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# The space of scalarly integrable functions for a Fréchet-space-valued measure

R. del Campo [a,*,1], W.J. Ricker [b]

[a] Departamento Matemática Aplicada I, EUITA, Ctra. de Utrera Km. 1, 41013-Sevilla, Spain
[b] Math.-Geogr. Fakultät, Katholische Universität Eichstätt-Ingolstadt, D-85072 Eichstätt, Germany

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ABSTRACT

The space $L^1_w(\nu)$ of all scalarly integrable functions with respect to a Fréchet-space-valued vector measure $\nu$ is shown to be a complete Fréchet lattice with the $\sigma$-Fatou property which contains the (traditional) space $L^1(\nu)$, of all $\nu$-integrable functions. Indeed, $L^1(\nu)$ is the $\sigma$-order continuous part of $L^1_w(\nu)$. Every Fréchet lattice with the $\sigma$-Fatou property and containing a weak unit in its $\sigma$-order continuous part is Fréchet lattice isomorphic to a space of the kind $L^1_w(\nu)$.



## 1. Introduction

Let $X$ be a Fréchet space and $\nu : \Sigma \to X$ be a vector measure (i.e. $\nu$ is $\sigma$-additive), where $(\Omega, \Sigma)$ is a measurable space. A measurable function $f : \Omega \to \mathbb{R}$ is said to be $\nu$-*integrable* if

(I-1) $f$ is *scalarly* $\nu$-*integrable*, that is, $f$ is integrable with respect to the scalar measure $\langle \nu, x^* \rangle : A \mapsto \langle \nu(A), x^* \rangle$, for each $x^* \in X^*$ (the continuous dual space of $X$), and

(I-2) for each $A \in \Sigma$ there exists an element $\int_A f\, d\nu \in X$ such that

$$\left\langle \int_A f\, d\nu, x^* \right\rangle = \int_A f\, d\langle \nu, x^* \rangle, \quad x^* \in X^*.$$

For $X$ a Banach space, the space $L^1(\nu)$, consisting of all (equivalence classes of) $\nu$-integrable functions equipped with the norm

$$\|f\|_\nu := \sup_{x^* \in B_{X^*}} \int_\Omega |f|\, d\left|\langle \nu, x^* \rangle\right|, \quad f \in L^1(\nu), \tag{1}$$

is also a Banach space. Here $B_{X^*}$ is the closed unit ball of $X^*$ and $|\langle \nu, x^* \rangle|$ is the variation measure of $\langle \nu, x^* \rangle$, for $x^* \in X^*$. The space $L^1(\nu)$ was introduced and investigated in [14], even for vector measures taking values in a locally convex Hausdorff

* Corresponding author.
*E-mail addresses:* rcampo@us.es (R. del Campo), werner.ricker@ku-eichstaett.de (W.J. Ricker).

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space (briefly, lcHs). Since then, $L^1(\nu)$ has been intensively studied in the Banach setting by many authors; see, e.g. [4,25] and the references therein.

G. Stefansson introduced the space $L^1_w(\nu)$ of all (classes of) scalarly $\nu$-integrable functions and showed that the same functional (1) above is a norm in $L^1_w(\nu)$ [28]. This is to be interpreted to mean that if $f$ is scalarly $\nu$-integrable, then $\|f\|_\nu < \infty$ and that (1) specifies a norm in $L^1_w(\nu)$. Actually, Stefansson also showed that $L^1_w(\nu)$ is complete for $\|\cdot\|_\nu$. It is then clear that $L^1(\nu)$ is a closed subspace of $L^1_w(\nu)$.

In contrast to $L^1(\nu)$, the Banach space $L^1_w(\nu)$ has received relatively little attention up to now; see, for example, [12], [25, Chapter 3], [28]. Recently however, new features of $L^1_w(\nu)$ have emerged, based on the theory of Banach function spaces, which indicate the importance of this space in its own right [3,5]. In particular, it is closely related to the "optimal domain property" for certain kernel operators.

For a *Fréchet space* $X$ and a vector measure $\nu : \Sigma \to X$, the corresponding (Fréchet) space $L^1(\nu)$ is well understood; see, e.g. [8–11,14,21,23,24,26] and the references therein. Although the notion of an *individual* scalarly $\nu$-integrable function already occurs in the Fréchet and even more general lcHs setting [14,16,17], there is no study made there or elsewhere (as far as we aware) of what the corresponding *space* $L^1_w(\nu)$ should be and what properties it might have. Our aim here is to address this deficiency: we introduce and investigate various properties of $L^1_w(\nu)$, for $\nu$ a Fréchet space-valued measure, and identify its connection to $L^1(\nu)$.

Let us indicate a sample of some of the results. In Section 2 we establish that $L^1_w(\nu)$ is always complete, even if $X$ is merely a metrizable lcHs. Even for normed spaces, this seems to have been overlooked. In contrast, $L^1(\nu)$ is typically not complete if the metrizable lcHs $X$ fails to be complete. In Section 3 it is shown that every Fréchet lattice $F$ with the $\sigma$-Fatou property and possessing a weak unit in its $\sigma$-order continuous part is Fréchet lattice isomorphic to $L^1_w(\nu)$ for a suitable vector measure $\nu$. This shows how extensive the family of all spaces of the type $L^1_w(\nu)$ is within the class of all Fréchet lattices.

## 2. The space $L^1_w(\nu)$

Let $X$ be a metrizable lcHs generated by a fundamental sequence of increasing seminorms $(\|\cdot\|^{(n)})_{n\in\mathbb{N}}$. The sets $B_n := \{x \in X\colon \|x\|^{(n)} \leqslant 1\}$ form a fundamental sequence of zero neighbourhoods for $X$ and their polars $B_n^\circ := \{x^* \in X^*\colon |\langle x, x^*\rangle| \leqslant 1, \forall x \in B_n\}$, for $n \in \mathbb{N}$, are absolutely convex [19, Theorem 23.5]. Moreover, $\{B_n^\circ\colon n \in \mathbb{N}\}$ is a fundamental sequence of bounded sets in the strong dual $X^*_\beta$ (i.e., each bounded set in $X^*_\beta$ is contained in a multiple of $B_n^\circ$ for some $n \in \mathbb{N}$). In addition, each set $B_n^\circ$, for $n \in \mathbb{N}$, is a *Banach disc* [19, Lemma 25.5], that is, the linear hull $Lin(B_n^\circ) = \bigcup_{t>0} tB_n^\circ$ of $B_n^\circ$ (formed in $X^*$) is a Banach space when equipped with its Minkowski functional

$$\|x^*\|_{B_n^\circ} := \inf\{s > 0\colon\ x^* \in sB_n^\circ\}, \quad x^* \in Lin(B_n^\circ).$$

For each $n \in \mathbb{N}$, the *local Banach space* $X_n$ is the completion of the quotient $X/M_n$ endowed with the quotient norm induced by $\|\cdot\|^{(n)}$, where $M_n := \{x \in X\colon \|x\|^{(n)} = 0\}$. Let $\pi_n : X \to X_n$ be the quotient map. Then its dual map $\pi_n^*$ is an isometric bijection between the Banach spaces $X_n^*$ and $Lin(B_n^\circ)$ [19, Remark 24.5(b)].

Given a vector measure $\nu : \Sigma \to X$, defined on a measurable space $(\Omega, \Sigma)$, a set $A \in \Sigma$ is $\nu$-*null* if $\nu(B) = 0$ for all $B \in \Sigma$ with $B \subseteq A$. Let $L^0(\nu)$ denote the $\sigma$-Dedekind complete Riesz space of all (classes, modulo $\nu$-a.e., of) scalar-valued, $\Sigma$-measurable functions defined on $\Omega$, with respect to the $\nu$-a.e. pointwise order [18, pp. 126–127]. For each $n \in \mathbb{N}$, define a $[0,\infty]$-valued seminorm $\|\cdot\|_\nu^{(n)}$ in $L^0(\nu)$ by

$$\|f\|_\nu^{(n)} := \sup_{x^* \in B_n^\circ} \int_\Omega |f|\, d|\langle \nu, x^*\rangle|, \quad f \in L^0(\nu). \tag{2}$$

The map $\nu_n : \Sigma \to X_n$ given by

$$\nu_n(A) := \pi_n\big(\nu(A)\big), \quad A \in \Sigma, \tag{3}$$

is a Banach-space-valued vector measure. Observe that $A \in \Sigma$ is $\nu$-null if and only if it is $\nu_n$-null for all $n \in \mathbb{N}$.

**Proposition 2.1.** *Let $X$ be a metrizable lcHs, $\nu$ an $X$-valued measure and $f \in L^0(\nu)$. Then $f \in L^1_w(\nu)$ if and only if $\|f\|_\nu^{(n)} < \infty$, for $n \in \mathbb{N}$.*

**Proof.** Let $f$ satisfy $\|f\|_\nu^{(n)} < \infty$, for all $n \in \mathbb{N}$. Given $x^* \in X^*$, there exist $m \in \mathbb{N}$ and $C > 0$ such that $|\langle x, x^*\rangle| \leqslant C\|x\|^{(m)}$ for all $x \in X$, i.e. $C^{-1}x^* \in B_m^\circ$, and so

$$\int_\Omega |f|\, d|\langle \nu, x^*\rangle| = C \int_\Omega |f|\, d|\langle \nu, C^{-1}x^*\rangle| \leqslant C\|f\|_\nu^{(m)} < \infty.$$

Conversely, if $f \in L^1_w(\nu)$, then $f$ is scalarly $\nu_n$-integrable, for $n \in \mathbb{N}$, since $\langle \nu, \pi_n^*(\xi^*)\rangle = \langle \nu_n, \xi^*\rangle$ as measures, for $\xi^* \in X_n^*$. Furthermore,

$$\|f\|_{\nu_n} := \sup_{\xi^* \in B_{X_n^*}} \int_\Omega |f|\, d|\langle \nu_n, \xi^*\rangle| = \sup_{\xi^* \in B_{X_n^*}} \int_\Omega |f|\, d|\langle \nu, \pi_n^*(\xi^*)\rangle|$$

and hence,

$$\|f\|_\nu^{(n)} := \sup_{x^*\in B_n^\circ} \int_\Omega |f|\, d\big|\langle \nu, x^*\rangle\big| = \|f\|_{\nu_n} \tag{4}$$

where, for the last equality, we use the fact that $\pi_n^*$ is an isometry and so $\pi_n^*(B_{X_n^*}) = B_n^\circ$. But, $X_n$ is a Banach space and so $\|f\|_{\nu_n} < \infty$. □

**Corollary 2.2.** *Let $X$ be a metrizable lcHs and $\nu$ be an $X$-valued vector measure. Then, $L^1_w(\nu)$ is an ideal in $L^0(\nu)$ and the restricted functionals $\|\cdot\|_\nu^{(n)} : L^1_w(\nu) \to [0,\infty)$ given by (2), for each $n \in \mathbb{N}$, are an increasing sequence of Riesz seminorms which turn $L^1_w(\nu)$ into a metrizable, locally convex-solid Riesz space.*

**Proof.** As $B_{n+1} \subseteq B_n$, the seminorms $(\|\cdot\|_\nu^{(n)})_{n\in\mathbb{N}}$ are increasing in $L^1_w(\nu)$. Also, if $g \in L^0(\nu)$ and $f \in L^1_w(\nu)$ with $|g| \leqslant |f|$, then $g \in L^1_w(\nu)$ and $\|g\|_\nu^{(n)} \leqslant \|f\|_\nu^{(n)}$, for $n \in \mathbb{N}$. This shows that $L^1_w(\nu)$ is an ideal in $L^0(\nu)$ and each $\|\cdot\|_\nu^{(n)}$ is a Riesz seminorm in $L^1_w(\nu)$.

Suppose that $f \in L^1_w(\nu)$ satisfies $\|f\|_\nu^{(n)} = 0$ for all $n \in \mathbb{N}$. According to (4) we have $f \in L^1_w(\nu_n)$ with $\|f\|_{\nu_n} = \|f\|_\nu^{(n)} = 0$ for all $n \in \mathbb{N}$. That is, the set $A := \{w \in \Omega\colon\ |f(w)| > 0\}$ is $\nu_n$-null for each $n \in \mathbb{N}$. Hence, $A$ is a $\nu$-null set and so $f = 0$ in $L^1_w(\nu)$. Now general theory can be invoked to conclude that $L^1_w(\nu)$ becomes a metrizable, locally convex-solid Riesz space when equipped with the topology induced by the seminorms $(\|\cdot\|_\nu^{(n)})_{n\in\mathbb{N}}$; see, for example, [1, Theorem 6.1], [19, Lemma 22.5]. □

Functions $f$ from $L^1_w(\nu)$ differ from those of $L^1(\nu)$ in that not all of their "integrals" belong to $X$. For $X$ a Banach space, given any $A \in \Sigma$ there always exists a "generalized integral" $x_A^{**} \in X^{**}$ satisfying

$$\langle x^*, x_A^{**}\rangle = \int_A f\, d\langle \nu, x^*\rangle, \quad x^* \in X^*;$$

see [5,16,28], for example. For $X$ a metrizable lcHs, we now show that a similar phenomenon occurs. First some notation: given $f \in L^1_w(\nu)$ and $A \in \Sigma$, define a linear functional $(\mathrm{w})\text{-}\int_A f\, d\nu : X^* \to \mathbb{R}$ by

$$(\mathrm{w})\text{-}\int_A f\, d\nu : x^* \mapsto \int_A f\, d\langle \nu, x^*\rangle, \quad x^* \in X^*. \tag{5}$$

The continuous dual space $(X_\beta^*)^*$ of $X_\beta^*$ is denoted simply by $X^{**}$.

**Proposition 2.3.** *Let $X$ be a metrizable lcHs and $\nu$ be an $X$-valued vector measure. For each $f \in L^1_w(\nu)$ and $A \in \Sigma$, the linear functional $(\mathrm{w})\text{-}\int_A f\, d\nu$ given by (5) belongs to $X^{**}$.*

**Proof.** Since $L^1_w(\nu)$ is an ideal in $L^0(\nu)$ (see Corollary 2.2), it suffices to consider $0 \leqslant f \in L^1_w(\nu)$. Fix $A \in \Sigma$. Select $\Sigma$-simple functions $0 \leqslant f_k \uparrow f$ pointwise on $\Omega$. Given $x^* \in X^*$, we have $f \in L^1(|\langle \nu, x^*\rangle|)$ and so the Dominated convergence theorem for scalar measures yields

$$\lim_{k\to\infty} \int_A f_k\, d\langle \nu, x^*\rangle = \int_A f\, d\langle \nu, x^*\rangle = \left\langle x^*, (\mathrm{w})\text{-}\int_A f\, d\nu \right\rangle.$$

Hence, $C := \{\int_A f_k\, d\nu\}$ is a bounded set in $X$. From the previous formula, we have $|\langle x^*, (\mathrm{w})\text{-}\int_A f\, d\nu\rangle| \leqslant 1$ for all $x^* \in C^\circ$, which is a neighbourhood of zero in $(X_\beta^*)^*$. So, $(\mathrm{w})\text{-}\int_A f\, d\nu \in X^{**}$. □

**Remark 2.4.** If $X$ is weakly sequentially complete, then it follows that $(\mathrm{w})\text{-}\int_A f\, d\nu \in X$, for every $A \in \Sigma$ and every $f \in L^1_w(\nu)$. That is, $L^1_w(\nu) = L^1(\nu)$ in this case. Actually, whenever $X$ does not contain an isomorphic copy of the Banach space $c_0$, it is known that $L^1_w(\nu) = L^1(\nu)$; see, for example, [14, p. 31], [17, Theorem 5.1].

If $X$ is a Fréchet space, then it is known that $L^1(\nu)$ is also a Fréchet space for every $X$-valued vector measure $\nu$ ([10], [14, Chapter 4, Theorems 4.1 and 7.1]). The same is true of $L^1_w(\nu)$, even without completeness of $X$!

**Theorem 2.5.** *Let $X$ be a metrizable lcHs and $\nu$ be an $X$-valued vector measure. Then $L^1_w(\nu)$ is complete and, in particular, is a Fréchet lattice. If, in addition, $X$ is a Fréchet space, then the Fréchet lattice $L^1_w(\nu)$ contains $L^1(\nu)$ as a closed subspace.*

**Proof.** Fix $n \in \mathbb{N}$. Let $\nu_n : \Sigma \to X_n$ be the *Banach*-space-valued vector measure given by (3). Rybakov's theorem states that there exists $\xi_n^* \in X_n^*$ such that $|\langle \nu_n, \xi_n^* \rangle|$ is a control measure for $\nu_n$ (i.e. $\nu_n$ and $|\langle \nu_n, \xi_n^* \rangle|$ have the same null sets) [6, p. 268]. In particular, $|\langle \nu_n, \xi^* \rangle| \ll |\langle \nu_n, \xi_n^* \rangle|$, for every $\xi^* \in X_n^*$. Since $\pi_n^*$ is a bijection from $X_n^*$ onto $Lin(B_n^\circ)$, it follows that the linear functional $x_n^* := \pi_n^*(\xi_n^*) \in Lin(B_n^\circ) \subseteq X^*$ satisfies

$$\left|\langle \nu, x^* \rangle\right| \ll \left|\langle \nu, x_n^* \rangle\right|, \quad x^* \in Lin\left(B_n^\circ\right). \tag{6}$$

Let $\mu_n := |\langle \nu, x_n^* \rangle|$, for $n \in \mathbb{N}$. Then

$$\mu := \sum_{n=1}^{\infty} \frac{\mu_n}{2^n(1 + \mu_n(\Omega))} \tag{7}$$

is a positive, finite control measure for $\nu$; this follows from (6) and the fact that $\{B_n^\circ\}_{n \in \mathbb{N}}$ is a fundamental sequence of bounded sets in $X_\beta^*$.

Let $\tau_u$ and $\tau$ denote the topology of converge in measure in $L^0(\mu)$ and the topology in $L^0(\mu)$ defined (in a standard way) by the (extended valued) seminorms (2), respectively. Then $(L^0(\mu), \tau_u)$ is a complete metrizable topological vector space and $(L^0(\mu), \tau)$ is a Hausdorff topological vector group. Let $f_k \xrightarrow{\tau} 0$. Since $\int_\Omega |f_k| \, d\mu_n \leqslant \|f_k\|_\nu^{(n)}$ for all $n \in \mathbb{N}$, we get $f_k \to 0$ in $\mu_n$-measure for each $n \in \mathbb{N}$. Consequently, $f_k \xrightarrow{\tau_u} 0$ and so $\tau_u \subseteq \tau$. On the other hand, it follows from the (classical) Fatou lemma that the closed $\|\cdot\|_\nu^{(n)}$-balls centred at 0 are $\tau_u$-closed. Therefore, $(L^0(\mu), \tau)$ is complete [15, Section 18.4(4)]. Noting that $L_w^1(\nu)$ is a closed subspace of $(L^0(\mu), \tau)$, also $L_w^1(\nu)$ is complete. □

**Remark 2.6.** Let $\widehat{X}$ denote the completion of the metrizable lcHs $X$ and let $\hat{\nu}$ denote the $X$-valued vector measure $\nu$ when interpreted as taking its values in $\widehat{X}$. Then $L_w^1(\nu) = L_w^1(\hat{\nu})$ as vector spaces with $\|\cdot\|_\nu^{(n)} = \|\cdot\|_{\hat{\nu}}^{(n)}$, for each $n \in \mathbb{N}$. This explains why $L_w^1(\nu)$ is always complete, independent of whether $X$ is complete or not.

For $L^1(\nu)$ the situation is different. Indeed, $L^1(\hat{\nu})$ is always complete but, $L^1(\nu)$ may fail to be complete if $X$ is not complete; explicit examples can be found in [22,27], for instance. Since $L^1(\nu)$ has the relative topology from the *complete* space $L_w^1(\nu)$, we see that the completion $(L^1(\nu))\widehat{}$ is the closure of $L^1(\nu)$ in the Fréchet space $L_w^1(\nu) = L_w^1(\hat{\nu})$. On the other hand, $L^1(\hat{\nu})$ is always complete in itself and has the relative topology from $L_w^1(\hat{\nu})$. Since the $\Sigma$-simple functions are dense in both $L^1(\nu)$ and $L^1(\hat{\nu})$, by the Dominated convergence theorem [16, Theorem 2.2], we see that $L^1(\hat{\nu})$ is also the closure of $L^1(\nu)$ in $L_w^1(\hat{\nu}) = L_w^1(\nu)$. So, $(L^1(\nu))\widehat{}$ can also be identified with $L^1(\hat{\nu})$.

Let $F$ be a Fréchet lattice with topology generated by a fundamental sequence of Riesz seminorms $\{q_n\}_{n \in \mathbb{N}}$. We say that $F$ has the *$\sigma$-Fatou property* if, for every increasing sequence $(u_k)_k$ contained in the positive cone $F^+$ (of $F$) which is topologically bounded in $F$, the element $u := \sup u_k$ exists in $F^+$ and $q_n(u_k) \uparrow_k q_n(u)$, for each $n \in \mathbb{N}$. This terminology is not "standard"; e.g. in [1, p. 94] such an $F$ is called a $\sigma$-Nakano space.

**Theorem 2.7.** *Let $\nu$ be a vector measure taking its values in a metrizable lcHs. Then $L_w^1(\nu)$ is a Fréchet lattice with the $\sigma$-Fatou property.*

**Proof.** Let $\mu$ be any control measure for $\nu$. Fix $n \in \mathbb{N}$. Observe that $\|\cdot\|_\nu^{(n)}$, as given by (2), is a classical function seminorm in $L^0(\mu)$ in the sense of [29, §63]. Since the norm of the $L^1$-space of any positive measure has the Fatou property and $\|\cdot\|_\nu^{(n)}$ is the supremum of such norms (see (2)), it is known that $\|\cdot\|_\nu^{(n)}$ also has the Fatou property (in the sense of [29, §65]); see [29, §65, Theorem 4].

Now, let $(u_k)_k$ be any positive, increasing, topologically bounded sequence in $L_w^1(\nu)$. Then, Theorem 3 of [29, §65] implies that $\|u\|_\nu^{(n)} \leqslant \sup_k \|u_k\|_\nu^{(n)} < \infty$, for $n \in \mathbb{N}$, where $u = \sup_k u_k = \lim_k u_k$ (pointwise). Hence, $u \in L_w^1(\nu)$ by Proposition 2.1. Moreover, $\|u_k\|_\nu^{(n)} \uparrow \|u\|_\nu^{(n)}$ because $\|\cdot\|_\nu^{(n)}$ has the Fatou property as a function seminorm. □

## 3. A representation theorem for Fréchet lattices

Let us begin with a summary of some fundamental properties of spaces of the kind $L^1(\nu)$.

Let $(F, \tau)$ be a Fréchet lattice. A positive element $e$ in $F$ is called a *weak unit* if, for every $u \in F$ we have $u \wedge (ne) \uparrow u$ [13, 14I]. Note, for any vector measure $\nu$ with values in a metrizable lcHs, that the constant function $\chi_\Omega$ is a weak unit for both $L_w^1(\nu)$ and $L^1(\nu)$.

We say that $F$ has a *Lebesgue* (resp. *$\sigma$-Lebesgue*) *topology*, if $u_\alpha \downarrow 0$ implies $u_\alpha \xrightarrow{\tau} 0$ in $F$ (resp. $u_k \downarrow 0$ implies $u_k \xrightarrow{\tau} 0$ in $F$) [1, Definition 8.1]. It is a direct consequence of the Dominated convergence theorem for vector measures [14, p. 30], [16], that if $\nu$ is any vector measure with values in a Fréchet space, then $L^1(\nu)$ always has a $\sigma$-Lebesgue topology. Actually, $L^1(\nu)$ even has a Lebesgue topology. To see this, let $\mu$ be given by (7) and recall that the classical Riesz space $L^0(\mu) = L^0(\nu)$ is always Dedekind complete [18, Example 23.3(iv)]. Since $L^1(\nu)$ is an ideal in $L^0(\nu)$, it follows that $L^1(\nu)$ is also Dedekind complete [18, Theorem 25.2]. It is well known that this property of $L^1(\nu)$, together with a $\sigma$-Lebesgue topology, imply that $L^1(\nu)$ has a Lebesgue topology [1, Theorem 17.9].

The above three properties of $L^1(\nu)$, namely, Dedekind completeness, having a Lebesgue topology and possessing a weak unit, are known to characterize a large class of Fréchet lattices.

**Proposition 3.1.** *Let $(F,\{q_n\}_{n\in\mathbb{N}})$ be a Dedekind complete Fréchet lattice with a Lebesgue topology and having a weak unit $e\in F^+$. Then there is a vector measure $\nu:\Sigma\to F^+$ such that the integration map $I_\nu:L^1(\nu)\to F$, defined by $f\mapsto\int_\Omega f\,d\nu$, for $f\in L^1(\nu)$, is a Fréchet lattice isomorphism of $L^1(\nu)$ onto $F$ satisfying $I_\nu(\chi_\Omega)=e$ and*

$$q_n\big(I_\nu(f)\big)=\|f\|_\nu^{(n)},\quad f\in L^1(\nu),\ n\in\mathbb{N}. \tag{8}$$

For $F$ a Banach lattice, Proposition 3.1 occurs in [2]. In the setting of a Fréchet lattice $F$ we refer to [7, Proposition 2.4(vi)], after reading its proof carefully, together with p. 364 of [7]. A similar but, somewhat different proof of Proposition 3.1 occurs in [21, Theorem 1.22]. Unlike in [7], the proof given in [21] does not rely on the theory of band projections.

An element $u$ of a Fréchet lattice $(F,\tau)$ is *$\sigma$-order continuous* if it has the property that $u_k\xrightarrow{\tau}0$ as $k\to\infty$ for every sequence $(u_k)_k\subseteq F^+$ satisfying $|u|\geqslant u_k\downarrow 0$. The *$\sigma$-order continuous part $F_a$* of $F$ is the collection of all $\sigma$-order continuous elements of $F$; it is a closed ideal in $F$ [30, pp. 331–332], and clearly has a $\sigma$-Lebesgue topology.

**Theorem 3.2.** *For any vector measure $\nu$ taking values in a Fréchet space, $(L^1_w(\nu))_a=L^1(\nu)$.*

**Proof.** As already noted, $L^1(\nu)$ has a $\sigma$-Lebesgue topology. Since $L^1(\nu)$ has the relative topology from $L^1_w(\nu)$, we have $L^1(\nu)\subseteq(L^1_w(\nu))_a$. On the other hand, let $f\in(L^1_w(\nu))_a$ and assume (without loss of generality) that $f\geqslant 0$. Select $\Sigma$-simple functions $(s_k)_k$ such that $0\leqslant s_k\uparrow f$ ($\nu$-a.e.). Then $0\leqslant f-s_k\leqslant|f|$ for all $k$ and $(f-s_k)\downarrow 0$. Hence, $(f-s_k)_k$ converges to 0 in $L^1_w(\nu)$, that is, $(s_k)_k$ converges to $f$ in $L^1_w(\nu)$. But, $(s_k)_k\subseteq L^1(\nu)$ with $L^1(\nu)$ closed in $L^1_w(\nu)$. Hence, $f\in L^1(\nu)$ and so $(L^1_w(\nu))_a\subseteq L^1(\nu)$. □

It is known that every *Banach lattice E* having the $\sigma$-Fatou property and a weak unit which belongs to $E_a$, is Banach lattice isomorphic to $L^1_w(\nu)$ for some vector measure $\nu$ taking values in $E_a^+$ [3, Theorem 2.5]. Our final result extends this fact to the Fréchet lattice setting. The proof proceeds along the lines of that of Theorem 2.5 in [3] but, with various differences due to the more general setting.

**Theorem 3.3.** *Let $(F,\{q_n\}_{n\in\mathbb{N}})$ be any Fréchet lattice with the $\sigma$-Fatou property and possessing a weak unit $e$ which belongs to $F_a$. Then there exists an $F_a^+$-valued vector measure $\nu$ such that $F$ is Fréchet lattice isomorphic to $L^1_w(\nu)$ via an isomorphism $T:L^1_w(\nu)\to F$ which satisfies $T(\chi_\Omega)=e$ and*

$$q_n(Tf)=\|f\|_\nu^{(n)},\quad f\in L^1_w(\nu),\ n\in\mathbb{N}.$$

*Moreover, the restriction map $T|_{L^1(\nu)}=I_\nu$.*

**Proof.** The proof proceeds via a series of steps.

(i) Since $F$ satisfies the $\sigma$-Fatou property, $F$ is $\sigma$-Dedekind complete and hence, so is $F_a$. Therefore, $F_a$ is a $\sigma$-Dedekind complete Fréchet lattice with a $\sigma$-Lebesgue topology. Theorem 17.9 in [1] then guarantees that $F_a$ has a Lebesgue topology and is Dedekind complete. Since $e$ is also a weak unit of $F_a$, Proposition 3.1 ensures that there exists a measurable space $(\Omega,\Sigma)$ and a positive vector measure $\nu:\Sigma\to F_a^+$ such that $F_a$ is Fréchet lattice isomorphic to $L^1(\nu)$ via the integration map $T:=I_\nu$. Moreover, (8) is also satisfied.

(ii) We extend $T$ to $L^1_w(\nu)^+$. Given $0\leqslant f\in L^1_w(\nu)$, choose $\Sigma$-simple functions $0\leqslant f_k\uparrow f$. Since $f_k\in L^1(\nu)$, we have $0\leqslant x_k:=Tf_k\in F_a$ and $q_n(x_k)=\|f_k\|_\nu^{(n)}\leqslant\|f\|_\nu^{(n)}$ for all $k\in\mathbb{N}$ and all $n\in\mathbb{N}$. Hence, $(x_k)_k$ is an increasing, topologically bounded sequence in $F_a\subseteq F$. By the $\sigma$-Fatou property of $F$ the element $x:=\sup_k x_k$ exists in $F^+$ and $q_n(x)=\lim_k q_n(x_k)$. Define $Tf:=x\geqslant 0$. Fix $n\in\mathbb{N}$. By the $\sigma$-Fatou property of $L^1_w(\nu)$ and (8) we have

$$q_n(Tf)=\lim_k q_n(x_k)=\lim_k q_n(Tf_k)=\lim_k\|f_k\|_\nu^{(n)}=\|f\|_\nu^{(n)}.$$

Let us see that this extension of $T$ is well defined. First note that if $0\leqslant h_k\uparrow h$ in $L^1(\nu)$, then $h_k\to h$ in $L^1(\nu)$. Hence, by continuity, $Th_k\to Th$ in $F$ with $(Th_k)_k$ increasing and, consequently, $Th=\sup_k Th_k$ in $F$ [1, Theorem 5.6(iii)]. Now, let $f\in L^1_w(\nu)$ and suppose that $f_k$ and $g_k$ are $\Sigma$-simple functions such that $0\leqslant f_k\uparrow f$ and $0\leqslant g_k\uparrow f$. Then $f_k\wedge g_m\uparrow_k g_m$ for all $m\in\mathbb{N}$ and so $Tg_m=\sup_k T(f_k\wedge g_m)$. Likewise, $Tf_k=\sup_m T(f_k\wedge g_m)$. Therefore, $\sup_k Tf_k=\sup_{m,k}T(f_k\wedge g_m)=\sup_{m,k}Tf_k\wedge Tg_m=\sup_m Tg_m$.

(iii) T is positive, additive and positively homogeneous on $L^1_w(\nu)^+$ and so, can be uniquely extended to a positive linear map $T:L^1_w(\nu)\to F$ in a standard way. Indeed, $T$ is clearly positive and $T(\alpha f)=\alpha Tf$ for all $\alpha\in[0,\infty)$ and $f\in L^1_w(\nu)^+$. To check additivity, let $0\leqslant f,g\in L^1_w(\nu)$ and choose $\Sigma$-simple functions $0\leqslant f_k\uparrow f$ and $0\leqslant g_j\uparrow g$. Define $x_k:=Tf_k$ and $y_j:=Tg_j$ for all $k,j\in\mathbb{N}$. By the definition of $T(f+g)$, $T(f)$ and $T(g)$ and [1, Theorem 1.6] we have

$$T(f+g)=\sup(x_k+y_j)=\sup x_k+\sup y_j=Tf+Tg.$$

(iv) T is a Riesz space homomorphism on $L^1_w(\nu)$ (equivalently, $|Tf| = T|f|$ for all $f \in L^1_w(\nu)$ [1, Theorem 1.17]). Suppose first that $0 \leqslant f, g \in L^1_w(\nu)$ satisfy $f \wedge g = 0$. Choose simple functions $0 \leqslant f_k \uparrow f$ and $0 \leqslant g_k \uparrow g$. Then $f_k \wedge g_k = 0$ for all $k \in \mathbb{N}$. Since $T$ is a Riesz space isomorphism of $L^1(\nu)$ onto $F_a$, it follows that $Tf_k \wedge Tg_k = T(f_k \wedge g_k) = T(0) = 0$ for all $k \in \mathbb{N}$ and hence, via [18, Theorem 15.3], that $Tf \wedge Tg = (\sup Tf_k) \wedge (\sup Tg_k) = \sup(Tf_k \wedge Tg_k) = 0$. For arbitrary $f \in L^1_w(\nu)$ we have that $f^+, f^- \in L^1_w(\nu)^+$ with $f^+ \wedge f^- = 0$ and so the previous argument yields $Tf^+ \wedge Tf^- = 0$. Hence, $T$ is a Riesz space homomorphism on $L^1_w(\nu)$ [20, Proposition 1.3.11].

(v) Fix $n \in \mathbb{N}$. Let $f \in L^1_w(\nu)$. From (iv) it follows that $q_n(Tf) = q_n(|Tf|) = q_n(T|f|)$. But, in (ii) it was shown that $T$ satisfies $q_n(T|f|) = \||f|\|^{(n)}_\nu = \|f\|^{(n)}_\nu$ (because $|f| \in L^1_w(\nu)^+$). Therefore,

$$q_n(Tf) = \|f\|^{(n)}_\nu, \quad f \in L^1_w(\nu).$$

Since this holds for every $n \in \mathbb{N}$, we see that $T$ is also injective.

(vi) $T$ is surjective. Fix $x \in F^+$. Since $e$ is a weak unit of $F$, we have $x_k \uparrow x$, where $x_k = x \wedge (ke) \geqslant 0$, for $k \in \mathbb{N}$. Moreover, $q_n(x_k) \uparrow_k q_n(x)$, for $n \in \mathbb{N}$, as $F$ has the $\sigma$-Fatou property. Since $e \in F_a$ and $F_a$ is an ideal, it is clear that $(x_k)_k \subseteq F_a$. But, $T$ is a Riesz space isomorphism of $L^1(\nu)$ onto $F_a$ and so there is an increasing sequence $(f_k)_k \subseteq L^1(\nu)^+$ such that $x_k = Tf_k$ for $k \in \mathbb{N}$. Fix $n \in \mathbb{N}$. By (v) it follows that $\|f_k\|^{(n)}_\nu = q_n(Tf_k) = q_n(x_k) \leqslant q_n(x)$, for $k \in \mathbb{N}$, and so $\sup_k \|f_k\|^{(n)}_\nu \leqslant q_n(x) < \infty$. Hence, the $\sigma$-Fatou property of $L^1_w(\nu)$ ensures that $f = \sup f_k \in L^1_w(\nu)$ and $\|f_k\|^{(n)}_\nu \uparrow \|f\|^{(n)}_\nu$. From the definition of the extension we have $Tf = x$. For arbitrary $x \in F$ there exist $f, g \in L^1_w(\nu)^+$ such that $x^+ = Tf$ and $x^- = Tg$. So, $x = T(f - g)$. □

**Example 3.4.** Any increasing sequence $A = (a_n)_n$ of functions $a_n : \mathbb{N} \to (0, \infty)$ is called a Köthe matrix on $\mathbb{N}$, where increasing means $0 < a_n \leqslant a_{n+1}$ pointwise on $\mathbb{N}$, for each $n \in \mathbb{N}$. The *Köthe echelon space* $\lambda_\infty(A)$ is the vector space

$$\lambda_\infty(A) := \{x \in \mathbb{R}^{\mathbb{N}}: \ a_n x \in \ell_\infty \text{ for all } n = 1, 2, \ldots\},$$

equipped with the increasing sequence of solid Riesz seminorms

$$\|x\|_k := \sup_{m \in \mathbb{N}} a_k(m)|x_m|, \quad x = (x_m) \in \lambda_\infty(A).$$

Of course, the order in $\lambda_\infty(A)$ is the pointwise order on $\mathbb{N}$. Then $\lambda_\infty(A)$ is a Fréchet lattice and $(\lambda_\infty(A))_a$ is the *proper* closed ideal

$$\lambda_0(A) := \{x \in \lambda_\infty(A): \ a_n x \in c_0 \text{ for all } n = 1, 2, \ldots\}.$$

It is routine to check that $\lambda_\infty(A)$ has the $\sigma$-Fatou property and contains a weak unit $e \in \lambda_0(A)^+$. Indeed, any $e \in \lambda_0(A)^+$ satisfying $e_m > 0$ for all $m \in \mathbb{N}$ suffices. According to Theorem 3.3, the space $\lambda_\infty(A)$ is Fréchet lattice isomorphic to $L^1_w(\nu)$ for some vector measure $\nu$. However, since $\lambda_\infty(A)$ does not have a Lebesgue topology, it cannot be Fréchet lattice isomorphic to $L^1(\eta)$ for any vector measure $\eta$.

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