Multiplication operators on spaces of integrable functions with respect to a vector measure

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Abstract

We study continuity and other properties related to some kind of compactness of multiplication operators between different spaces of pth power integrable scalar functions with respect to a vector measure.

Keywords: Vector measure; Integrable function; Multiplication operator; Weakly compact operator

1. Introduction

The study of multiplication operators

$$M_g : f \in \mathcal{F} \to M_g(f) := gf \in \mathcal{G} \tag{1}$$

between function spaces $\mathcal{F}$ and $\mathcal{G}$ has a very long history, mainly when the spaces $\mathcal{F}$ and $\mathcal{G}$ are spaces of continuous, holomorphic or analytic functions. But there has been relatively little study of multiplication operators between *Banach measurable functions spaces*. Takagi and Yokouchi [15] studied continuity and closedness of range of multiplication and composition operators between different L^p spaces over a σ-finite measure space. More recently Sirotkin [13] characterizes when a multiplication operator on a Banach function space is *compact-friendly*, a concept related to the problem of existence of invariant subspaces. See [1] and [2] for more information on this topic and its relationship with multiplication operators. We consider here multiplication operators between spaces of pth power

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integrable functions with respect to a vector measure. These spaces of integrable functions provide an interesting tool to obtain representation theorems for a large class of abstract Banach lattices. See [5,6,8].

The aim of the present paper is to describe several properties of multiplication operators (1) when $\mathcal{F}$ and $\mathcal{G}$ are different L^p-spaces of a vector measure in terms of the function g, as continuity and some others related to differents kinds of compactness. Let us start with some notation and essential definitions about integration with respect to vector measures.

Let $m: \Sigma \to X$ be a vector measure defined on a σ-algebra of subsets Σ of a nonempty set Ω; this will always mean that m is countably additive on Σ with values in a real Banach space X. We denote by X' its dual space, and by $X'' := (X')'$. Also $B_1(X)$ denotes the unit ball of X. The semivariation of m is the set function $\|m\|: \Sigma \to [0, \infty)$ defined by

$$\|m\|(A) := \sup\{|\langle m, x'\rangle|(A) \colon x' \in B_1(X')\}, \quad A \in \Sigma,$$

where $|\langle m, x'\rangle|$ is the total variation measure of the scalar measure $\langle m, x'\rangle$ given by $\langle m, x'\rangle(A) := \langle m(A), x'\rangle$, for all $A \in \Sigma$. A set $A \in \Sigma$ is called m-null if $\|m\|(A) = 0$. Let $L^0(m)$ be the space of all $\mathbb{R}$-valued Σ-measurable functions on Ω. Two functions $f, g \in L^0(m)$ are identified if they are equal m-a.e., that is, if $\{w \in \Omega\colon f(w) \neq g(w)\}$ is an m-null set. A function $f \in L^0(m)$ is called *weakly integrable* (with respect to m) if $f \in L^1(|\langle m, x'\rangle|)$, for all $x' \in X'$. In this case (see [14, Corollary 3]) for each $A \in \Sigma$ there exists an element $\int_A f\,dm \in X''$ (called the *weak integral* of f over A) such that $\langle \int_A f\,dm, x'\rangle = \int_A f\,d\langle m, x'\rangle$, for all $x' \in X'$. The space $L^1_w(m)$ of all (equivalence classes of) weakly integrable functions becomes a Banach lattice when it is endowed with the natural order m-a.e., and the norm

$$\|f\|_{L^1_w(m)} := \sup\left\{\int_\Omega |f|\,d|\langle m, x'\rangle| \colon x' \in B_1(X')\right\}, \quad f \in L^1_w(m).$$

We say that a weakly integrable function f is *integrable* (with respect to m) if the vector $\int_A f\,dm \in X$, for all $A \in \Sigma$ (see [9,10]). The set $L^1(m)$ of all (equivalence classes of) integrable functions becomes an order continuous closed lattice ideal of $L^1_w(m)$. In general, we have $L^1(m) \subsetneq L^1_w(m)$. Associated with the Banach lattice $L^1(m)$ is the integration operator $I: L^1(m) \to X$ given by $f \mapsto \int_\Omega f\,dm$. The operator I is always linear and continuous.

Now, if $1 < p < \infty$, we say that a measurable function $f: \Omega \to \mathbb{R}$ *is weakly p-integrable* (with respect to m) if $|f|^p \in L^1_w(m)$, and *p-integrable* with respect to m if $|f|^p \in L^1(m)$. We denote by $L^p(m)$ the space of (equivalence classes of) p-integrable functions and by $L^p_w(m)$ the space of (equivalence classes of) weakly p-integrable functions. Obviously we have $L^p(m) \subseteq L^p_w(m)$. The natural norm for both spaces is given by

$$\|f\|_{L^p_w(m)} := \sup\left\{\left(\int_\Omega |f|^p\,d|\langle m, x'\rangle|\right)^{\frac{1}{p}} \colon x' \in B_1(X')\right\}, \quad f \in L^p_w(m).$$

We know neither $L^p(m)$ nor $L^p_w(m)$ are reflexive spaces even if $p > 1$. See [8] for a detailed study of the relationship between the spaces $L^p(m)$ and $L^p_w(m)$. In particular, the inclusion $L^p_w(m) \subseteq L^1(m)$ holds for all $p > 1$. Moreover this embedding operator is L-weakly compact (see below for the definition).

We also consider the space $L^\infty(m)$ of (equivalence classes of) essentially bounded functions (modulo m-a.e.) equipped with the supremum norm $\|\cdot\|_{L^\infty(m)}$. The inclusion $L^\infty(m) \subseteq L^1(m)$ holds and $\|f\|_{L^1(m)} \leqslant \|f\|_{L^\infty(m)}\|m\|(\Omega)$, for all f in $L^\infty(m)$.

Our references for Banach lattices are [3,11,12]. Nevertheless, let us introduce some notation and recall some basic definitions about the classes of operators that we will consider through this paper. If E and F are Banach lattices we denote by $\mathcal{B}(E, F)$ the Banach space of all linear and continuous operators from E into F. We will consider several operator classes in $\mathcal{B}(E, F)$ (see [12, 3.6.9]); namely an operator $T \in \mathcal{B}(E, F)$ is said to be in:

- $\mathcal{S}(E, F)$, if $T(B_1(E))$ is *approximately order bounded* in F, that is, for every $\varepsilon > 0$ there exists $0 \leqslant g \in F$ such that $T(f) \in [-g, g] + \varepsilon B_1(F)$, for all $f \in B_1(E)$. Operators in $\mathcal{S}(E, F)$ are called *semi-compact* operators.
- $\mathcal{L}(E, F)$, if $T(B_1(E))$ is *L-weakly compact* in F, that is, $\|g_n\|_F \to 0$ for every disjoint sequence $(g_n)_n$ contained in the solid hull of $T(B_1(E))$. Operators in $\mathcal{L}(E, F)$ are called *L-weakly compact* operators.
- $\mathcal{M}(E, F)$, if $\|T(f_n)\|_F \to 0$ for all disjoint sequences $(f_n)_n$ in $B_1(E)$. Operators in $\mathcal{M}(E, F)$ are called *M-weakly compact* operators.

Finally we denote by $\mathcal{K}(E,F)$ and $\mathcal{W}(E,F)$ the ideals of compact and weakly compact operators, respectively. It is known (see [12, Proposition 3.6.12]) that $\mathcal{L}(E,F) \subseteq \mathcal{W}(E,F)$ and $\mathcal{M}(E,F) \subseteq \mathcal{W}(E,F)$. Moreover $\mathcal{L}(E,F) \subseteq \mathcal{S}(E,F)$ and these classes coincide if F has order continuous norm (see [12, Proposition 3.6.10]).

For a given function $g \in L^0(m)$, we can always consider the multiplication operator

$$M_g : f \in L^0(m) \to M_g(f) = gf \in L^0(m).$$

In the following sections of this work we will study multiplication operators from $L^p(m)$ to $L^1(m)$, with $p > 1$, in Section 2 and also from $L^p(m)$ to $L^p(m)$, with $p \geqslant 1$ in Section 3. The behavior of the multiplication operator from $L^1(m)$ into $L^p(m)$, with $p > 1$, is quite different and will be considered elsewhere.

2. Multiplication operators from L^p into L^1, with $p > 1$

We begin by proving some basic facts which will be used throughout what follows.

Lemma 1. *If $p, q > 1$ are conjugated exponents, then*

(A) $L^q(m) \cdot L^p(m) = L^q_w(m) \cdot L^p(m) = L^q(m) \cdot L^p_w(m) = L^1(m)$.
(B) $L^q_w(m) \cdot L^p_w(m) = L^1_w(m)$.

Proof. By symmetry on the exponents p and q, (A) can be reduced to prove $L^q(m) \cdot L^p(m) = L^q_w(m) \cdot L^p(m) = L^1(m)$. Since $L^q(m) \subseteq L^q_w(m)$, we have $L^q(m) \cdot L^p(m) \subseteq L^q_w(m) \cdot L^p(m)$. Moreover, given $g \in L^q_w(m)$ and $f \in L^p(m)$ let us see that $gf \in L^1(m)$. Taking a sequence $(\varphi_n)_n$ of simple functions such that $\|f - \varphi_n\|_{L^p(m)} \to 0$, we get $g\varphi_n \in L^q_w(m) \subseteq L^1(m)$, for all $n = 1, 2, \ldots,$ and using Hölder's inequality yields

$$\|gf - g\varphi_n\|_{L^1_w(m)} = \big\|g(f - \varphi_n)\big\|_{L^1_w(m)} \leqslant \|g\|_{L^q_w(m)} \|f - \varphi_n\|_{L^p(m)}.$$

Therefore, $\|gf - g\varphi_n\|_{L^1_w(m)} \to 0$ and, since $L^1(m)$ is closed in $L^1_w(m)$ we conclude that $gf \in L^1(m)$. Finally, if $f \in L^1(m)$, then

$$f = \operatorname{sign}(f)|f| = \operatorname{sign}(f)|f|^{\frac{1}{q}} \cdot |f|^{\frac{1}{p}} = \big(\operatorname{sign}(f)|f|^{\frac{1}{q}}\big) \cdot |f|^{\frac{1}{p}},$$

with $(\operatorname{sign}(f)|f|^{\frac{1}{q}}) \in L^q(m)$ and $|f|^{\frac{1}{p}} \in L^p(m)$.

The proof of (B) is immediate. $\square$

For conjugated exponents $p, q > 1$ and $g \in L^q_w(m)$, Lemma 1 guarantees that the multiplication operator $M_g : L^p(m) \to L^1(m)$ is well-defined. Moreover, if $g \in L^q(m)$, then the multiplication operator $M_g : L^p_w(m) \to L^1(m)$ is also well-defined. In fact, both these operators are continuous:

Lemma 2. *If $p, q > 1$ are conjugated exponents and $g \in L^q(m)$, then*

1) $M_g \in \mathcal{B}(L^p_w(m), L^1(m))$.
2) $M_g \in \mathcal{B}(L^p(m), L^1(m))$.

In both cases, the norm of the operator M_g coincides with $\|g\|_{L^q(m)}$.

Proof. Part 2) follows directly from part 1) taking into account the continuity of the inclusion $L^p(m) \subseteq L^p_w(m)$.

Let us prove part 1) by the Closed Graph Theorem. Assume that $(f_n)_n \subset L^p_w(m)$ is a sequence such that $f_n \to f$ in $L^p_w(m)$, and $M_g(f_n) = gf_n \to h$ in $L^1(m)$. Using a Rybakov control measure [7, Theorem IX.2.2] for m and replacing the sequences by subsequences if necessary, we can certainly assume that $f_n \to f$ and $gf_n \to h$ pointwise m-a.e. Thus, we have $gf_n \to gf$ pointwise m-a.e. and hence $h = gf$. This proves that the graph of M_g is closed and so $M_g \in \mathcal{B}(L^p_w(m), L^1(m))$.

Moreover, $\|M_g(f)\|_{L^1(m)} \leqslant \|f\|_{L^p_w(m)} \|g\|_{L^q(m)}$ for all $f \in L^p_w(m)$, by Hölder's inequality and, if we consider the function $h := \frac{|g|^{q-1}}{\|g\|^{q/p}_{L^q(m)}}$, then it is easy to show that $h \in L^p(m) \subset L^p_w(m)$ and $\|M_g(h)\|_{L^1(m)} = \|g\|_{L^q(m)}$. Therefore $\|M_g\| = \|g\|_{L^q(m)}$. $\square$

We will also need the following lemma. Its proof can be found in [8].

Lemma 3. *Let $q > 1$ and $(g_n)_n$ be a norm bounded, positive, increasing sequence in $L^q_w(m)$. Then $g := \sup_n g_n$ exists and $g \in L^q_w(m)$.*

Theorem 4. *Let $p, q > 1$ be conjugated exponents and let $g \in L^0(m)$. The following conditions are equivalent*:

1) $g \in L^q_w(m)$.
2) $M_g \in \mathcal{B}(L^p(m), L^1(m))$.
3) $M_g \in \mathcal{B}(L^p(m), L^1_w(m))$.
4) $M_g \in \mathcal{B}(L^p_w(m), L^1_w(m))$.

Moreover, in such a case, the norm of the operator M_g coincides with $\|g\|_{L^q_w(m)}$.

Proof. Let us consider the measurable sets $A_n := \{w \in \Omega\colon |g(w)| \leqslant n\}$ and the bounded functions $0 \leqslant g_n := |g|\chi_{A_n}$, for all $n = 1, 2, \dots$. It is clear that $g_n \in L^q(m) \subset L^q_w(m)$, $g_n \uparrow |g|$ pointwise m-a.e., $|g| = \sup_n g_n$ and, by Lemma 2, $\|g_n\|_{L^q(m)} = \|M_{g_n}\|$, for all $n = 1, 2, \dots$.

The proof of 1) $\Rightarrow$ 2) is analogous to the one of Lemma 2. Let us see the implication 2) $\Rightarrow$ 1). If $f \in L^p(m)$, then $g_n f \in L^1(m)$, and by hypothesis 2), $|g|f \in L^1(m)$. Moreover, $g_n f \to |g|f$ pointwise m-a.e. and, since $L^1(m)$ has order continuous norm, it follows that $g_n f \to |g|f$ in $L^1(m)$, that is, $M_{g_n}(f) \to M_{|g|}(f)$ in $L^1(m)$. Therefore, the Banach–Steinhaus Theorem guarantees that $\sup_n \|g_n\|_{L^q(m)} = \sup_n \|M_{g_n}\| < \infty$, and hence, Lemma 3 gives $|g| \in L^q_w(m)$, that is, $g \in L^q_w(m)$.

The implication 1) $\Rightarrow$ 4) follows from part (B) of Lemma 1 and the Closed Graph Theorem, and 4) $\Rightarrow$ 3) is evident. We now pass to prove 3) $\Rightarrow$ 1). Given $f \in L^p(m)$ and $n = 1, 2, \dots$, we have

$$\big\|M_{g_n}(f)\big\|_{L^1_w(m)} = \|g_n f\|_{L^1_w(m)} = \big\||g_n||f|\big\|_{L^1_w(m)} \leqslant \big\||g||f|\big\|_{L^1_w(m)} = \big\|M_g(f)\big\|_{L^1_w(m)} \leqslant \|M_g\|\|f\|_{L^p(m)}.$$

Therefore, $\|M_{g_n}\| \leqslant \|M_g\|$, for all $n = 1, 2, \dots$, and hence

$$\sup_n \|g_n\|_{L^q(m)} = \sup_n \|M_{g_n}\| < \infty.$$

Applying Lemma 3, we conclude that $|g| \in L^q_w(m)$, that is, $g \in L^q_w(m)$.

Let us see now that $\|M_g\| = \|g\|_{L^q_w(m)}$. On the one hand, from Hölder's inequality we have $\|M_g(f)\|_{L^1(m)} \leqslant \|g\|_{L^q_w(m)}\|f\|_{L^p(m)}$, for all $f \in L^p(m)$, and hence $\|M_g\| \leqslant \|g\|_{L^q_w(m)}$. On the other hand, since $L^q_w(m)$ has the Fatou property [6, Proposition 1] it follows that $\|g\|_{L^q_w(m)} = \sup_n \|g_n\|_{L^q(m)} = \sup_n \|M_{g_n}\| \leqslant \|M_g\|$. Thus, $\|M_g\| = \|g\|_{L^q_w(m)}$ as claimed. □

Theorem 5. *Let $p, q > 1$ be conjugated exponents and let $g \in L^0(m)$. The following conditions are equivalent*:

1) $g \in L^q(m)$.
2) $M_g \in \mathcal{B}(L^p_w(m), L^1(m))$.

Proof. 1) $\Rightarrow$ 2). This is proved in Lemma 2.

2) $\Rightarrow$ 1). Since the inclusion $L^p(m) \subseteq L^p_w(m)$ is continuous, the last theorem gives $g \in L^q_w(m)$. Let us see that, in fact, $g \in L^q(m)$. Since $g \in L^q_w(m)$ we have $|g|^{\frac{q}{p}} \in L^p_w(m)$, and hence $M_g(|g|^{\frac{q}{p}}) = g|g|^{\frac{q}{p}} \in L^1(m)$. Therefore, $|g||g|^{\frac{q}{p}} \in L^1(m)$, but $|g||g|^{\frac{q}{p}} = |g|^q$. Thus, we conclude that $g \in L^q(m)$. □

Remark 6. It is clear that it is not possible to add any other (equivalent) continuity condition in the last theorem. A glance to Theorems 4 and 5 makes evident that the continuity of the multiplication operator possesses some kind of asymmetry with respect to the domains of definition. However, this asymmetry disappears when we study compact conditions instead of continuity conditions on the multiplication operator, as we next see.

Theorem 7. *Let $p, q > 1$ be conjugated exponents and let $g \in L^0(m)$. The following conditions are equivalent*:

1) $g \in L^q(m)$.
2) $M_g \in \mathcal{B}(L^p_w(m), L^1(m))$.
3) $M_g \in \mathcal{L}(L^p_w(m), L^1(m))$.
4) $M_g \in \mathcal{L}(L^p(m), L^1(m))$.
5) $M_g \in \mathcal{L}(L^p_w(m), L^1_w(m))$.
6) $M_g \in \mathcal{L}(L^p(m), L^1_w(m))$.
7) $M_g \in \mathcal{M}(L^p_w(m), L^1(m))$.
8) $M_g \in \mathcal{M}(L^p(m), L^1(m))$.
9) $M_g \in \mathcal{M}(L^p_w(m), L^1_w(m))$.
10) $M_g \in \mathcal{M}(L^p(m), L^1_w(m))$.
11) $M_g \in \mathcal{W}(L^p_w(m), L^1(m))$.
12) $M_g \in \mathcal{W}(L^p(m), L^1(m))$.
13) $M_g \in \mathcal{W}(L^p_w(m), L^1_w(m))$.
14) $M_g \in \mathcal{W}(L^p(m), L^1_w(m))$.
15) $M_g \in \mathcal{S}(L^p_w(m), L^1(m))$.
16) $M_g \in \mathcal{S}(L^p(m), L^1(m))$.
17) $M_g \in \mathcal{S}(L^p_w(m), L^1_w(m))$.
18) $M_g \in \mathcal{S}(L^p(m), L^1_w(m))$.

Proof. The equivalence 1) $\Leftrightarrow$ 2) is just Theorem 5 and the implication 3) $\Rightarrow$ 2) is evident.

1) $\Rightarrow$ 3). We already know that $M_g \in \mathcal{B}(L^p_w(m), L^1(m))$ so we only need to show that $M_g(B_1(L^p_w(m)))$ is an L-weakly compact set in $L^1(m)$. Since $M_g(B_1(L^p_w(m)))$ is norm-bounded and solid in $L^1(m)$ it is sufficient to prove that $\|h_n\|_{L^1(m)} \to 0$ for every disjoint sequence $(h_n)_n \subseteq M_g(B_1(L^p_w(m)))$. Let us consider the disjoint measurable sets $A_n := \{w \in \Omega\colon h_n(w) \neq 0\}$, for $n = 1, 2, \ldots$. Thus, $h_n = M_g(f_n) = gf_n = gf_n\chi_{A_n} = g\chi_{A_n} f_n$ for some sequence $(f_n)_n \subseteq B_1(L^p_w(m))$. From Hölder's inequality we deduce that $\|h_n\|_{L^1(m)} = \|M_g(f_n)\|_{L^1(m)} = \|g\chi_{A_n} f_n\|_{L^1(m)} \leqslant \|g\chi_{A_n}\|_{L^q(m)}\|f_n\|_{L^p_w(m)} \leqslant \|g\chi_{A_n}\|_{L^q(m)}$, but $\|g\chi_{A_n}\|_{L^q(m)} \to 0$ since $(g\chi_{A_n})_n$ is an order bounded disjoint sequence in $L^q(m)$ and the space $L^q(m)$ is order continuous.

3) $\Rightarrow$ 4). The inclusion $L^p(m) \subseteq L^p_w(m)$ is continuous and the composition of a continuous operator (to the right) with an L-weakly compact operator (to the left) is an L-weakly compact operator.

Implications 5) $\Rightarrow$ 6), 11) $\Rightarrow$ 12), 13) $\Rightarrow$ 14) and 17) $\Rightarrow$ 18) follow by the same argument.

4) $\Rightarrow$ 1). In particular, 4) implies that $M_g \in \mathcal{B}(L^p(m), L^1(m))$ and thus Theorem 4 yields $g \in L^q_w(m)$. To prove that $g \in L^q(m)$, let us consider the measurable sets $A_k := \{w \in \Omega\colon k-1 \leqslant |g(w)| < k\}$, for $k = 1, 2, \ldots,$ and let us denote by B_n the set $\bigcup_{k=1}^n A_k$, for all $n = 1, 2, \ldots$. Note that $(A_k)_k$ is a disjoint sequence of measurable sets and $g_n := |g|\chi_{B_n} \uparrow |g|$ pointwise m-a.e. It is sufficient to show that $(g_n)_n$ is a Cauchy sequence in $L^q(m)$. Suppose, contrary to our claim, that there exist $\varepsilon > 0$ and two increasing sequences $(n_k)_k$ and $(m_k)_k$ in $\mathbb{N}$, with $m_1 < n_1 < m_2 < n_2 < \cdots < m_k < n_k < \cdots$, such that $\|g_{n_k} - g_{m_k}\|_{L^q(m)} \geqslant \varepsilon$, for all $k = 1, 2, \ldots$. Let us consider the disjoint measurable sets $C_k := \bigcup_{i=m_k+1}^{n_k} A_i$, and the disjoint measurable functions $f_k := \frac{|g|^{q-1}}{\|g\|^{q/p}_{L^q_w(m)}}\chi_{C_k}$ for $k = 1, 2, \ldots$.

On the one hand, since $g_{n_k} - g_{m_k} = |g|\chi_{C_k}$, we obtain

$$\|g\chi_{C_k}\|_{L^q(m)} > \varepsilon, \quad k = 1, 2, \ldots. \tag{2}$$

On the other hand, since $q = p(q-1)$, it follows that $|f_k|^p = \frac{|g|^q}{\|g\|^q_{L^q_w(m)}}\chi_{C_k}$ and hence $\|f_k\|_{L^p(m)} \leqslant 1$, that is, $(f_k)_k$ is in the unit ball of $L^p(m)$. By hypothesis we deduce that $\|M_g(f_k)\|_{L^1(m)} \to 0$, but

$$M_g(f_k) = gf_k = g\frac{|g|^{q-1}}{\|g\|^{\frac{q}{p}}_{L^q_w(m)}}\chi_{C_k} = \operatorname{sign}(g)\frac{|g|^q}{\|g\|^{\frac{q}{p}}_{L^q_w(m)}}\chi_{C_k},$$

and hence $\||g|^q\chi_{C_k}\|_{L^1(m)} = \|g\|^{\frac{q}{p}}_{L^q_w(m)}\|M_g(f_k)\|_{L^1(m)}$. Therefore,

$$\|g\chi_{C_k}\|_{L^q(m)} = \|g\|^{\frac{1}{p}}_{L^q_w(m)}\|M_g(f_k)\|^{\frac{1}{q}}_{L^1(m)} \to 0,$$

contrary to (2).

3) $\Rightarrow$ 5). It is obvious, since $L^1(m) \subseteq L^1_w(m)$ with coinciding norms.

Implications 11) $\Rightarrow$ 13), 15) $\Rightarrow$ 17) and 16) $\Rightarrow$ 18) follow by the same reasoning.

6) $\Rightarrow$ 4). If $M_g \in \mathcal{L}(L^p(m), L^1_w(m))$, then, in particular, we have $M_g \in \mathcal{B}(L^p(m), L^1_w(m))$. Theorem 4 implies $M_g \in \mathcal{B}(L^p(m), L^1(m))$ and hence $M_g(L^p(m)) \subseteq L^1(m)$ which gives $M_g \in \mathcal{L}(L^p(m), L^1(m))$.

Implications 14) $\Rightarrow$ 12) and 18) $\Rightarrow$ 16) can be deduced by the same arguments.

3) $\Rightarrow$ 7). If $(f_n)_n \subseteq B_1(L^p_w(m))$ is a disjoint sequence, then so is $(gf_n)_n \subseteq M_g(B_1(L^p_w(m)))$, and consequently $\|M_g(f_n)\|_{L^1(m)} \to 0$.

7) $\Rightarrow$ 3). Let $G := \{w \in \Omega\colon\ g(w) \neq 0\}$ be the support of g. If $(gf_n)_n$ is a disjoint sequence in $M_g(B_1(L^p_w(m)))$, with $(f_n)_n \subseteq B_1(L^p_w(m))$, then $(\chi_G f_n)_n$ is a disjoint sequence in $B_1(L^p_w(m))$. Moreover, $M_g(\chi_G f_n) = g\chi_G f_n = gf_n$ and hence, by hypothesis we deduce that $\|gf_n\|_{L^1(m)} = \|M_g(\chi_G f_n)\|_{L^1(m)} \to 0$.

The same arguments used to prove equivalence 3) $\Leftrightarrow$ 7) apply to the equivalences 4) $\Leftrightarrow$ 8), 5) $\Leftrightarrow$ 9), and 6) $\Leftrightarrow$ 10).

3) $\Rightarrow$ 11). Every L-weakly compact operator is weakly compact.

11) $\Rightarrow$ 2). Every weakly compact operator is continuous.

12) $\Rightarrow$ 1). If $M_g \in \mathcal{W}(L^p(m), L^1(m))$, then, in particular, we have $M_g \in \mathcal{B}(L^p(m), L^1(m))$ and hence Theorem 4 yields $g \in L^q_w(m)$. To see that, in fact, $g \in L^q(m)$, let us define the measurable sets

$$A_k := \{w \in \Omega\colon\ k-1 \leqslant |g(w)|^q < k\}, \quad k = 1, 2, \ldots,$$

and the sequence $(S_n)_n$ given by

$$S_n := \sum_{k=1}^{n} \int_{A_k} |g|^q\,dm = \sum_{k=1}^{n} \int_{\Omega} |g|^q \chi_{A_k}\,dm = \int_{\Omega} \sum_{k=1}^{n} |g|^q \chi_{A_k}\,dm = \int_{\Omega} |g| \sum_{k=1}^{n} |g|^{q-1} \chi_{A_k}\,dm. \tag{3}$$

Writing $f_n := \operatorname{sign}(g) \sum_{k=1}^{n} |g|^{q-1} \chi_{A_k}$, we have $S_n = \int_\Omega gf_n\,dm$, for $n = 1, 2, \ldots$, that is, $S_n = I \circ M_g(f_n)$, where I is the integration operator

$$I : h \in L^1(m) \to I(h) := \int_{\Omega} h\,dm \in X.$$

The continuity of the integration operator and the weak compactness of multiplication operator guarantee that $I \circ M_g \in \mathcal{W}(L^p(m), X)$. Since $|f_n|^p \leqslant |g|^q$, it is easy to check that inequality $\|f_n\|_{L^p(m)} \leqslant \|g\|^{\frac{q}{p}}_{L^q_w(m)}$ holds, that is, the sequence $(f_n)_n$ is included in a multiple of $B_1(L^p(m))$ and hence $(S_n)_n$ is contained in a relatively weakly compact subset of X. Therefore, there exists a subsequence $(S_{n_k})_k$ which is weakly convergent to some x_0 in X.

In addition, from (3) we deduce that, for every $x' \in X'$,

$$\begin{aligned}
\langle S_n, x' \rangle &= \left\langle \sum_{k=1}^{n} \int_{A_k} |g|^q\,dm, x' \right\rangle = \sum_{k=1}^{n} \int_{A_k} |g|^q\,d\langle m, x' \rangle \to \sum_{k=1}^{\infty} \int_{A_k} |g|^q\,d\langle m, x' \rangle \\
&= \int_{\Omega} |g|^q\,d\langle m, x' \rangle = \left\langle \int_{\Omega} |g|^q\,dm, x' \right\rangle
\end{aligned}$$

and hence $(S_n)_n$ converges in the weak* topology of X'' to $\int_\Omega |g|^q\,dm \in X''$. Since the weak* topology of X'' coincides in X with the weak topology of X, we conclude that $\int_\Omega |g|^q\,dm = x_0 \in X$.

Given any measurable set A, we now apply the above argument again, with g replaced by $g\chi_A$, to obtain $\int_A |g|^q\,dm \in X$, for all $A \in \Sigma$. Thus, we have proved that $|g|^q \in L^1(m)$, that is, $g \in L^q(m)$, as claimed.

Finally, since semi-compact operators and L-weakly compact operators coincide when the final space has order continuous norm, we obtain the equivalences 3) $\Leftrightarrow$ 15), and 4) $\Leftrightarrow$ 16). □

3. Multiplication operators from L^p into L^p, with $p \geqslant 1$

Theorem 8. *Let $g \in L^0(m)$, the following assertions are equivalent.*

1) $g \in L^\infty(m)$.
2) $M_g \in \mathcal{B}(L^p(m), L^p(m))$, *for all* $p \geqslant 1$.
3) $M_g \in \mathcal{B}(L^p(m), L^p(m))$, *for some* $p \geqslant 1$.
4) $M_g \in \mathcal{B}(L^p_w(m), L^p_w(m))$, *for all* $p \geqslant 1$.

5) $M_g \in \mathcal{B}(L^p_w(m), L^p_w(m))$, *for some* $p \geqslant 1$.

6) $M_g \in \mathcal{B}(L^p(m), L^p_w(m))$, *for all* $p \geqslant 1$.

7) $M_g \in \mathcal{B}(L^p(m), L^p_w(m))$, *for some* $p \geqslant 1$.

Moreover, in case of equivalence, we have $\|M_g\| = \|g\|_{L^\infty(m)}$.

Proof. 1) $\Rightarrow$ 2). If $p \geqslant 1$, it is well known that $L^\infty(m) \cdot L^p(m) \subseteq L^p(m)$, and the operator M_g is well-defined. The Closed Graph Theorem assures that M_g is continuous. Moreover, the following inequality holds $\|M_g(f)\|_{L^p(m)} \leqslant \|g\|_{L^\infty(m)}\|f\|_{L^p(m)}$, for all f in $L^p(m)$.

2) $\Rightarrow$ 3). It is obvious.

3) $\Rightarrow$ 1). Let g be in $L^0(m)$ and suppose that $M_g \in \mathcal{B}(L^p(m), L^p(m))$ for some $p \geqslant 1$, thus $\|M_g\| < \infty$. We shall prove that $|g| \leqslant \|M_g\|$ m-a.e. Thus we will obtain that $g \in L^\infty(m)$ and also the inequality $\|g\|_{L^\infty(m)} \leqslant \|M_g\|$. It suffices to prove that, if $\varepsilon > 0$, then the set $A_\varepsilon := \{w \in \Omega\colon\ |g(w)| > \|M_g\| + \varepsilon\}$, is m-null, that is, $\|m\|(A_\varepsilon) = 0$. To this end we consider the increasing sequence (to Ω) of measurable sets $A_n := \{w \in \Omega\colon\ |g(w)| \leqslant n\}$, for all $n = 1, 2, \ldots$, and the sequence of bounded functions $f_n := \operatorname{sign}(g)\chi_{A_n \cap A_\varepsilon}$, for $n = 1, 2, \ldots$. Note that $\|f_n\|_{L^p(m)} = (\|m\|(A_n \cap A_\varepsilon))^{\frac{1}{p}}$. By the continuity of the operator M_g we deduce that

$$\big\|M_g(f_n)\big\|_{L^p(m)} \leqslant \|M_g\|\big(\|m\|(A_n \cap A_\varepsilon)\big)^{\frac{1}{p}}, \quad n = 1, 2, \ldots. \tag{4}$$

On the other hand we have that $M_g(f_n) = |g|\chi_{A_n \cap A_\varepsilon}$ and in this way we have

$$\begin{aligned}
\big\|M_g(f_n)\big\|_{L^p(m)} &= \left(\sup\left\{\int_\Omega |g|^p \chi_{A_n \cap A_\varepsilon}\, d\big|\langle m, x'\rangle\big|\colon\ x' \in B_1(X')\right\}\right)^{\frac{1}{p}} \\
&\geqslant \left(\sup\left\{\int_{A_n \cap A_\varepsilon} \big(\|M_g\| + \varepsilon\big)^p\, d\big|\langle m, x'\rangle\big|\colon\ x' \in B_1(X')\right\}\right)^{\frac{1}{p}} \\
&= \big(\|M_g\| + \varepsilon\big)\big(\sup\big\{\big|\langle m, x'\rangle\big|(A_n \cap A_\varepsilon)\colon\ x' \in B_1(X')\big\}\big)^{\frac{1}{p}} \\
&= \big(\|M_g\| + \varepsilon\big)\big(\|m\|(A_n \cap A_\varepsilon)\big)^{\frac{1}{p}}.
\end{aligned} \tag{5}$$

From (4) and (5) we obtain that

$$\big(\|M_g\| + \varepsilon\big)\big(\|m\|(A_n \cap A_\varepsilon)\big)^{\frac{1}{p}} \leqslant \|M_g\|\big(\|m\|(A_n \cap A_\varepsilon)\big)^{\frac{1}{p}}, \tag{6}$$

for all $n = 1, 2, \ldots$. Since $\|m\|(A_n \cap A_\varepsilon) \to \|m\|(A_\varepsilon)$, taking limit when n tends to infinity in the inequality (6) we obtain

$$\big(\|M_g\| + \varepsilon\big)\big(\|m\|(A_\varepsilon)\big)^{\frac{1}{p}} \leqslant \|M_g\|\big(\|m\|(A_\varepsilon)\big)^{\frac{1}{p}}. \tag{7}$$

But the inequality (7) only holds whenever $\|m\|(A_\varepsilon) = 0$, as we wanted to prove.

The equivalences 1) $\Leftrightarrow$ 4) $\Leftrightarrow$ 5), and 1) $\Leftrightarrow$ 6) $\Leftrightarrow$ 7) can be established in a similar way. □

Remark 9. Note that if $g \in L^0(m)$ and $p \geqslant 1$ are such that M_g belongs to $\mathcal{B}(L^p(m), L^p_w(m))$, then $g \in L^\infty(m)$ and this implies that M_g belongs actually to $\mathcal{B}(L^p(m), L^p(m))$.

In the sequel we will study which conditions assure that the multiplication operator $M_g\colon L^p_w(m) \to L^p(m)$ is continuous. As we may see, these conditions will be, as in Theorem 7, equivalent to the weak compactness of the operator M_g, considered between different spaces of functions $L^p(m)$ and $L^p_w(m)$, for $p > 1$. To this end we will use the following general construction. For a fixed set $G \in \Sigma$ we consider the measurable space (G, Σ_G), where $\Sigma_G := \{A \in \Sigma\colon\ A \subseteq G\}$, and the vector measure $m_G\colon A \in \Sigma_G \to m_G(A) := m(A) \in X$.

Consider the space of measurable functions $L^0(m_G)$, the extension map,

$$\mathcal{E}_G\colon f \in L^0(m_G) \to \mathcal{E}_G(f) \in L^0(m),$$

defined by $\mathcal{E}_G(f) = f$ in G and $\mathcal{E}_G(f) = 0$ in $\Omega \setminus G$, and the restriction map,

$$\mathcal{R}_G : f \in L^0(m) \to \mathcal{R}_G(f) \in L^0(m_G),$$

defined by $\mathcal{R}_G(f) = f$. It is not difficult to establish the following properties:

(P1) $\mathcal{E}_G(\mathcal{R}_G(f)) = \chi_G f$, for all $f \in L^0(m)$.
(P2) $\mathcal{R}_G(\mathcal{E}_G(f)) = f$, for all $f \in L^0(m_G)$.
(P3) $\mathcal{R}_G(g \cdot \mathcal{E}_G(f)) = \mathcal{R}_G(g) \cdot f$, for all $g \in L^0(m)$ and $f \in L^0(m_G)$.
(P4) $\mathcal{E}_G \in \mathcal{B}(L^p(m_G), L^p(m))$, for all $p \geqslant 1$, and $\|\mathcal{E}_G(f)\|_{L^p(m)} = \|f\|_{L^p(m_G)}$, for all $f \in L^p(m_G)$.
(P5) $\mathcal{R}_G \in \mathcal{B}(L^p(m), L^p(m_G))$, for all $p \geqslant 1$, and $\|\mathcal{R}_G(f)\|_{L^p(m_G)} = \|\chi_G f\|_{L^p(m)} \leqslant \|f\|_{L^p(m)}$, for all $f \in L^p(m)$.

Note that $L^p(m_G)$ can be identified with the band in $L^p(m)$ which contains the functions that are null outside the set G. We can also replace the spaces L^p by L^p_w, in properties (P4)–(P5).

Theorem 10. *Let $p \geqslant 1$. For a function $g \in L^0(m)$, we denote by $G_n := \{w \in \Omega\colon\ |g(w)| \geqslant \frac{1}{n}\}$, for all $n = 1, 2, \ldots.$ The following assertions are equivalent*:

1) $g \in L^\infty(m)$ *and* $L^p_w(m_{G_n}) = L^p(m_{G_n})$, *for all* $n = 1, 2, \ldots.$
2) $M_g \in \mathcal{B}(L^p_w(m), L^p(m))$.

Proof. Suppose that $M_g \in \mathcal{B}(L^p_w(m), L^p(m))$. Particularly, M_g belongs to $\mathcal{B}(L^p_w(m), L^p_w(m))$ and by Theorem 8 we know that g is in $L^\infty(m)$. Note that $h := \frac{1}{\mathcal{R}_{G_n}(g)} \in L^\infty(m_{G_n})$ since the inequality $|\mathcal{R}_{G_n}(g)| \geqslant \frac{1}{n}$ holds, and thus $M_h \in \mathcal{B}(L^p(m_{G_n}), L^p(m_{G_n}))$. Now we must prove that $L^p_w(m_{G_n}) \subseteq L^p(m_{G_n})$. We claim that the composition

$$L^p_w(m_{G_n}) \xrightarrow{\mathcal{E}_{G_n}} L^p_w(m) \xrightarrow{M_g} L^p(m) \xrightarrow{\mathcal{R}_{G_n}} L^p(m_{G_n}) \xrightarrow{M_h} L^p(m_{G_n})$$

is simply the inclusion map. Indeed, if $f \in L^p_w(m_{G_n})$ we have that

$$M_h \mathcal{R}_{G_n} M_g \mathcal{E}_{G_n}(f) = M_h \mathcal{R}_{G_n}\big(g\, \mathcal{E}_{G_n}(f)\big) = M_h \mathcal{R}_{G_n}(g) f = f.$$

Conversely, we suppose now that conditions in 1) hold. By Theorem 8 we have that $M_g \in \mathcal{B}(L^p_w(m), L^p_w(m))$. Consider now the sequence $(M_{g\chi_{G_n}})_n$ of multiplication operators. We claim that $M_{g\chi_{G_n}}$ belongs to $\mathcal{B}(L^p_w(m), L^p(m))$, for each $n = 1, 2, \ldots.$ Indeed, according to the hypothesis 1) we have that the composition

$$L^p_w(m) \xrightarrow{M_g} L^p_w(m) \xrightarrow{\mathcal{R}_{G_n}} L^p_w(m_{G_n}) = L^p(m_{G_n}) \xrightarrow{\mathcal{E}_{G_n}} L^p(m)$$

is a continuous operator. But, bearing in mind the property (P1), we get $\mathcal{E}_{G_n}\mathcal{R}_{G_n} M_g(f) = \mathcal{E}_{G_n}\mathcal{R}_{G_n}(gf) = gf\chi_{G_n} = M_{g\chi_{G_n}}(f)$, for each $f \in L^p(m)$.

We consider now the set $G := \{w \in \Omega\colon\ g(w) \neq 0\}$ and denote by $C_n := G \setminus G_n$, for all $n = 1, 2, \ldots.$ Then we have

$$\begin{aligned} \|M_g - M_{g\chi_{G_n}}\|_{\mathcal{B}(L^p_w(m), L^p_w(m))} &= \|M_{g\chi_{C_n}}\|_{\mathcal{B}(L^p_w(m), L^p_w(m))} \\ &= \sup\big\{\|g\chi_{C_n} f\|_{L^p_w(m)}\colon\ \|f\|_{L^p_w(m)} \leqslant 1\big\} \\ &\leqslant \|g\chi_{C_n}\|_{L^\infty(m)} \leqslant \frac{1}{n} \to 0. \end{aligned}$$

Therefore M_g is the uniform limit in $\mathcal{B}(L^p_w(m), L^p_w(m))$ of the operators $M_{g\chi_{G_n}} \in \mathcal{B}(L^p_w(m), L^p(m))$. Since $L^p(m)$ is closed in $L^p_w(m)$ we conclude that M_g belongs to $\mathcal{B}(L^p_w(m), L^p(m))$. □

Remark 11. In general the hypothesis 1) of the previous theorem cannot be relaxed by $L^p(m_G) = L^p_w(m_G)$, even when the measure m is atomless. Let us consider a sequence of finite atomless positive scalar measure spaces $(\Omega_n, \Sigma_n, \mu_n)$ such that $(\mu_n(\Omega_n)_n) \in c_0$, the space of null-sequences, and denote by $L^2(\mu_n)$ the corresponding space of square integrable functions. Also consider the c_0-sum of the spaces $(L^2(\mu_n))_n$, that is, the Banach lattice $E := (L^2(\mu_1) \oplus L^2(\mu_2) \oplus \cdots)_{c_0}$ of all the sequences of functions $(f_n)_n$ satisfying that $(\|f_n\|_{L^2(\mu_n)})_n \in c_0$ with the norm $\|(f_n)_n\| := \|(\|f_n\|_{L^2(\mu_n)})_n\|_{c_0}$. This norm is order continuous and $(\chi_{\Omega_n})_n$ is a weak order unit in the c_0-sum E. Define the measurable space (Ω, Σ), where $\Omega := \bigsqcup_{n \geqslant 1} \Omega_n$ and

$$\Sigma := \{A \subseteq \Omega\colon\ A \cap \Omega_n \in \Sigma_n,\ n = 1, 2, \ldots\}.$$

The function $m : A \in \Sigma \to m(A) \in E$, given by $m(A) := (\chi_{A \cap \Omega_n})_n$, defines a countably additive positive vector measure such that $L^1(m) = E$. In fact, the elements of this space $L^1(m)$ are (equivalence classes modulo m-a.e. of) functions $f : \Omega \to \mathbb{R}$ such that $(\|f\chi_{\Omega_n}\|_{L^2(\mu_n)})_n \in c_0$. Similarly the space $L^1_w(m)$ consists of (equivalence classes modulo m-a.e. of) functions $f : \Omega \to \mathbb{R}$ such that $(\|f\chi_{\Omega_n}\|_{L^2(\mu_n)})_n \in \ell_\infty$, the space of bounded sequences. From here, we can get for all $p \geqslant 1$ that

$$L^p(m) = \{f \in L^0(m)\colon\ (\|f\chi_{\Omega_n}\|_{L^{2p}(\mu_n)})_n \in c_0\} \quad \text{and}$$
$$L^p_w(m) = \{f \in L^0(m)\colon\ (\|f\chi_{\Omega_n}\|_{L^{2p}(\mu_n)})_n \in \ell_\infty\}.$$

Moreover, it is not difficult to see that

$$\|f\|_{L^p_w(m)} = \sup\left\{\left(\int_{\Omega_n} |f|^{2p}\, d\mu_n\right)^{\frac{1}{2p}} : n = 1, 2, \ldots\right\}, \quad f \in L^p_w(m).$$

Now consider the function $g := \sum_{n=1}^\infty \frac{1}{n}\chi_{\Omega_n} \in L^\infty(m)$. Thus it is clear that for this function, $G = \Omega$, and so $L^p(m_G) = L^p(m) \neq L^p_w(m) = L^p_w(m_G)$. But the operator $M_g \in \mathcal{B}(L^p_w(m), L^p(m))$, for all $p \geqslant 1$, as we can see easily.

In the proof of the following theorem, where we characterize the weak compactness of the operator M_g defined between different spaces $L^p(m)$ and $L^p_w(m)$, the reflexivity of the spaces $L^p(m_{G_n})$ will play an important role. If $p > 1$, it is well known that $L^p(m)$ is reflexive if and only if $L^p_w(m)$ is reflexive, and that happens if and only if $L^p(m) = L^p_w(m)$. See [8, Corollary 3.10]. For $p = 1$ these equivalences are not true, but it remains true the equality $L^1_w(m) = L^1(m)$, if $L^1(m)$ is reflexive. See the proof of the implication $(e) \Rightarrow (f)$ in the cited corollary.

Theorem 12. *Let $p \geqslant 1$. For a function g in $L^0(m)$, we denote by $G_n := \{w \in \Omega\colon\ |g(w)| \geqslant \frac{1}{n}\}$, for all $n = 1, 2, \ldots$. The following assertions are equivalent*:

1) $g \in L^\infty(m)$ *and* $L^p(m_{G_n})$ *is reflexive for all* $n = 1, 2, \ldots$.
2) $M_g \in \mathcal{W}(L^p_w(m), L^p(m))$.
3) $M_g \in \mathcal{W}(L^p_w(m), L^p_w(m))$.
4) $M_g \in \mathcal{W}(L^p(m), L^p_w(m))$.
5) $M_g \in \mathcal{W}(L^p(m), L^p(m))$.

If $p > 1$, these conditions are also equivalent to

6) $M_g \in \mathcal{B}(L^p_w(m), L^p(m))$.

Proof. 1) $\Rightarrow$ 2). Since $g \in L^\infty(m)$, by Theorem 8 we have that $M_g \in \mathcal{B}(L^p_w(m), L^p_w(m))$. Consider now the sequence $(M_{g\chi_{G_n}})_n$ of multiplication operators defined in $L^p_w(m)$. For each $n = 1, 2, \ldots$, we claim that $M_{g\chi_{G_n}}$ belongs to $\mathcal{W}(L^p_w(m), L^p(m))$. Indeed, note that the composition

$$L^p_w(m) \xrightarrow{M_g} L^p_w(m) \xrightarrow{\mathcal{R}_{G_n}} L^p_w(m_{G_n}) = L^p(m_{G_n}) \xrightarrow{\mathcal{E}_{G_n}} L^p(m)$$

is a weakly compact operator, because $L^p(m_{G_n})$ is reflexive and, particularly, $L^p_w(m_{G_n}) = L^p(m_{G_n})$. But, for each $f \in L^p_w(m)$, having in mind the property (P1), we have that $\mathcal{E}_{G_n}\mathcal{R}_{G_n}M_g(f) = \mathcal{E}_{G_n}\mathcal{R}_{G_n}(gf) = gf\chi_{G_n} = M_{g\chi_{G_n}}(f)$. Let us consider now the set $G := \{w \in \Omega\colon\ g(w) \neq 0\}$ and denote $C_n := G \setminus G_n$, for all $n = 1, 2, \ldots$. Thus we have

$$\begin{aligned}\|M_g - M_{g\chi_{G_n}}\|_{\mathcal{B}(L^p_w(m),L^p_w(m))} &= \|M_{g\chi_{C_n}}\|_{\mathcal{B}(L^p_w(m),L^p_w(m))} \\ &= \sup\big\{\|g\chi_{C_n} f\|_{L^p_w(m)}\colon \ \|f\|_{L^p_w(m)} \leqslant 1\big\} \\ &\leqslant \|g\chi_{C_n}\|_{L^\infty(m)} \leqslant \frac{1}{n} \to 0.\end{aligned}$$

Therefore M_g is the uniform limit in $\mathcal{B}(L^p_w(m), L^p_w(m))$ of the operators $M_{g\chi_{G_n}}$ which belong to $\mathcal{W}(L^p_w(m), L^p(m))$. Since $L^p(m)$ is closed in $L^p_w(m)$ we conclude that $M_g \in \mathcal{W}(L^p_w(m), L^p(m))$.

The implications 2) $\Rightarrow$ 3), and 3) $\Rightarrow$ 4) are evident.

4) $\Rightarrow$ 5). If $M_g \in \mathcal{W}(L^p(m), L^p_w(m))$, then $M_g \in \mathcal{B}(L^p(m), L^p_w(m))$ and by Theorem 8 we obtain that $g \in L^\infty(m)$. Therefore, $M_g(L^p(m)) \subseteq L^p(m)$ and $M_g \in \mathcal{W}(L^p(m), L^p(m))$.

5) $\Rightarrow$ 1). If $M_g \in \mathcal{W}(L^p(m), L^p(m))$, then in particular we have $M_g \in \mathcal{B}(L^p(m), L^p(m))$ and by Theorem 8 we know that $g \in L^\infty(m)$. Note that $h := \frac{1}{\mathcal{R}_{G_n}(g)} \in L^\infty(m_{G_n})$ since the inequality $|\mathcal{R}_{G_n}(g)| \geqslant \frac{1}{n}$ holds, and hence $M_h \in \mathcal{B}(L^p(m_{G_n}), L^p(m_{G_n}))$. We shall now prove that the identity operator in $L^p(m_{G_n})$ is weakly compact and therefore $L^p(m_{G_n})$ would be reflexive. Note that the composition

$$L^p(m_{G_n}) \xrightarrow{\mathcal{E}_{G_n}} L^p(m) \xrightarrow{M_g} L^p(m) \xrightarrow{\mathcal{R}_{G_n}} L^p(m_{G_n}) \xrightarrow{M_h} L^p(m_{G_n})$$

is weakly compact. But, for each $f \in L^p(m_{G_n})$ we have that

$$M_h \mathcal{R}_{G_n} M_g \mathcal{E}_{G_n}(f) = M_h \mathcal{R}_{G_n}\big(g\mathcal{E}_{G_n}(f)\big) = M_h \mathcal{R}_{G_n}(g) f = f.$$

If $p > 1$, the equivalence 1) $\Leftrightarrow$ 6) is just Theorem 10, having in mind that $L^p(m_{G_n})$ is reflexive if and only if $L^p(m_{G_n}) = L^p_w(m_{G_n})$. □

Remark 13. As we noted in the Remark 11, in general, the hypothesis of the reflexivity of $L^p(m_{G_n})$, for all $n = 1, 2, \ldots,$ of the previous theorem cannot be replaced by the reflexivity of $L^p(m_G)$, even when the measure m is atomless.

Corollary 14. *Let m be an atomless vector measure with σ-finite variation. Let $g \in L^0(m)$ such that the operator $M_g \in \mathcal{W}(L^1(m), L^1(m))$. Then $g = 0$.*

Proof. It is sufficient to prove that $\|m\|(G) = 0$ where G is the set $\{w \in \Omega\colon\ g(w) \neq 0\}$. Suppose, on the contrary, that $\|m\|(G) > 0$. Then there is some $n \geqslant 1$ such that $\|m\|(G_n) > 0$, where $G_n := \{w \in \Omega\colon\ |g(w)| \geqslant \frac{1}{n}\}$. By the previous theorem we have that $L^1(m_{G_n})$ is reflexive. On the other hand, it is easy to prove that m_{G_n} is also an atomless vector measure with σ-finite variation. By the remark after [4, Theorem 4] we conclude that $L^1(m_{G_n})$ is not reflexive. This contradiction establishes the result. □

Remark 15. (1) From the equivalences of Theorem 12 it follows that Corollary 14 remains valid if we consider multiplication operators in $\mathcal{W}(L^1(m), L^1_w(m))$, $\mathcal{W}(L^1_w(m), L^1(m))$ or in $\mathcal{W}(L^1_w(m), L^1_w(m))$.

(2) In general, as we will show in the following examples, it is not possible to obtain a similar result to the previous corollary neither for $p > 1$, nor for vector measures m with no σ-finite variation.

(E1) The vector measure m, defined on the σ-algebra of the Borel subsets of [0, 1], by $m(A) := \chi_A \in L^2[0, 1]$, has no σ-finite variation. It is well known that $L^1(m) = L^1_w(m) = L^2[0, 1]$ and every function in $L^\infty[0, 1]$ defines a weakly compact multiplication operator.

(E2) If m is the Lebesgue measure on the interval [0, 1] we have that $L^p(m) = L^p_w(m) = L^p[0, 1]$, for all $p \geqslant 1$, and again, every function in $L^\infty[0, 1]$ defines a weakly compact multiplication operator whenever $p > 1$.

The characterization of the compactness of the multiplication operator is similar to that obtained for the weak compactness in Theorem 12.

Theorem 16. *Fix $p \geqslant 1$ and let g be a function in $L^0(m)$, we denote by $G_n := \{w \in \Omega\colon\ |g(w)| \geqslant \frac{1}{n}\}$, for all $n = 1, 2, \ldots.$ The following assertions are equivalent*:

1) $g \in L^\infty(m)$ *and* $\dim(L^p(m_{G_n})) < \infty$, *for all* $n = 1, 2, \ldots$.
2) $M_g \in \mathcal{K}(L_w^p(m), L^p(m))$.
3) $M_g \in \mathcal{K}(L_w^p(m), L_w^p(m))$.
4) $M_g \in \mathcal{K}(L^p(m), L_w^p(m))$.
5) $M_g \in \mathcal{K}(L^p(m), L^p(m))$.

Furthermore, in this case, the following sets $\{\frac{1}{(\|m\|(A))^{\frac{1}{p}}} \int_A g\, dm\colon A \in \Sigma\}$, $\{\int_A g\, dm\colon A \in \Sigma\}$ *and* $m(\Sigma_G) := \{m(A)\colon A \in \Sigma, A \subseteq G\}$ *are relatively compact in* X.

Proof. The proof of the equivalences 1)–5) follows the lines of the those of Theorem 12.

In order to prove the last statement we must have in mind that, whenever the equivalence is true, the composition of the multiplication operator M_g with the integration operator, that is, $\mathcal{I}_g : f \in L^p(m) \to \mathcal{I}_g(f) := \int_\Omega gf\, dm \in X$ is a compact operator, therefore the set $\mathcal{I}_g(B_1(L^p(m)))$ is relatively compact in X. Since $\|\chi_A\|_{L^p(m)} = (\|m\|(A))^{\frac{1}{p}}$, for all $A \in \Sigma$; it is easy to prove that

$$\left\{\frac{1}{(\|m\|(A))^{\frac{1}{p}}} \int_A g\, dm\colon A \in \Sigma\right\} \subseteq \mathcal{I}_g\bigl(B_1\bigl(L^p(m)\bigr)\bigr) \quad \text{and}$$

$$\left\{\int_A g\, dm\colon A \in \Sigma\right\} \subseteq \bigl(\|m\|(\Omega)\bigr)^{\frac{1}{p}} \mathcal{I}_g\bigl(B_1\bigl(L^p(m)\bigr)\bigr).$$

To finish, we shall prove that the set $m(\Sigma_G)$ is relatively compact in X. To this end, for each $n = 1, 2, \ldots$, let us consider the following decomposition:

$$m(A) = m(A \cap G_n) + m\bigl(A \cap (G \setminus G_n)\bigr), \quad A \in \Sigma_G. \tag{8}$$

Note that, since $\|\chi_{A\cap G_n}\|_{L^p(m_{G_n})} = (\|m\|(A \cap G_n))^{\frac{1}{p}} \leqslant (\|m\|(G))^{\frac{1}{p}}$, we get that the set $\{\chi_{A\cap G_n}\colon A \in \Sigma_G\}$ is bounded in $L^p(m_{G_n})$. Since the integration map $I : L^p(m_{G_n}) \to X$ is continuous and $\dim(L^p(m_{G_n})) < \infty$, we conclude that $\{m(A \cap G_n)\colon A \in \Sigma_G\}$ is relatively compact in X for all $n = 1, 2, \ldots$. On the other hand, since $\chi_{G_n} \to \chi_G$ in $L^1(m)$ it follows that $\|m\|(G \setminus G_n) \to 0$. Therefore, for a fixed $\varepsilon > 0$, there is some n_0 such that $\|m\|(G \setminus G_{n_0}) < \varepsilon$. Thus, by (8) we obtain that $m(\Sigma_G) \subseteq \{m(A \cap G_{n_0})\colon A \in \Sigma_G\} + \varepsilon B_1(X)$ and, consequently, $m(\Sigma_G)$ is relatively compact in X. □

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