Interpolation of Vector Measures

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Abstract Let (Ω, Σ) be a measurable space and $m_0 : \Sigma \to X_0$ and $m_1 : \Sigma \to X_1$ be positive vector measures with values in the Banach Köthe function spaces X_0 and X_1. If $0 < \alpha < 1$, we define a new vector measure $[m_0, m_1]_\alpha$ with values in the *Calderón lattice interpolation* space $X_0^{1-\alpha}X_1^{\alpha}$ and we analyze the space of integrable functions with respect to measure $[m_0, m_1]_\alpha$ in order to prove suitable extensions of the classical Stein–Weiss formulas that hold for the complex interpolation of L^p-spaces. Since each p-convex order continuous Köthe function space with weak order unit can be represented as a space of p-integrable functions with respect to a vector measure, we provide in this way a technique to obtain representations of the corresponding complex interpolation spaces. As applications, we provide a Riesz–Thorin theorem for spaces of p-integrable functions with respect to vector measures and a formula for representing the interpolation of the injective tensor product of such spaces.

Keywords Interpolation, Banach function space, vector measure

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1 Introduction

One of the classical areas on interpolation theory is the investigation of complex interpolation spaces between L^p-spaces with different measures, where the theorem by Stein and Weiss [1] plays a central role. One of its extension reads as follows. Let μ be a positive scalar measure and assume that $1 \leq p_0, p_1 < \infty$. Then we have, with equal norms,

$$[L^{p_0}(f_0\mu), L^{p_1}(f_1\mu)]_\theta = L^p(f_0^{1-\alpha} f_1^{\alpha}\mu), \quad 0 < \theta < 1, \tag{1.1}$$

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where $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $\alpha := \frac{p\,\theta}{p_1}$, see [2, Theorem 5.5.3]. Observe that the weighted measure $f_0^{1-\alpha} f_1^{\alpha} \mu$ is a certain *interpolated measure* of the weighted measures $f_0 \mu$ and $f_1 \mu$. Later, Calderón considers the extension of (1.1) to the context of L^p-spaces of Banach vector-valued functions, see [3] for details. In this paper we analyze what happens with the formula (1.1) in the context of integration of scalar functions with respect to vector measures. Indeed, for two given countably additive vector measures m_0 and m_1 we show how to construct a new countably additive vector measure $[m_0, m_1]_\alpha$, that we call the *interpolated vector measure*, which allows us how relate its associated space of p-integrable functions with the corresponding lattice interpolation space of the spaces of p-integrable functions with respect to m_0 and m_1; all this, under a certain compatibility requirement between the measures. Indeed, the aim of the paper is to study when the following result holds:

Let $1 \le p_0, p_1 < \infty$ and $0 < \theta < 1$. Consider the corresponding spaces of integrable functions $L^{p_0}(m_0)$ and $L^{p_1}(m_1)$ with respect to the vector measures m_0 and m_1. Then $[L^{p_0}(m_0), L^{p_1}(m_1)]_\theta = L^p([m_0, m_1]_\alpha)$ holds isometrically, where $\frac{1}{p} := \frac{(1-\theta)}{p_0} + \frac{\theta}{p_1}$ and $\alpha := \frac{p\,\theta}{p_1}$.

The spaces $L^1(m)$ of integrable functions with respect to a vector measure m define a broad class of Köthe function spaces. In fact, integration with respect to a vector measure can be used as a representation technique for order continuous Banach lattices with weak order unit, since every such space can be written as an $L^1(m)$ of a vector measure m (see [4, Theorem 8]). The same representation procedure can be applied using L^p spaces of a vector measure when the function space is moreover p-convex (see [5, 6] for the definition and main properties of such spaces). Therefore, we provide our interpolation formulas as a general procedure for representing the lattice interpolation space of couples of (order continuous Banach lattices with weak order unit) Köthe function spaces. Moreover, it is enough to look at the proofs of the results that relate spaces of integrable functions with general Banach lattices (see [4, Theorem 8] and [5, Proposition 2.4]) to understand that we can use positive vector measures for representing the corresponding class of Banach lattices. In this sense, the requirement of positivity for the vector measures that we deal with is not a strong restriction for the aim of representation of Banach lattices.

But let us provide in what follows in a systematic way the *assumptions* that are necessary for defining the setting in which our interpolation technique can be applied. Our framework is the theory of Köthe function spaces (see p. 28 and subsequent in [7]). A Köthe function space on a complete σ-finite measure space (Θ, Λ, η) is a Banach lattice X, consisting of equivalence classes, modulo equality η-a.e., of locally integrable, real valued functions on Θ, that satisfies

(1) if $|f| \le |g|$ η-a.e. with f measurable and $g \in X$, then $f \in X$ and $\|f\|_X \le \|g\|_X$, and

(2) for every $A \in \Lambda$ of finite measure, the characteristic function χ_A of A is an element of X.

If X is a Köthe function space we denote by $X^\times$ the Köthe dual space of X. It is well known that $X^\times$ coincides with the dual space X' if X has order-continuous norm. See p. 29 on [7].

For a couple of Köthe function spaces (X_0, X_1) over the same measure space (Θ, Λ, η), and $0 < \alpha < 1$, consider the *Calderón lattice interpolation* space $X(\alpha) := X_0^{1-\alpha} X_1^{\alpha}$ of the spaces X_0 and X_1 defined as the set of all $x \in L^0(\eta)$ for which $|x| \le x_0^{1-\alpha} x_1^{\alpha}$, for some $0 \le x_0 \in X_0$, and $0 \le x_1 \in X_1$. The norm of x in $X(\alpha)$ is defined by

$$\|x\|_{X(\alpha)} := \inf \left\{ \|x_0\|_{X_0}^{1-\alpha} \|x_1\|_{X_1}^{\alpha} : |x| \le x_0^{1-\alpha} x_1^{\alpha}, 0 \le x_0 \in X_0, 0 \le x_1 \in X_1 \right\}.$$

Endowed with this norm, $X(\alpha)$ becomes a Köthe function space over the same measure space. See [8, 9] or [10, IV §1.11]. From the definition of the norm we have immediately the following inequality

$$\left\| x_0^{1-\alpha} x_1^{\alpha} \right\|_{X(\alpha)} \le \|x_0\|_{X_0}^{1-\alpha} \|x_1\|_{X_1}^{\alpha}, \quad 0 \le x_0 \in X_0, \ 0 \le x_1 \in X_1. \tag{1.2}$$

It is well known that this space coincides with the complex interpolation space $[X_0, X_1]_\alpha$ of the involved spaces X_0 and X_1 (see [3, 13.6] or [10, Theorem §1.14] and Remark after) if at least one of the spaces X_0 or X_1 has order continuous norm. As the reader will see, this assumption of order continuity of the interpolated space $X(\alpha)$ provides the order properties that are necessary for the definition of the interpolated measure $[m_0, m_1]_\alpha$.

In what follows we also need some relation between the dual of the space $X(\alpha)$ and the duals of spaces X_0 and X_1. It is known that $\left(X_0^{1-\alpha} X_1^{\alpha}\right)' = (X_0')^{1-\alpha} (X_1')^{\alpha}$, where the equality means coincidence of sets (as classes of functions on a *certain strange space of measure outside* of (Θ, Λ, η)) as well as equality of norms, provided that $X_0 \cap X_1$ is dense in both spaces X_0 and X_1. See [8] or the comments after the proof of [9, Theorem 1]. A consequence of this fact is that the equality $\left(X_0^{1-\alpha} X_1^{\alpha}\right)^\times = \left(X_0^\times\right)^{1-\alpha} \left(X_1^\times\right)^{\alpha}$ holds for the Köthe duals of the spaces X_0 and X_1. Equality means also equality of norms. In addition, if we assume order continuity of the norm on both spaces X_0 and X_1, we know that the dual and the Köthe dual coincide, and the duality in the lattice interpolation space $X(\alpha)$ can be *represented* by means of integrals of products of functions with respect to the measure η. In particular, we have the following result. See [8] or [11, Theorem 1].

Lemma 1.1 *Suppose X_0 and X_1 are Köthe function spaces with order continuous norms. Then for (Köthe) duals we have the following coincidence of sets $\left(X_0^{1-\alpha} X_1^{\alpha}\right)^\times = \left(X_0^\times\right)^{1-\alpha} \cdot \left(X_1^\times\right)^{\alpha}$, as well as equality of norms. In particular, for all $0 \le x' \in X(\alpha)^\times$ and every $\varepsilon > 0$, there exist two elements $0 \le x_0' \in X_0^\times$ and $0 \le x_1' \in X_1^\times$ such that $\|x'\|_{X(\alpha)^\times} \le \|x_0'\|_{X_0^\times}^{1-\alpha} \|x_1'\|_{X_1^\times}^{\alpha} \le \|x'\|_{X(\alpha)^\times} + \varepsilon$, and moreover*

$$\left\langle x', x_0^{1-\alpha} x_1^{\alpha} \right\rangle \le \int_\Theta \left(x_0' \cdot x_0\right)^{1-\alpha} \cdot \left(x_1' \cdot x_1\right)^{\alpha} \, d\eta, \tag{1.3}$$

for all $0 \le x_0 \in X_0$ and $0 \le x_1 \in X_1$.

As we said before, we will deal with a particular subclass of Köthe function spaces, the one defined by the spaces of integrable functions with respect to a vector measure. We briefly recall the definitions in what follows. Let X be a Banach space and m an X-valued countably additive vector measure on the measurable space (Ω, Σ). Let μ be a Rybakov measure for m (see [12, Ch. IX]), that is, a scalar measure $\langle m, x' \rangle$, where $x' \in X'$, defined by $\langle m, x' \rangle(A) := \langle m(A), x' \rangle$, $A \in \Sigma$, and such that m is absolutely continuous with respect to the variation of $\langle m, x' \rangle$. The space $L^1(m)$ of integrable functions with respect to m is an order-continuous Köthe function space over μ with weak order unit, and its elements are the (classes of μ-a.e.) measurable functions $f : \Omega \to \mathbb{R}$ that are scalarly integrable, that is, integrable with respect to each scalar measure $\langle m, x' \rangle$, $x' \in X'$, and satisfy also that for every $A \in \Sigma$ there is an element $\int_A f dm \in X$ such that

$$\left\langle \int_A f dm, x' \right\rangle = \int_A f d\langle m, x' \rangle, \quad x' \in X'.$$

The reader can find the definitions and fundamental results concerning the space $L^1(m)$ in [4, 13–15]. If $1 \leq p < \infty$, the space of p-integrable functions with respect to m is defined as the linear space of real measurable functions that satisfy $|f|^p \in L^1(m)$, with the norm

$$\|f\|_{L^p(m)} := \sup \left\{ \left(\int_\Omega |f|^p \, d|\langle m, x' \rangle| \right)^{\frac{1}{p}} : x' \in B_{X'} \right\}, \quad f \in L^p(m).$$

It is also an order-continuous Köthe function space with weak order unit over any Rybakov measure for m, and the simple functions are dense in it. See [5, 6] for more information about this space. In the case where X is a Banach lattice and m is a positive vector measure, the norm can be directly computed by means of the formula

$$\|f\|_{L^p(m)} = \left\| \int_\Omega |f|^p \, dm \right\|_X^{\frac{1}{p}}, \quad f \in L^p(m).$$

Consider a couple of Köthe function spaces (X_0, X_1) over the same measure space (Θ, Λ, η), a measurable space (Ω, Σ), and a pair of countably additive positive vector measures $m_0 : \Sigma \to X_0$ and $m_1 : \Sigma \to X_1$. Let $1 \leq p_0, p_1 < \infty$. In order to assure that the couple $(L^{p_0}(m_0), L^{p_1}(m_1))$ of spaces of p-integrable functions with respect to the vector measures m_0 and m_1 is a *compatible couple*, we need our second fundamental assumption on the measures m_0 and m_1. They must be *equivalent,* that is, they have a common Rybakov control measure μ. This property is not connected with the range space of the measures m_0 and m_1, but with the sets of the σ-algebra Σ where the semivariation of the measures is zero. In particular, in our setting, two equivalent vector measures have the same null sets. Under this assumption we know that classes a.e. with respect to m_0, m_1 and μ coincide, and also we can embed continuously both spaces $L^{p_0}(m_0)$ and $L^{p_1}(m_1)$ in the same topological vector space $L^0(\mu)$ of real-valued measurable (classes of μ-a.e) functions on Ω. Moreover, for every $0 < \alpha < 1$ the equality

$$[L^{p_0}(m_0), L^{p_1}(m_1)]_\alpha = (L^{p_0}(m_0))^{1-\alpha} (L^{p_1}(m_1))^\alpha \tag{1.4}$$

holds, since $L^{p_0}(m_0)$ and $L^{p_1}(m_1)$ have order continuous norms. At this point let us mention that (when the measures m_0 and m_1 coincide) the space $[L^{p_0}(m), L^{p_1}(m)]_\alpha$, with $0 < \alpha < 1$ and $1 \leq p_0, p_1 \leq \infty$, and also others related interpolation spaces have been deeply studied in [16].

2 Interpolation of Countably Additive Positive Vector Measures

Let us start this section with a motivation, based on the formula (1.1), of the construction that follows. Consider a couple of positive (scalar) finite measures μ_0, and μ_1 on the same measurable space (Ω, Σ) that are absolutely continuous with respect to another positive finite measure μ. In this case, the Radon–Nikodym theorem gives two functions $0 \leq f_0, f_1 \in L^1(\mu)$ such that $\mu_0(A) = \int_A f_0 \, d\mu$ and $\mu_1(A) = \int_A f_1 \, d\mu$, for every $A \in \Sigma$. Let $0 < \theta < 1 \leq p_0, p_1 < \infty$, and denote $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $\alpha := \frac{p\theta}{p_1}$. Observe that $0 < \alpha < 1$. Consider the *interpolated measure* $[\mu_0, \mu_1]_\alpha : \Sigma \to \mathbb{R}$ defined by the formula

$$[\mu_0, \mu_1]_\alpha(A) := \inf \left\{ \sum_{B \in \pi} \mu_0(A \cap B)^{1-\alpha} \, \mu_1(A \cap B)^\alpha : \pi \in \Pi(\Omega) \right\}, \tag{2.1}$$

where $\Pi(\Omega)$ is the set of all finite Σ-partitions of Ω. It is not difficult to prove that $[\mu_0, \mu_1]_\alpha$ is actually the measure with density $f_0^{1-\alpha} f_1^\alpha$, that is, $[\mu_0, \mu_1]_\alpha(A) = \int_A f_0^{1-\alpha} f_1^\alpha \, d\mu$, for $A \in \Sigma$. Thus, the formula of Stein–Weiss (1.1) reads as

$$[L^{p_0}(\mu_0), L^{p_1}(\mu_1)]_\theta = L^p\left([\mu_0, \mu_1]_\alpha\right), \quad 0 < \theta < 1. \tag{2.2}$$

In the case of two arbitrary positive scalar measures μ_0 and μ_1 it is clear that they are absolutely continuous with respect to the measure $\mu := \mu_0 + \mu_1$, and so it is always possible to find Radon–Nikodým derivatives $f_0, f_1 \in L^1(\mu)$ such that $\mu_0(A) = \int_A f_0 \, d\mu$ and $\mu_1(A) = \int_A f_1 \, d\mu$, for all $A \in \Sigma$. This means that the formula (2.1) always defines a positive scalar measure such that $[\mu_0, \mu_1]_\alpha(A) \le \mu_0(A)^{1-\alpha} \mu_1(A)^\alpha$, for all $A \in \Sigma$ and $[\mu_0, \mu_1]_\alpha(A) = 0$ if and only $\mu_0(A) = 0$ or (and) $\mu_1(A) = 0$. In what follows the following lemma will be useful.

Lemma 2.1 *Let $0 \le \varphi, \phi$ be simple functions. Then*

$$\int_\Omega \varphi^{1-\alpha} \phi^\alpha \, d\left[\mu_0, \mu_1\right]_\alpha \le \left(\int_\Omega \varphi \, d\mu_0\right)^{1-\alpha} \left(\int_\Omega \phi \, d\mu_1\right)^\alpha. \tag{2.3}$$

Proof We know that $[\mu_0, \mu_1]_\alpha$ is actually the measure with density $f_0^{1-\alpha} f_1^\alpha$, with respect to a positive measure μ, so Hölder's inequality gives

$$\begin{aligned} \int_\Omega \varphi^{1-\alpha} \phi^\alpha \, d\left[\mu_0, \mu_1\right]_\alpha &= \int_\Omega \varphi^{1-\alpha} \phi^\alpha f_0^{1-\alpha} f_1^\alpha \, d\mu = \int_\Omega (\varphi f_0)^{1-\alpha} (\phi f_1)^\alpha \, d\mu \\ &\le \left(\int_\Omega \varphi f_0 \, d\mu\right)^{1-\alpha} \left(\int_\Omega \phi f_1 \, d\mu\right)^\alpha = \left(\int_\Omega \varphi \, d\mu_0\right)^{1-\alpha} \left(\int_\Omega \phi \, d\mu_1\right)^\alpha. \quad \square \end{aligned}$$

However, this argument that provides a representation of a couple of positive scalar measures defined on the same σ-algebra with respect to their sum fails in the case of vector measures, even if they are positive and take their values on the same Köthe function space. The following easy example shows this.

Example 1 Let $([0,1], \mathscr{M}, \lambda)$ be the Lebesgue measure space. Consider the vector measures

$$\begin{aligned} m_0 &: A \in \mathscr{M} \longrightarrow m_0(A) := (\lambda(A), 0) \in \mathbb{R}^2, \\ m_1 &: A \in \mathscr{M} \longrightarrow m_1(A) := (0, \lambda(A)) \in \mathbb{R}^2. \end{aligned}$$

It is easy to see that $L^1(m_0 + m_1) = L^1[0,1]$ isometrically and

$$\int_{[0,1]} f d(m_0 + m_1) = \left(\int_{[0,1]} f d\lambda, \int_{[0,1]} f d\lambda\right), \quad f \in L^1(m_0 + m_1).$$

Moreover it is clear that $m_0\left([0,1]\right) = (1,0)$ and $m_1\left([0,1]\right) = (0,1)$. This makes evident that there are no functions $f_0, f_1 \in L^1(m_0 + m_1)$ such that

$$\int_\Omega f_0 \, d(m_0 + m_1) = m_0(\Omega) \quad \text{and} \quad \int_\Omega f_1 \, d(m_0 + m_1) = m_1(\Omega).$$

What is more, if we compute (2.1) formally for the measures m_0 and m_1 we obtain trivially $[m_0, m_1]_\alpha(A) = (0,0)$ for all $A \in \mathscr{M}$.

This fact — the nonexistence of Radon–Nikodým derivatives in the vector valued case — becomes the main difference between the scalar- and the vector-valued theory regarding interpolation of the corresponding spaces of integrable functions. The aim of this section is to provide a vector-valued version of the representation formula (2.1), and to show some of its properties. The main result is the following equality (see Proposition 2.6 below)

$$[f_0 m_0, f_1 m_1]_\alpha(A) = \int_A f_0^{1-\alpha} f_1^\alpha \, d\,[m_0, m_1]_\alpha, \quad A \in \Sigma,$$

if m_0 and m_1 are adequate vector measures and $0 \le f_0 \in L^1(m_0)$, $0 \le f_1 \in L^1(m_1)$; that will become a basic tool for the calculations of Section 3.

Let $0 < \alpha < 1$ and consider two countably additive positive vector measures $m_0 : \Sigma \to X_0$ and $m_1 : \Sigma \to X_1$ on the same measurable space (Ω, Σ) such that the lattice interpolation space $X(\alpha) := X_0^{1-\alpha} X_1^\alpha$ is order-continuous. Let $A \in \Sigma$ and $\pi \in \Pi(\Omega)$ be the sets of finite measurable partitions of Ω. Denote $C_\pi(A) := \sum_{B \in \pi} m_0(A \cap B)^{1-\alpha}\, m_1(A \cap B)^\alpha$. This element $C_\pi(A)$ is a well-defined element of $X(\alpha)$, since m_0 and m_1 are positive. Now consider the set $\mathscr{C}(A) := \{C_\pi(A) : \pi \in \Pi(\Omega)\}$ of $X(\alpha)$. With the natural order by refinement in $\Pi(\Omega)$, a pointwise evaluation using Hölder's inequality gives that $C_{\pi_2}(A) \le C_{\pi_1}(A)$ if $\pi_1 \le \pi_2$ in $\Pi(\Omega)$, so $\mathscr{C}(A)$ defines a downward directed positive net in $X(\alpha)$. The order-continuity of the interpolation space gives directly that the limit $[m_0, m_1]_\alpha(A) := \lim_\pi C_\pi(A)$ exists in the norm of $X(\alpha)$ for every $A \in \Sigma$, and in fact is given by the infimum $\inf \mathscr{C}(A)$. Thus, the map

$$[m_0, m_1]_\alpha : A \in \Sigma \to [m_0, m_1]_\alpha(A) := \lim_\pi C_\pi(A) = \inf \mathscr{C}(A) \in X(\alpha)$$

is well defined. The definition of $[m_0, m_1]_\alpha$ makes clear that

$$[m_0, m_1]_\alpha(A) \le m_0(A)^{1-\alpha} m_1(A)^\alpha, \quad A \in \Sigma. \tag{2.4}$$

Note that this formula must be read as an inequality η-a.e. between elements of the Köthe function space $X(\alpha)$. Moreover, having in mind (1.2), we get

$$\|[m_0, m_1]_\alpha(A)\|_{X(\alpha)} \le \|m_0(A)\|_{X_0}^{1-\alpha} \|m_1(A)\|_{X_1}^\alpha, \quad A \in \Sigma. \tag{2.5}$$

Lemma 2.2 *The map $[m_0, m_1]_\alpha$ defines a countably additive positive vector measure that satisfies*

$$\langle [m_0, m_1]_\alpha(A), x' \rangle \le \langle m_0(A), x_0' \rangle^{1-\alpha} \langle m_1(A), x_1' \rangle^\alpha, \quad A \in \Sigma \tag{2.6}$$

for every $0 \le x' \in X(\alpha)'$ such that $x' \le (x_0')^{1-\alpha}(x_1')^\alpha$, with $0 \le x_0' \in X_0'$ and $0 \le x_1' \in X_1'$. In particular, we have

$$\langle [m_0, m_1]_\alpha, x' \rangle \le [\langle m_0, x_0' \rangle, \langle m_1, x_1' \rangle]_\alpha. \tag{2.7}$$

Proof Let us show first that it is finitely additive. Consider two disjoint sets $A, B \in \Sigma$. Since $\{C_\pi(A \cup B) : \pi \in \Pi(\Omega)\}$ is downward directed, we can always choose a partition $\pi \in \Pi(\Omega)$ such that

$$\sum_{C \in \pi} m_0\left((A \cup B) \cap C\right)^{1-\alpha} m_1\left((A \cup B) \cap C\right)^\alpha = \sum_{C \in \pi,\, C \subseteq A} m_0\left(A \cap C\right)^{1-\alpha} m_1\left(A \cap C\right)^\alpha$$

$$+ \sum_{C\in\pi,\, C\subseteq B} m_0\left(B\cap C\right)^{1-\alpha} m_1\left(B\cap C\right)^{\alpha}.$$

Therefore, we have $[m_0,m_1]_\alpha(A)+[m_0,m_1]_\alpha(B)\le[m_0,m_1]_\alpha(A\cup B)$. On the other hand, if $\pi_1\in\Pi(\Omega)$ and $\pi_2\in\Pi(\Omega)$, consider the partition

$$\pi := \{A\cap C : C\in\pi_1\}\cup\{B\cap C : C\in\pi_2\}\cup\{\Omega\setminus(A\cup B)\}.$$

For this partition we have

$$\sum_{C\in\pi} m_0\left((A\cup B)\cap C\right)^{1-\alpha} m_1\left((A\cup B)\cap C\right)^{\alpha} = \sum_{C\in\pi_1} m_0\left(A\cap C\right)^{1-\alpha} m_1\left(A\cap C\right)^{\alpha} + \sum_{C\in\pi_2} m_0\left(B\cap C\right)^{1-\alpha} m_1\left(B\cap C\right)^{\alpha}.$$

This implies that $[m_0,m_1]_\alpha(A\cup B)\le[m_0,m_1]_\alpha(A)+[m_0,m_1]_\alpha(B)$. The countable additivity now follows directly from (2.5).

The formula (2.6) is a direct consequence of the following calculations. Now, consider $0\le x'\le(x_0')^{1-\alpha}(x_1')^{\alpha}\in(X_0')^{1-\alpha}(X_1')^{\alpha}$, where $0\le x_0'\in X_0'$ and $0\le x_1'\in X_1'$. Then, since x' is positive, we obtain from Hölder's inequality and the relation (2.4)

$$\begin{aligned}
\langle[m_0,m_1]_\alpha(A),x'\rangle &\le \left\langle m_0(A)^{1-\alpha}m_1(A)^{\alpha},x'\right\rangle \le \int_\Theta \left(m_0(A)^{1-\alpha}m_1(A)^{\alpha}\cdot(x_0')^{1-\alpha}(x_1')^{\alpha}\right)\,d\eta \\
&= \int_\Theta \left(m_0(A)\cdot x_0'\right)^{1-\alpha}\cdot\left(m_1(A)\cdot x_1'\right)^{\alpha}\,d\eta \\
&\le \left(\int_\Theta m_0(A)\cdot x_0'\,d\eta\right)^{1-\alpha}\left(\int_\Theta m_1(A)\cdot x_1'\,d\eta\right)^{\alpha} \\
&= \left\langle m_0(A),x_0'\right\rangle^{1-\alpha}\left\langle m_1(A),x_1'\right\rangle^{\alpha}, \quad A\in\Sigma.
\end{aligned}$$

This finishes the proof. □

Remark 2.3 For a couple of equivalent measures m_0 and m_1, from the formula (2.5) we deduce the following inequality for the semivariations:

$$\left\|[m_0,m_1]_\alpha\right\|(A)\le\left(\|m_0\|(A)\right)^{1-\alpha}\left(\|m_1\|(A)\right)^{\alpha}, \quad A\in\Sigma. \tag{2.8}$$

In particular, this tell us that a null set for the measures m_0 and m_1 (they are equivalent) is also a null set for the interpolated measure $[m_0,m_1]_\alpha$.

Example 2 Let $([0,1],\mathscr{M},\lambda)$ be Lebesgue measure space, a couple of real numbers $1\le s_1\le s_0<\infty$ and a function $0<g\in L^t[0,1]$, where $\frac{1}{s_0}+\frac{1}{t}=\frac{1}{s_1}$. Consider the positive countably additive vector measures

$$\begin{aligned}
m_0 : A\in\mathscr{M} &\longrightarrow m_0(A):=\chi_A\in L^{s_0}[0,1],\\
m_1 : A\in\mathscr{M} &\longrightarrow m_1(A):=g\cdot\chi_A\in L^{s_1}[0,1].
\end{aligned}$$

Let us compute explicitly the interpolated vector measure $[m_0,m_1]_\alpha$, for $0<\alpha<1$. Consider a partition $\pi\in\Pi(\Omega)$ and a set $A\in\mathscr{M}$. Then we have

$$C_\pi(A)=\sum_{B\in\pi} m_0(A\cap B)^{1-\alpha}m_1(A\cap B)^{\alpha}=\sum_{B\in\pi}\chi_{A\cap B}\cdot g^{\alpha}=g^{\alpha}\cdot\chi_A.$$

Consequently, $[m_0, m_1]_\alpha(A) = g^\alpha \cdot \chi_A$ for all $A \in \mathscr{M}$. In this case the measure $[m_0, m_1]_\alpha$ takes values in the space $(L^{s_0}[0,1])^{1-\alpha}(L^{s_1}[0,1])^\alpha = L^s[0,1]$, where $\frac{1}{s} = \frac{(1-\alpha)}{s_0} + \frac{\alpha}{s_1}$. Observe that $\alpha s < t$, and then $g^\alpha \in L^s[0,1]$, since g belongs to $L^t[0,1]$.

Example 3 Consider a couple of order-continuous Köthe function spaces X_0 and X_1 over the σ-finite measure space (Θ, Λ, η). Take a pair of positive unconditionally convergent series $\sum_{n=1}^\infty f_n$ in X_0 and $\sum_{n=1}^\infty g_n$ in X_1. Then we can define, on the σ-algebra $\mathscr{P}(\mathbb{N})$ of all subsets of the natural numbers $\mathbb{N}$, the vector measures

$$m_0 : A \in \mathscr{P}(\mathbb{N}) \longrightarrow m_0(A) := \sum_{n\in A} f_n \in X_0,$$
$$m_1 : A \in \mathscr{P}(\mathbb{N}) \longrightarrow m_1(A) := \sum_{n\in A} g_n \in X_1.$$

If $0 < \alpha < 1$, note that $\sum_{n=1}^\infty f_n^{1-\alpha} g_n^\alpha$ defines also a positive unconditionally convergent series in $X_0^{1-\alpha} X_1^\alpha$, as a consequence of the application pointwise of Hölder's inequality on series as

$$\sum_{n\in A} f_n^{1-\alpha} g_n^\alpha \le \left(\sum_{n\in A} f_n\right)^{1-\alpha} \left(\sum_{n\in A} g_n\right)^\alpha, \quad A \in \mathscr{P}(\mathbb{N}).$$

Thus, the expression $m_\alpha : A \in \mathscr{P}(\mathbb{N}) \longrightarrow m_\alpha(A) := \sum_{n\in A} f_n^{1-\alpha} g_n^\alpha \in X_0^{1-\alpha} X_1^\alpha$ provides also a positive vector measure. Let us prove that the interpolated vector measure $[m_0, m_1]_\alpha$ and m_α coincide. If $A \in \mathscr{P}(\mathbb{N})$, then knowing that $[m_0, m_1]_\alpha$ is a countably additive measure, we have

$$[m_0, m_1]_\alpha(A) = \sum_{n\in A} [m_0, m_1]_\alpha(\{n\}) = \sum_{n\in A} f_n^{1-\alpha} g_n^\alpha = m_\alpha(A).$$

Lemma 2.4 *Let $m_0 : \Sigma \to X_0$ and $m_1 : \Sigma \to X_1$ be a couple of equivalent positive countably additive vector measures on (Ω, Σ). Let $0 \le \varphi, \phi$ be simple functions. Then*

$$\|\varphi^{1-\alpha}\phi^\alpha\|_{L^1([m_0,m_1]_\alpha)} \le \|\varphi\|_{L^1(m_0)}^{1-\alpha} \|\phi\|_{L^1(m_1)}^\alpha. \tag{2.9}$$

Proof Note that $\|\varphi^{1-\alpha}\phi^\alpha\|_{L^1([m_0,m_1]_\alpha)} = \| \int_\Omega \varphi^{1-\alpha}\phi^\alpha \, d\,[m_0, m_1]_\alpha \|_{X(\alpha)}$ since the measure $[m_0, m_1]_\alpha$ is positive. Then there is an element $0 \le x' \in X(\alpha)'$, with $\|x'\|_{X(\alpha)'} \le 1$, such that

$$\begin{aligned} \left\| \int_\Omega \varphi^{1-\alpha}\phi^\alpha \, d\,[m_0, m_1]_\alpha \right\|_{X(\alpha)} &= \left\langle \int_\Omega \varphi^{1-\alpha}\phi^\alpha \, d\,[m_0, m_1]_\alpha \,, x' \right\rangle \\ &= \int_\Omega \varphi^{1-\alpha}\phi^\alpha \, d\,\langle [m_0, m_1]_\alpha, x'\rangle \,. \end{aligned}$$

Given $\varepsilon > 0$, we know from Lemma 1.1 that there exist two elements $0 \le x_0' \in X_0'$ and $0 \le x_1' \in X_1'$ such that $x' \le (x_0')^{1-\alpha}(x_1')^\alpha$, and $\|x'\|_{X(\alpha)'} \le \|x_0'\|_{X_0'}^{1-\alpha} \|x_1'\|_{X_1'}^\alpha \le \|x'\|_{X(\alpha)'} + \varepsilon$. By inequalities (2.7) from Lemma 2.2 and (2.3) from Lemma 2.1 we obtain

$$\begin{aligned} \int_\Omega \varphi^{1-\alpha}\phi^\alpha \, d\,\langle [m_0, m_1]_\alpha, x'\rangle &\le \int_\Omega \varphi^{1-\alpha}\phi^\alpha \, d\,[\langle m_0, x_0'\rangle \,, \langle m_1, x_1'\rangle]_\alpha \\ &\le \left(\int_\Omega \varphi \, d\,\langle m_0, x_0'\rangle\right)^{1-\alpha} \left(\int_\Omega \phi \, d\,\langle m_1, x_1'\rangle\right)^\alpha \\ &\le \|\varphi\|_{L^1(m_0)}^{1-\alpha} \, \|x_0'\|_{X_0'}^{1-\alpha} \, \|\phi\|_{L^1(m_1)}^\alpha \, \|x_1'\|_{X_0'}^\alpha \\ &\le \|\varphi\|_{L^1(m_0)}^{1-\alpha} \, \|\phi\|_{L^1(m_1)}^\alpha \, (\|x'\|_{X(\alpha)'} + \varepsilon) \end{aligned}$$

$$\leq \|\varphi\|_{L^1(m_0)}^{1-\alpha} \|\phi\|_{L^1(m_1)}^{\alpha} (1+\varepsilon)\,.$$

Since ε was arbitrary we obtain $\|\varphi^{1-\alpha}\phi^{\alpha}\|_{L^1([m_0,m_1]_\alpha)} \leq \|\varphi\|_{L^1(m_0)}^{1-\alpha} \|\phi\|_{L^1(m_1)}^{\alpha}$. □

Lemma 2.5 *Let $m_0 : \Sigma \to X_0$ and $m_1 : \Sigma \to X_1$ be a couple of equivalent positive countably additive vector measures on (Ω, Σ). Let $0 \leq \varphi_0, \varphi_1, \phi_0, \phi_1$ simple functions. Then*

$$\begin{aligned} &\left\|\varphi_0^{1-\alpha}\phi_0^{\alpha} - \varphi_1^{1-\alpha}\phi_1^{\alpha}\right\|_{L^1([m_0,m_1]_\alpha)} \\ &\quad \leq \|\varphi_0\|_{L^1(m_0)}^{1-\alpha} \|\phi_0-\phi_1\|_{L^1(m_1)}^{\alpha} + \|\varphi_0-\varphi_1\|_{L^1(m_0)}^{1-\alpha} \|\phi_1\|_{L^1(m_1)}^{\alpha}\,. \end{aligned} \tag{2.10}$$

Proof To obtain (2.10) we apply the inequality (2.9) of Lemma 2.4 and also that $|s^\alpha - t^\alpha| \leq |s-t|^\alpha$ for all numbers $0 \leq s, t$, and $0 < \alpha < 1$.

$$\begin{aligned} &\left\|\varphi_0^{1-\alpha}\phi_0^{\alpha} - \varphi_1^{1-\alpha}\phi_1^{\alpha}\right\|_{L^1([m_0,m_1]_\alpha)} \\ &\quad \leq \left\|\varphi_0^{1-\alpha}\phi_0^{\alpha} - \varphi_0^{1-\alpha}\phi_1^{\alpha}\right\|_{L^1([m_0,m_1]_\alpha)} + \left\|\varphi_0^{1-\alpha}\phi_1^{\alpha} - \varphi_1^{1-\alpha}\phi_1^{\alpha}\right\|_{L^1([m_0,m_1]_\alpha)} \\ &\quad \leq \left\|\varphi_0^{1-\alpha}\left|\phi_0-\phi_1\right|^{\alpha}\right\|_{L^1([m_0,m_1]_\alpha)} + \left\|\left|\varphi_0-\varphi_1\right|^{1-\alpha}\phi_1^{\alpha}\right\|_{L^1([m_0,m_1]_\alpha)} \\ &\quad \leq \|\varphi_0\|_{L^1(m_0)}^{1-\alpha} \|\phi_0-\phi_1\|_{L^1(m_1)}^{\alpha} + \|\varphi_0-\varphi_1\|_{L^1(m_0)}^{1-\alpha} \|\phi_1\|_{L^1(m_1)}^{\alpha}\,. \end{aligned}$$ □

Let $0 \leq f_0 \in L^1(m_0)$ and $0 \leq f_1 \in L^1(m_1)$. Then the formula

$$(f_k m_k)(A) := \int_A f_k \, dm_k, \quad A \in \Sigma,$$

defines a countably additive vector measure for $k = 0, 1$ (see [14]). Since the functions are also positive, we have a couple of countably additive positive vector measures. Thus, under the adequate requirements on the Köthe function spaces where the vector measures are defined, we obtain that $[f_0m_0, f_1m_1]_\alpha$ is a positive countably additive vector measure. Recall that we are under the assumptions that have been indicated in Section 1 for the couple of Köthe function spaces (X_0, X_1).

Proposition 2.6 *Let $m_0 : \Sigma \to X_0$ and $m_1 : \Sigma \to X_1$ be a couple of equivalent positive countably additive vector measures on (Ω, Σ). Let $0 \leq f_0 \in L^1(m_0)$ and $0 \leq f_1 \in L^1(m_1)$. Then the function $f_0^{1-\alpha}f_1^{\alpha}$ is $[m_0, m_1]_\alpha$-integrable, and*

$$[f_0m_0, f_1m_1]_\alpha(A) = \int_A f_0^{1-\alpha} f_1^{\alpha} \, d\,[m_0, m_1]_\alpha\,, \quad A \in \Sigma. \tag{2.11}$$

Proof Let us show first that the function $f_0^{1-\alpha}f_1^{\alpha}$ is $[m_0, m_1]_\alpha$-integrable. Note now that since $0 \leq f_0 \in L^1(m_0)$, there is a sequence of simple functions $(\varphi_n)_n$ such that $0 \leq \varphi_n \to f_0$ in $L^1(m_0)$ and pointwise, and $0 \leq \varphi_n \leq f_0$. In the same way, there is another sequence of simple functions $(\phi_n)_n$ such that $0 \leq \phi_n \to f_1$ in $L^1(m_1)$ and pointwise, and $0 \leq \phi_n \leq f_1$. Thus, $\varphi_n^{1-\alpha}\phi_n^{\alpha} \to f_0^{1-\alpha}f_1^{\alpha}$ pointwise, and by (2.10) of Lemma 2.5 applied to the simple functions $0 \leq \varphi_n, \varphi_m, \phi_n, \phi_m$ we obtain

$$\begin{aligned} &\left\|\varphi_n^{1-\alpha}\phi_n^{\alpha} - \varphi_m^{1-\alpha}\phi_m^{\alpha}\right\|_{L^1([m_0,m_1]_\alpha)} \\ &\quad \leq \|\varphi_n\|_{L^1(m_0)}^{1-\alpha} \|\phi_n-\phi_m\|_{L^1(m_1)}^{\alpha} + \|\varphi_n-\varphi_m\|_{L^1(m_0)}^{1-\alpha} \|\phi_m\|_{L^1(m_1)}^{\alpha} \\ &\quad \leq \|f_0\|_{L^1(m_0)}^{1-\alpha} \|\phi_n-\phi_m\|_{L^1(m_1)}^{\alpha} + \|\varphi_n-\varphi_m\|_{L^1(m_0)}^{1-\alpha} \|f_1\|_{L^1(m_1)}^{\alpha}\,. \end{aligned}$$

Therefore, $\left(\varphi_n^{1-\alpha}\phi_n^{\alpha}\right)_n$ is a Cauchy sequence in $L^1([m_0,m_1]_\alpha)$, and so $f_0^{1-\alpha}f_1^{\alpha} \in L^1([m_0,m_1]_\alpha)$. Moreover, we have that

$$\int_\Omega f_0^{1-\alpha}f_1^{\alpha}\,d\,[m_0,m_1]_\alpha = \lim_n \int_\Omega \varphi_n^{1-\alpha}\phi_n^{\alpha}\,d\,[m_0,m_1]_\alpha \tag{2.12}$$

in $X(\alpha)$. Let us show now that

$$\lim_n\,[\varphi_n m_0,\phi_n m_1]_\alpha\,(\Omega) = [f_0 m_0, f_1 m_1]_\alpha\,(\Omega). \tag{2.13}$$

Clearly $[\varphi_n m_0,\phi_n m_1]_\alpha(\Omega) \le [f_0 m_0, f_1 m_1]_\alpha(\Omega)$, for all $n = 1,2,\dots$. In order to show the equality, consider an arbitrary partition $\pi \in \Pi(\Omega)$; the same kind of calculations that we have used above in Lemma 2.5 give

$$\begin{aligned}
&\left\| \sum_{B\in\pi} (f_0 m_0(B))^{1-\alpha}(f_1 m_1(B))^{\alpha} - \sum_{B\in\pi}(\varphi_n m_0(B))^{1-\alpha}(\phi_n m_1(B))^{\alpha}\right\|_{X(\alpha)} \\
&\quad = \left\| \sum_{B\in\pi} (f_0 m_0(B))^{1-\alpha}\left((f_1 m_1(B))^{\alpha} - (\phi_n m_1(B))^{\alpha}\right)\right\|_{X(\alpha)} \\
&\qquad + \left\| \sum_{B\in\pi} \left((f_0 m_0(B))^{1-\alpha} - (\varphi_n m_0(B))^{1-\alpha}\right)(\phi_n m_1(B))^{\alpha}\right\|_{X(\alpha)} \\
&\quad \le \left\| \sum_{B\in\pi} f_0 m_0(B)\right\|_{X_0}^{1-\alpha} \left\| \sum_{B\in\pi} (f_1 m_1(B)) - (\phi_n m_1(B))\right\|_{X_1}^{\alpha} \\
&\qquad + \left\| \sum_{B\in\pi} (f_0 m_0(B)) - (\varphi_n m_0(B))\right\|_{X_0}^{1-\alpha} \left\| \sum_{B\in\pi} \phi_n m_1(B)\right\|_{X_1}^{\alpha} \\
&\quad \le \|f_0\|_{L^1(m_0)}^{1-\alpha}\|f_1-\phi_n\|_{L^1(m_1)}^{\alpha} + \|f_0-\varphi_n\|_{L^1(m_0)}^{1-\alpha}\|f_1\|_{L^1(m_1)}^{\alpha} \to 0
\end{aligned}$$

whenever $n\to\infty$. This means that the formula (2.13) holds. Consequently, if formula (2.11) in the statement of the proposition holds for simple functions, then it must hold for each couple of functions $0 \le f_0 \in L^1(m_0)$ and $0 \le f_1 \in L^1(m_1)$ since by (2.12) and (2.13),

$$\begin{aligned}
\int_\Omega f_0^{1-\alpha}f_1^{\alpha}\,d\,[m_0,m_1]_\alpha &= \lim_n \int_\Omega \varphi_n^{1-\alpha}\phi_n^{\alpha}\,d\,[m_0,m_1]_\alpha = \lim_n\,[\varphi_n m_0,\phi_n m_1]_\alpha\,(\Omega) \\
&= [f_0 m_0, f_1 m_1]_\alpha\,(\Omega).
\end{aligned}$$

Let us prove now (2.11) for simple functions; it is enough to do it for $A=\Omega$, since if the formula (2.11) holds for Ω and $A\in\Sigma$, then we can consider the functions $g_0 = f_0\chi_A$ and $g_1 = f_1\chi_A$, in which case, $g_0^{1-\alpha}g_1^{\alpha} = \chi_A f_0^{1-\alpha}f_1^{\alpha}$, and then $\int_\Omega g_0^{1-\alpha}g_1^{\alpha}\,d\,[m_0,m_1]_\alpha = \int_A f_0^{1-\alpha}f_1^{\alpha}\,d\,[m_0,m_1]_\alpha$. Note also that $g_0 m_0(B) = \int_B g_0\,dm_0 = \int_B f_0\chi_A\,dm_0 = f_0 m_0(A\cap B)$. Analogously, $g_1 m_1(B) = f_1 m_1(A\cap B)$. Therefore, $[g_0 m_0, g_1 m_1]_\alpha\,(\Omega) = [f_0\chi_A m_0, f_1\chi_A m_1]_\alpha\,(\Omega) = [f_0 m_0, f_1 m_1]_\alpha\,(A)$. Consider two positive simple functions φ_0 and φ_1. It is always possible to find coefficients $0 \le a_1,\dots,a_n \in \mathbb{R}$ and $0 \le b_1,\dots,b_n\in\mathbb{R}$, and disjoint subsets $A_1, A_2,\dots,A_n \in \Sigma$ such that $\varphi_0 = \sum_{k=1}^n a_k\chi_{A_k}$ and $\varphi_1 = \sum_{k=1}^n b_k\chi_{A_k}$. For these functions, clearly, we have

$$\int_\Omega \varphi_0^{1-\alpha}\varphi_1^{\alpha}\,d\,[m_0,m_1]_\alpha = \sum_{k=1}^n a_k^{1-\alpha}b_k^{\alpha}\,[m_0,m_1]_\alpha\,(A_k).$$

On the other hand, $[\varphi_0 m_0, \varphi_1 m_1]_\alpha(\Omega) = \sum_{k=1}^n [\varphi_0 m_0, \varphi_1 m_1]_\alpha(A_k)$, but

$$\begin{aligned}
[\varphi_0 m_0, \varphi_1 m_1]_\alpha(A_k) &= \inf\left\{ \sum_{B\in\pi} (\varphi_0 m_0(B\cap A_k))^{1-\alpha} (\varphi_1 m_1(B\cap A_k))^\alpha : \pi\in\Pi(\Omega)\right\} \\
&= \inf\left\{ \sum_{B\in\pi} \left(\int_{B\cap A_k} \varphi_0\, dm_0\right)^{1-\alpha} \left(\int_{B\cap A_k} \varphi_1\, dm_1\right)^\alpha : \pi\in\Pi(\Omega)\right\} \\
&= \inf\left\{ \sum_{B\in\pi} a_k^{1-\alpha} (m_0(B\cap A_k))^{1-\alpha} b_k^\alpha (m_1(B\cap A_k))^\alpha : \pi\in\Pi(\Omega)\right\} \\
&= a_k^{1-\alpha} b_k^\alpha \inf\left\{ \sum_{B\in\pi} (m_0(B\cap A_k))^{1-\alpha} (m_1(B\cap A_k))^\alpha : \pi\in\Pi(\Omega)\right\} \\
&= a_k^{1-\alpha} b_k^\alpha [m_0, m_1](A_k).
\end{aligned}$$

Thus $\int_\Omega \varphi_0^{1-\alpha}\varphi_1^\alpha\, d[m_0, m_1]_\alpha = [\varphi_0 m_0, \varphi_1 m_1]_\alpha(\Omega)$. This with the arguments before clearly implies that formula (2.11) holds for simple functions. Therefore, it holds for each couple of functions. This finishes the proof. □

Corollary 2.7 *Let $m_0 : \Sigma \to X_0$ and $m_1 : \Sigma \to X_1$ be a couple of equivalent positive countably additive vector measures on (Ω, Σ). Then the inclusion*

$$\left(L^1(m_0)\right)^{1-\alpha}\left(L^1(m_1)\right)^\alpha \subseteq L^1\left([m_0, m_1]_\alpha\right)$$

is continuous for all $0 < \alpha < 1$.

Proof Let $f \in \left(L^1(m_0)\right)^{1-\alpha}\left(L^1(m_1)\right)^\alpha$. Then there exist two functions $0 \le f_0 \in L^1(m_0)$ and $0 \le f_1 \in L^1(m_1)$ such that $|f| \le f_0^{1-\alpha} f_1^\alpha$. The above Proposition 2.6 assures that $f \in L^1([m_0, m_1]_\alpha)$. The comments in Remark 2.3 assure that there is no conflict with the classes, so the inclusion is well defined. Moreover, from (2.11) we have

$$\begin{aligned}
\|f\|_{L^1([m_0,m_1]_\alpha)} \le \left\|f_0^{1-\alpha} f_1^\alpha\right\|_{L^1([m_0,m_1]_\alpha)} &= \left\| \int_\Omega f_0^{1-\alpha} f_1^\alpha\, d[m_0, m_1]_\alpha \right\|_{X(\alpha)} \\
&= \|[f_0 m_0, f_1 m_1]_\alpha(\Omega)\|_{X(\alpha)} \le \|f_0 m_0(\Omega)\|_{X_0}^{1-\alpha} \|f_1 m_1(\Omega)\|_{X_1}^\alpha \\
&= \|f_0\|_{L^1(m_0)}^{1-\alpha} \|f_1\|_{L^1(m_1)}^\alpha .
\end{aligned}$$

By the definition of the norm in $\left(L^1(m_0)\right)^{1-\alpha}\left(L^1(m_1)\right)^\alpha$ we obtain

$$\|f\|_{L^1([m_0,m_1]_\alpha)} \le \|f\|_{(L^1(m_0))^{1-\alpha}(L^1(m_1))^\alpha} .$$ □

However, in general we cannot establish the existence of an isometry between these spaces. In fact, the situation might be dramatic: $[m_0, m_1]_\alpha$ *can be the trivial measure* 0, *even if* m_0 *and* m_1 *are non-trivial.* Indeed, the vector measures m_0 and m_1 defined in Example 1 provide a simple proof of this fact. In this case a direct calculation shows that $[m_0, m_1]_\alpha = 0$, for all $0 < \alpha < 1$. However, we obviously have that $(L^1(m_0))^{1-\alpha}(L^1(m_1))^\alpha = (L^1[0,1])^{1-\alpha}(L^1[0,1])^\alpha = L^1[0,1]$. In this case the inclusion is simply the zero map. This situation motivates us to introduce the following definition that we will analyze and exploit in the next section.

Definition 2.8 *Let $0 < \alpha < 1$. A couple of equivalent vector measures m_0 and m_1 is said to be α-compatible if the following equality holds*: $(L^1(m_0))^{1-\alpha}(L^1(m_1))^\alpha = L^1([m_0, m_1]_\alpha)$.

In other words, two vector measures are α-compatible if the space $L^1([m_0, m_1]_\alpha)$ of integrable functions with respect to the interpolated vector measure coincides with the interpolation space $[L^1(m_0), L^1(m_1)]_\alpha$.

3 Interpolation Formulas for α-Compatible Vector Measures

In this section we consider two α-compatible vector measures m_0 and m_1 with values in two order-continuous Köthe function spaces X_0 and X_1. We provide a generalization of the complex interpolation formulas (1.1) for L^p-spaces. In particular, we prove, for two functions $0 \le f_0 \in L^1(m_0)$ and $0 \le f_1 \in L^1(m_1)$, the following equality:

$$[L^{p_0}(f_0 m_0), L^{p_1}(f_1 m_1)]_\theta = L^p\left(f_0^{1-\alpha} f_1^\alpha [m_0, m_1]_\alpha\right), \quad 0 < \theta < 1$$

for all $1 \le p_0, p_1 < \infty$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $\alpha = \frac{p\,\theta}{p_1}$.

As the statement above shows, the α-compatibility relation becomes the keystone of our technique for obtaining interpolation formulas for $L^p(m)$-spaces.

The α-compatibility relation is clearly satisfied for any couple of positive scalar finite measures. Moreover, obviously, each vector measure is α-compatible with itself. Of course, the measures m_0 and m_1 in Example 1 are not α-compatible. Let us start this section with some examples that illustrate that there are also other situations in which α-compatible couples of vector measures appear.

Example 4 Let $1 \le p_0, p_1 < \infty$, and consider a finite measure space (Ω, Σ, μ) and the vector measures

$$m_0 : A \in \Sigma \to m_0(A) := \chi_A \in L^{p_0}(\mu),$$
$$m_1 : A \in \Sigma \to m_1(A) := \chi_A \in L^{p_1}(\mu).$$

These vector measures are obviously countably additive, and the corresponding spaces of integrable functions are $L^1(m_0) = L^{p_0}(\mu)$ and $L^1(m_1) = L^{p_1}(\mu)$. Let $0 < \alpha < 1$. The complex interpolation formula for L^p-spaces of scalar measures gives $(L^{p_0}(\mu))^{1-\alpha}(L^{p_1}(\mu))^\alpha = L^p(\mu)$, where $\frac{1}{p} = \frac{1-\alpha}{p_0} + \frac{\alpha}{p_1}$. A direct calculation shows that the interpolated vector measure $[m_0, m_1]_\alpha$ takes values in $L^p(\mu)$ and is given by $[m_0, m_1]_\alpha(A) = \chi_A$, for all $A \in \Sigma$. Then obviously

$$\left(L^1(m_0)\right)^{1-\alpha}\left(L^1(m_1)\right)^\alpha = (L^{p_0}(\mu))^{1-\alpha}(L^{p_1}(\mu))^\alpha = L^p(\mu) = L^1([m_0, m_1]_\alpha).$$

Example 5 Now we show that the interpolated vector measure given in Example 2 provides also an α-compatible couple of vector measures. Recall that $\frac{1}{s_0} + \frac{1}{t} = \frac{1}{s_1}$, $m_0(A) = \chi_A \in L^{s_0}[0,1]$ and $m_1(A) = g \cdot \chi_A \in L^{s_1}([0,1])$ for all $A \in \mathscr{M}$. In this case, $L^1(m_0) = L^{s_0}[0,1]$ and $L^1(m_1) = L^{s_1}(g^{s_1}\lambda)$. Therefore, if $\frac{1}{s} = \frac{1-\alpha}{s_0} + \frac{\alpha}{s_1}$ and $\beta = \frac{s\,\alpha}{s_1}$, the complex interpolation formula for L^p-spaces of scalar measures gives

$$\left(L^1(m_0)\right)^{1-\alpha}\left(L^1(m_1)\right)^\alpha = (L^{s_0}[0,1])^{1-\alpha}(L^{s_1}(g^{s_1}\lambda))^\alpha = L^s(g^{\beta\, s_1}) = L^s(g^{s\,\alpha}\lambda).$$

This clearly coincides (isometrically) with $L^1([m_0, m_1]_\alpha)$, since the measure of Example 2 is given, as we computed there, by $[m_0, m_1]_\alpha(A) = g^\alpha \cdot \chi_A \in L^s([0,1])$, for all $A \in \mathscr{M}$, and so $L^1([m_0, m_1]_\alpha) = L^s(g^{s\,\alpha}\lambda)$.

In what follows we also show that in particular, if two positive vector measures m_0 and m_1 with values in the same Köthe function space X satisfy that there is a measure m on X and functions $0 < f_0, f_1 \in L^1(m)$ such that $m_0 = f_0 m$ and $m_1 = f_1 m$, then m_0 and m_1 are α-compatible for every $0 < \alpha < 1$. The reader can find in [17] the conditions that assure the existence of functions $f_0, f_1 \in L^1(m)$ with the mentioned property (see also Corollary 3.4 below).

In the proof of the next result we use the following well-known fact: *For any positive vector measure m and any function $0 \le f \in L^1(m)$, a measurable function g belongs to $L^1(fm)$ if and only if gf belongs to $L^1(m)$; moreover, in this case, $\|g\|_{L^1(fm)} = \|gf\|_{L^1(m)}$.*

Theorem 3.1 *Let $m_0 : \Sigma \to X_0$ and $m_1 : \Sigma \to X_1$ be two α-compatible vector measures, and consider two functions $0 \le f_0 \in L^1(m_0)$ and $0 \le f_1 \in L^1(m_1)$. Then*

$$\left(L^1(f_0 m_0)\right)^{1-\alpha} \left(L^1(f_1 m_1)\right)^{\alpha} = L^1(f_0^{1-\alpha} f_1^{\alpha} [m_0, m_1]_\alpha).$$

Proof Take the function $f := f_0^{1-\alpha} f_1^{\alpha}$. We already know by Corollary 2.7 that $f \in L^1([m_0, m_1]_\alpha)$ and also from (2.11) we know that $\int_A f_0^{1-\alpha} f_1^{\alpha} \, d[m_0, m_1]_\alpha = [f_0 m_0, f_1 m_1]_\alpha(A)$, for all $A \in \Sigma$, that is, $[f_0 m_0, f_1 m_1]_\alpha = f_0^{1-\alpha} f_1^{\alpha} \, [m_0, m_1]_\alpha$. Consider now a function

$$g \in \left(L_1(f_0 m_0)\right)^{1-\alpha} \left(L_1(f_1 m_1)\right)^{\alpha}$$

and let $\varepsilon > 0$. Then there are functions $0 \le u_0 \in L^1(f_0 m_0)$ and $0 \le u_1 \in L^1(f_1 m_1)$ such that $|g| \le u_0^{1-\alpha} u_1^{\alpha}$ and $\|u_0\|_{L^1(f_0 m_0)}^{1-\alpha} \|u_1\|_{L^1(f_1 m_1)}^{\alpha} \le \|g\|_{(L_1(f_0 m_0))^{1-\alpha} (L_1(f_1 m_1))^{\alpha}} + \varepsilon$. Then we have $|g| f \le u_0^{1-\alpha} u_1^{\alpha} f = h_0^{1-\alpha} h_1^{\alpha}$, where $h_0 := u_0 f_0$ and $h_1 := u_1 f_1$. Note that $h_0 \in L^1(m_0)$ since $u_0 \in L^1(f_0 m_0)$. Analogously $h_1 \in L^1(m_1)$. This means that

$$gf \in \left(L_1(m_0)\right)^{1-\alpha} \left(L_1(m_1)\right)^{\alpha} = L^1\left([m_0, m_1]_\alpha\right)$$

by the α-compatibility assumption for the measures m_0 and m_1. Thus $g \in L^1(f_0^{1-\alpha} f_1^{\alpha} [m_0, m_1]_\alpha)$. Moreover, using again the α-compatibility we have

$$\begin{aligned}
\|g\|_{L^1(f\,[m_0,m_1]_\alpha)} &= \|gf\|_{L^1([m_0,m_1]_\alpha)} = \|gf\|_{(L_1(m_0))^{1-\alpha}(L_1(m_1))^{\alpha}} \\
&\le \|h_0\|_{L^1(m_0)}^{1-\alpha} \|h_1\|_{L^1(m_1)}^{\alpha} = \|u_0 f_0\|_{L^1(m_0)}^{1-\alpha} \|u_1 f_1\|_{L^1(m_1)}^{\alpha} \\
&= \|u_0\|_{L^1(f_0 m_0)}^{1-\alpha} \|u_1\|_{L^1(f_1 m_1)}^{\alpha} \le \|g\|_{(L_1(f_0 m_0))^{1-\alpha}(L_1(f_1 m_1))^{\alpha}} + \varepsilon.
\end{aligned}$$

This means that $\|g\|_{L^1\left(f_0^{1-\alpha} f_1^{\alpha} [m_0,m_1]_\alpha\right)} \le \|g\|_{(L_1(f_0 m_0))^{1-\alpha}(L_1(f_1 m_1))^{\alpha}}$, since $\varepsilon > 0$ was taken arbitrarily.

Now we prove the converse. Let $g \in L^1(f\,[m_1, m_0]_\alpha)$. Recall that this is the same to say that $gf \in L^1([m_0, m_1]_\alpha)$ and

$$\|g\|_{L^1(f\,[m_0,m_1]_\alpha)} = \|gf\|_{L^1([m_0,m_1]_\alpha)} = \|gf\|_{(L^1(m_0))^{1-\alpha}(L^1(m_1))^{\alpha}} . \tag{3.1}$$

The last equality follows from the α-compatibility assumption on the measures m_0 and m_1. Now, if $\varepsilon > 0$, there are functions $0 \le u_0 \in L^1(m_0)$ and $0 \le u_1 \in L^1(m_1)$ such that $|g|\, f \le u_0^{1-\alpha} u_1^{\alpha}$ and

$$\|u_0\|_{L^1(m_0)}^{1-\alpha} \|u_1\|_{L^1(m_1)}^{\alpha} \le \|gf\|_{(L^1(m_0))^{1-\alpha}(L^1(m_1))^{\alpha}} + \varepsilon. \tag{3.2}$$

Since $u_0 \in L^1(m_0)$, we obtain $h_0 := \frac{u_0}{f_0} \in L^1(f_0 m_0)$. The fact that $u_1 \in L^1(m_1)$ gives also $h_1 := \frac{u_1}{f_1} \in L^1(f_1 m_1)$. Therefore, we have $|g| \le h_0^{1-\alpha} h_1^{\alpha}$. Consequently, $g \in \big(L^1(f_0 m_0)\big)^{1-\alpha} \cdot \big(L^1(f_1 m_1)\big)^{\alpha}$. Moreover using (3.1) and (3.2) we obtain

$$\begin{aligned}\|g\|_{(L^1(f_0 m_0))^{1-\alpha}(L^1(f_1 m_1))^{\alpha}} &\le \|h_0\|_{L^1(f_0 m_0)}^{1-\alpha} \|h_1\|_{L^1(f_1 m_1)}^{\alpha} = \|u_0\|_{L^1(m_0)}^{1-\alpha} \|u_1\|_{L^1(m_1)}^{\alpha} \\ &\le \|g\|_{L^1(f[m_0,m_1]_\alpha)} + \epsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we obtain $\|g\|_{(L^1(f_0 m_0))^{1-\alpha}(L^1(f_1 m_1))^{\alpha}} \le \|g\|_{L^1(f_0^{1-\alpha} f_1^{\alpha}[m_0,m_1]_\alpha)}$. With these all arguments together we get $(L^1(f_0 m_0))^{1-\alpha}(L^1(f_1 m_1))^{\alpha} = L^1(f_0^{1-\alpha} f_1^{\alpha}[m_0,m_1]_\alpha)$, and the equality of norms in these spaces. □

In what follows we show that this approach can also be used to characterize the class of couples of vector measures for which the interpolation formulas hold.

Theorem 3.2 *Let $1 \le p_0, p_1 < \infty$, and $\alpha = \frac{p\,\theta}{p_1}$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $0 < \theta < 1$. Consider two α-compatible vector measures $m_0 : \Sigma \to X_0$ and $m_1 : \Sigma \to X_1$, and let $0 \le f_0 \in L^1(m_0)$, and $0 \le f_1 \in L^1(m_1)$. Then $[L^{p_0}(f_0 m_0), L^{p_1}(f_1 m_1)]_\theta = L^p\left(f_0^{1-\alpha} f_1^{\alpha}[m_0,m_1]_\alpha\right)$.*

Proof Consider first the case $f_0 = f_1 = \chi_\Omega$. Let $g \in [L^{p_0}(m_0), L^{p_1}(m_1)]_\theta$ and $\varepsilon > 0$. Then by the definition of the interpolation space there are functions $0 \le u_0 \in L^{p_0}(m_0)$ and $0 \le u_1 \in L^{p_1}(m_1)$ such that $|g| \le u_0^{1-\theta} u_1^{\theta}$ and $\|u_0\|_{L^{p_0}(m_0)}^{1-\theta} \|u_1\|_{L^{p_1}(m_1)}^{\theta} \le \|g\|_{[L^{p_0}(m_0), L^{p_1}(m_1)]_\theta} + \varepsilon$. Note that $h_0 := u_0^{p_0} \in L^1(m_0)$ and $h_1 := u_1^{p_1} \in L^1(m_1)$. Thus, $|g| \le h_0^{\frac{1-\theta}{p_0}} h_1^{\frac{\theta}{p_1}}$, and therefore $|g|^p \le h_0^{\frac{(1-\theta)p}{p_0}} h_1^{\frac{\theta p}{p_1}} = h_0^{1-\alpha} h_1^{\alpha}$. Moreover we know that $\|u_0\|_{L^{p_0}(m_0)}^{1-\theta} = \|u_0^{p_0}\|_{L^1(m_0)}^{\frac{1-\theta}{p_0}} = \|h_0\|_{L^1(m_0)}^{\frac{(1-\alpha)p}{}}$, which implies $\|h_0\|_{L^1(m_0)}^{1-\alpha} = \|u_0\|_{L^{p_0}(m_0)}^{\frac{1-\theta}{p}}$. Similarly, $\|h_1\|_{L^1(m_1)}^{\alpha} = \|u_1\|_{L^{p_1}(m_1)}^{\frac{\theta}{p}}$. Thus, $|g|^p \le |h_0|^{1-\alpha} |h_1|^{\alpha}$ and

$$\|h_0\|_{L^1(m_1)}^{1-\alpha} \|h_1\|_{L^1(m_1)}^{\alpha} \le \left(\|g\|_{[L^{p_0}(m_0), L^{p_1}(m_1)]_\theta} + \varepsilon\right)^{\frac{1}{p}}, \tag{3.3}$$

which implies that $|g|^p \in [L^1(m_0), L^1(m_1)]_\alpha$, which is equal to $L^1([m_0,m_1]_\alpha)$ in order to the α-compatibility assumption. Moreover, by the definition of the norms and (3.3) we obtain

$$\|g\|_{L^p([m_0,m_1]_\alpha)} = \||g|^p\|_{L^1([m_0,m_1]_\alpha)}^{\frac{1}{p}} = \||g|^p\|_{[L^1(m_0),L^1(m_1)]_\alpha}^{\frac{1}{p}} \le \|g\|_{[L^{p_0}(m_0), L^{p_1}(m_1)]_\theta} + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we obtain $\|g\|_{L^p([m_0,m_1]_\alpha)} \le \|g\|_{[L^{p_0}(m_0), L^{p_1}(m_1)]_\theta}$.

Let us prove now the converse. Let $g \in L^p([m_0,m_1]_\alpha)$. Then

$$|g|^p \in L^1\left([m_0,m_1]_\alpha\right) = \left[L^1(m_0), L^1(m_1)\right]_\alpha.$$

The last equality follows from the α-compatibility for the measures m_0 and m_1. Let $\varepsilon > 0$. Then there are $0 \le u_0 \in L^1(m_0)$ and $0 \le u_1 \in L^1(m_1)$ such that $|g|^p \le u_0^{1-\alpha} u_1^{\alpha}$ and

$$\|u_0\|_{L^1(m_0)}^{1-\alpha} \|u_1\|_{L^1(m_1)}^{\alpha} \le \||g|^p\|_{[L^1(m_0),L^1(m_1)]_\alpha} + \varepsilon. \tag{3.4}$$

Thus, $|g|^p \le u_0^{\frac{(1-\theta)p}{p_0}} u_1^{\frac{\theta p}{p_1}}$, and then $|g| \le u_0^{\frac{1-\theta}{p_0}} u_1^{\frac{\theta}{p_1}}$. Now we define $h_0 := u_0^{\frac{1}{p_0}} \in L^{p_0}(m_0)$ and $h_1 := u_1^{\frac{1}{p_1}} \in L^{p_1}(m_1)$. Clearly, $|g| \le h_0^{1-\theta} h_1^{\theta}$, and moreover

$$\begin{aligned}\|h_0\|_{L^{p_0}(m_0)} &= \|h_0^{p_0}\|_{L^1(m_0)}^{\frac{1}{p_0}} = \|u_0\|_{L^1(m_0)}^{\frac{1}{p_0}}, \\ \|h_1\|_{L^{p_1}(m_1)} &= \|h_1^{p_1}\|_{L^1(m_1)}^{\frac{1}{p_1}} = \|u_1\|_{L^1(m_1)}^{\frac{1}{p_1}}.\end{aligned}$$

Thus, by (3.4), we obtain $\|h_0\|_{L^{p_0}(m_0)}^{1-\theta}\|h_1\|_{L^{p_1}(m_1)}^{\theta} \leq (\||g|^p\|_{[L^1(m_0),L^1(m_1)]_\alpha}+\varepsilon)^{\frac{1}{p}}$. Consequently, $g \in [L^{p_0}(m_0), L^{p_1}(m_1)]_\theta$ and $\|g\|_{[L^{p_0}(m_0),L^{p_1}(m_1)]_\theta} \leq \|g\|_{L^p([m_0,m_1]_\alpha)}$. We conclude that

$$[L^{p_0}(m_0), L^{p_1}(m_1)]_\theta = L^p([m_0, m_1]_\alpha).$$

This proves the result for $f_0 = f_1 = \chi_\Omega$.

The general case is given from this just using Theorem 3.1; if m_0 and m_1 are α-compatible, then $f_0 m_0$ and $f_1 m_1$ are also α-compatible, so we have

$$[L^{p_0}(f_0 m_0), L^{p_1}(f_1 m_1)]_\theta = L^p\left([f_0 m_0, f_1 m_1]_\alpha\right).$$

Moreover we know that $[f_0 m_0, f_1 m_1]_\alpha = f_0^{1-\alpha} f_1^\alpha [m_0, m_1]_\alpha$. Therefore,

$$[L^{p_0}(f_0 m_0), L^{p_1}(f_1 m_1)]_\theta = L^p\left([f_0 m_0, f_1 m_1]_\alpha\right) = L^p(f_0^{1-\alpha} f_1^\alpha [m_0, m_1]_\alpha).$$

This finishes the proof. □

The following corollary is a direct consequence of Proposition 2.6 and Theorem 3.2. Recall that the measures $f_0 m$ and $f_1 m$ are α-compatible for every $0 < \alpha < 1$ and $0 \leq f_0, f_1 \in L^1(m)$.

Corollary 3.3 *Let m be a countably additive positive vector measure with values in a Köthe function space, and consider $0 \leq f_0, f_1 \in L^1(m)$. Then $[L^{p_0}(f_0 m), L^{p_1}(f_1 m)]_\theta = L^p(f_0^{1-\alpha} f_1^\alpha\, m)$, where $1 \leq p_0, p_1 < \infty$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\alpha = \frac{p\,\theta}{p_1}$.*

We say that a countably additive vector measure $n : \Sigma \to X$, with values in a Banach space X, is *scalarly uniformly absolutely continuous* with respect to another countably additive vector measure $m : \Sigma \to X$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x' \in X'$ and each $A \in \Sigma$ the inequality $|\langle m, x'\rangle|(A) < \delta$ yields $|\langle n, x'\rangle|(A) < \varepsilon$. Theorem 1 in [17] and the results above provide the following interpolation formula. Note that the requirement of equivalence of m_0 and m_1 implies that we can restrict the σ-algebra to assure that the functions f_0 and f_1 are in fact μ-almost everywhere positive for a Rybakov measure μ of m. Moreover, Theorem 1 in [17] gives that the corresponding Radon–Nikoým derivatives of m_0 and m_1 with respect to m belong to $L^\infty(\mu) \subseteq L^1(m)$; furthermore the results in that paper should provide Radon–Nikodým derivatives just belonging to $L^1(m)$.

Corollary 3.4 *Let m_0 and m_1 be a couple of equivalent countably additive positive vector measures that are scalarly uniformly absolutely continuous with respect to the positive countably additive vector measure $m : \Sigma \to X$, with values in a Köthe function space X. Let μ be a Rybakov control measure for m. Then there are functions $0 \leq f_0, f_1 \in L^\infty(\mu)$ such that*

$$[L^{p_0}(m_0), L^{p_1}(m_1)]_\theta = L^p(f_0^{1-\alpha} f_1^\alpha\, m),$$

where $1 \leq p_0, p_1 < \infty$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\alpha = \frac{\theta p}{p_1}$.

Let us finish the paper by writing two direct applications of our results in interpolation theory. The first one provides a Riesz–Thorin Theorem for spaces of p-integrable functions with respect to a vector measure, and is a direct consequence of the properties of the complex interpolation method (see for instance [2]). The second one provides a formula for interpolation of injective tensor products of such spaces and is a direct consequence of the results of [18] (see also [19]).

Corollary 3.5 *Let* $0 < \theta < 1 \leq p_0, p_1 < \infty$, *and* $\alpha = \frac{p\,\theta}{p_1}$, *where* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. *Let* $1 \leq q_0, q_1 < \infty$, *and* $\beta = \frac{q\,\theta}{q_1}$, *where* $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. *Let* m_0 *and* m_1 *be a couple of* α*-compatible vector measures, and similarly let* n_0 *and* n_1 *be another couple of* β*-compatible vector measures. Consider an operator* $T : L^{p_0}(m_0) + L^{p_1}(m_1) \longrightarrow L^{q_0}(n_0) + L^{q_1}(n_1)$ *that is also well defined and continuous when restricted to* $T : L^{p_0}(m_0) \cap L^{p_1}(m_1) \longrightarrow L^{q_0}(n_0) \cap L^{q_1}(n_1)$. *Then* $T : L^p\left([m_0, m_1]_\alpha\right) \longrightarrow L^q\left([n_0, n_1]_\beta\right)$ *is well defined and continuous.*

Corollary 3.6 *Under the assumptions of Corollary* 3.5, *if all the spaces in the left hand side of the following formula are* 2*-concave, then*

$$\left[L^{p_0}(m_0)\hat{\otimes}_\varepsilon L^{q_0}(n_0), L^{p_1}(m_1)\hat{\otimes}_\varepsilon L^{q_1}(n_1)\right]_\theta = L^p\left([m_0, m_1]_\alpha\right)\hat{\otimes}_\varepsilon L^q\left([n_0, n_1]_\beta\right).$$

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