the corresponding statements with the Fréchet spaces $L^p(\nu)$ and $L^p_{\mathrm{w}}(\nu)$ in place of the complete pseudo-metrizable lc-spaces $\mathcal{L}^p(\nu)$ and $\mathcal{L}^p_{\mathrm{w}}(\nu)$. Define $L^0(\nu) := \mathcal{L}^0(\nu)/\mathcal{N}(\nu)$.

A vector measure $\nu : \Sigma \to X$ is called *σ-decomposable* if Σ admits countably infinite many pairwise disjoint non-ν-null sets ([25], p. 129). For X a Banach space the natural inclusion $L^p(\nu) \subseteq L^q(\nu)$ is *proper* whenever $1 \leqslant q < p$, ([25], Proposition 3.28).

PROPOSITION 2.5. *Let X be a Fréchet space not admitting a continuous norm. There exists an X-valued, σ-decomposable measure ν such that*

$$L^p_{\mathrm{w}}(\nu) = L^p(\nu) = L^q_{\mathrm{w}}(\nu) = L^q(\nu), \quad 1 \leqslant q \leqslant p < \infty.$$

Proof. By a classical result of Bessaga and Pelczynski ([3] and Theorem 7.2.7 of [18]) X contains a complemented subspace isomorphic to the Fréchet sequence space $\omega := \mathbb{R}^{\mathbb{N}}$, equipped with the seminorms $q_n(x) := \max_{1 \leqslant j \leqslant n} |x_j|$, for $x = (x_1, x_2, \ldots) \in \omega$. So, it suffices to establish the result for $X = \omega$. Let $\Omega = \mathbb{N}$ and $\Sigma = 2^\Omega$. Define $\nu : \Sigma \to \omega$ by $\nu(A) = \chi_A$, for $A \in \Sigma$. Then ν is a vector measure with $\emptyset$ as its only ν-null set and $\mathcal{L}^0(\nu) = \mathbb{R}^{\mathbb{N}}$. Observe that $\omega^* = \{\xi \in \mathbb{R}^{\mathbb{N}} :$ supp(ξ) is finite$\}$ with duality $\langle x, \xi \rangle = \sum_{n=1}^{\infty} x_n \xi_n$ (a finite sum), for each $x \in \omega$ and $\xi \in \omega^*$. It is routine to check that $L^p_{\mathrm{w}}(\nu) = L^p(\nu) = L^0(\nu) \simeq \omega$, for $1 \leqslant p < \infty$, with equality as vector spaces and topologically. ∎

In *Banach spaces* X, the continuous inclusion of $L^p_{\mathrm{w}}(\nu)$ into $L^1(\nu)$ is weakly compact for all $p > 1$ ([15], Proposition 3.3). So, the restriction to $L^p(\nu)$ of the (continuous) integration map $I_\nu : L^1(\nu) \to X$ (i.e., $f \mapsto I_\nu(f) := \int_\Omega f \mathrm{d}\nu$ for $f \in L^1(\nu)$), is also weakly compact. For X a Fréchet space, this may fail. Via (2.2), $I_\nu : L^1(\nu) \to X$ is still continuous and so, by Corollary 2.3, also the restriction $I_\nu : L^p(\nu) \to X$ is continuous for all $p \geqslant 1$. The problem lies with weak compactness.

A continuous linear map T from a lc-space Y into a Fréchet space X is *weakly compact* (respectively *compact*) if there is a neighbourhood U of $0 \in Y$ such that the closure of $T(U)$ is weakly compact (respectively compact) in X.

PROPOSITION 2.6. *Let X be any Fréchet space which does not admit a continuous norm. Then there exists an X-valued measure ν such that:*

(i) *the continuous inclusion $L^p_{\mathrm{w}}(\nu) \subseteq L^1(\nu)$ is not weakly compact, for every $p > 1$;*

(ii) *the continuous integration map $I_\nu : L^p(\nu) \to X$ is not weakly compact, for every $p \geqslant 1$.*

Proof. As in the proof of Proposition 2.5 it suffices to consider $X = \omega$ and $\nu : \Sigma \to \omega$ as given there. Since $L^p_{\mathrm{w}}(\nu) = L^1(\nu) \simeq \omega$ for all $p \geqslant 1$, the natural inclusion $L^p_{\mathrm{w}}(\nu) \subseteq L^1(\nu)$ is the identity map on ω. If this map was weakly compact, then ω would have a bounded neighbourhood of 0 and hence, would be normable (which is not so). This establishes (i). It is also routine to check that the integration map $I_\nu : L^p(\nu) \to \omega$ is the identity map on ω and so the same argument yields (ii). ∎

For a Banach-space-valued measure $\nu : \Sigma \to X$ it is known, for each $1 < p < \infty$, that the restriction of the integration map $I_\nu : L^p_{\mathrm{w}}(\nu) \to X$ (well defined by Proposition 3.1 of [15]) is a compact operator if and only if the range $R(\nu) := \{\nu(A) : A \in \Sigma\}$ is a relatively compact subset of X ([15], Theorem 3.6 and [25], Proposition 3.56(I)). What if X is a Fréchet space? According to Proposition 2.4 the restriction map $I_\nu : L^p_{\mathrm{w}}(\nu) \to X$ is again well defined. If this map is compact, then there exists $n \in \mathbb{N}$ such that $I_\nu(W_n)$ is relatively compact in X, where $W_n := \{f \in L^p_{\mathrm{w}}(\nu) : \|f\|^{(n)}_{\nu,p} \leqslant 1\}$ is a basic neighbourhood of 0 in $L^p_{\mathrm{w}}(\nu)$. Since $(\||f|^p\|^{(n)}_\nu)^{1/p} = \|f\|^{(n)}_{\nu,p}$, for every $f \in L^1_{\mathrm{w}}(\nu)$, we have

$$\|\chi_A\|^{(n)}_{\nu,p} = (\||\chi_A|^p\|^{(n)}_\nu)^{1/p} = (\|\chi_A\|^{(n)}_\nu)^{1/p} \leqslant (\|\chi_\Omega\|^{(n)}_\nu)^{1/p}, \quad A \in \Sigma.$$

Accordingly, $R(\nu) \subseteq (\|\chi_\Omega\|^{(n)}_\nu)^{1/p} \cdot I_\nu(W_n)$, which shows that $R(\nu)$ is necessarily a relatively compact subset of X. Unfortunately, the converse statement does not hold for general non-normable X.

PROPOSITION 2.7. *Let X be any Fréchet space which does not admit a continuous norm. There exists an X-valued measure ν such that:*

(i) *$R(\nu)$ is a relatively compact subset of X, but*

(ii) *the restricted integration map $I_\nu : L^p_{\mathrm{w}}(\nu) \to X$ fails to be compact for every $1 < p < \infty$.*

Proof. As in the proof of Proposition 2.5 it suffices to consider $X = \omega$ and $\nu : \Sigma \to X$ as given there. Since $R(\nu)$ is a bounded subset of ω (being relatively weakly compact ([19], p. 76) and ω is a Montel space, it follows that $R(\nu)$ is a relatively compact subset of ω, i.e., (i) holds. The same argument as in the proof of Proposition 2.6, with "compact" in place of "weakly compact" establishes part (ii). ∎

3. IDEAL PROPERTIES OF THE INTEGRATION MAP I_ν

The aim of this section is to investigate how certain ideal properties of I_ν influence the nature of its domain space $L^1(\nu)$.

Let $\nu : \Sigma \to X$ be a Fréchet-space-valued vector measure and ν_n, for $n \in \mathbb{N}$, be as in (2.3). According to (2.4) we have $\mathcal{N}(\nu_{n+1}) \subseteq \mathcal{N}(\nu_n)$ and hence, via Lemma 2.1(iv), that $L^1(\nu_{n+1}) \subseteq L^1(\nu_n)$, $n \in \mathbb{N}$, with a continuous inclusion.

We point out that the equality $L^1(\nu) = L^1_{\mathrm{w}}(\nu)$ is equivalent to $L^1(\nu)$ (or $L^1_{\mathrm{w}}(\nu)$) being weakly sequentially complete; see Proposition 3.4 of [5].

THEOREM 3.1. *Let $\nu : \Sigma \to X$ be a Fréchet-space-valued measure. Then the integration map $I_\nu : L^1(\nu) \to X$ is weakly compact if and only if there exists $r \in \mathbb{N}$ such that for all $k \geqslant r$ we have:*

(i) *$I_{\nu_k} : L^1(\nu_k) \to X_k$ is weakly compact, and*

(ii) *$L^1(\nu) = L^1(\nu_k) = L^1(\nu_r)$, with equality as* lc-*spaces.*

In this case, $L^1(\nu)$ is necessarily a Banach space.

First we require a preliminary lemma. Recall that a convex balanced subset B of a lc-space X is a *Banach disc* if $X_B := \mathrm{span}(B)$, equipped with its Minkowski functional, is a Banach space ([22], p. 267).

LEMMA 3.2. *Let X be a Fréchet space and $\nu : \Sigma \to X$ be a vector measure. Then the integration map $I_\nu : L^1(\nu) \to X$ is weakly compact if and only if there exist a Banach disc $B \subseteq X$ and a vector measure $\mu : \Sigma \to X_B$ such that:*

(i) *$X_B \hookrightarrow X$ continuously via the natural injection J;*

(ii) *$L^1(\mu) = L^1(\nu)$ as* lc-*spaces;*

(iii) *$I_\mu : L^1(\mu) \to X_B$ is weakly compact; and*

(iv) *$I_\nu = J \circ I_\mu$ (i.e. ν factors through X_B via μ and J).*

Proof. Clearly (i)–(iv) imply that I_ν is weakly compact.

Conversely, let I_ν be weakly compact. We adapt the proof of Lemma 3.2 in [23]. Take a convex balanced neighbourhood V of 0 in $L^1(\nu)$ with $A := \overline{I_\nu(V)}$ weakly compact in X. Then choose a weakly compact Banach disc B of X such that $A \subseteq B$ and A is weakly compact in X_B ([27], p. 422 Lemma). Then (i) follows from Corollary 23.14 of [22]; see also its proof. Since the range $R(I_\nu) \subseteq X_B$, let $I_\nu^{(B)} : L^1(\nu) \to X_B$ denote I_ν considered as being X_B-valued. Arguing as in [23] the map $I_\nu^{(B)}$ is weakly compact (hence, continuous). Moreover, $\mu : \Sigma \to X_B$ defined by $E \mapsto I_\nu^{(B)}(\chi_E)$, for $E \in \Sigma$, is a vector measure satisfying (ii), (iv); see the proof in [23]. Since $L^1(\nu) = L^1(\mu)$ as lc-spaces, V is also a 0-neighbourhood in the Banach space $L^1(\mu)$ and so, is contained in a multiple of the unit ball in

$L^1(\mu)$. Moreover, $I_\mu(V) = I_\nu^{(B)}(V) \subseteq A$ is relatively weakly compact in X_B from which (iii) follows. ∎

Proof of Theorem 3.1. Suppose that I_ν is weakly compact. By Lemma 3.2, $L^1(\nu)$ is a Banach space. According to p. 25 of [17] there exists $r \in \mathbb{N}$ such that (ii) of Theorem 3.1 holds. Then $I_{\nu_k} = \pi_k \circ I_\nu$ is weakly compact, for all $k \geqslant r$, after noting that the domain $\mathcal{D}(I_{\nu_k}) = L^1(\nu_k) = L^1(\nu) = \mathcal{D}(I_\nu)$ by (ii). Hence, (i) is valid.

Suppose that (i), (ii) of Theorem 3.1 hold. By (ii), $L^1(\nu)$ is a Banach space. Then apply the version of Lemma 2.3 of [23] with "compact" replaced throughout by "weakly compact" (the "same" proof applies), together with (i), to conclude that I_ν is weakly compact. ∎

A version of Theorem 3.1 is known for *compactness* of the integration map I_ν ([23], Theorem 2).

For Banach spaces the following result occurs in Corollary 2.3 of [7].

COROLLARY 3.3. *The Fréchet space $L^1(\nu)$ is weakly sequentially complete whenever the integration map $I_\nu : L^1(\nu) \to X$ is weakly compact.*

Proof. Let $r \in \mathbb{N}$ be as in Theorem 3.1. Then $I_{\nu_r} : L^1(\nu_r) \to X_r$ is weakly compact. Since X_r is a Banach space, $L^1(\nu_r)$ is weakly sequentially complete ([7], Corollary 2.3). By (ii) of Theorem 3.1, $L^1(\nu)$ is also weakly sequentially complete (and a Banach space). ∎

COROLLARY 3.4. *Let X be a Fréchet space such that each Banach space X_k, for $k \in \mathbb{N}$, is reflexive. Then an X-valued measure ν satisfies I_ν is weakly compact if and only if $L^1(\nu)$ is a Banach space.*

Proof. If I_ν is weakly compact, then Theorem 3.1 implies that $L^1(\nu)$ is a Banach space. Conversely, suppose $L^1(\nu)$ is a Banach space. Then $\pi_k \circ I_\nu : L^1(\nu) \to X_k$ is weakly compact from the Banach space $L^1(\nu)$ into the Banach space X_k, for $k \in \mathbb{N}$. By the "weakly compact" version of Lemma 2.3 in [23] it follows that I_ν is weakly compact. ∎

We point out that if each Banach space X_k, $k \in \mathbb{N}$, is reflexive, then X itself is necessarily reflexive ([22], Proposition 25.15).

For Y, Z Fréchet spaces, a continuous linear map $T : Y \to Z$ is called *completely continuous* (or Dunford–Pettis) if it maps weakly convergent sequences in Y to convergent sequences in Z. For Banach spaces, such operators form a classical operator ideal. If Z is a Fréchet–Montel space, then every continuous linear map $T : Y \to Z$ is completely continuous. The same is true whenever Z has the Schur property.

LEMMA 3.5. *For a continuous linear operator $T : Y \to Z$ between Fréchet spaces Y, Z the following assertions are equivalent:*

(i) *T maps weakly compact subsets of Y to compact subsets of Z;*
(ii) *T is completely continuous;*
(iii) *T maps weakly Cauchy sequences in Y to convergent sequences in Z.*

Proof. (i) $\Leftrightarrow$ (ii) is known ([16], p. 43), and depends on the fact that a subset of a Fréchet space is weakly compact if and only if it is weakly sequentially compact ([16], pp. 30-31 and (1) p. 39; [20], (9) p. 318).

(iii) $\Rightarrow$ (ii). Immediate.

(ii) $\Rightarrow$ (iii). For Banach spaces, see p. 333 of [2]. Suppose there is a weak Cauchy sequence $\{y_n\}_{n=1}^{\infty} \subseteq Y$ such that $\{T(y_n)\}_{n=1}^{\infty}$ is *not* convergent in Z. With d denoting a translation invariant metric in Z which determines its given Fréchet space topology, there is $\varepsilon>0$ and positive integers $p(1)<q(1)<p(2)<q(2)< \cdots$ such that $d(T(y_{p(n)}), T(y_{q(n)})) > \varepsilon$ for $n \in \mathbb{N}$. It is routine to check that $\{y_{p(n)} - y_{q(n)}\}_{n=1}^{\infty}$ is weakly convergent to $0 \in Y$. Since T is also continuous when Y, Z have their weak topology, it follows that $T(y_{p(n)} - y_{q(n)}) \to 0$ weakly in Z. Now, (ii) implies that $\{T(y_{p(n)} - y_{q(n)})\}_{n=1}^{\infty}$ is convergent in Z (to 0 by the previous sentence). This contradicts $d(T(y_{p(n)}), T(y_{q(n)})) > \varepsilon$ for $n \in \mathbb{N}$. So, no such weak Cauchy sequence $\{y_n\}_{n=1}^{\infty}$ exists. ∎

Corollary 3.3 shows that compactness/weak compactness of I_ν has a strong influence on $L^1(\nu)$, i.e., it is weakly sequentially complete. We will see in Section 4 that this, in turn, forces *all* spaces $L^p(\nu)$ and $L^p_{\mathrm{w}}(\nu)$, for $1 \leqslant p < \infty$, to be weakly sequentially complete. The complete continuity of I_ν has the same effect on $L^1(\nu)$.

THEOREM 3.6. *Let X be a Fréchet space and ν be an X-valued measure. If the integration map* $I_\nu : L^1(\nu) \to X$ *is completely continuous, then* $L^1_{\mathrm{w}}(\nu) = L^1(\nu)$.

The proof proceeds via a series of lemmata, using the notation of Theorem 3.6. In particular, X is always a Fréchet space and $\nu : \Sigma \to X$ a (fixed) vector measure with Σ a σ-algebra of subsets of Ω.

Let $L^1(\nu)_\sigma$ denote $L^1(\nu)$ with its weak topology $\sigma(L^1(\nu), L^1(\nu)^*)$. The strong bidual $L^1(\nu)^{**}$ is a Fréchet space which contains $L^1(\nu)$ as a closed subspace via the canonical embedding $J : L^1(\nu) \to L^1(\nu)^{**}$ ([22], Corollary 25.10). The space $L^1(\nu)^{**}$ equipped with its weak-* topology $\sigma(L^1(\nu)^{**}, L^1(\nu)^*)$ is denoted by $L^1(\nu)^{**}_{\sigma^*}$; it is a *sequentially complete* Hausdorff lcs ([20], (3) p. 396). The following fact is routine to establish.

LEMMA 3.7. *The embedding* $J : L^1(\nu)_\sigma \to L^1(\nu)^{**}_{\sigma^*}$ *is a topological isomorphism onto its range* $J(L^1(\nu))$.

Given $f \in L^1(\nu)$, define $m_f : \Sigma \to L^1(\nu)$ by $m_f(A) := f\chi_A$, for $A \in \Sigma$. If $A_n \downarrow \emptyset$ in Σ, then $f\chi_{A_n} \to 0$ pointwise ν-a.e. on Ω and $|f\chi_{A_n}| \leqslant |f| \in L^1(\nu)$, for $n \in \mathbb{N}$. By the dominated convergence theorem ([19], p. 30), $m_f(A_n) = f\chi_{A_n} \to 0$ in $L^1(\nu)$ as $n \to \infty$, i.e., m_f is σ-additive in the Fréchet space $L^1(\nu)$.

LEMMA 3.8. *Let $\{f_n\}_{n=1}^{\infty} \subseteq L^1(\nu)$ be a weak Cauchy sequence, i.e., Cauchy in the* lcs $L^1(\nu)_\sigma$.

(i) *For each $A \in \Sigma$, the following limit exists in the Hausdorff* lcs $L^1(\nu)^{**}_{\sigma^*}$:

$$\xi(A) := \lim_{n\to\infty} J(f_n\chi_A). \tag{3.1}$$

(ii) *The set function $\xi : \Sigma \to L^1(\nu)^{**}_{\sigma^*}$ defined by* (3.1) *is a vector measure and satisfies $\mathcal{N}_0(\nu) \subseteq \mathcal{N}_0(\xi)$.*

Proof. (i) Fix $A \in \Sigma$. It is clear from (2.1), with $p = 1$, that the operator of multiplication by χ_A is continuous from $L^1(\nu)$ to itself. It is then also continuous from $L^1(\nu)_\sigma$ to itself and hence, $\{f_n\chi_A\}_{n=1}^{\infty}$ is a weak Cauchy sequence in $L^1(\nu)$. By Lemma 3.7 the limit (3.1) exists in the sequentially complete lcs $L^1(\nu)^{**}_{\sigma^*}$.

(ii) For $k \in \mathbb{N}$ fixed the discussion prior to Lemma 3.8 ensures that the set function $m_{f_k} : \Sigma \to L^1(\nu)$ is a vector measure and hence, is also σ-additive when considered to be $L^1(\nu)_\sigma$-valued. Lemma 3.7 implies that $J \circ m_{f_k} : \Sigma \to L^1(\nu)^{**}_{\sigma^*}$ is also σ-additive. Given $u \in L^1(\nu)^*$ we have via (3.1) that $\langle u, \xi\rangle(A) = \lim_{n\to\infty} \langle u, (J \circ m_{f_n})(A)\rangle$, for each $A \in \Sigma$. Since $\langle u, J \circ m_{f_n}\rangle$ is a scalar measure, for each $n \in \mathbb{N}$, the Vitali–Hahn–Saks theorem ensures that $\langle u, \xi\rangle$ is σ-additive. But, the continuous seminorms generating the topology of $L^1(\nu)^{**}_{\sigma^*}$ are given by $\eta \mapsto |\langle u, \eta\rangle|$, for $\eta \in L^1(\nu)^{**}_{\sigma^*}$, as u varies in $L^1(\nu)^*$. Accordingly, ξ is σ-additive as an $L^1(\nu)^{**}_{\sigma^*}$-valued set function.

To see that $\mathcal{N}_0(\nu) \subseteq \mathcal{N}_0(\xi)$, let $A \in \mathcal{N}_0(\nu)$. For each $B \in \Sigma$ with $B \subseteq A$ the functions $\{f_n\chi_B : n \in \mathbb{N}\}$ are ν-null. Hence, $J(f_n\chi_B) = 0$ for $n \in \mathbb{N}$. Then (3.1) implies that $\xi(B) = 0$. So, $A \in \mathcal{N}_0(\xi)$. ∎

LEMMA 3.9. *Let I_ν be completely continuous and $\{f_n\}_{n=1}^{\infty} \subseteq L^1(\nu)_\sigma$ be Cauchy.*

(i) *For each $A \in \Sigma$, the following limit exists in X:*

$$m(A) := \lim_{n\to\infty} I_\nu(f_n\chi_A). \tag{3.2}$$

(ii) *The set function $m : \Sigma \to X$ defined by* (3.2) *is a vector measure and satisfies $\mathcal{N}_0(\nu) \subseteq \mathcal{N}_0(m)$.*

Proof. (i) For $A \in \Sigma$, the sequence $\{f_n\chi_A\}_{n=1}^{\infty}$ is Cauchy in $L^1(\nu)_\sigma$ (c.f. proof of Lemma 3.8(i)). So, by complete continuity the limit (3.2) exists.

(ii) Since $f_n \in L^1(\nu)$, the Orlicz–Pettis theorem ensures that $u_n : A \mapsto \int_A f_n d\nu = I_\nu(f_n\chi_A)$ is a σ-additive, X-valued measure on Σ for each $n \in \mathbb{N}$. In particular, for each $x^* \in X^*$, the scalar valued set functions $\langle u_n, x^*\rangle$, for $n \in \mathbb{N}$, are σ-additive. By the Vitali–Hahn–Saks theorem, also $\langle m, x^*\rangle(A) := \langle m(A), x^*\rangle = \lim_{n\to\infty} \langle u_n(A), x^*\rangle$, for $A \in \Sigma$, is σ-additive. By the Orlicz–Pettis theorem m is σ-additive in X.

To verify that $\mathcal{N}_0(\nu) \subseteq \mathcal{N}_0(m)$, adapt that part of the argument in the proof of Lemma 3.8(ii) showing that $\mathcal{N}_0(\nu) \subseteq \mathcal{N}_0(\xi)$. ∎

Since X is a Fréchet space, there exists a sequence $\{x_j^*\}_{j=1}^\infty \subseteq X^*$ (fixed from now on) which satisfies $\mathcal{N}_0(\nu) = \bigcap_{j=1}^\infty \mathcal{N}_0(\langle \nu, x_j^* \rangle)$; see the proof of Theorem 2.5 in [4], for example. Fix $j \in \mathbb{N}$. It is clear from (2.1), with $p = 1$, and the fact that $x_j^* \in \alpha B_m^\circ$ for some $\alpha > 0$ and $m \in \mathbb{N}$, that the natural identity map from $L^1(\nu)$ into the Banach space $L^1(\langle \nu, x_j^* \rangle)$ is continuous and hence, also continuous from $L^1(\nu)_\sigma$ into $L^1(\langle \nu, x_j^* \rangle)_\sigma$. So given a Cauchy sequence $\{f_n\}_{n=1}^\infty \subseteq L^1(\nu)_\sigma$ (fixed henceforth), it is also Cauchy in the sequentially complete lcs $L^1(\langle \nu, x_j^* \rangle)_\sigma$. Accordingly, there exists $\varphi_j \in L^1(\langle \nu, x_j^* \rangle)$ such that $f_n \to \varphi_j$ weakly in $L^1(\langle \nu, x_j^* \rangle)$ as $n \to \infty$. By the weak Banach–Saks property of $L^1(\langle \nu, x_j^* \rangle)$ ([29]) there exists a subsequence $\{f_{n(k)}^{(j)}\}_{k=1}^\infty$ of $\{f_n\}_{n=1}^\infty$ whose arithmetic means $N^{-1} \sum_{k=1}^N f_{n(k)}^{(j)} \to \varphi_j$ in the *norm* of $L^1(\langle \nu, x_j^* \rangle)$ as $N \to \infty$. These arithmetic means admit a subsequence

(3.3)
$$g_{N(\ell)}^{(j)} := \frac{1}{N(\ell)} \sum_{k=1}^{N(\ell)} f_{n(k)}^{(j)}, \quad \ell \in \mathbb{N},$$

converging $\langle \nu, x_j^* \rangle$-a.e. to φ_j. So, there exists $B_j \in \Sigma$ with $(\Omega \backslash B_j) \in \mathcal{N}_0(\langle \nu, x_j^* \rangle)$ and $g_{N(\ell)}^{(j)} \to \varphi_j$ pointwise on B_j as $\ell \to \infty$. Set $A_1 := B_1$ and $A_k := B_k \backslash \bigcup_{j=1}^{k-1} B_j$, for $k \geqslant 2$. Then $\bigcup_{j=1}^k A_j = \bigcup_{j=1}^k B_j$, for $k \in \mathbb{N}$, and

(3.4)
$$\lim_{\ell \to \infty} g_{N(\ell)}^{(j)} \chi_{A_j} = \varphi_j \chi_{A_j}, \quad \text{pointwise on } \Omega, \quad \forall j \in \mathbb{N}.$$

LEMMA 3.10. *In the setting of the above construction and with I_ν assumed to be completely continuous we have the following facts:*

(i) *The sets $\{A_k\}_{k=1}^\infty$ are pairwise disjoint with $(\Omega \backslash \bigcup_{k=1}^\infty A_k) \in \mathcal{N}_0(\nu)$.*

(ii) *For each $j \in \mathbb{N}$ the function $\varphi_j \chi_{A_j}$ belongs to $L^1(\nu)$ with*

(3.5)
$$\int_A \varphi_j \chi_{A_j} \, d\nu = m(A \cap A_j), \quad A \in \Sigma, \text{ and}$$

(3.6)
$$J(\varphi_j \chi_{A_j}) = \xi(A_j), \quad \text{as elements of } L^1(\nu)_{\sigma^*}^{**}.$$

Proof. (i) The sets $\{A_k\}_{k=1}^\infty$ are pairwise disjoint by construction. Also

$$|\langle \nu, x_j^* \rangle|(\Omega \backslash \textstyle\bigcup_{k=1}^\infty A_k) \leqslant |\langle \nu, x_j^* \rangle|(\Omega \backslash B_j) = 0, \quad \forall j \in \mathbb{N},$$

and so $(\Omega \backslash \bigcup_{k=1}^\infty A_k) \in \bigcap_{j=1}^\infty \mathcal{N}_0(\langle \nu, x_j^* \rangle) = \mathcal{N}_0(\nu)$.

(ii) Fix $j \in \mathbb{N}$. For each $A \in \Sigma$ it follows from (3.2) that $m(A \cap A_j) = \lim_{N\to\infty} I_\nu\Big(N^{-1}\sum_{k=1}^{N} f^{(j)}_{n(k)}\chi_{A_j}\chi_A\Big)$ in X and hence, from (3.3), that also

$$\text{(3.7)} \qquad \lim_{\ell\to\infty}\int_A g^{(j)}_{N(\ell)}\chi_{A_j}\,d\nu = \lim_{\ell\to\infty} I_\nu(g^{(j)}_{N(\ell)}\chi_{A_j}\chi_A) = m(A\cap A_j).$$

Fix $x^* \in X^*$. As a consequence of (3.7) the scalar measures $\lambda_\ell(A) := \int_A g^{(j)}_{N(\ell)}\chi_{A_j}\,d\langle\nu,x^*\rangle$ (recall that $g^{(j)}_{N(\ell)}\chi_{A_j} \in L^1(\nu)$, for $\ell \in \mathbb{N}$) have a limit for each $A \in \Sigma$ namely, $\lim_{\ell\to\infty}\lambda_\ell(A) = \langle m(A\cap A_j), x^*\rangle$. It follows from (3.4) and Lemma 2.3 of [21] that $\varphi_j\chi_{A_j} \in L^1(\langle\nu,x^*\rangle)$ and

$$\text{(3.8)} \qquad \int_A \varphi_j\chi_{A_j}\,d\langle\nu,x^*\rangle = \langle m(A\cap A_j), x^*\rangle, \quad A \in \Sigma.$$

Since $x^* \in X^*$ is arbitrary, we have $\varphi_j\chi_{A_j} \in L^1_{\mathrm{w}}(\nu)$. Moreover, for each $A \in \Sigma$ it follows from (3.8) that the vector $m(A\cap A_j) \in X$ satisfies

$$\langle m(A\cap A_j), x^*\rangle = \int_A \varphi_j\chi_{A_j}\,d\langle\nu,x^*\rangle, \quad x^* \in X^*.$$

According to the definition of ν-integrability we can conclude that $\varphi_j\chi_{A_j} \in L^1(\nu)$, for each $j \in \mathbb{N}$, and that (3.5) is valid.

Again fix $j \in \mathbb{N}$. Since $\{g^{(j)}_{N(\ell)}\chi_{A_j}\}_{\ell=1}^\infty \subseteq L^1(\nu)$, Lemma 2.1(ii) yields that $\{g^{(j)}_{N(\ell)}\chi_{A_j}\}_{\ell=1}^\infty \subseteq L^1(\nu_n)$, for all $n \in \mathbb{N}$. Moreover, (3.7) implies that $\lim_{\ell\to\infty}\int_A g^{(j)}_{N(\ell)}\chi_{A_j}d\nu_n = \pi_n(m(A\cap A_j))$ exists in X_n, for each $A \in \Sigma$ and $n \in \mathbb{N}$. Via (3.4) it follows from Theorem 2.2.8 of [26] that

$$\lim_{\ell\to\infty}\int_A g^{(j)}_{N(\ell)}\chi_{A_j}\,d\nu_n = \int_A \varphi_j\chi_{A_j}\,d\nu_n, \quad A \in \Sigma,$$

with the limit in X_n existing *uniformly* for $A \in \Sigma$. Hence, (2.2) and the identities $\|f\|^{(n)}_\nu = \|f\|_{\nu_n}$, valid for all $n \in \mathbb{N}$ and $f \in L^1(\nu)$, imply

$$\text{(3.9)} \qquad \lim_{\ell\to\infty} g^{(j)}_{N(\ell)}\chi_{A_j} = \varphi_j\chi_{A_j}, \quad \text{in the Fréchet space } L^1(\nu).$$

To establish (3.6), note that (3.9) implies $g^{(j)}_{N(\ell)}\chi_{A_j} \to \varphi_j\chi_{A_j}$ in $L^1(\nu)_\sigma$ as $\ell \to \infty$ and hence, by Lemma 3.7, that $J(g^{(j)}_{N(\ell)}\chi_{A_j}) \to J(\varphi_j\chi_{A_j})$ in $L^1(\nu)^{**}_{\sigma^*}$ as $\ell \to \infty$. On the other hand, it follows from (3.1) that $J\Big(N^{-1}\sum_{k=1}^{N} f^{(j)}_{n(k)}\chi_{A_j}\Big) \to \xi(A_j)$ in $L^1(\nu)^{**}_{\sigma^*}$ as $N \to \infty$. Via (3.3), also $J(g^{(j)}_{N(\ell)}\chi_{A_j}) \to \xi(A_j)$ in $L^1(\nu)^{**}_{\sigma^*}$ as $\ell \to \infty$. So, (3.6) is valid. ∎

Proof of Theorem 3.6. To show $L^1(\nu)$ is weakly sequentially complete (equivalent to $L^1(\nu) = L^1_{\mathrm{w}}(\nu)$) let $\{f_n\}_{n=1}^\infty$ be a weak Cauchy sequence in $L^1(\nu)$. In the notation of the above construction, define $f := \sum_{j=1}^{\infty} \varphi_j \chi_{A_j}$ pointwise on Ω (recall the sets $A_j, j \in \mathbb{N}$, are pairwise disjoint). The aim is to show that $f \in L^1(\nu)$ and $f_n \to f$ in $L^1(\nu)_\sigma$.

By Lemma 3.10(ii), each function $\psi_r := \sum_{j=1}^{r} \varphi_j \chi_{A_j} \in L^1(\nu)$, for $r \in \mathbb{N}$. Moreover, $\psi_r \to f$ pointwise on Ω as $r \to \infty$. By Lemma 3.9(ii) and Lemma 3.10(i) the set $\Omega \backslash (\bigcup_{j=1}^\infty A_j)$ is m-null. Due to (3.5), the σ-additivity of m (c.f. Lemma 3.9(ii)), and the fact that $\Omega \backslash (\bigcup_{j=1}^\infty A_j) \in \mathcal{N}_0(m)$, we can conclude, for each $A \in \Sigma$, that the sequence

$$\int_A \psi_r \mathrm{d}\nu = \sum_{j=1}^{r} \int_A \varphi_j \chi_{A_j} \mathrm{d}\nu = \sum_{j=1}^{r} m(A \cap A_j) = m(A \cap (\textstyle\bigcup_{j=1}^r A_j)), \quad r \in \mathbb{N},$$

converges in X to $m(A \cap (\bigcup_{j=1}^\infty A_j)) = m(A)$ as $r \to \infty$. Repeating the argument used to establish (3.9) it follows that $f \in L^1(\nu)$ with $\int_A f \mathrm{d}\nu = m(A)$, for $A \in \Sigma$ and that $\psi_r \to f$ in $L^1(\nu)$ as $r \to \infty$. In particular, $\psi_r \to f$ in $L^1(\nu)_\sigma$ and so, by Lemma 3.7, $J(\psi_r) \to J(f)$ in $L^1(\nu)^{**}_{\sigma^*}$. On the other hand, applying Lemma 3.8(ii) and (3.6) yields

$$\xi(\Omega) = \sum_{j=1}^{\infty} \xi(A_j) = \sum_{j=1}^{\infty} J(\varphi_j \chi_{A_j}) = \lim_{r \to \infty} J(\psi_r) = J(f)$$

with the limit existing in $L^1(\nu)^{**}_{\sigma^*}$. In view of (3.1), with $A := \Omega$, we conclude that $J(f_n - f) \to 0$ in $L^1(\nu)^{**}_{\sigma^*}$ which, by Lemma 3.7, implies that $f_n \to f$ in $L^1(\nu)_\sigma$ as $n \to \infty$. ∎

REMARK 3.11. (i) If there is a *single* (Rybakov) functional $x_1^* \in X^*$ such that $\mathcal{N}_0(\nu) = \mathcal{N}_0(\langle \nu, x_1^* \rangle)$, then clearly the proof of Theorem 3.6 can be simplified. For Fréchet spaces X which admit a continuous norm (i.e., all Banach spaces and many non-normable Fréchet spaces), *every* X-valued measure has such a Rybakov functional, [12].

(ii) By Corollary 3.3 and Theorem 3.6, any one of compactness, weak compactness or complete continuity of I_ν force $L^1(\nu)$ to be weakly sequentially complete. The converse fails. The following example is over $\mathbb{C}$ rather than $\mathbb{R}$. But, via the usual complexification of Banach (and Fréchet) lattices and function spaces over $\mathbb{R}$ (cf. [14], Chapters 2 and 3 of [25] and [26]), this causes no difficulties in passing to spaces over $\mathbb{C}$. If G is any infinite compact abelian group and λ any $\mathbb{C}$-valued, regular Borel measure on G, then the convolution operator $C_\lambda : L^1(G) \to L^1(G)$, i.e., $f \mapsto \lambda * f$, for $f \in L^1(G)$, is continuous and induces the $L^1(G)$-valued measure $m_\lambda : A \mapsto C_\lambda(\chi_A)$. For *every* such λ the space $L^1(m_\lambda) = L^1(G)$ and so $L^1(m_\lambda)$ is weakly sequentially complete ([25], Proposition 7.35). But, I_{m_λ} is compact (= weakly compact) if and only if λ is absolutely continuous with respect to Haar measure whereas I_{m_λ} is completely continuous if and only if the Fourier–Stieltjes transform $\widehat{\lambda}$ of λ belongs to $c_0(\Gamma)$, with Γ the dual group of G ([25], Remark 7.36). Since there always exist λ with $\widehat{\lambda} \notin c_0(\Gamma)$, for such λ we see that $L^1(m_\lambda)$ is weakly sequentially complete, but I_{m_λ} is neither compact, weakly compact or completely continuous.

The Fréchet space $L^1(\nu)$ can also be weakly sequentially complete for a different reason. A Fréchet-space-valued measure ν is said to have *finite variation* if $|\nu_n|(\Omega) < \infty$ for each $n \in \mathbb{N}$, where ν_n, for $n \in \mathbb{N}$, is given by (2.3).

We have seen that $L^1(\nu_{n+1}) \subseteq L^1(\nu_n)$, for $n \in \mathbb{N}$, with a continuous inclusion. Similarly, $L^1(|\nu_{n+1}|) \subseteq L^1(|\nu_n|)$ continuously, for each $n \in \mathbb{N}$. Then $L^1(\nu) := \mathcal{L}^1(\nu)/\mathcal{N}(\nu)$ is the Fréchet space $\bigcap_{n=1}^{\infty} L^1(\nu_n)$ with the increasing sequence of *norms* $\{\|\cdot\|_{\nu_n}\}_{n=1}^{\infty}$. Define $L^1(|\nu|) := \bigcap_{n=1}^{\infty} L^1(|\nu_n|) = (\bigcap_{n=1}^{\infty} \mathcal{L}^1(|\nu_n|)/\mathcal{N}(\nu)$, equipped with the increasing sequence of *norms*

$$(3.10) \qquad |||f|||_{\nu}^{(n)} := \int_{\Omega} |f| \mathrm{d}|\nu_n|, \quad f \in \bigcap_{k=1}^{\infty} L^1(|\nu_k|), \quad n \in \mathbb{N}.$$

Then $L^1(|\nu|)$ is also a Fréchet space ([17], p. 17) and is continuously included in $L^1(\nu)$ ([23], Lemma 2.4; [24], Lemma 2).

PROPOSITION 3.12. *Let ν be any Fréchet-space-valued measure with finite variation. Then $L^1(|\nu|)$ is weakly sequentially complete.*

Proof. Since each lc-space $L^1(|\nu_n|)_\sigma$, $n \in \mathbb{N}$, is sequentially complete, the product lc-space $\prod_{n=1}^{\infty} L^1(|\nu_n|)_\sigma$ is also sequentially complete ([20], (2) p. 296). Moreover, we have $(\prod_{n=1}^{\infty} L^1(|\nu_n|))_\sigma = \prod_{n=1}^{\infty} L^1(|\nu_n|)_\sigma$ ([20], (3) p. 285) and so $\prod_{n=1}^{\infty} L^1(|\nu_n|)$ is weakly sequentially complete. But, $L^1(|\nu|)$ is topologically isomorphic to a closed (hence, also weakly closed) subspace of $\prod_{n=1}^{\infty} L^1(|\nu_n|)$; Lemma 25.4 of [22] or see the proof of (7) p. 208 in [20] as applied to $L^1(|\nu|)$ and recall (3.10). Accordingly, $L^1(|\nu|)$ is weakly sequentially complete. ∎

COROLLARY 3.13. *Let ν be any Fréchet-space-valued measure such that $L^1(\nu) = L^1(|\nu|)$ as vector spaces. Then $L^1(\nu)$ is weakly sequentially complete. In particular, $L^1(\nu) = L^1_{\mathrm{w}}(\nu)$.*

Proof. Since $\chi_\Omega \in L^1(\nu)$, also $\chi_\Omega \in L^1(|\nu|) = \bigcap_{k=1}^{\infty} L^1(|\nu_k|)$ and hence, ν has finite variation. As already noted, the natural inclusion of $L^1(|\nu|)$ into $L^1(\nu)$ is always continuous and injective. So, as it is also surjective, the open mapping theorem ensures that $L^1(\nu)$ and $L^1(|\nu|)$ are topologically isomorphic. Then apply Proposition 3.12. ∎

Whenever I_ν is a compact map the vector measure ν necessarily satisfies $L^1(\nu) = L^1(|\nu|)$ ([23], Theorems 1 and 2). The converse is false, even for Banach spaces; see Remark 3.11(ii).

Since $L^1(|\nu|)$ is a Fréchet AL-space, we point out that Corollary 3.13 also follows from Corollary 2.6 of [13].

It is instructive to analyze some non-trivial examples of vector measures ν in *non-normable* Fréchet spaces and the ideal properties of I_ν. So, let us consider Examples 4.1–4.4 in [23]; the vector measures presented there, all denoted by m, will here be denoted by ν.

For Examples 4.1 and 4.2 the Fréchet space X is, respectively, ω and s (rapidly decreasing sequences). In both examples it is shown that $L^1(\nu)$ is non-normable, satisfies $L^1(\nu) = L^1(|\nu|)$ but, I_ν is not compact. So, I_ν cannot be weakly compact either (ω and s are Montel). As already noted, whenever X is a Montel, I_ν is completely continuous.

In the case of Example 4.4 in [23], where X is the reflexive Fréchet space $\ell^{p+} = \bigcap_{q>p} \ell^q$, $1 < p < \infty$, it is shown that $L^1(\nu)$ is a *Banach* space, satisfies $L^1(\nu) = L^1(|\nu|)$ but, I_ν is not compact. Since each Banach space $X_k = \ell^{p_k}$ for some $p < p_k < \infty$, it is reflexive. By Corollary 3.4, I_ν is weakly compact. The claim is that I_ν is also completely continuous. To see this, let $\{f_n\}_{n=1}^{\infty}$ be a null sequence in $L^1(\nu)_\sigma$ and observe that $I_\nu(f_n) \to 0$ in X if and only if $(\pi_k \circ I_\nu)(f_n) = I_{\nu_k}(f_n) \to 0$ in the Banach space X_k as $n \to \infty$, for each $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$. Since $L^1(\nu) = L^1(\nu_k)$ as lc spaces ([23], p. 226) also $f_n \to 0$ in $L^1(\nu_k)_\sigma$ as $n \to \infty$. Moreover, it follows from p. 225 of [23] that $I_{\nu_k} : L^1(\nu_k) = L^1(|\nu_k|) \to X_k$ is Bochner representable. So, by Corollary 9(c) in p. 56 of [11], ν_k has relatively compact range in X_k. Hence, I_{ν_k} is completely continuous ([25], Corollary 2.43) and so $I_{\nu_k}(f_n) \to 0$ in X_k as $n \to \infty$. This shows that $I_\nu(f_n) \to 0$ in X, i.e., I_ν is completely continuous.

For Example 4.3 of [23], where $X = \ell^{p+}, 1 < p < \infty$, the inclusion $L^1(|\nu|) \subsetneq L^1(\nu)$ is *proper* with $L^1(|\nu|)$ a Banach space whereas $L^1(\nu)$ is non-normable. Since the X_k, $k \in \mathbb{N}$, are all reflexive, Corollary 3.4 yields that I_ν is not weakly compact (hence, not compact). To show I_ν is *not* completely continuous is more involved. The notation below is as in [23] except that m there is here denoted by ν.

Set $h_n := (n+1)n\alpha_n^{-1}\chi_{F(n)} \geqslant 0$, for $n \in \mathbb{N}$. With $\{e_n\}_{n=1}^{\infty}$ being the standard basis vectors of X, fix $x = \sum_{n=1}^{\infty} x(n)e_n$. Since $\nu_k : \Sigma \to X_k := \ell^{p_k}$ is a *positive* vector measure, for each $k \in \mathbb{N}$, it follows from Lemma 3.13 of [25] and pairwise disjointness of the sets $\{F(n)\}_{n=1}^{\infty}$ that

$$\begin{aligned}
&\Big\| \sum_{n=1}^{N} x(n)h_n - \sum_{n=1}^{M} x(n)h_n \Big\|_{\nu_k} \\
&= \Big\| \sum_{n=M+1}^{N} x(n)h_n \Big\|_{\nu_k} = \Big\| \int_{\Omega} \Big| \sum_{n=M+1}^{N} x(n)h_n \Big| d\nu_k \Big\|_{X_k} \\
&= \Big\| \sum_{n=M+1}^{N} |x(n)| \int_{\Omega} h_n d\nu_k \Big\|_{X_k} = \Big\| \sum_{n=M+1}^{N} |x(n)| e_n \Big\|_{X_k} = \Big\| \sum_{n=M+1}^{N} x(n) e_n \Big\|_{X_k},
\end{aligned} \tag{3.11}$$

for all $M < N$ in $\mathbb{N}$. So, $\{\sum_{n=1}^{N} x(n)h_n\}_{N=1}^{\infty}$ is Cauchy in $L^1(\nu)$, with limit h, say. For each $j \in \mathbb{N}$, multiplication by $\chi_{F(j)}$ is a continuous operator from $L^1(\nu)$ into itself, which implies that

$$h\chi_{F(j)} = \lim_{N\to\infty} \sum_{n=1}^{N} x(n)h_n\chi_{F(j)} = x(j)h_j \quad (\text{in } L^1(\nu)).$$

Accordingly, $h = \sum_{n=1}^{\infty} x(n)h_n$ pointwise on Ω and with the series converging in $L^1(\nu)$. By continuity of I_ν we can conclude that

$$I_\nu(h) = \sum_{n=1}^{\infty} x(n)I_\nu(h_n) = \sum_{n=1}^{\infty} x(n)e_n = x \quad (\text{in } X). \tag{3.12}$$

Moreover, since also $\sum_{n=1}^{\infty} x(n)h_n = h$ in $L^1(\nu_k)$ and $x = \sum_{n=1}^{\infty} x(n)e_n$ in $X_k = \ell^{p_k}$, for each $k \in \mathbb{N}$, where we now interpret $\{e_n\}_{n=1}^{\infty}$ as the canonical basis in X_k, it follows from (3.12) that

$$\|h\|_{\nu_k} = \lim_{N\to\infty} \Big\| \sum_{n=1}^{N} x(n)h_n \Big\|_{\nu_k} = \lim_{N\to\infty} \Big\| \sum_{n=1}^{N} x(n)e_n \Big\|_{X_k} = \|x\|_{X_k}; \tag{3.13}$$

here we have used $\Big\| \sum_{n=1}^{N} x(n)h_n \Big\|_{\nu_k} = \Big\| \sum_{n=1}^{N} x(n)e_n \Big\|_{X_k}$ for each $N \in \mathbb{N}$, which can be verified as in (3.11).

Define $\Phi : X \to L^1(\nu)$ by $\Phi(x) := \sum_{n=1}^{\infty} x(n)h_n$, for $x \in X$. Then Φ is injective and, by (3.13) continuous since, for each $k \in \mathbb{N}$, we have

$$\|\Phi(x)\|_{\nu_k} = \|x\|_{X_k}, \quad x \in X. \tag{3.14}$$

The range of Φ is precisely the subspace $W \subseteq L^1(\nu)$ given by

$$W := \Big\{ h \in L^1(\nu) : h = \sum_{n=1}^{\infty} x(n)h_n \text{ pointwise, for some } x \in X \Big\}.$$

From (3.14) we see that $\|h\|_{\nu_k} = \|\Phi^{-1}(h)\|_{X_k}$, $h \in W$, for $k \in \mathbb{N}$, that is, $\Phi^{-1} : W \to X$ is continuous when W is equipped with the relative topology from $L^1(\nu)$. So,

Φ is a topological isomorphism of X onto W, i.e., the restriction $I_\nu|_W = \Phi^{-1}$ is a surjective isomorphism of W onto X. Since $h_n \to 0$ in W_σ (as $e_n \to 0$ in X_σ and $h_n = \Phi^{-1}(e_n)$, $n \in \mathbb{N}$), but $I_\nu|_W(h_n) = e_n \nrightarrow 0$ in the Fréchet topology of X, the operator $I_\nu|_W$ is not completely continuous. So, if $H : W \to L^1(\nu)$ denotes the canonical inclusion, then the identity $I_\nu|_W = I_\nu \circ H : W \to X$ implies that I_ν cannot be completely continuous either.

Accordingly, for Example 4.3 of [23] the integration map I_ν is *neither* compact, weakly compact or completely continuous and it fails to satisfy $L^1(\nu) = L^1(|\nu|)$. Nevertheless, since the reflexive space X cannot contain an isomorphic copy of c_0 we still have $L^1(\nu) = L^1_{\mathrm{w}}(\nu)$ ([19], p. 31 Theorem 1) that is, $L^1(\nu)$ is weakly sequentially complete. For a Banach space example exhibiting these features (i.e., those of Example 4.3 in [23]) we refer to Example 3.26(ii) of [25].

4. LATTICE PROPERTIES OF $L^p_{\mathrm{w}}(\nu)$ AND $L^p(\nu)$

Let (F, τ) be a metrizable lc-solid Riesz space with a fundamental sequence of Riesz seminorms $\{q_n\}_{n\in\mathbb{N}}$ ([1], Chapter 2, Section 6). Recall that τ is called a *Lebesgue* (respectively *σ-Lebesgue*) *topology*, if $u_\alpha \downarrow 0$ implies $u_\alpha \xrightarrow{\tau} 0$ in F (respectively $u_k \downarrow 0$ implies $u_k \xrightarrow{\tau} 0$ in F) ([1], Chapter 3). The space F has the *Fatou* (respectively *σ-Fatou*) *property* if, for every increasing net $\{u_\alpha\}_\alpha$ (respectively increasing sequence $\{u_k\}_k$) in the positive cone F^+ of F which is topologically bounded in F, the element $u := \sup u_\alpha$ exists in F^+ and $q_n(u_\alpha) \uparrow_\alpha q_n(u)$ (respectively $u := \sup u_k$ exists in F^+ and $q_n(u_k) \uparrow_k q_n(u)$), for $n \in \mathbb{N}$.

Let (Ω, Σ, μ) be a σ-finite measure space, $\mathcal{M} := L^0(\mu)$, and $\{\rho_n\}_{n\in\mathbb{N}}$ be a fundamental (i.e. $\bigcap_{n\in\mathbb{N}} \rho_n^{-1}(\{0\}) = \{0\}$) increasing sequence of function seminorms on $\mathcal{M}$ (see Chapter 15 of [31] for the definition of a function seminorm). The *metrizable function space* induced by $\{\rho_n\}_{n\in\mathbb{N}}$ is the locally solid, metrizable lcs

$$L_{\{\rho_n\}} := \{f \in \mathcal{M} : \rho_n(f) < \infty, \ \forall n \in \mathbb{N}\}$$

equipped with the topology induced by $\{\rho_n\}_{n\in\mathbb{N}}$; see Lemma 22.5 of [22]. If $L_{\{\rho_n\}}$ is also complete, then it is called a *Fréchet function space* (briefly, F.f.s.). Given a F.f.s. there is no distinction between using nets or sequences when specifying either a Lebesgue topology or the Fatou property ([5], Section 2). A function seminorm ρ in $\mathcal{M}$ is said to have the *Fatou property* if $\rho(u_k) \uparrow \rho(u)$ whenever $0 \leqslant u_k \uparrow u$ in $\mathcal{M}$.

Let X be a Fréchet space and $\nu : \Sigma \to X$ be a measure. It was noted in Theorem 4.5.11(ii) of [26] that $L^p(\nu)$, $L^p_{\mathrm{w}}(\nu)$ are Fréchet lattices for the pointwise order. Actually, they are "better" than just being Fréchet lattices. Let μ be any control measure for ν (e.g., the one in the proof of Theorem 2.5 in [4]). It was shown in Example 1 of [5] that both $L^1_{\mathrm{w}}(\nu), L^1(\nu)$ are F.f.s.' relative to (Ω, Σ, μ).

The next result shows the same is true of $L^p_{\mathrm{w}}(\nu)$ $L^p(\nu)$ and that they have special properties.

THEOREM 4.1. *Let X be a metrizable* lcs, $\nu : \Sigma \to X$ *be a vector measure and μ be a control measure for ν. Then, for each $1 \leqslant p < \infty$, the increasing sequence of functions seminorms $\{(\rho_\nu)_n^{(p)}\}_{n\in\mathbb{N}}$ defined, relative to (Ω, Σ, μ), by*

$$(\rho_\nu)_n^{(p)}(f) := \|f\|_{\nu,p}^{(n)} \quad f \in L^0(\mu) = L^0(\nu), \quad n \in \mathbb{N},$$

makes $L^p_{\mathrm{w}}(\nu)$ a F.f.s. with the Fatou property.

Moreover, if X is a Fréchet space, then $L^p(\nu)$ is a F.f.s. for the topology $\tau^{(p)}$ induced by the increasing sequence of function seminorms $\{(\widetilde{\rho}_\nu)_n^{(p)}\}_{n\in\mathbb{N}}$ defined, relative to (Ω, Σ, μ), by

$$(\widetilde{\rho}_\nu)_n^{(p)}(f) := \begin{cases} (\rho_\nu)_n^{(p)}(f) & \text{if } f \in L^p(\nu), \\ \infty & \text{if } f \in L^0(\mu)\backslash L^p(\nu) = L^0(\nu)\backslash L^p(\nu), \end{cases}$$

and $\tau^{(p)}$ is a Lebesgue topology.

Proof. By Example 1 of [5] we know that $L^1_{\mathrm{w}}(\nu) = L_{\{(\rho_\nu)_n^{(1)}\}}$ and that $L^1(\nu) = L_{\{(\widetilde{\rho}_\nu)_n^{(1)}\}}$. Hence, $L^p_{\mathrm{w}}(\nu) = L_{\{(\rho_\nu)_n^{(p)}\}}$ and $L^p(\nu) = L_{\{(\widetilde{\rho}_\nu)_n^{(p)}\}}$.

According to Section 65, Theorem 4 of [31] and the formula (2.1), the function seminorm $(\rho_\nu)_n^{(p)}$ has the Fatou property for each $n \in \mathbb{N}$ (since the norm of the L^p-space of any positive measure has the Fatou property). Then $L_{\{\rho_n\}}$ is a F.f.s. with the Fatou property ([5], Theorem 2.4).

To check that $\tau^{(p)}$ is a Lebesgue topology, let $\{u_k\}_k$ be a sequence in $L^p(\nu)^+$ with $u_k \downarrow 0$. Then each $u_k^p \in L^1(\nu)^+$ and $u_k^p \downarrow 0$. Fix $n \in \mathbb{N}$. As $\tau^{(1)}$ is a Lebesgue topology ([4], Section 3) it follows that $(\widetilde{\rho}_\nu)_n^{(1)}(u_k^p) \to 0$ as $k \to \infty$. Hence, also $(\widetilde{\rho}_\nu)_n^{(p)}(u_k) \to 0$ as $k \to \infty$. As $n \in \mathbb{N}$ is arbitrary, this shows that $\tau^{(p)}$ is a Lebesgue topology. ∎

For ν a Banach-space-valued measure, Theorem 4.1 is known. Indeed, that $L^p_{\mathrm{w}}(\nu)$ has the Fatou property occurs in Proposition 1 of [10], Proposition 2.1 and Lemma 3.8 of [15], and for $\tau^{(p)}$ being a Lebesgue topology (i.e., the norm is order continuous) we refer to p. 291 of [10], and Proposition 2.1 of [15].

The *Lorentz function seminorm* ρ_{L}, associated to any function seminorm $\rho : \mathcal{M} \to [0, \infty]$, is defined by

$$\rho_{\mathrm{L}}(u) := \inf\left\{ \lim_k \rho(u_k) : u_k \in \mathcal{M}^+, u_k \uparrow u \right\}, \quad u \in \mathcal{M}^+,$$

in which case $\rho_{\mathrm{L}} \leqslant \rho$. Moreover, ρ_{L} is the largest function seminorm with the Fatou property which is majorized by ρ ([31], Chapter 15, Section 66). Let $\{\rho_n\}_{n\in\mathbb{N}}$ be any increasing fundamental sequence of function seminorms on $\mathcal{M}$. According to Definition 2.6 of [5] the *Fatou completion* of the metrizable function space

$L_{\{\rho_n\}}$ is defined as the F.f.s. $(L_{\{\rho_n\}})^{\mathrm{F}} := L_{\{(\rho_n)_{\mathrm{L}}\}}$ (i.e., the *minimal* F.f.s. in $L^0(\mu)$ with the Fatou property and containing $L_{\{\rho_n\}}$ continuously, as $(\rho_n)_{\mathrm{L}} \leqslant \rho_n$, $\forall n \in \mathbb{N}$). It is known that $(L^1(\nu))^{\mathrm{F}} = L^1_{\mathrm{w}}(\nu)$ ([5], Theorem 3.1); see p. 191 of [8] for Banach spaces. For the notion of the *σ-order continuous part* $(L_{\{\rho_n\}})_a$ of $L_{\{\rho_n\}}$ see pp. 331–332 of [32]. It is known that $(L^1_{\mathrm{w}}(\nu))_a = L^1(\nu)$ ([4], Theorem 3.2); for Banach spaces see p. 192 of [8]. The above relationships remain valid for all L^p-spaces; for Banach spaces see p. 289 and p. 291 of [10].

THEOREM 4.2. *Let ν be a Fréchet space-valued measure and $p \geqslant 1$. Then we have $(L^p(\nu))^{\mathrm{F}} = L^p_{\mathrm{w}}(\nu)$ and $(L^p_{\mathrm{w}}(\nu))_a = L^p(\nu)$.*

Proof. Let μ be any control measure for ν. Fix $n \in \mathbb{N}$. Clearly $(\rho_\nu)^{(p)}_n \leqslant (\widetilde{\rho}_\nu)^{(p)}_n$ in $L^0(\mu)$ with the function seminorms $(\rho_\nu)^{(p)}_n$, $n \in \mathbb{N}$, having the Fatou property; see Theorem 4.1. By maximality of the Lorentz seminorm $((\widetilde{\rho}_\nu)^{(p)}_n)_{\mathrm{L}}$ it follows that $(\rho_\nu)^{(p)}_n \leqslant ((\widetilde{\rho}_\nu)^{(p)}_n)_{\mathrm{L}}$ in $L^0(\mu)$. Hence, $(L^p(\nu))^{\mathrm{F}} \subseteq L^p_{\mathrm{w}}(\nu)$. On the other hand, given $f \in L^p_{\mathrm{w}}(\nu)^+$, choose Σ-simple functions $0 \leqslant s_k \uparrow f$. Since $((\widetilde{\rho}_\nu)^{(p)}_n)_{\mathrm{L}} \leqslant (\widetilde{\rho}_\nu)^{(p)}_n$ with $\{s_k\}_k \subseteq L^p(\nu)$, we have, for each $n \in \mathbb{N}$, that

$$((\widetilde{\rho}_\nu)^{(p)}_n)_{\mathrm{L}}(s_k) \leqslant (\widetilde{\rho}_\nu)^{(p)}_n(s_k) = (\rho_\nu)^{(p)}_n(s_k) \leqslant (\rho_\nu)^{(p)}_n(f) < \infty, \quad k \in \mathbb{N}.$$

Accordingly, $\{s_k\}_k$ is topologically bounded in $(L^p(\nu))^{\mathrm{F}}$ with $s_k \uparrow f$. By the Fatou property of $(L^p(\nu))^{\mathrm{F}}$ we conclude that $f \in (L^p(\nu))^{\mathrm{F}}$. This shows that $(L^p(\nu))^{\mathrm{F}} = L^p_{\mathrm{w}}(\nu)$, with equality as vector spaces and also topologically (by the open mapping theorem).

As already noted, $\tau^{(p)}$ is a Lebesgue topology for $L^p(\nu)$. Since $L^p(\nu)$ has the relative topology from $L^p_{\mathrm{w}}(\nu)$, we have $L^p(\nu) \subseteq (L^p_{\mathrm{w}}(\nu))_a$. Conversely, let $f \in (L^p_{\mathrm{w}}(\nu))_a$ and assume $f \geqslant 0$. Choose Σ-simple functions $\{s_k\}_k$, with $0 \leqslant s_k \uparrow f$ (ν-a.e.). Then $0 \leqslant (f - s_k) \leqslant f$ for all k with $(f - s_k) \downarrow 0$. By definition of $f \in (L^p_{\mathrm{w}}(\nu))_a$ this implies that $\{f - s_k\}_k$ converges to 0 in $L^p_{\mathrm{w}}(\nu)$, i.e., $\{s_k\}_k$ converges to f in $L^p_{\mathrm{w}}(\nu)$. But, $\{s_k\}_k \subseteq L^p(\nu)$ with $L^p(\nu)$ closed in $L^1_{\mathrm{w}}(\nu)$. So, $f \in L^p(\nu)$. Since every $f \in (L^p_{\mathrm{w}}(\nu))_a$ has a decomposition $f = f^+ - f^-$ with $f^+, f^- \in (L^p_{\mathrm{w}}(\nu))_a$, we have $(L^p_{\mathrm{w}}(\nu))_a \subseteq L^p(\nu)$. Thus, $(L^p_{\mathrm{w}}(\nu))_a = L^p(\nu)$. ∎

The previous results yield information about $L^p(\nu)$, $L^p_{\mathrm{w}}(\nu)$; for $p = 1$ see Proposition 3.4 of [5]. For the Banach space version of the following result see Proposition 3 of [10], Corollary 3.10 of [15]. A Fréchet lattice F is a KB-*space* if every topologically bounded, increasing sequence in F^+ is convergent.

COROLLARY 4.3. *Let ν be a Fréchet-space-valued measure and $p \geqslant 1$. Then the following statements are equivalent:*

(i) $L^1_{\mathrm{w}}(\nu) = L^1(\nu)$.

(ii) $L^p_{\mathrm{w}}(\nu) = L^p(\nu)$.

(iii) *The topology of $L^p_{\mathrm{w}}(\nu)$ is Lebesgue.*

(iv) *$L^p(\nu)$ has the Fatou property.*

(v) $L^p_{\rm w}(\nu)$ *is a* KB-*space.*

(vi) $L^p(\nu)$ *is a* KB-*space.*

(vii) $L^p_{\rm w}(\nu)$ *contains no lattice copy of* c_0.

(viii) $L^p(\nu)$ *contains no lattice copy of* c_0.

(ix) $L^p_{\rm w}(\nu)$ *is weakly sequentially complete.*

(x) $L^p(\nu)$ *is weakly sequentially complete.*

Proof. (i) $\Leftrightarrow$ (ii) Evident from the definitions.

(ii) $\Leftrightarrow$ (iii) Theorem 4.1 and the second equality in Theorem 4.2.

(ii) $\Leftrightarrow$ (iv) Theorem 4.1 and the first equality in Theorem 4.2.

(iii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (ix) are inmediate from Lemma 3.3 of [5] since $L^p_{\rm w}(\nu)$ always has the Fatou property; see Theorem 4.1.

(iv) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (viii) $\Leftrightarrow$ (x) follow from Lemma 3.3 of [5] since $\tau^{(p)}$ is always a Lebesgue topology for $L^p(\nu)$; see Theorem 4.1. ∎

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