ABSTRACT RIEMANN INTEGRABILITY AND MEASURABILITY

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Abstract. We prove that the spectral sets of any positive abstract Riemann integrable function are measurable but (at most) a countable amount of them. In addition, the integral of such a function can be computed as an improper classical Riemann integral of the measures of its spectral sets under some weak continuity conditions which in fact characterize the integral representation.

Keywords: finitely additive integration, localized convergence, integral representation, weak continuity conditions, horizontal integration

1. Introduction

Given a Loomis system (X, B, I) without any continuity conditions for the basic integral I, Díaz Carrillo and Muñoz Rivas introduced in [13] a general extension process using a suitable *localized convergence*, thus obtaining the class $R_1(B, I)$ of *abstract Riemann integrable functions.* This extension process subsumes μ-Riemann [18], Riemann-Loomis [22], and Dunford-Schwartz [15] integrations.

The classical problem of the relation between the integrability of a function f and the measurability of its spectral sets, $f^{-1}(]r, +\infty[)$, for $r \in \mathbb{R}$, has been widely studied by several authors in those measure-theoretic contexts (see [23] and [24], for instance).

We now carry on this discussion in our functional setting and prove that every abstract integrable function is *quasi-measurable*, that is, all its spectral sets are measurable but at most a countable amount of them.

Moreover, under some weak continuity conditions, we are able to obtain a formula that allows to reconstruct the integral of an integrable function through the measures of its spectral sets.

Our results generalize the previous corresponding ones from the measure-theoretic point of view to the functional context in which the integral need not even be induced by any finitely additive measure.

Section 2 is devoted to introducing notation and preliminary results to make the paper self-contained. In Section 3, the additional conditions we need are defined and their interactions with the localized convergence are presented. The measurability of the spectral sets of an integrable function is discussed in Section 4. The last section shows that, under stonian and lower and upper continuity conditions, it is possible to obtain the integral of $f \in R_1(B, I)$ by adding the measures of its spectral sets. In fact, these conditions characterize integral representation in this new situation (see [9], [17], [19] and [21]).

2. Preliminaries

For $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, where $\mathbb{R}$ is the real line, we extend the usual addition in $\mathbb{R}$ to $\overline{\mathbb{R}}$ by the conventions $r + s := 0$ if $r = -s \in \{-\infty, +\infty\}$ and $r - s := r + (-s)$.

We also set $a \vee b := \max\{a, b\}$, $a \wedge b := \min\{a, b\}$, $a^+ := a \vee 0$ and $a^- := -(a \wedge 0)$.

Given an arbitrary nonempty set X, let $\overline{\mathbb{R}}^X$ consist of all functions defined on X with values in $\overline{\mathbb{R}}$. All operations and relations in $\overline{\mathbb{R}}^X$ are defined pointwise, with the conventions $\inf \emptyset := +\infty$ and $\sup \emptyset := -\infty$.

A functional $T\colon \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ will be called *subadditive* if $T(f + g) \leqslant T(f) + T(g)$ for all $f, g \in \overline{\mathbb{R}}^X$ unless $T(f) = -T(g) = +\infty$ and $T(f) = -T(g) = -\infty$. The notion of a *superadditive* functional is introduced in the completely dual way.

A triple (X, B, I) is called a *Loomis system* if X is a nonempty set, $B \subseteq \mathbb{R}^X$ is a vector lattice of real functions and $I\colon B \to \mathbb{R}$ is a positive (i.e., $I(h) \geqslant 0$ for all $h \in B$ with $h \geqslant 0$) linear functional. We set $+B := \{h \in B\colon\ h \geqslant 0\}$.

Given (X, Ω, μ) with μ a finitely additive measure and Ω a ring, we call (X, B_Ω, I_μ) the *induced Loomis system*, where B_Ω is the vector lattice of μ-simple functions,

$$B_\Omega := \Big\{ h \in \mathbb{R}^X :\ h = \sum_{i=1}^{n} a_i \chi_{A_i},\ a_i \in \mathbb{R},\ A_i \in \Omega,\ \mu([h \neq 0]) < +\infty \Big\},$$

and I_μ is its canonical elementary integral given by

$$I_\mu(h) := \sum_{i=1}^{n} a_i \mu(A_i),\ \forall h \in B_\Omega.$$

2.1. Proper Riemann integration. Following Loomis in [22] we extend the elementary functional I on B to the functionals I^+ and I^- (oscillation integrals) over the class $\overline{\mathbb{R}}^X$ of extended real valued functions,

$$I^-(f) := \inf\{I(h)\colon\ h \in B, h \geqslant f\} \text{ (upper functional of } I),$$
$$I^+(f) := \sup\{I(h)\colon\ h \in B, h \leqslant f\} \text{ (lower functional of } I),$$

which evidently verify $I^-(f) = -I^+(-f)$, are positively homogeneous and monotone, I^- is subadditive and I^+ is superadditive. We also consider the class of the *properly Riemann integrable functions*

$$R_{\text{prop}}(B, I) = \{f \in \mathbb{R}^X :\ I^+(f) = I^-(f) \in \mathbb{R}\},$$

which is a vector lattice where the functional $I := I^+ = I^-$ is linear and positive, i.e., it is an integral which extends the initial I.

For every $f \in \mathbb{R}^X$, the following statements are equivalent:

(i) $f \in R_{\text{prop}}(B, I)$.

(ii) $\forall \varepsilon > 0\ \exists h, g \in B$ such that $h \leqslant f \leqslant g$ and $I(g - h) < \varepsilon$.

(iii) $\forall \varepsilon > 0\ \exists h \in B$ such that $I^-(|f - h|) < \varepsilon$.

Hence, $R_{\text{prop}}(B, I)$ is the closure of the vector lattice B with respect to the integral seminorm $I^-(|\cdot|)$ (see [4] and [27]).

Note that the particular case of a Loomis system induced by the semiring $\{[a, b[\colon -\infty < a \leqslant b < +\infty\}$ and the finitely additive measure $\mu([a, b[) = b - a$, leads to the classical Riemann integrable functions (as in [4] and [18, p. 216]).

2.2. Abstract Riemann integration. Since for proper Riemann integration there are no satisfactory Lebesgue convergence type theorems to make a consistent integration theory, Díaz Carrillo and Muñoz Rivas introduced in [13] the class $R_1(B, I)$ of the *abstract Riemann integrable functions* as

$$R_1(B, I) := \{f \in \overline{\mathbb{R}}^X :\ \exists\{h_n\} \text{ in } B,\ I\text{-Cauchy; } \{h_n\} \longrightarrow f(I^-)\}$$

where $\{h_n\}$ I-Cauchy means that $I(|h_n - h_m|) \to 0$, for $n, m \to +\infty$, and $\{h_n\} \longrightarrow f(I^-)$ means that $\{I^-(|f_n - f| \wedge h)\} \to 0$, $\forall h \in +B$. This notion of *local I-convergence* allowed them to obtain convergence theorems for $R_1(B, I)$ (see theorems 2.3, 2.4 and 2.7 in [13]).

Moreover, for each $f \in R_1(B, I)$ we set $I(f) := \lim\limits_{n \to +\infty} I(h_n)$ for any I-Cauchy sequence $\{h_n\}$ in B such that $\{h_n\} \longrightarrow f(I^-)$.

The definition does not depend on the particular sequence $\{h_n\}$ and no confusion arises with this notation since $R_{\text{prop}}(B, I) \subseteq R_1(B, I)$ with coinciding integrals I.

There are several useful characterizations for the class $R_1(B,I)$. On the one hand, we have

$$R_1(B,I) = \{f \in \overline{\mathbb{R}}^X : \; I^+(|f|) < +\infty, f^{\pm} \wedge h \in R_{\mathrm{prop}}, \; \forall h \in +B\},$$

which, in particular, says that

$$+R_1(B,I) = \{f \in \overline{\mathbb{R}}^X : \; I^+(f) < +\infty, f \wedge h \in R_{\mathrm{prop}}, \; \forall h \in +B\};$$

furthermore, $I(f) = I^+(f)$ for all $f \in +R_1(B,I)$.

On the other hand, given $f \in \overline{\mathbb{R}}^X$, the *localized functional* I_l^- in the sense of Schäfke (see [28]) is defined as

$$I_l^-(f) := \sup\{I^-(f \wedge h) \colon \; h \in +B\}.$$

It is easily verified that I_l^- is positively homogeneous, monotone and subadditive. Moreover, $(I_l^-)_l = I_l^-$, $I^+ \leqslant I_l^- \leqslant I^-$ and $I_l^-(f) = I^-(f)$ if $f \leqslant h$ for some $h \in +B$.

Theorem 2 in [14] guarantees that $R_1(B,I)$ is the closure of B in $\overline{\mathbb{R}}^X$ with respect to the integral seminorm $I_l^-(|\cdot|)$, and $I_l^-(f) = I(f)$, for all $f \in R_1(B,I)$ (I is the only I_l^--continuous extension of I from B to $R_1(B,I)$).

It is also possible to provide another description of the class $R_1(B,I)$ by means of the *upper and lower essential functionals* due to Anger and Portenier (see [3]),

$$I^\bullet(f) := \inf_{v \in +B} \sup_{u \in +B} I^-((f \wedge u) \vee v), \quad \forall f \in \overline{\mathbb{R}}^X,$$
$$I_\bullet(f) := -I^\bullet(-f), \; \forall f \in \overline{\mathbb{R}}^X.$$

Evidently $I^\bullet$ coincides with I_l^- on the positive functions and therefore, theorems 4.4 in [3] and 5 in [11] guarantee that $R_1(B,I)$ can be represented in the following way:

$$R_1(B,I) = \{f \in \overline{\mathbb{R}}^X : \; I^\bullet(f) = I_\bullet(f) \in \mathbb{R}\}.$$

Finally, we define the oscillation integrals for $R_1(B,I)$: $\forall f \in \overline{\mathbb{R}}^X$,

$$I^*(f) := \inf\{I(g) \colon \; g \in R_1(B,I), g \geqslant f\}, \tag{1}$$
$$I_*(f) := \sup\{I(g) \colon \; g \in R_1(B,I), g \leqslant f\}, \tag{2}$$

which verify that $I^*(f) = -I_*(-f)$, both I^* and I_* are positively homogeneous and increasing, I^* is subadditive and I_* is superadditive, and both extend I on $R_1(B, I)$.

In fact, in [2, Cor. 3.9] it is proved that the extension process for the initial Loomis system $(X, R_1(B,I), I)$ through oscillation integrals I_* and I^* is closed; i.e.

$$R_1(B,I) = \{f \in \overline{\mathbb{R}}^X : I_*(f) = I^*(f) \in \mathbb{R}\}.$$

Since $B \subseteq R_1(B,I)$ we have that $I^+ \leqslant I_* \leqslant I^* \leqslant I^-$. Moreover, if $f \geqslant 0$ and there exists $h \in B$ such that $f \leqslant h$, then $I^-(f) = I^*(f)$.

3. Additional conditions

It is worth pointing out that abstract Riemann integration coincides with classical Daniell integration [8] when monotone continuity is assumed, but in order to obtain the previous integral extension process, we have not used any additional condition on the initial vector lattice B, nor on the linear functional I defined on it, and hence it allows to subsume most of finitely additive integration theories.

Nevertheless, weak continuity conditions (on the Loomis system) need to be introduced if we want to obtain the representation result we desire.

Definition 3.1. The vector lattice B is *stonian* if $f \wedge 1 \in B$, $\forall f \in B$ (equivalently, $f \wedge r \in B$, $\forall f \in B$, $\forall r \in \mathbb{R}$).

Definition 3.2. A Loomis system (X, B, I) is called C_∞ or *upper continuous* if

$$\lim_{r \to +\infty} I^*(f - f \wedge r) = 0, \ \forall f \in +B,$$

and it is called C_0 or *lower continuous* if

$$\lim_{r \to 0} I^*(f \wedge r) = 0, \ \forall f \in +B.$$

The stonian condition on B is hereditary for the class $R_1(B,I)$, as is said in

Lemma 3.3. *If B is stonian, then $R_1(B,I)$ is stonian too.*

Proof. For $f \in R_1(B,I)$, there exists an I-Cauchy sequence $\{h_n\}$ in B, and $\{h_n\} \to f(I^-)$. Thus $\{h_n \wedge 1\} \to f \wedge 1(I^-)$, and $\{h_n \wedge 1\}$ is I-Cauchy, because the inequality $|h_n \wedge 1 - h_m \wedge 1| \leqslant |h_n - h_m|$ is valid for all $n, m \in \mathbb{N}$. Therefore, $f \wedge 1 \in R_1(B,I)$; i.e., $R_1(B,I)$ is stonian. $\square$

We now study conditions under which C_0 and C_∞ are hereditary from B to $R_1(B,I)$. We call attention to the fact that, while for C_0 the stonian condition is necessary for the initial vector lattice B, for C_∞ it is possible to obtain the heritage directly for the class $R_1(B,I)$.

Lemma 3.4. *For an arbitrary Loomis system* (X, B, I) *we have:*

(i) *If* B *is* C_0 *and stonian, then* $R_1(B, I)$ *is* C_0.

(ii) *If* B *is* C_∞, *then* $R_1(B, I)$ *is* C_∞.

P r o o f. (i) Let $f \in +R_1(B, I)$ and $f_n := f \wedge 1/n$, $\forall n \in \mathbb{N}$. Since B is stonian so is $R_1(B, I)$. Therefore, $f_n \in R_1(B, I)$, $\forall n \in \mathbb{N}$. Clearly, $\{f_n\}$ converges uniformly to 0 on X and, since B is C_0, it is easy to check that $\{f_n\} \to 0(I^-)$. Moreover, $|f_n| \leqslant f \in R_1(B, I)$ for all $n \in \mathbb{N}$ and hence the Dominated Convergence Theorem (see [13, theorem 2.7]), guarantees that $I(f_n) = I(f \wedge 1/n) \to I(0) = 0$.

Thus, given $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $I(f_m) < \varepsilon$, and hence, given $0 < r < 1/m$, we have $I(f \wedge r) \leqslant I(f_M) \leqslant \varepsilon$ which proves that $R_1(B, I)$ is C_0.

(ii) Let us consider $f \in +R_1(B, I)$ and $\varepsilon > 0$. Since $I^+(f) \leqslant I(f) < +\infty$, there exists $h \in +B$ with $h \leqslant f$ such that $I(f) - \frac{1}{4}\varepsilon < I(h)$ and so $I(f - h) < \frac{1}{4}\varepsilon$.

Moreover, since B is C_∞, there exists $s > 0$ such that $I^*(h - h \wedge r) < \frac{1}{2}\varepsilon$, for all $r \geqslant s$. Then, given $r \geqslant s$, we have

$$I^*(f - f \wedge r) = I^*(|f - f \wedge r|) \leqslant I(f - h) + I^*(h - h \wedge r) + I^*(|h \wedge r - f \wedge r|)$$

and using the Birkhoff inequalities we deduce that

$$I^*(f - f \wedge r) \leqslant I(f - h) + I^*(h - h \wedge r) + I(f - h) < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.$$

□

Condition C_∞ on $R_1(B, I)$ means that it is possible to approximate the elements in $R_1(B, I)$ through their I^--local upper truncations with constants, i.e, if $R_1(B, I)$ is C_∞, then $\{f \wedge r_n\} \to f(I^-)$ for all $\{r_n\} \to +\infty$, with $\{r_n\} \subset \mathbb{R}^+$ and for all $f \in R_1(B, I)$. In fact, under the additional condition that $R_1(B, I)$ is stonian, the converse is true by the Dominated Convergence Theorem, [13, Th. 2.7]. Analogously, C_0 condition on $R_1(B, I)$ says that $\{f \wedge r_n\} \to 0(I^-)$ for all $\{r_n\} \to 0$, with $\{r_n\} \subseteq \mathbb{R}^+$ and for all $f \in R_1(B, I)$.

Some examples of Loomis systems which are C_∞ but are not C_0 and viceversa can be found in [19].

4. Measurability for abstract Riemann integration

First of all, we must make precise the meaning of measurability in this functional context: Given an arbitrary Loomis system (X, B, I) we consider the finite measure space (X, Ω, μ) induced by $R_1(B, I)$, that is,

$$\Omega := \{A \subseteq X\colon\ \chi_A \in R_1(B, I)\} \quad \text{and} \quad \mu(A) := I(\chi_A),\ \forall A \in \Omega.$$

Therefore, the measurable sets will be those sets whose characteristic function is abstract Riemann integrable, and their measure will be the integral of this function.

In addition, the spectral sets of a function are defined in the following way:

Definition 4.1. Given $f \in \overline{\mathbb{R}}^X$, the *spectral sets* of f are the sets $[f \geqslant r] := \{x \in X\colon\ f(x) \geqslant r\}$ and $[f > r] := \{x \in X\colon\ f(x) > r\}$, with $r \in \mathbb{R}$.

With these definitions, the classical notion of measurability can be formulated in this functional context as those functions whose all spectral sets are measurable. Unfortunately, this kind of measurability does not have a good behavior with respect to Riemann-type integration since there are integrable functions (even classical Riemann integrable functions) such that they are not measurable in this sense (see [16]).

Nonetheless, employing this slightly modified version of the classical notion of measurability we obtain some results in this direction:

Definition 4.2. Given a ring $\mathscr{R}$ in X and $f \in \overline{\mathbb{R}}^X$, we say that f is quasi-$\mathscr{R}$-measurable if there is a countable set N in $\mathbb{R}$ such that $[f \geqslant r], [f \geqslant r] \in \mathscr{R}$ for all $r \in \mathbb{R} \setminus N$.

With this new terminology (inspired by Maharam [24]), Ridder proved in [26] that integrability and quasi-measurability are equivalent properties for the particular case of proper Riemann integration with respect to a finite finitely additive measure. In her paper, Maharam introduced an improper integration theory with respect to a finitely additive measure and extended this result to this new situation (see [24, Th. 5.1]). Finally, Luxemburg proved in [23, Th. 4.10] that integrable functions are quasi-measurable for Dunford-Schwartz integration.

Using the previously mentioned description of abstract Riemann integration as the essential integration of Anger and Portenier and some techniques employed before in [5], we have now proved that, under the stonian condition, every integrable function is quasi-measurable, that is:

Theorem 4.3. *If B is stonian, then every abstract Riemann integrable function is quasi-Ω-measurable.*

P r o o f. Let $f \in R_1(B, I)$ and let $r_0 < r_1 < r_2 < \ldots < r_n$ be real numbers. For each $k \in \{1, \ldots, n\}$, set $f_k := (f \wedge r_k - f \wedge r_{k-1})/(r_k - r_{k-1})$, $A_k := [f \geqslant r_k]$ and $B_k := [f > r_k]$. Since $R_1(B, I)$ is stonian we have $f_k \in R_1(B, I), \forall k \in \{1, \ldots, n\}$. Moreover, $\chi_{A_k} \leqslant f_k \leqslant \chi_{A_{k-1}}$ and $\chi_{B_k} \leqslant f_k \leqslant \chi_{B_{k-1}}$, $\forall k \in \{1, \ldots, n\}$. Therefore we can restrict ourselves to proving the measurability of the A_k's except for a countable amount of them (the reasoning is analogous for the B_k's).

From $\chi_{A_k} \leqslant f_k \leqslant \chi_{A_{k-1}}$ we deduce that

$$\begin{aligned}
&I^\bullet(\chi_{A_k}),\ I_\bullet(\chi_{A_k}) \in \mathbb{R},\ \forall k \in \{1, \ldots, n\},\\
&I^\bullet(\chi_{A_k}) - I_\bullet(\chi_{A_k}) \leqslant I(f_k) - I(f_{k+1}),\ \forall k \in \{1, \ldots, n-1\},\\
&I^\bullet(\chi_{A_n}) - I_\bullet(\chi_{A_n}) \leqslant I(f_n).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sum_{k=1}^{n}[I^\bullet(\chi_{A_k}) - I_\bullet(\chi_{A_k})] &\leqslant \sum_{k=1}^{n-1}[I^\bullet(\chi_{A_k}) - I_\bullet(\chi_{A_k})] + [I^\bullet(\chi_{A_n}) - I_\bullet(\chi_{A_n})]\\
&\leqslant \sum_{k=1}^{n-1}[I(f_k) - I(f_{k+1})] + I(f_n) = I(f_1) - I(f_n) + I(f_n) = I(f_1).
\end{aligned}$$

Thus, we have obtained the inequality

$$\sum_{k=1}^{n}[I^\bullet(\chi_{A_k}) - I_\bullet(\chi_{A_k})] \leqslant I(f_1) \tag{3}$$

Let N be the set of all real numbers r such that $\chi_{A_r} \notin R_1(B, I)$, where $A_r := [f \geqslant r]$. It is clear that N can be written as

$$N = \bigcup_{a \in \mathbb{Q},\ \varepsilon \in \mathbb{Q}^+} \{r > a\colon\ I^\bullet(\chi_{A_r}) - I_\bullet(\chi_{A_r}) \geqslant \varepsilon\}.$$

Assume that for some $a \in \mathbb{Q}$ and some $\varepsilon \in \mathbb{Q}^+$ there exists an infinite number of $r \in \mathbb{R}$ with $r > a$ and $I^\bullet(\chi_{A_r}) - I_\bullet(\chi_{A_r}) \geqslant \varepsilon$. Then for each $n \in \mathbb{N}$ we can select real numbers $r_2 < \ldots < r_n$ among them and therefore, applying inequality (3) (with $r_0 = a - 1$ and $r_1 = a$), we conclude that

$$I(f_1) \geqslant \sum_{k=2}^{n}[I^\bullet(\chi_{A_k}h) - I_\bullet(\chi_{A_k})] \geqslant \sum_{k=2}^{n} \varepsilon = (n-1)\varepsilon,\ \forall n \in \mathbb{N}.$$

Letting $n \to +\infty$, we obtain that $I(f_1) = +\infty$, but this contradicts the fact that $f_1 \in R_1(B, I)$.

We have proved that for each $a \in \mathbb{Q}$ and for each $\varepsilon \in \mathbb{Q}^+$ there exists a finite number of real numbers r such that $r > a$ and $I^\bullet(\chi_{A_r}) - I_\bullet(\chi_{A_r}) \geqslant \varepsilon$. Therefore N is countable and $\chi_{A_r} \in R_1(B, I)$ for all $r \in \mathbb{R} \setminus N$. $\square$

Theorem 4.3 says that if B is stonian and $f \in R_1(B, I)$, then the set $T(f) := \{t \in \mathbb{R}\colon\ [f \geqslant t] \in \Omega\}$ is co-countable and hence dense. The converse of this theorem is still an open question, but at least with the aid of the Dominated Convergence Theorem for the class $R_1(B, I)$ ([13, th. 2.7]) we have proved that, in this case, both the properties are equivalent for $T(f)$, the same as it occurs for Maharam integration theory with respect to a finitely additive measure (see [24, cor. 5.2]).

Proposition 4.4. *Let $f \in R_1(B, I)$ and $T(f) = \{t \in \mathbb{R}\colon\ [f \geqslant t] \in \Omega\}$. The following statements are equivalent:*

(i) *$T(f)$ is co-countable.*

(ii) *$T(f)$ is dense in $\mathbb{R}$.*

Proof. (i) $\Rightarrow$ (ii) is evident. Therefore assume that $T(f)$ is dense and let us see that, in fact, it is co-countable.

Consider the function $F\colon\ \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ given by $F(t) := I_*(\chi_{[f \geqslant t]})$ for all $t \in \mathbb{R}$, which is, clearly, decreasing.

Let $C(F)$ be the set of all continuity points of F. It is well known that, because of the monotony of F, $C(F)$ is co-countable (see, for example, [6]), so it is enough to show that $C(F) \subseteq T(f)$, and then $T(f)$ will be a co-countable set.

Let $t_0 \in C(F)$. Since $T(f)$ is dense in $\mathbb{R}$ we are able to choose an increasing sequence $\{t_n\} \to t_0$, with $t_n \in T(f)$, for all $n \in \mathbb{N}$ and hence $\{F(t_n)\} \to F(t_0)$, i.e., $\{I_*(\chi_{[f \geqslant t_n]})\} \to I_*(\chi_{[f \geqslant t_0]})$.

Setting $g_n := \chi_{[f \geqslant t_n]}$ for all $n \in \mathbb{N} \cup \{0\}$, we have that the sequence $\{g_n\}$ is decreasing, $I_*(g_n) \to I_*(g_0)$ and $g_n \in R_1(B, I)$ for all $n \in \mathbb{N}$. Moreover, $|g_n| \leqslant g_1 \in R_1(B, I)$, that is, $\{g_n\}$ is dominated by an element belonging to $R_1(B, I)$.

Thus, for any $h \in +B$, we have that $I^-(|g_n - g_0| \wedge h) = I^*(|g_n - g_0| \wedge h) \leqslant I^*(|g_n - g_0|) = I^*(g_n - g_0) \leqslant I^*(g_n) - I_*(g_0) = I(g_n) - I_*(g_0) \to 0$, which says that $\{g_n\} \to g_0(I^-)$ and hence, applying the Dominated Convergence Theorem for the class $R_1(B, I)$, we deduce that $g_0 \in R_1(B, I)$, that is, $t_0 \in T(f)$. $\square$

Let us now see some first nice consequences from Theorem 4.3. To this end we define μ^* and μ_* to be, respectively, the *outer* and the *inner measures* of μ, i.e.,

$$\mu^*(A) = \inf\{\mu(C)\colon\ A \subseteq C,\ C \in \Omega\}$$
$$\mu_*(A) = \sup\{\mu(D)\colon\ D \subseteq A,\ D \in \Omega\}$$

which, as is well known, are monotone and extend μ. Moreover, μ^* is subadditive.

Using the above theorem, we can deduce some conditions under which the outer and the inner measures μ^* and μ_* coincide with the functionals I^* and I_* on characteristic functions, respectively.

Corollary 4.5. *If B is stonian then $\mu^*(A) = I^*(\chi_A)$, $\forall A \subseteq X$.*

Proof. On one hand, $I^*(\chi_A) = \inf\{I(g)\colon\ g \in R_1(B,I), g \geqslant \chi_A\} \leqslant \inf\{I(\chi_C)\colon \chi_C \in R_1(B,I), \chi_C \geqslant \chi_A\} = \inf\{\mu(C)\colon\ C \in \Omega, C \supseteq A\} = \mu^*(A)$.

On the other hand, if $I^*(\chi_A) = +\infty$, the other inequality is trivial. Thus, without loss of generality, we can assume that $I^*(\chi_A) < +\infty$. Therefore, there exists $g \in R_1(B,I), g \geqslant \chi_A$, and for any such g, Lemma 3.3 and Theorem 4.3 guarantee the existence of a strictly increasing sequence $\{r_n\}$ in $\mathbb{R}^+$ which converges to 1 such that $[g \geqslant r_n] \in \Omega$, for all $n \in \mathbb{N}$. Since $g \geqslant \chi_A$ and $r_n \leqslant 1$, it follows that $A \subseteq [g \geqslant r_n]$, and consequently

$$\mu^*(A) \leqslant \mu([g \geqslant r_n]) = I(\chi_{[g \geqslant r_n]}) \leqslant I(\frac{g}{r_n}) = \frac{1}{r_n} I(g), \ \forall n \in \mathbb{N}.$$

Thus, $\mu^*(A) \leqslant I(g)$ for all $g \in R_1(B,I)$ with $g \geqslant \chi_A$, which shows that $\mu^*(A) \leqslant I^*(\chi_A)$. $\square$

Corollary 4.6. *If B is stonian and C_0 then $\mu_*(A) = I_*(\chi_A)$, $\forall A \subseteq X$.*

Proof. On the one hand, $\mu_*(A) = \sup\{\mu(D)\colon\ D \in \Omega,\ D \subseteq A\} = \sup\{I(\chi_D)\colon \chi_D \in R_1(B,I),\ \chi_D \leqslant \chi_A\} \leqslant \sup\{I(g)\colon\ g \in R_1(B,I),\ g \leqslant \chi_A\} = I_*(\chi_A)$.

On the other hand, if $I_*(\chi_A) = -\infty$ the result immediately follows. Thus, we assume that $I^*(\chi_A) > -\infty$. Therefore, for arbitrary $g \in B$ with $g \leqslant \chi_A$, Lemma 3.3 and Theorem 4.3 allow us to consider a strictly decreasing sequence $\{r_n\} \subseteq]0,1[$ convergent to 0 such that $[g \geqslant r_n] \in \Omega$ for all $n \in \mathbb{N}$.

Since $g \leqslant \chi_A$ and $r_n > 0$, it follows that $[g \geqslant r_n] \subseteq A$, and hence for every $n \in \mathbb{N}$ we have

$$\begin{aligned} I(g) \leqslant\ & I^*(g \wedge r_n) + I^*(g - g \wedge r_n) = I^*(g \wedge r_n) + I^*((g - r_n)\chi_{[g \geqslant r_n]}) \\ & \leqslant I^*(g \wedge r_n) + I^*((1 - r_n)\chi_{[g \geqslant r_n]}) = I^*(g \wedge r_n) + (1 - r_n) I(\chi_{[g \geqslant r_n]}) \\ & = I^*(g \wedge r_n) + (1 - r_n)\mu([g \geqslant r_n]) \leqslant I^*(g \wedge r_n) + (1 - r_n)\mu_*(A). \end{aligned}$$

Letting $n \to +\infty$ and using Lemma 3.4 we deduce that $I(g) \leqslant \mu_*(A)$ for all $g \in R_1(B,I)$ with $g \leqslant \chi_A$, i.e., $\mu_*(A) \geqslant I_*(\chi_A)$. $\square$

In addition, we are able to characterize the C_0 and C_∞ conditions by means of spectral sets:

Proposition 4.7. *Given an arbitrary Loomis system (X, B, I) we have*

(i) *B is $C_\infty \Leftrightarrow \lim\limits_{r\to+\infty} I^*(h\chi_{[h\geqslant r]}) = 0,\ \forall h \in +B$;*

(ii) *B is $C_0 \Leftrightarrow \lim\limits_{r\to 0} I^*(h\chi_{[h\leqslant r]}) = 0,\ \forall h \in +B$.*

Proof. (i) ($\Rightarrow$) Assume that B is C_∞. Given $r > 0$ and $n \in \mathbb{N}$, we have

$$h\chi_{[h\geqslant r]} \leqslant \frac{h}{n} + \Big(h - h \wedge \frac{r}{n}\Big).$$

Let $\varepsilon > 0$. Since $I(h) \in \mathbb{R}^+$, there exists $m \in \mathbb{N}$ such that $I(h)/m < \frac{1}{2}\varepsilon$, and from $\lim\limits_{r\to+\infty} I^*(h - h \wedge r/m) = 0$ we find a δ such that $I^*(h - h \wedge r/m) < \frac{1}{2}\varepsilon$ for all $r < \delta$. Thus, given $r < \delta$ we have

$$I^*(h\chi_{[h\geqslant r]}) \leqslant I^*\Big(\frac{h}{m} + h - h \wedge \frac{r}{m}\Big) \leqslant \frac{I(h)}{m} + I^*\Big(h - h \wedge \frac{r}{m}\Big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which says that $\lim\limits_{r\to+\infty} I^*(h\chi_{[h\geqslant r]}) = 0$.

($\Leftarrow$) Assume that $\lim\limits_{r\to+\infty} I^*(h\chi_{[h\geqslant r]}) = 0$. Since $0 \leqslant h - h \wedge r \leqslant h\chi_{[h\geqslant r]}$, it follows that $\lim\limits_{r\to+\infty} I^*(h - h \wedge r) = 0$, that is, B is C_∞.

(ii) ($\Rightarrow$) Assume now that B is C_0. From $0 \leqslant h\chi_{[h\leqslant r]} \leqslant h \wedge r$ we deduce that $0 \leqslant I^*(h\chi_{[h\leqslant r]}) \leqslant I^*(h \wedge r)$, which implies that $\lim\limits_{r\to 0} I^*(h\chi_{[h\leqslant r]}) = 0$.

($\Leftarrow$) Finally, assume that $\lim\limits_{r\to 0} I^*(h\chi_{[h\leqslant r]}) = 0$. For any $n \in \mathbb{N}$ and $r \in \mathbb{R}^+$ we have $0 \leqslant h \wedge r \leqslant h/n + h\chi_{[h\leqslant nr]}$. Thus, given $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $I(h)/m < \frac{1}{2}\varepsilon$, and then, by hypothesis, there exists $\delta > 0$ such that $I^*(h\chi_{[h\leqslant mr]}) < \frac{1}{2}\varepsilon$ for all $r < \delta$.

Therefore, given $r < \delta$ we have $0 \leqslant I^*(h \wedge r) \leqslant I^*(h/m + h\chi_{[h\leqslant mr]}) \leqslant I(h)/m + I^*(h\chi_{[h\leqslant mr]}) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$, which proves that B is C_0. □

The proof remains valid if we consider $R_1(B, I)$ instead of B. Thus, using Lemmas 3.3 and 3.4, we can modify Proposition 4.7 in the following manner:

Corollary 4.8. *Given an arbitrary Loomis system (X, B, I), we have*

$$B \text{ is } C_\infty \Leftrightarrow \lim_{r\to+\infty} I^*(f\chi_{[f\geqslant r]}) = 0,\ \forall f \in +R_1(B, I).$$

Moreover, if B is stonian, then

$$B \text{ is } C_0 \Leftrightarrow \lim_{r\to 0} I^*(f\chi_{[f\leqslant r]}) = 0,\ \forall f \in +R_1(B, I).$$

Corollary 4.8 allows to generalize theorem 4.11 in [23] to our functional context as an immediate consequence:

Corollary 4.9. *If B is C_∞, then*

$$\lim_{r\to+\infty} rI^*(\chi_{[f\geqslant r]}) = 0,\ \forall f \in +R_1(B,I).$$

P r o o f. By Corollary 4.8, $\lim\limits_{r\to+\infty} I^*(f\chi_{[f\geqslant r]}) = 0$. Therefore, the inequality

$$0 \leqslant rI^*(\chi_{[f\geqslant r]}) = I^*(r\chi_{[f\geqslant r]}) \leqslant I^*(f\chi_{[f\geqslant r]})$$

leads to $\lim\limits_{r\to+\infty} rI^*(\chi_{[f\geqslant r]}) = 0$. □

Another interesting problem related to upper continuity is the one of determining when the indefinite integrals of an integrable function are absolutely continuous in this functional context. It is well known that they are always absolutely continuous for Dunford-Schwartz integration with respect to a finitely additive measure (see theorem 4.9 in [23]).

We have recently extended this result to abstract Riemann integration in the following way (see [1] for the details): If (X,B,I) is a C_∞ Loomis system, $f \in +R_1(B,I)$ and μ_f is the finite additive measure induced by $\mu_f(A) := I(f\chi_A)$, $\forall A \in \Omega$, then

$$\forall\ \varepsilon > 0,\ \exists\ \delta > 0\colon\ A \in \Omega,\ \mu(A) < \delta \Rightarrow \mu_f(A) < \varepsilon.$$

5. Representation of the integral

A long time after Ridder had proved the equivalence between integrability and quasi-measurability for proper Riemann integration with respect to a finite finitely additive measure, Topsøe checked in [29] that the formula

$$\text{(4.)}\qquad \int f\,\mathrm{d}\mu = \int_0^{+\infty} \mu([f\geqslant r])\,\mathrm{d}r,$$

which allows to compute the integral of a function by summing the measures of its spectral sets, holds for any positive integrable function f.

Later, Maharam extended this formula to her improper integration theory with respect to a finitely additive measure (see [24]) and Luxemburg did the same (in [23]) for Dunford-Schwartz integration.

The aim of this section is to carry formula (4) from the measure-theoretic context to our functional setting in such a way that the results of Maharam and Luxemburg will be, at least, partially generalized. Thus, the main result we are looking for is the expression of the integral I on the class $R_1(B,I)$ as an improper Riemann integral in the classical sense (Corollary 5.2).

We must point out that the idea of integration behind the right-hand member of formula (4) can be found in literature under the name of *horizontal integration* and its background goes back to the work by Choquet [7], which can be also considered the starting point of the so-called *non-additive integration theory.* For a systematic study of this type of theory we refer to the book by König [21] which collects and develops the main results in this direction due to Choquet, Topsøe, Kindler (see also [20]) and Dennemberg (see also [9]), among others.

We begin by proving an integral representation for the positive elements in B. From now on, given a function f we assume F to be the function $F\colon\, \mathbb{R}^+ \longrightarrow \overline{\mathbb{R}}$ defined by $F(r) := I_*(\chi_{[f \geqslant r]})$. Note that F is a decreasing function.

Proposition 5.1. *If B is stonian, C_0 and C_∞, then*

$$I(f) = \int_0^{+\infty} I_*(\chi_{[f\geqslant r]})\,\mathrm{d}r = \int_0^{+\infty} I^*(\chi_{[f\geqslant r]})\,\mathrm{d}r, \quad \forall f \in +B.$$

P r o o f. We only give the proof for I_* (the same reasoning applies to I^*). Let $f \in B$ and $0 < r < t$. It is clear that

$$\chi_{[f\geqslant t]} \leqslant \frac{f\wedge t - f \wedge r}{t-r} \leqslant \chi_{[f\geqslant r]}.$$

Since $f \wedge t$, $f \wedge r \in B$ we have $I((f\wedge t - f\wedge r)/(t-r)) = (I(f\wedge t) - I(f\wedge r))/(t-r)$. Thus, $F(r) = I_*(\chi_{[f\geqslant r]}) \leqslant I((f\wedge t - f \wedge r)/(t-r)) < +\infty$ and hence the decreasing function F goes in fact from $\mathbb{R}^+$ into $\mathbb{R}^+$. Therefore F is Riemann integrable on every interval in $\mathbb{R}^+$.

Moreover,

$$I_*(\chi_{[f\geqslant t]}) \leqslant \frac{I(f\wedge t) - I(f\wedge r)}{t-r} \leqslant I_*(\chi_{[f\geqslant r]}),$$

that is,

$$(t-r)I_*(\chi_{[f\geqslant t]}) \leqslant I(f\wedge t) - I(f \wedge r) \leqslant (t-r)I_*(\chi_{[f\geqslant r]}).$$

Given a partition $\{r_0, r_1, \ldots, r_{n-1}, r_n\}$ of the interval $[u, v]$ with $0 < u < v$, applying the last inequality on each subinterval (i.e., for $t = r_i$ and $r = r_{i-1}$, $\forall i = 1, 2, \ldots, n$), and adding over i, we deduce that:

$$\sum_{i=1}^{n}(r_i - r_{i-1})I_*(\chi_{[f\geqslant r_i]}) \leqslant I(f\wedge v) - I(f\wedge u) \leqslant \sum_{i=1}^{n}(r_i - r_{i-1})I_*(\chi_{[f\geqslant r_{i-1}]}).$$

Letting $n \to +\infty$, we obtain

$$\int_u^v I_*(\chi_{[f\geqslant r]})\,\mathrm{d}r = I(f\wedge v) - I(f\wedge u) = \int_u^v I_*(\chi_{[f\geqslant r]})\,\mathrm{d}r$$

and, since B is stonian, C_0 and C_∞, letting $u \to 0$ and $v \to +\infty$, we deduce that

$$I(f) = \int_0^{+\infty} I^*(\chi_{[f \geqslant r]})\,\mathrm{d}r = \int_0^{+\infty} I_*(\chi_{[f \geqslant r]})\,\mathrm{d}r.$$

□

Keeping in mind that if $f \in +R_1(B, I)$ then the function $I(\chi_{[f \geqslant r]})$, in the variable r, is defined for all $r \in \mathbb{R}^+$, except for a countable set (see Theorem 4.3) and that this function coincides with $I_*(\chi_{[f \geqslant r]})$ on all its domain, we obtain the following corollary with the desirable integral representation for the positive elements of $R_1(B, I)$.

Corollary 5.2. *If B is stonian, C_0 and C_∞, then*

$$I(f) = \int_0^{+\infty} I(\chi_{[f \geqslant r]})\,\mathrm{d}r, \quad \forall f \in +R_1(B, I).$$

Theorem 4.13 in [23] is a particular case of this result. In fact, Corollary 5.2 can be generalized in the following way (which also subsumes item (3) of Theorem 5.4 in [24]):

Theorem 5.3. *If B is stonian, C_0 and $C_{+\infty}$, then*

$$I_*(f) = \int_0^{+\infty} I_*(\chi_{[f \geqslant r]})\,\mathrm{d}r, \quad \forall f \in \overline{\mathbb{R}}_+^X.$$

P r o o f. Let us fix $\alpha := I_*(f) = \sup\{I(h)\colon\ h \leqslant f, h \in R_1(B, I)\}$. We first assume that $I_*(\chi_{[f \geqslant s]}) = +\infty$ for some $s \in \mathbb{R}^+$.

On the one hand, since F is decreasing and $F(s) = +\infty$, we have $F(r) = +\infty$ for all $r \in]0, s]$, which implies $\int_0^{+\infty} I_*(\chi_{[f \geqslant r]})\,\mathrm{d}r = +\infty$.

But, on the other hand, for each $M > 0$ there exists $h \in +R_1$ with $h \leqslant \chi_{[f \geqslant s]}$ such that $I(h) \geqslant \frac{M}{s}$. Setting $g := sh \in +R_1$ we deduce that $g \leqslant s\chi_{[f \geqslant s]} \leqslant f$ and $I(g) = sI(h) \geqslant M$. Therefore, $I_*(f) \geqslant I(g) \geqslant M$ and the arbitrariness of M yields $I_*(f) = +\infty$.

We have proved that if $I_*(\chi_{[f \geqslant s]}) = +\infty$ for some $s \in \mathbb{R}^+$, then

$$\int_0^{+\infty} I_*(\chi_{[f \geqslant r]})\,\mathrm{d}r = +\infty = I_*(f).$$

Assume now that $I_*(\chi_{[f \geqslant r]}) < +\infty$ for all $r \in \mathbb{R}^+$ and let $h \in +R_1$ with $h \leqslant f$. It follows from $[h \geqslant r] \subseteq [f \geqslant r]$, that $I_*(\chi_{[h \geqslant r]}) \leqslant I_*(\chi_{[f \geqslant r]})$ and hence, by Corollary 5.2, we have

$$I(h) = \int_0^{+\infty} I_*(\chi_{[h \geqslant r]})\,\mathrm{d}r \leqslant \int_0^{+\infty} I_*(\chi_{[f \geqslant r]})\,\mathrm{d}r, \quad \forall h \in +R_1, h \leqslant f,$$

which leads to

$$\alpha \leqslant \int_0^{+\infty} I_*(\chi_{[f\geqslant r]})\,\mathrm{d}r.$$

In order to prove the opposite inequality, let $u, v \in \mathbb{R}$ be such that $0 < u < v < +\infty$. Given $\varepsilon > 0$, by definition of the Riemann integral on $[u, v]$ there exists a partition $P \in \mathscr{P}([u, v])$, $P = \{r_0, \dots, r_n\}$, such that

$$\int_u^v I_*(\chi_{[f\geqslant r]})\,\mathrm{d}r \leqslant \sum_{i=1}^n (r_i - r_{i-1}) I_*(\chi_{[f\geqslant r_i]}) + \frac{\varepsilon}{2}.$$

Since $I_*(\chi_{[f\geqslant r_i]}) = \sup\{I(h)\colon\ h \in +B, h \leqslant \chi_{[f\geqslant r_i]}\} < +\infty$ for all $i = 1, \dots, n$, given $\varepsilon/2(v-u)$, there exists $h_i \in +R_1$, $h_i \leqslant \chi_{[f\geqslant r_i]}$ such that

$$I_*(\chi_{[f\geqslant r_i]}) \leqslant I(h_i) + \frac{\varepsilon}{2(v-u)}, \qquad \forall i = 1, \dots, n.$$

Thus,

$$\begin{aligned}
\int_u^v I_*(\chi_{[f\geqslant r_i]})\,\mathrm{d}r &\leqslant \sum_{i=1}^n (r_i - r_{i-1}) I_*(\chi_{[f\geqslant r_i]}) + \frac{\varepsilon}{2} \ \leqslant \\
&\leqslant \sum_{i=1}^n (r_i - r_{i-1})\Big(I(h_i) + \frac{\varepsilon}{2(v-u)}\Big) + \frac{\varepsilon}{2} \ = \\
&\leqslant \sum_{i=1}^n (r_i - r_{i-1}) I(h_i) + \varepsilon.
\end{aligned}$$

Setting $h := \sum_{i=1}^n (r_i - r_{i-1}) h_i$, we have $h \in +R_1$, $h \leqslant (f-u)^+ \wedge (v-u) \leqslant f$ and $\int_u^v I_*(\chi_{[f\geqslant r]})\,\mathrm{d}r \leqslant I(h) + \varepsilon$. Therefore,

$$\int_u^v I_*(\chi_{[f\geqslant r]})\,\mathrm{d}r \leqslant I_*(f) = \alpha,\ \forall u, v \in \mathbb{R},\ 0 < u < v < +\infty,$$

which implies that

$$\int_0^{+\infty} I_*(\chi_{[f\geqslant r]})\,\mathrm{d}r \leqslant \alpha.$$

$\square$

A natural question arises at this moment: Is it possible to obtain a representation of this type for the integral without any continuity condition? The answer consists in the negative since both the weak continuity conditions C_0 and C_∞ are necessary conditions, that is, we have

Proposition 5.4. *If (X,B,I) is a stonian Loomis system such that*

$$I(f)=\int_0^{+\infty} I(\chi_{[f\geqslant r]})\,\mathrm{d}r,\quad \forall f\in +B,$$

then B is C_0 and C_∞.

P r o o f. For every $t,r<0$ the set $[f\wedge t\geqslant r]$ coincides with $[f\geqslant r]$ if $r\leqslant t$ and it is the empty set if $r>t$. Thus, we have

$$I(f\wedge t)=\int_0^{+\infty} I(\chi_{[f\wedge t\geqslant r]})\,\mathrm{d}r=\int_0^{t} I(\chi_{[f\geqslant r]})\,\mathrm{d}r.$$

Therefore,

$$\lim_{t\to 0} I(f\wedge t)=\lim_{t\to 0}\int_0^{t} I(\chi_{[f\geqslant r]})\,\mathrm{d}r=0$$

and

$$\lim_{t\to +\infty} I(f\wedge t)=\lim_{t\to +\infty}\int_0^{t} I(\chi_{[f\geqslant r]})\,\mathrm{d}r=\int_0^{+\infty} I(\chi_{[f\geqslant r]})\,\mathrm{d}r=I(f),$$

which proves that B satisfies the C_0 and C_∞ conditions. $\square$

We summarize the results of this section in

Corollary 5.5. *Let (X,B,I) be a stonian Loomis system. The following assertions are equivalent:*

(i) *B is C_0 and C_∞.*

(ii) *$R_1(B,I)$ is C_0 and C_∞.*

(iii) *$I(f)=\int_0^{+\infty} I(\chi_{[f\geqslant r]})\,\mathrm{d}r,\ \forall f\in +R_1(B,I)$.*

References

[1] *E. de Amo, R. del Campo and M. Díaz Carrillo*: Absolute continuity theorems for abstract Riemann integration. Czech. Math. J. *57* (2007), 793–807.

[2] *E. de Amo and M. Díaz Carrillo*: Local and improper Daniell-Loomis integrals. Rend. Circ. Mat. Palermo *54* (2005), 329–342.

[3] *B. Anger and C. Portenier*: Radon Integrals. Progress in Math. Vol. 103, Birkhäuser, Boston, 1992.

[4] *G. Aumann*: Integralerweiterungen mittels Normen. Arch. Math. *3* (1952), 441–450.

[5] *P. Bobillo Guerrero and M. Díaz Carrillo*: Fonctions fortement-mesurables et mesurables par rapport a un systeme de Loomis. Bull. Soc. Roy. Sci. Ličge *55* (1987), 467–471.

[6] *C. W. Burrill*: Measure, Integration and Probability. McGraw-Hill, 1972.

[7] *G. Choquet*: Theory of capacities. Ann. Inst. Fourier, Grenoble *5* (1953/54), 131–295.

[8] *P. J. Daniell*: A general form of integral. Ann. of Math. *19* (1917/18), 279–294.

[9] *D. Denneberg*: Non-Additive Measure and Integral. Kluwer, 1994.

[10] *M. Díaz Carrillo and H. Günzler*: Abstract Daniell-Loomis spaces. Bull. Austral. Math. Soc. *53* (1996), 135–142.
[11] *M. Díaz Carrillo and H. Günzler*: Daniell-Loomis integrals. Rocky Mount. J. Math. *27* (1997), 1075–1087.
[12] *M. Díaz Carrillo and H. Günzler*: Local integral metrics and Daniell-Loomis integrals. Bull. Austral. Math. Soc. *48* (1993), 411–426.
[13] *M. Díaz Carrillo and P. Muñoz Rivas*: Finitely additive integration: integral extension with local-convergence. Ann. Sci. Math. Québec *17* (1993), 145–154.
[14] *M. Díaz Carrillo and P. Muñoz Rivas*: Positive linear functionals and improper integration. Ann. Sci. Math. Québec *18* (1994), 149–156.
[15] *N. Dunford and J. T. Schwartz*: Linear Operartors, part I, General Theory. Interscience, New-York, 1957.
[16] *O. Frink*: Jordan measure and Riemann integration. Ann. of Math. *34* (1933), 518–526.
[17] *G. Greco*: Sulla reppresentazione di funzionali mediante integrali. Rend. Sem. Mat. Padova *66* (1982), 21–42.
[18] *H. Günzler*: Integration. Bibliogr. Institut, Mannheim, 1985.
[19] *H. Günzler*: Linear Functionals which are Integrals. Rend. Sem. Mat. Fis. Milano *43* (1973), 167–176.
[20] *J. Kindler*: A Mazur-Orlicz theorem for submodular set functions. Journ. Math. Analysis Appl. *120* (1986), 533–564.
[21] *H. König*: Measure and Integration. An advanced course in basic procedures and applications. Springer, 1997.
[22] *L. H. Loomis*: Linear functionals and content. Amer. J. Math. *76* (1954), 168–182.
[23] *W. A. J. Luxemburg*: Integration with respect to finitely additive measures. Stud. Econ., Theory 2 (1991), 109–150.
[24] *D. Maharam*: Jordan fields and improper integrals. J. Math. Anal. and Appl. *133* (1988), 163–194.
[25] *W. F. Pfeffer*: Integrals and Measures. Dekker, New-York, 1977.
[26] *J. Ridder*: Over de Integraldefinities van Riemann en Lebesgue. Christiann Huygens *4* (1925/26), 246–250 ; nd correction, [redacted]d. *5* (1927), 205.
[27] *F. W. Schäfke*: Integrationstheorie I. J. Reine Angew. Math. *244* (1970), 154–176.
[28] *F. W. Schäfke*: Lokale Integralnormen and verallgemeinerte uneigentlich Riemann-Stiltjes-Integrals. J. Reine Angew. Math. *289* (1977), 118–134.
[29] *F. Topsøe*: On constructions of measures. Proceedings of the Conference on Topology and Measure I, Zinnowitz, 1974, Part 2, 343–381, Ernst-Moritz-Arndt Univ., Greifswald, 1978.