THE FATOU COMPLETION OF A FRÉCHET FUNCTION SPACE AND APPLICATIONS

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Abstract

Given a metrizable locally convex-solid Riesz space of measurable functions we provide a procedure to construct a minimal Fréchet (function) lattice containing it, called its Fatou completion. As an application, we obtain that the Fatou completion of the space $L^1(\nu)$ of integrable functions with respect to a Fréchet-space-valued measure ν is the space $L^1_w(\nu)$ of scalarly ν-integrable functions. Further consequences are also given.

1. Introduction

Let X be a Banach space and ν be a (σ-additive) X-valued measure. Then the Banach space $L^1(\nu)$ of all ν-integrable functions is a closed subspace (typically proper) of the Banach space $L^1_w(\nu)$ of all scalarly ν-integrable functions [20]. Both $L^1(\nu)$ and $L^1_w(\nu)$ are Banach function spaces (relative to a control measure for ν and the pointwise almost everywhere order), with the distinction that $L^1(\nu)$ has order continuous norm (that is, a Lebesgue topology), whereas $L^1_w(\nu)$ always has the σ-Fatou property [6, 7]. Actually, $L^1(\nu)$ is the order continuous part of $L^1_w(\nu)$, that is, the largest ideal inside $L^1_w(\nu)$ with each element having order continuous norm, and $L^1_w(\nu)$ is the σ-Fatou completion of $L^1(\nu)$, that is, the minimal Banach function space which has the σ-Fatou property and contains $L^1(\nu)$ [7].

What is the situation when ν takes its values in a Fréchet space X? It has been known for some time that $L^1(\nu)$ is complete, that is, is a Fréchet space [12]. In a recent article [8] it was shown that $L^1_w(\nu)$ is also complete and contains $L^1(\nu)$ as a

closed subspace. It is worth pointing out that the strong dual of a Fréchet space (or lattice or function space) is typically not a *Fréchet* space (or lattice or function space) and so certain duality arguments used in the Banach setting are not available in the Fréchet setting. It is also shown in [8] that both $L^1(\nu)$ and $L^1_w(\nu)$ are Fréchet function spaces with $L^1(\nu)$ having a Lebesgue topology, $L^1_w(\nu)$ having the σ-Fatou property and $L^1(\nu)$ being the order continuous part of $L^1_w(\nu)$. The aim of this note is to establish the 'missing link', namely, that $L^1_w(\nu)$ is the σ-Fatou completion of $L^1(\nu)$. In this regard, we develop various aspects of the theory of Fréchet *function spaces* (especially in relation to Lorentz function seminorms), which are not available in the literature in the form needed here; this, of interest in its own right, is done in Section 2. With these techniques we are able to establish close connections between $L^1(\nu)$ and $L^1_w(\nu)$ which are known in the Banach space setting [7]. Namely, in Section 3 it is shown that the following are equivalent: $L^1(\nu) = L^1_w(\nu)$; $L^1(\nu)$ has the σ-Fatou property; $L^1_w(\nu)$ is order continuous; $L^1(\nu)$ is weakly sequentially complete; and $L^1_w(\nu)$ is weakly sequentially complete.

2. Fréchet function spaces

Let (Ω, Σ, μ) be a σ-finite measure space and $\mathcal{M} := L^0(\mu)$ be the space of all Σ-measurable, finite $\mathbb{R}$-valued μ-a.e. functions on Ω. Functions in $\mathcal{M}$ differing on a μ-null set are identified. We define $\mathcal{M}^+ := \{f \in \mathcal{M} \mid f \geq 0\}$, where the notation $f \leq g$ means that f and g are $\mathbb{R}$-valued and $f(x) \leq g(x)$ for μ-a.e. $x \in \Omega$. According to [22, Ch. 15], a *function seminorm* in $\mathcal{M}$ is any function $\rho : \mathcal{M}^+ \to [0, \infty]$ satisfying:

(i) $\rho(u) = 0$ if $u = 0$ (μ-a.e.);
(ii) $\rho(au) = a\rho(u)$ for all $a \geq 0$ and $u \in \mathcal{M}^+$;
(iii) $\rho(u + v) \leq \rho(u) + \rho(v)$ for all $u, v \in \mathcal{M}^+$;
(iv) $\rho(u) \leq \rho(v)$ whenever $u, v \in \mathcal{M}^+$ satisfy $u \leq v$.

One can then extend ρ to the whole of $\mathcal{M}$ (the extension is again denoted by ρ) by setting $\rho(f) := \rho(|f|)$, for any $f \in \mathcal{M}$. The closed unit ball $B_\rho := \{f \in \mathcal{M} \mid \rho(f) \leq 1\}$ is *solid*, that is, if $f \in B_\rho$ and $g \in \mathcal{M}$ satisfy $|g| \leq |f|$, then also $g \in B_\rho$. If ρ has the additional property that $\rho(f) = 0$ if and only if $f = 0$ (μ-a.e.), then it is called a *function norm*. In this case $L_\rho := \{f \in \mathcal{M} \mid \rho(f) < \infty\}$ is a normed space (with ρ as its norm) and is an *ideal* in $\mathcal{M}$, that is, $f \in \mathcal{M}$ and $g \in L_\rho$ with $|f| \leq |g|$ implies that $f \in L_\rho$ and $\rho(f) \leq \rho(g)$; see [22, Ch. 15] for all of these facts. Any normed space of this kind is called a *Köthe function space* and, if it is complete, a *Banach function space*.

Let $\{\rho_n\}_{n \in \mathbb{N}}$ be any increasing sequence of function seminorms in $\mathcal{M}$. We always assume that $\{\rho_n\}_{n \in \mathbb{N}}$ is *fundamental*, meaning that if $f \in \mathcal{M} \setminus \{0\}$, then there exists $m \in \mathbb{N}$ such that $\rho_m(f) \neq 0$. We then define

$$L_{\{\rho_n\}} := \{f \in \mathcal{M} \mid \rho_n(f) < \infty, \ \forall n \in \mathbb{N}\} = \bigcap_{n \in \mathbb{N}} L_{\rho_n}.$$

The following fact is immediate from [15, Lemma 22.5]. For the notion of a *locally solid topology* we refer to [3, p. 33].

LEMMA 2.1. *If $\{\rho_n\}_{n\in\mathbb{N}}$ is any increasing fundamental sequence of function seminorms, then $L_{\{\rho_n\}}$ is a metrizable, locally solid, locally convex Hausdorff space for the topology induced by $\{\rho_n\}_{n\in\mathbb{N}}$. Moreover, if $f \in \mathcal{M}$ and $g \in L_{\{\rho_n\}}$ satisfy $|f| \le |g|$, then $f \in L_{\{\rho_n\}}$ and $\rho_n(f) \le \rho_n(g)$ for all $n \in \mathbb{N}$.*

Any locally convex Hausdorff space $L_{\{\rho_n\}}$ as given by Lemma 2.1 is called a (locally solid, metrizable) *function space* and, if complete, a *Fréchet function space* (F.f.s.).

Let $\Omega = [0, 1]$ be equipped with Lebesgue measure μ. For each $n \in \mathbb{N}$, let $A_n = [0, 1/n]$ and define ρ_n on $L^0(\mu)^+$ by

$$\rho_n(f) := \begin{cases} \int_0^{1/n} f \, d\mu & \text{if } f\chi_{A_n} = f \ (\mu\text{-a.e.}) \\ \infty & \text{otherwise.} \end{cases}$$

Then $\{\rho_n\}_{n\in\mathbb{N}}$ is an increasing fundamental sequence of function norms for which $L_{\{\rho_n\}} = \{0\}$. Let $\rho := \rho_2$ be as above. Every Borel set $A \subseteq [1/2, 1]$ with $\mu(A) > 0$ satisfies $\rho(\chi_A) = \infty$, that is, $[1/2, 1]$ is a ρ-purely infinite set [22, Ch. 15, Section 67]. In this case, the carrier of the ideal $L_\rho \subseteq L^0(\mu)$ is not all of Ω, but rather $[0, 1/2]$. To avoid pathologies of the above kind, we call an increasing fundamental sequence of function seminorms $\{\rho_n\}_{n\in\mathbb{N}}$ *determining* if there exists $u \in L_{\{\rho_n\}}^+$ satisfying $u > 0$ pointwise μ-a.e. on Ω. It is useful to exhibit some examples in the nonnormable setting.

EXAMPLE 1. Let X be a metrizable locally convex Hausdorff space generated by an increasing fundamental sequence of seminorms $(\|\cdot\|^{(n)})_{n\in\mathbb{N}}$ in X and with continuous dual space X^*. For each $n \in \mathbb{N}$, let

$$B_n := \{x \in X : \|x\|^{(n)} \le 1\},$$
$$B_n^\circ := \{x^* \in X^* : |\langle x, x^*\rangle| \le 1, \ \forall x \in B_n\}.$$

Given a (σ-additive) vector measure $\nu : \Sigma \to X$, defined on a measurable space (Ω, Σ), a set $A \in \Sigma$ is called ν-*null* if $\nu(B) = 0$ for all $B \in \Sigma$ with $B \subseteq A$. Let $L^0(\nu)$ denote the Riesz space of all (equivalence classes modulo equality ν-a.e. of) scalar-valued, Σ-measurable functions defined on Ω, with respect to the ν-a.e. pointwise order. For each $n \in \mathbb{N}$, define a $[0, \infty]$-valued seminorm $\|\cdot\|_\nu^{(n)}$ in $L^0(\nu)$ by

$$\|f\|_\nu^{(n)} := \sup_{x^*\in B_n^\circ} \int_\Omega |f| \, d|\langle \nu, x^*\rangle| \quad \forall f \in L^0(\nu), \tag{2.1}$$

where $|\langle \nu, x^*\rangle|$ is the variation of the signed measure $\langle \nu, x^*\rangle : A \mapsto \langle \nu(A), x^*\rangle$ for all $x^* \in X^*$. If $\|f\|_\nu^{(n)} < \infty$ for all $n \in \mathbb{N}$, then f is called *scalarly ν-integrable*. This is equivalent to $\int_\Omega |f| \, d|\langle \nu, x^*\rangle| < \infty$ for all $x^* \in X^*$ [8, Proposition 2.1]. The space of all such functions $f \in L^0(\nu)$ is denoted by $L_w^1(\nu)$. Given a control measure μ for ν [12, pp. 19–21], we define the function seminorms

$$\rho_n(f) := \|f\|_\nu^{(n)} \quad \forall f \in L^0(\mu) = L^0(\nu) \ \forall n \in \mathbb{N}, \tag{2.2}$$

in which case $L_{\{\rho_n\}} = L_w^1(\nu)$. So $L_w^1(\nu)$ is a F.f.s. [8, Theorem 2.5].

A function $f \in L^1_w(\nu)$ such that for each $A \in \Sigma$ there exists $\int_A f \, d\nu \in X$ satisfying $\langle \int_A f \, d\nu, \, x^* \rangle = \int_A f \, d\langle \nu, x^* \rangle$ for all $x^* \in X^*$ is called ν-*integrable*. The subspace of $L^1_w(\nu)$ consisting of all ν-integrable functions is denoted by $L^1(\nu)$. If X is a Fréchet space and, for all $n \in \mathbb{N}$,

$$\widetilde{\rho}_n(f) := \begin{cases} \rho_n(f) & \text{if } f \in L^1(\nu) \\ \infty & \text{if } f \in L^0(\mu) \setminus L^1(\nu) = L^0(\nu) \setminus L^1(\nu), \end{cases} \tag{2.3}$$

then $L_{\{\widetilde{\rho}_n\}} = L^1(\nu)$ is also a F.f.s. [8, Theorem 2.5]. If X is not complete, then the $\{\widetilde{\rho}_n\}_{n \in \mathbb{N}}$ are typically *not* function seminorms [17, Section 3]. Since the constant function χ_Ω is positive μ-a.e. and belongs to $L_{\{\rho_n\}}$ and to $L_{\{\widetilde{\rho}_n\}}$, both $\{\rho_n\}_{n \in \mathbb{N}}$ and $\{\widetilde{\rho}_n\}_{n \in \mathbb{N}}$ are determining.

EXAMPLE 2. Let ρ be any determining function norm in $L^0(\mu)$, with (Ω, Σ, μ) a σ-finite, positive measure space. A Köthe matrix $A = (a_n)_n$ on Ω is any sequence of functions $a_n \in \mathcal{M}^+$ which satisfy $0 < a_n \le a_{n+1} < \infty$ (μ-a.e. on Ω), for each $n \in \mathbb{N}$. Define the normed *Köthe function space* $L_\rho(a_n) := \{f \in \mathcal{M} \mid \rho(a_n f) < \infty\}$, in which case $L_\rho(a_{n+1}) \subseteq L_\rho(a_n)$ and $L_{\{\rho_n\}} = \bigcap_{n=1}^\infty L_\rho(a_n)$ is a locally solid, metrizable function space, where $\rho_n(f) := \rho(a_n f)$, $f \in \mathcal{M}$, for each $n \in \mathbb{N}$, is an increasing fundamental sequence of function norms. Such spaces have been treated in [4, 18], for example. If Ω is a countable set and μ is counting measure, then $L_{\{\rho_n\}}$ corresponds to a classical *Köthe echelon space*. For instance, with $\rho(x) := (\sum_{i \in \Omega} |x_i|^p)^{1/p}$ and $p \in [1, \infty)$ fixed, such spaces are traditionally denoted by $\lambda_p(A)$ [15, Ch. 27].

Further examples include $L_{p-} := \bigcap_{1 \le r < p} L^r([0, 1])$ for $p \in (1, \infty)$ [5], $\ell^{p+} := \bigcap_{p < q} \ell^q$ for $p \in [1, \infty)$ [16], and $L^p_{\text{loc}}(\mathbb{R})$, that is, pth power locally integrable functions on $\mathbb{R}$ for $p \in [1, \infty)$ [1, 2].

Let (F, τ) be a metrizable locally convex-solid Riesz space generated by a fundamental sequence of Riesz seminorms $\{q_n\}_{n \in \mathbb{N}}$ [3, Theorem 6.1]. Then F has a *Lebesgue (σ-Lebesgue) topology* if $u_\alpha \downarrow 0$ implies $u_\alpha \xrightarrow{\tau} 0$ in F ($u_k \downarrow 0$ implies that $u_k \xrightarrow{\tau} 0$ in F) [3, Ch. 3]. The space F has the *Fatou (σ-Fatou) property* if, for every increasing net $(u_\alpha)_\alpha$ (increasing sequence $(u_k)_k$) in the positive cone F^+ of F that is topologically bounded in F, the element $u := \sup u_\alpha$ exists in F^+ and $q_n(u_\alpha) \uparrow_\alpha q_n(u)$ ($u := \sup u_k$ exists in F^+ and $q_n(u_k) \uparrow_k q_n(u)$) for all $n \in \mathbb{N}$. This notion is not 'standard'; for example, in [3, p. 94] such a space F is called a Nakano (σ-Nakano) space.

An element u of a Fréchet lattice F is σ-*order continuous* if it has the property that $u_k \xrightarrow{\tau} 0$ as $k \to \infty$ for every sequence $(u_k)_k \subseteq F^+$ satisfying $|u| \ge u_k \downarrow 0$. The σ-*order continuous part* F_a of F consists of the collection of all σ-order continuous elements of F; it is a closed ideal in F [23, pp. 331–332], and clearly has a σ-Lebesgue topology.

If $F = L_{\{\rho_n\}}$ is a F.f.s., then there is no distinction between using nets and sequences for the Lebesgue and Fatou properties. Indeed, $L^0(\mu)$ is a σ-*Dedekind complete* Riesz

space [14, pp. 126–127], and F is an ideal in $L^0(\mu)$; see Lemma 2.1. So F is also σ-Dedekind complete [14, Theorem 25.2]. Hence whenever F has a σ-Lebesgue topology, it also has a Lebesgue topology [3, Theorem 17.9]. The converse is obvious. Concerning the Fatou and σ-Fatou properties, we need a preliminary result. For the notion of an *order basis* in a general Riesz space we refer to [14, Section 28] and for the concept of *order separability* we refer to [14, Section 23]. A normed lattice possessing these two properties has the Fatou property whenever it has the σ-Fatou property [23, Theorem 113.2]. The same is valid for Fréchet lattices.

PROPOSITION 2.2. *Any order separable Fréchet lattice F with a countable order basis and the σ-Fatou property has the Fatou property.*

PROOF. Since F has the σ-Fatou property, it follows from [14, Theorem 23.2(ii)] and [3, Theorem 5.4(i)] that F is σ-Dedekind complete. By order separability, F is then Dedekind complete [14, Theorem 23.6].

Let $\{b_n\}_{n\in\mathbb{N}}$ be an increasing, positive order basis of F [14, p. 161]. Consider a net $0 \le u_\lambda \uparrow$ that is topologically bounded in F. We need to show that $\sup_\lambda u_\lambda$ exists in F and that $q_n(\sup_\lambda u_\lambda) = \sup_\lambda q_n(u_\lambda)$ for all $n \in \mathbb{N}$. The proof proceeds as for normed spaces [23, Theorem 113.2]. Set $v_{\lambda,k} := u_\lambda \wedge kb_k$ for all λ and all $k \in \mathbb{N}$. For $k \in \mathbb{N}$ fixed, $v_{\lambda,k} \le kb_k$ for all λ and so $s_k := \sup_\lambda v_{\lambda,k}$ exists in F (by Dedekind completeness). Also, $s_k = \sup_j v_{\lambda_j,k}$ for some increasing sequence $(v_{\lambda_j,k})_j$, by order separability of F [14, Theorem 23.2(iii)].

For all $k \in \mathbb{N}$, since $v_{\lambda_j,k} \uparrow_j s_k$, the σ-Fatou property of F implies that $q_n(s_k) = \sup_j q_n(v_{\lambda_j,k})$ for all $n \in \mathbb{N}$. Hence $(s_k)_k$ is an increasing sequence that is topologically bounded in F since, for all $n \in \mathbb{N}$,

$$q_n(s_k) = \sup_j q_n(v_{\lambda_j,k}) \le \sup_j q_n(u_{\lambda_j}) \le \sup_\lambda q_n(u_\lambda) < \infty \quad \forall k \in \mathbb{N}.$$

By the σ-Fatou property of F, it follows that $s := \sup_k s_k$ exists in F and that

$$q_n(s) = \sup_k q_n(s_k) \le \sup_\lambda q_n(u_\lambda) \quad \forall n \in \mathbb{N}. \tag{2.4}$$

Since $\{b_n\}_{n\in\mathbb{N}}$ is an order basis, we have $s = \sup_\lambda u_\lambda$ [14, Theorem 28.2]. Thus $u_\lambda \uparrow s$. Moreover, $u_\lambda \le s$ implies that $q_n(u_\lambda) \le q_n(s)$ for all n and λ which, combined with (2.4), yields $q_n(s) = \sup_\lambda q_n(u_\lambda)$ for all $n \in \mathbb{N}$. So F has the Fatou property. □

COROLLARY 2.3. *In every Fréchet function space, the σ-Fatou property and the Fatou property are equivalent.*

PROOF. Let $L_{\{\rho_n\}}$ be a F.f.s. (over (Ω, Σ, μ)) with the σ-Fatou property. Note that $L^0(\mu)$ is a Dedekind complete, order separable Riesz space [14, pp. 126–127]. Since $L_{\{\rho_n\}}$ is an ideal in $L^0(\mu)$ (see Lemma 2.1) it follows that $L_{\{\rho_n\}}$ is order separable [10, Theorem 18.C]. Observe that $\chi_\Omega \in L^0(\mu)^+$ satisfies $\sup_k(f \wedge k\chi_\Omega) = f$ for all $f \in L^0(\mu)$, and hence that $\{\chi_\Omega\}$ is a countable order basis of $L^0(\mu)$. Applying the last

statement in [14, Theorem 29.3] to $L := L^0(\mu)$, we see that the order separable ideal $L_{\{\rho_n\}}$ of $L^0(\mu)$ has a countable order basis. According to Proposition 2.2, F has the Fatou property. □

A function seminorm ρ in $\mathcal{M} = L^0(\mu)$ has the *Fatou property* if $\rho(u_k) \uparrow \rho(u)$ whenever $0 \leq u_k \uparrow u$ in $\mathcal{M}$ [22, Ch. 15]. An increasing fundamental sequence of function seminorms $\{\rho_n\}_{n\in\mathbb{N}}$ is said to have the Fatou property if each ρ_n, for $n \in \mathbb{N}$, has it.

THEOREM 2.4. *If $\{\rho_n\}_{n\in\mathbb{N}}$ is an increasing fundamental sequence of function seminorms with the Fatou property, then $L_{\{\rho_n\}}$ is a Fréchet function space with the Fatou property.*

PROOF. Let $(u_k)_k \subseteq L^+_{\{\rho_n\}}$ be increasing and topologically bounded. Fix $n_0 \in \mathbb{N}$. Since ρ_{n_0} is a function seminorm with the Fatou property and $\sup_k \rho_{n_0}(u_k) < \infty$, it follows from [22, Section 65, Theorem 3] that $\rho_{n_0}(u) \leq \sup_k \rho_{n_0}(u_k) < \infty$, where $u = \sup_k u_k = \lim_k u_k$ (pointwise). Hence $u \in L_{\{\rho_n\}}$. Moreover, $\rho_{n_0}(u_k) \uparrow \rho_{n_0}(u)$ as ρ_{n_0} has the Fatou property as a function seminorm. This shows that $L_{\{\rho_n\}}$ has the σ-Fatou property. Thus, Corollary 2.3 and [3, Theorem 13.9] guarantee that $L_{\{\rho_n\}}$ is complete, that is, a F.f.s. with the Fatou property. □

Recall that the *Lorentz function seminorm* ρ_L associated with any function seminorm ρ in $\mathcal{M} = L^0(\mu)$ is defined by

$$\rho_L(u) := \inf\left\{\lim_k \rho(u_k) \,\Big|\, u_k \in \mathcal{M}^+, \, u_k \uparrow u\right\} \quad \forall u \in \mathcal{M}^+; \tag{2.5}$$

it is the largest function seminorm with the Fatou property that is majorized by ρ [22, Ch. 15, Section 66]. The following property of ρ_L is established in the proof of Theorem 2 in [22, pp. 450–451].

FACT 1. *Given $u \in \mathcal{M}^+$, there exists a sequence $(u_k)_k \subseteq \mathcal{M}^+$ satisfying $u_k \uparrow u$ and $\rho(u_k) \uparrow \rho_L(u)$.*

LEMMA 2.5. *Let $\{\rho_n\}_{n\in\mathbb{N}}$ be an increasing sequence of function seminorms in $L^0(\mu)$. Then also $\{(\rho_n)_L\}_{n\in\mathbb{N}}$ is an increasing sequence of function seminorms. Moreover, $\{\rho_n\}_{n\in\mathbb{N}}$ is fundamental if and only if $\{(\rho_n)_L\}_{n\in\mathbb{N}}$ is fundamental.*

PROOF. The first statement follows from (2.5).

Concerning the second statement, suppose that $\{(\rho_n)_L\}$ is fundamental. Then it follows from $(\rho_n)_L \leq \rho_n$ that $\{\rho_n\}_{n\in\mathbb{N}}$ is also fundamental.

Conversely, assume that $\{\rho_n\}$ is fundamental. Suppose that there exists $0 < u \in L^0(\mu)^+$ satisfying $(\rho_n)_L(u) = 0$ for all $n \in \mathbb{N}$. Now, there is some $r > 0$ and a subset $A \in \Sigma$ of $\{w \in \Omega \mid u(w) > r\} \in \Sigma$ such that $0 < \mu(A) < \infty$. Since $r\chi_A \leq u$, we have $(\rho_n)_L(\chi_A) = 0$ for all $n \in \mathbb{N}$. We proceed to construct a decreasing sequence $(A_n)_n \subseteq \Sigma$ satisfying $A_n \subseteq A$ with $\mu(A_n) > \frac{1}{2}\mu(A)$ and $\rho_n(\chi_{A_n}) = 0$ for all $n \in \mathbb{N}$.

For each $n \geq 2$, define $\alpha_n := (1 + 2^n)/(2 + 2^n)$ and $\beta_n := \frac{1}{2} + 2^{-n}$. Observe that $\frac{1}{2} < \alpha_n < \alpha_{n+1} < 1$ with $\alpha_n \uparrow 1$ and that $\frac{1}{2} < \beta_{n+1} < \beta_n < 1$ with $\beta_n \downarrow \frac{1}{2}$. Moreover, $\alpha_n \beta_n = \beta_{n+1} > \frac{1}{2}$ for all $n \geq 2$.

Since $(\rho_1)_L(\chi_A) = 0$, Fact 1 ensures for us the existence of a sequence $(u_k)_k \subseteq \mathcal{M}^+$ with $u_k \uparrow \chi_A$ and $\rho_1(u_k) = 0$ for all $k \in \mathbb{N}$. Define $B_k := \{w \in \Omega \mid u_k(w) > \frac{1}{2}\}$ for all $k \in \mathbb{N}$, in which case $B_k \uparrow A$. Accordingly, $\mu(B_k) \uparrow \mu(A)$ and so there exists k_0 such that $\mu(B_{k_0}) > \beta_2 \mu(A)$. Moreover, from $\frac{1}{2} \chi_{B_{k_0}} \leq u_{k_0} \chi_{B_{k_0}} \leq u_{k_0}$, we conclude that $\rho_1(\chi_{B_{k_0}}) = 0$. Set $A_1 := B_{k_0} \subseteq A$ so that $\rho_1(\chi_{A_1}) = 0$ and $\mu(A_1) > \beta_2 \mu(A)$.

Observe that $(\rho_2)_L(\chi_{A_1}) \leq (\rho_2)_L(\chi_A) = 0$. Repeat the process with ρ_1 and χ_A replaced by ρ_2 and χ_{A_1}, to produce $A_2 \subseteq A_1$ such that $\rho_2(\chi_{A_2}) = 0$ and $\mu(A_2) > \alpha_2 \mu(A_1)$, that is, $\mu(A_2) > \alpha_2 \beta_2 \mu(A) = \beta_3 \mu(A)$.

Note that $(\rho_3)_L(\chi_{A_2}) \leq (\rho_3)_L(\chi_A) = 0$. Again repeat the process for ρ_3 and χ_{A_2} to produce $A_3 \subseteq A_2$ such that $\rho_3(\chi_{A_3}) = 0$ and $\mu(A_3) > \alpha_3 \mu(A_2)$, that is, $\mu(A_3) > \alpha_3 \beta_3 \mu(A) = \beta_4 \mu(A)$.

Proceed inductively so that at stage n one produces $A_n \subseteq A_{n-1}$ such that $\rho_n(\chi_{A_n}) = 0$ and $\mu(A_n) > \alpha_n \mu(A_{n-1})$, that is, $\mu(A_n) > \alpha_n \beta_n \mu(A) = \beta_{n+1} \mu(A) > \frac{1}{2} \mu(A)$. So the sequence of sets $(A_n)_n$ with the properties claimed above exists.

Finally, define $B := \bigcap_{n=1}^{\infty} A_n$. Then

$$\mu(B) = \lim_{n \to \infty} \mu(A_n) \geq \tfrac{1}{2} \mu(A) > 0.$$

But $\rho_n(\chi_B) \leq \rho_n(\chi_{A_n}) = 0$ for all $n \in \mathbb{N}$. Since $\{\rho_n\}$ is fundamental, it follows that $\chi_B = 0$ in $L^0(\mu)$, that is, $\mu(B) = 0$, which is a contradiction. $\qquad \square$

Let $\{\rho_n\}_{n \in \mathbb{N}}$ be any increasing fundamental sequence of function seminorms. By Lemma 2.5, the same is true of $\{(\rho_n)_L\}_{n \in \mathbb{N}}$. Hence Theorem 2.4 shows that $L_{\{(\rho_n)_L\}} = \bigcap_{n \in \mathbb{N}} L_{(\rho_n)_L}$ is a F.f.s. with the Fatou property. The inequality $(\rho_n)_L \leq \rho_n$ for all $n \in \mathbb{N}$, implies that $L_{\{\rho_n\}} \subseteq L_{\{(\rho_n)_L\}}$ with a continuous inclusion. Moreover, given $n \in \mathbb{N}$, if ρ is any function seminorm in $L^0(\mu)$ with the Fatou property such that $(\rho_n)_L \leq \rho \leq \rho_n$, then $\rho = (\rho_n)_L$ [22, Section 71, Theorems 2 and 3(d)]. Accordingly, $L_{\{(\rho_n)_L\}}$ is the *minimal* F.f.s. in $L^0(\mu)$ with the Fatou property and which contains $L_{\{\rho_n\}}$ continuously. By minimal we mean that if $\{\eta_n\}_{n \in \mathbb{N}}$ is any increasing fundamental sequence of function seminorms in $L^0(\mu)$ (for the same measure space (Ω, Σ, μ)) with the Fatou property and such that $\eta_n \leq \rho_n$ for all $n \in \mathbb{N}$, then $L_{\{(\rho_n)_L\}} \subseteq L_{\{\eta_n\}}$.

DEFINITION 2.6. $L_{\{(\rho_n)_L\}}$ is called the *Fatou completion* of $L_{\{\rho_n\}}$ and is denoted by $(L_{\{\rho_n\}})^F$.

3. Applications

As a first application of Theorem 2.4, observe that the completeness of $L_w^1(\nu)$ follows immediately since the function seminorms ρ_n given by (2.2), which induce this space, obviously have the Fatou property by [22, Section 65, Theorem 4]. For an alternative proof see [8, Theorem 2.5].

It is important to note that $(L_{\{\rho_n\}})^F$ is not necessarily obtained from $L_{\{\rho_n\}}$ in any topological sense. Indeed, if X is a Banach space and ν is an X-valued vector measure, then for the (single) function norm

$$\rho(f) := \begin{cases} \displaystyle\sup_{x^* \in B_{X^*}} \int_\Omega |f| \, d|\langle \nu, x^* \rangle| & \text{if } f \in L^1(\nu) \\ \infty & \text{if } f \in L^0(\nu) \setminus L^1(\nu), \end{cases}$$

we have $L_\rho = L^1(\nu)$ whereas $(L_\rho)^F = L_w^1(\nu)$ [7]. For certain ν, the space $L^1(\nu)$ can be a *proper* closed subspace of $L_w^1(\nu)$; [7], [12, p. 31]. So in the Banach space setting we see that $L_w^1(\nu)$ is the Fatou completion of $L^1(\nu)$. We show that the same result holds in Fréchet spaces.

THEOREM 3.1. *Let ν be any vector measure taking its values in a Fréchet space. Then the Fatou completion of $L^1(\nu)$ is precisely $L_w^1(\nu)$. Thus, $L^1(\nu)$ has the Fatou property if and only if $L^1(\nu) = L_w^1(\nu)$.*

PROOF. We use the notation of Example 1. Fix $n \in \mathbb{N}$. As noted above, ρ_n has the Fatou property and, clearly, $\rho_n \leq \widetilde{\rho}_n$ in $L^0(\mu)$. By the maximal property of the Lorentz seminorm we conclude that $\rho_n \leq (\widetilde{\rho}_n)_L$ in $L^0(\mu)$. On the other hand, let $f \in L_w^1(\nu)^+$. Choose Σ-simple functions $0 \leq s_k \uparrow f$, in which case $(s_k)_k \subseteq L^1(\nu) \subseteq L_w^1(\nu)$. By (2.5) applied to $\widetilde{\rho}_n$,

$$(\widetilde{\rho}_n)_L(f) \leq \lim_k \widetilde{\rho}_n(s_k) = \lim_k \rho_n(s_k) \leq \rho_n(f),$$

from which we can conclude that $(\widetilde{\rho}_n)_L(f) = \rho_n(f)$ for all $f \in L_w^1(\nu)$. These inequalities imply that $L_w^1(\nu) = L_{\{\rho_n\}} \subseteq L_{\{(\widetilde{\rho}_n)_L\}} = (L^1(\nu))^F$.

Let $f \in (L^1(\nu))^F$, that is, $(\widetilde{\rho}_n)_L(f) < \infty$, for all $n \in \mathbb{N}$. Fix $m \in \mathbb{N}$. By Fact 1 applied to ρ_m there exists $(u_k)_k \subseteq L^0(\mu)^+$ satisfying $u_k \uparrow f$ with $\widetilde{\rho}_m(u_k) \uparrow (\widetilde{\rho}_m)_L(f)$. Then $\sup_k \widetilde{\rho}_m(u_k) \leq (\widetilde{\rho}_m)_L(f) < \infty$. From the definition of $\widetilde{\rho}_m$, we see that $(u_k)_k \subseteq L^1(\nu) \subseteq L_w^1(\nu)$. Since ρ_m has the Fatou property, we conclude that

$$\rho_m(f) = \sup_k \rho_m(u_k) = \sup_k \widetilde{\rho}_m(u_k) < \infty.$$

But $m \in \mathbb{N}$ is arbitrary and so $f \in L_{\{\rho_n\}} = L_w^1(\nu)$. This shows that $L_w^1(\nu) = (L^1(\nu))^F$, with equality both as vector spaces and topologically.

If $L^1(\nu) = L_w^1(\nu)$, then $L^1(\nu)$ has the Fatou property. Conversely, assume that $L^1(\nu)$ has the Fatou property. It is always the case that $L^1(\nu) \subseteq L_w^1(\nu)$. Let $f \in L_w^1(\nu)^+$ and choose Σ-simple functions $0 \leq f_k \uparrow f$. As $(f_k)_k \subseteq L^1(\nu)$, it follows from (2.2) and (2.3), for each $n \in \mathbb{N}$, that

$$\widetilde{\rho}_n(f_k) = \rho_n(f_k) \leq \rho_n(f) < \infty \quad \forall k \in \mathbb{N}.$$

Hence $(f_k)_k$ is topologically bounded in $L^1(\nu)$. Since $L^1(\nu)$ has the Fatou property, it follows that $f = \sup_k f_k \in L^1(\nu)$. So $L^1(\nu) = L_w^1(\nu)$. □

For Banach spaces the above result is known [7], after observing that the second associate norm ρ'' of a function norm ρ is precisely ρ_L [22, p. 471].

The following result [8, Theorem 3.2], alluded to in the Introduction, is recorded for the sake of completeness.

THEOREM 3.2. *For any vector measure v taking values in a Fréchet space we have $(L^1_w(v))_a = L^1(v)$. In particular, $L^1_w(v)$ has a Lebesgue topology if and only if $L^1_w(v) = L^1(v)$.*

We provide further equivalent conditions for $L^1_w(v) = L^1(v)$ to hold.

A Fréchet lattice F is called a KB-*space* if every topologically bounded, increasing sequence in F^+ is convergent. If every topologically bounded, increasing sequence in F^+, with F a Fréchet lattice, has a supremum in F, then F is said to satisfy the σ-*Levi property*.

LEMMA 3.3. *For any Fréchet lattice F the following are equivalent.*

(i)   *F is a KB-space.*
(ii)   *F satisfies the σ-Levi property and has a σ-Lebesgue topology.*
(iii) *F has a σ-Lebesgue topology and the σ-Fatou property.*
(iv)   *F contains no lattice copy of c_0.*
(v)   *F is weakly sequentially complete.*

PROOF. The equivalence of (i) and (ii) is part of [9, Proposition 2.1].

To establish that (ii) implies (iii), let $\{q_n\}_{n \in \mathbb{N}}$ be a fundamental system of solid seminorms generating the topology τ of F. Let $(u_k)_k \subseteq F^+$ be an increasing and topologically bounded sequence. By the σ-Levi property of F, $u = \sup_k u_k$ exists in F^+. Since (ii) implies (i), there exists $v \in F$ with $u_k \xrightarrow{\tau} v$. By [3, Theorem 5.6(iii)], $v = \sup_k u_k$, that is, $u = v$. Fix $n \in \mathbb{N}$. The inequality $|q_n(u_k) - q_n(u)| \leq q_n(u_k - u)$, for each $k \in \mathbb{N}$, shows that $q_n(u_k) \uparrow_k q_n(u)$.

That (iii) implies (ii) is obvious.

To establish that (iv) implies (ii), observe that the completeness of F ensures the monotone completeness property (see [3, p. 45] for the definition). Then [9, Theorem 2.5] implies that F satisfies the σ-Levi property and has a σ-Lebesgue topology.

To see that (ii) implies (iv), again note that the completeness of F ensures the monotone completeness property. Then [9, Proposition 2.2 and Theorem 2.5] imply that F cannot contain a lattice copy of c_0.

That (v) implies (iv) is clear. For if F is weakly sequentially complete, then it cannot contain an isomorphic lattice copy of the Banach lattice c_0 [11, 19].

Finally, that (iv) implies (v) follows from [21, Theorem 1].   □

For Banach spaces the following result occurs in [7].

PROPOSITION 3.4. *Let v be any vector measure taking values in a Fréchet space. Then the following statements are equivalent.*

(i) $L_w^1(\nu) = L^1(\nu)$.

(ii) $L_w^1(\nu)$ *has a* σ-*Lebesgue topology.*

(iii) $L^1(\nu)$ *has the* σ-*Fatou property.*

(iv) $L_w^1(\nu)$ *is a* KB-*space.*

(v) $L^1(\nu)$ *is a* KB-*space.*

(vi) $L_w^1(\nu)$ *contains no lattice copy of* c_0.

(vii) $L^1(\nu)$ *contains no lattice copy of* c_0.

(viii) $L_w^1(\nu)$ *is weakly sequentially complete.*

(ix) $L^1(\nu)$ *is weakly sequentially complete.*

PROOF. The equivalence of (i) and (ii) occurs in the statement of Theorem 3.2 and the equivalence of (i) and (iii) occurs in the statement of Theorem 3.1.

The equivalences of (ii) and (iv), of (iv) and (vii), and of (vii) and (viii) are immediate from Lemma 3.3, since the space $L_w^1(\nu)$ always has the Fatou property.

Finally, the equivalences of (iii) and (v), of (v) and (vii), and of (vii) and (ix) are also immediate from Lemma 3.3 since the space $L^1(\nu)$ always has a Lebesgue topology. □

According to [12, p. 31] and [13, Theorem 5.1], condition (i) of Proposition 3.4 is satisfied whenever the Fréchet space in which ν takes its values does not contain an isomorphic copy of c_0.

An examination of the proof of Proposition 3.4 shows that it can be adapted to establish the following more general result. Recall that $L_{\{\rho_n\}}$ is *super order dense* in $(L_{\{\rho_n\}})^F$ means that for every element $0 \leq u \in (L_{\{\rho_n\}})^F$ there exists a sequence $(u_k)_k \subseteq L_{\{\rho_n\}}^+$ satisfying $u_k \uparrow u$ in $(L_{\{\rho_n\}})^F$ [3, Definition 1.9].

PROPOSITION 3.5. *Let* $L_{\{\rho_n\}}$ *be a Fréchet function space generated by the increasing fundamental sequence of function seminorms* $\{\rho_n\}_{n\in\mathbb{N}}$. *Assume, for each* $m \in \mathbb{N}$, *that the restriction of* $(\rho_m)_L$ *to* $L_{\{\rho_n\}}$ *coincides with* ρ_m, *that* $((L_{\{\rho_n\}})^F)_a = L_{\{\rho_n\}}$ *and that* $L_{\{\rho_n\}}$ *is super order dense in* $(L_{\{\rho_n\}})^F$. *Then the following assertions are equivalent.*

(i) $L_{\{\rho_n\}} = (L_{\{\rho_n\}})^F$.

(ii) $(L_{\{\rho_n\}})^F$ *has a* σ-*Lebesgue topology.*

(iii) $L_{\{\rho_n\}}$ *has the* σ-*Fatou property.*

(iv) $(L_{\{\rho_n\}})^F$ *is a* KB-*space.*

(v) $L_{\{\rho_n\}}$ *is a* KB-*space.*

(vi) $(L_{\{\rho_n\}})^F$ *contains no lattice copy of* c_0.

(vii) $L_{\{\rho_n\}}$ *contains no lattice copy of* c_0.

(viii) $(L_{\{\rho_n\}})^F$ *is weakly sequentially complete.*

(ix) $L_{\{\rho_n\}}$ *is weakly sequentially complete.*

We conclude with an application to function spaces induced by a Köthe matrix which was stated, without proof, in [4, p. 94].

COROLLARY 3.6. *In the notation of Example 2, let* ρ *be a determining function norm with the Fatou property. Then* $L_{\{\rho_n\}}$ *is complete and* $\{\rho_n\}_{n\in\mathbb{N}}$ *is also determining.*

PROOF. Since ρ has the Fatou property, so does each ρ_n, for $n \in \mathbb{N}$, that is, $\{\rho_n\}_{n \in \mathbb{N}}$ has the Fatou property. By Theorem 2.4, $L_{\{\rho_n\}}$ is complete.

Since ρ is determining, choose $u \in L_\rho$ that is positive μ-a.e. Then the sets $A_n := \{w \in \Omega \mid u(w) \geq n^{-1}\}$, for all $n \in \mathbb{N}$, satisfy $A_n \uparrow \Omega$. Let $A \in \Sigma$ satisfy $\mu(A) > 0$. Then $A \cap A_n \uparrow A$ and so there exists $m \in \mathbb{N}$ such that $\mu(A_m \cap A) > 0$. Moreover, $\chi_{A \cap A_m} \in L_\rho$ because $\chi_{A_m \cap A} \leq \chi_{A_m} \leq m \cdot u$ with $m \cdot u \in L_\rho$. So ρ is saturated [22, Ch. 15, Section 67]. It follows from [18, Lemma 1.1] that there is an increasing sequence $(\Omega(k))_k \subseteq \Sigma$ with $\bigcup_{k=1}^\infty \Omega(k) = \Omega$ (μ-a.e.) such that $(\chi_{\Omega(k)})_k \subseteq L_\rho$ and, for each $(k, n) \in \mathbb{N}^2$, there exists $\beta(k, n) > 0$ satisfying $a_n \chi_{\Omega(k)} \leq \beta(k, n) \chi_{\Omega(k)}$ for all $(k, n) \in \mathbb{N}^2$, pointwise on Ω. All these inequalities and the definition of ρ_n imply that $\rho_n(\chi_{\Omega(k)}) < \infty$ for all $k, n \in \mathbb{N}$, that is, $(\chi_{\Omega(k)})_k \subseteq L_{\{\rho_n\}}$. Since $L_{\{\rho_n\}}$ is metrizable, there exist positive numbers λ_k, for all $k \in \mathbb{N}$, such that $(\lambda_k \chi_{\Omega(k)})_k$ is a bounded sequence in $L_{\{\rho_n\}}$. Moreover, we can choose $0 < \alpha_k \leq \lambda_k$ with $\sum_k \alpha_k < \infty$. Then the series $\sum_k \alpha_k \chi_{\Omega(k)} = v$ is absolutely summable in $L_{\{\rho_n\}}$. Completeness of $L_{\{\rho_n\}}$ ensures that $v \in L_{\{\rho_n\}}^+$, with $v > 0$ (μ-a.e.) on Ω. So $\{\rho_n\}_{n \in \mathbb{N}}$ is determining. □

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