SEQUENTIAL DENSITY TECHNIQUES FOR AN APPROXIMATE RADON–NIKODÝM THEOREM

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Abstract: We deal with the problem of approximate representation of linear functionals and show how the sequential density techniques may be used for solution.

Keywords: Radon, Nikodým, absolute continuity, sequential density, approximate representation

1. Introduction

Recently, in the context of the Daniell (continuous) integrals, sufficient conditions were given in [1] for obtaining by a constructive method an "exact" Radon–Nikodým derivative v; i.e., $J(h) = I(hv)$ for an absolutely I-continuous linear functional J and all h in B. At the same time, without continuity conditions (a functional analog of the case of a finitely additive measure), and in the spirit of the paper by C. Fefferman [2], it was shown that $AC(I) \subset R(I)$, i.e., the set of all absolutely I-continuous linear functionals is (strictly) included in the set of all approximately I-representable linear functionals (see [3]). In the initial paper by de Amo, Chiţescu, and Díaz Carrillo [3], the hypothesis $1 \in B$ (i.e., B is a unital algebra) still plays a crucial role. This result was generalized more recently to the context of a Riesz algebra by Günzler in [4]. Some ideas and notation come from this paper, although the techniques are quite different.

In this work, we give an approximate Radon–Nikodým theorem by a new procedure which falls naturally into three parts: We first prove a basic representation theorem for certain continuous functionals (Theorem 3.1) and then, with the aid of a sequential density property (Corollary 4.2), we deduce that $AC(I) \subseteq R(I)$ (see Theorem 5.2) in a straight way which makes natural the role that the hypothesis of this result has played until now.

We make stress upon the naturalness and originality in proving Theorem 5.2: the completeness property for dual spaces provides convergence in the norm of operators where there was only pointwise convergence, and this is what allows us to obtain Theorem 5.2 as a natural extension of Theorem 3.1 via Corollary 4.2.

2. Preliminaries

Throughout this paper $\mathbb{R}$ will be the reals; $\mathbb{N}$, the naturals; $\varnothing$, the empty set; X will be an arbitrary nonempty set, and $B \subseteq \mathbb{R}^X$, an algebra of functions on X (i.e., a vector lattice or Riesz space closed under the pointwise product: $uv \in B$ for all $u, v \in B$). Let $+B := \{h \in B : h \geq 0\}$ be the positive cone of X. Moreover $h^+ := h \vee 0$ and $h^- := (-h) \wedge 0$ (and so $|h| = h^+ + h^-$).

Let B' be the set of all linear functionals on B, and we consider the partial order of B' given by

$$J \geq 0 \Leftrightarrow J(h) \geq 0, \quad h \in +B.$$

We let $B^\diamond$ denote the class of all linear functionals bounded on each interval $[u, v] := \{x \in B : u \leq x \leq v\}$, for $u, v \in B$, $u \leq v$ ("*relativement bornée*" in [5, p. 35], "*order bounded*" in [6, p. 150], "*interval bounded*" in [7, p. 169]), which is a Riesz space under the pointwise operations and order induced from B'.

A *Loomis algebra system* or *Loomis system* is a triple (X, B, I) where $X \neq \varnothing$ is an abstract set, $B \subseteq \mathbb{R}^X$ is an algebra of functions with pointwise operations and $I : B \to \mathbb{R}$ is a positive (i.e., $I(h) \geq 0$, $h \in +B$) linear functional.

For our purpose, we need the following formula (see [5]) for the meet $T \wedge S$ of two bounded linear functionals T and S:

$$(T \wedge S)(h) := \inf\{T(h_1) + S(h_2) : h_1, h_2 \in +B, h_1 + h_2 = h\} \quad \forall h \in +B.$$

From now on $I : B \to \mathbb{R}$ will denote a positive linear functional on B. This functional induces the following seminorms on B: $\quad I_1(h) := I(|h|)$ and $I_2(h) := \sqrt{I(h^2)} \quad \forall h \in B$.

The advantage of the seminorm I_2 lies in the fact that it proceeds from the semi-inner product ϕ defined on B by $\phi(h, g) := I(hg)$ $\forall h, g \in B$ and, therefore, it satisfies the Cauchy–Schwarz inequality:

$$|I(hg)| \leq I_2(h)I_2(g) \quad \forall h, g \in B.$$

These seminorms yield the corresponding spaces of all continuous linear functionals on B:

$$C_1(I) := \{J \in B' : \exists M \in \mathbb{R}\ \forall h \in B\ |J(h)| \leq MI_1(h)\};$$
$$C_2(I) := \{J \in B' : \exists M \in \mathbb{R}\ \forall h \in B\ |J(h)| \leq MI_2(h)\},$$

which are the dual spaces of (B, I_1) and (B, I_2). The elements in $C_1(I)$ and $C_2(I)$ are called I_1- and I_2-*continuous functionals*. Note that for positive J, I_1-continuity is equivalent to $J \leq MI$ for some $M \geq 0$. The I_1-continuity is generalized by the following definition (see [3, Example 1]):

Definition 2.1. For a given (X, B, I) Loomis system, $J \in B'$ is said to be *absolutely continuous with respect to* I, which is denoted by $J \ll I$, if

$$(\forall \varepsilon > 0\ \forall h \in +B\ \exists \delta > 0\ \forall k \in +B)\ k \leq h,\ I(k) < \delta \Rightarrow |J(k)| < \varepsilon.$$

Put $AC(I) := \{J \in B' : J \ll I\}$.

It is easy to show that $C_1(I) \subseteq AC(I) \subseteq B^{\diamond}$. In fact, they are (order) ideals in $B^{\diamond}$ and $AC(I)$ is even a band (with the notions in [8, p. 93]), but for our purposes we only need to know that they are Riesz subspaces of $B^{\diamond}$.

Moreover, there is no general inclusion relation between $C_1(I)$ and $C_2(I)$ (see Examples 6.3 and 6.5 at the end of this paper).

Finally, we consider the spaces $SR(I)$ and $R(I)$, which were introduced in [3], of strongly and approximately I-representable functions, respectively:

$$SR(I) := \{I_g : g \in B\},\ \text{where } I_g(h) := I(gh) \quad \forall h \in B;$$
$$R(I) := \{J \in B' : \exists v_n \in B\ \forall h \in B\ I_{v_n}(h) \to J(h)\}.$$

Obviously, $SR(I) \subseteq R(I)$. With this notation, an exact Radon–Nikodým theorem would claim that $AC(I) = SR(I)$. In [3, Example 2] it is seen that we cannot expect in general more than $AC(I) \subseteq R(I)$ (even on assuming that $1 \in B$).

3. A Representation Theorem for $C_2(I)$

In this section we prove a first approximate representation theorem for the class $C_2(I)$ of all I_2-continuous functionals via the classical Riesz–Fréchet theorem on Hilbert spaces. This is a modified version of Proposition 1 in [3].

Theorem 3.1. *Let (X, B, I) be a Loomis system and let $J \in B'$. If J is I_2-continuous then J is approximately I-representable and, moreover, the approximate derivative of J is an I_2-Cauchy sequence, i.e., for each $J \in C_2(I)$ there exists an I_2-Cauchy sequence $\{v_n\} \subseteq B$ such that $J(h) = \lim_{n\to\infty} I(hv_n)$ $\forall h \in B$. In particular, $C_2(I) \subseteq R(I)$.*

Proof. Put $K := \{h \in B : I_2(h) = 0\}$. By passing to equivalence classes in B modulo K, we obtain the quotient space $\widehat{B} := B/K$ under the induced norm $\widehat{I}_2(\hat{h}) := I_2(h)$ $\forall h \in \hat{h}$. In fact, $\widehat{B}$ is a prehilbertian space under the inner product $\widehat{\psi}(\hat{u}, \hat{v}) := I(uv)$ $\forall u \in \hat{u},\ v \in \hat{v}$.

The prehilbertian space $(\widehat{B}, \widehat{\psi})$ can be completed to a Hilbert space $(\overline{B}, \overline{\psi})$, and each element in $\widehat{B}$ can be identified with its image in $\overline{B}$, thus yielding the natural inclusion $\widehat{B} \subseteq \overline{B}$.

Hence, for each $\bar{h} \in \overline{B}$ there exists an I_2-Cauchy sequence $\{\hat{h}_n\}$ in $\widehat{B}$ such that $\hat{h}_n \to \bar{h}$ in $\overline{B}$, and the inner product in $\overline{B}$ is given by:

$$\overline{\psi}(\bar{u}, \bar{v}) := \lim_{n\to\infty} \widehat{\psi}(\hat{u}_n, \hat{v}_n) = \lim_{n\to\infty} I(u_n v_n) \quad \forall \bar{u}, \bar{v} \in \overline{B}.$$

Take $J \in C_2(I)$. Then there exists $M > 0$ such that $|J(h)| \le M I_2(h)$ $\forall h \in B$. It follows that the functional $\widehat{J} : \widehat{B} \to \mathbb{R}$ given by $\widehat{J}(\hat{h}) := J(h)$ $\forall h \in \hat{h}$ is well defined and, moreover,

$$|\widehat{J}(\hat{h})| = |J(h)| \le M I_2(h) = M \widehat{I}_2(\hat{h}).$$

Therefore, $\widehat{J}$ is a linear and continuous functional on $\widehat{B}$ and we may extend $\widehat{J}$ uniquely to a linear and continuous functional $\overline{J} : \overline{B} \to \mathbb{R}$ on the completion $\overline{B}$ of $\widehat{B}$. Now, the Riesz–Fréchet theorem guarantees the existence of one, and only one, element $\bar{v} \in \overline{B}$ such that $\overline{J}(\bar{h}) = \overline{\psi}(\bar{h}, \bar{v})$ $\forall \bar{h} \in \overline{B}$. In particular, there exists an I_2-Cauchy sequence $\{v_n\}$ in B such that

$$J(h) = \widehat{J}(\hat{h}) = \overline{J}(\hat{h}) = \overline{\psi}(\hat{h}, \bar{v}) = \lim_{n\to\infty} \widehat{\psi}(\hat{h}, \hat{v}_n) = \lim_{n\to\infty} I(h\, v_n) \quad \forall h \in B. \qquad \square$$

4. Sequential Density of $C_1(I)$ in $AC(I)$

This section with no technique external to our context is devoted to a direct proof of a result that is the cornerstone among the papers such as [3, 4], in order to prove their respective representation theorems. It is possible to obtain this property via a classical result in the context of Riesz space theory (as a consequence of [5, Corollary II.1.5]), on assuming that $AC(I)$ is the band generated by I in $B^{\diamond}$ (see [5, p. 37]). The advantage of the proof we give here (based upon the ideas of [9]) lies in the fact that we do not have to use the general theory (we avoid any result on bands), and so we are able to deduce this result in the context of function spaces.

Theorem 4.1. *Let (X, B, I) be a Loomis system and $J \in +AC(I)$. Then*

$$J(h) = \lim_{m\to\infty} (J \wedge mI)(h) \quad \forall h \in +B.$$

Proof. Let $h \in +B$ and $\varepsilon > 0$. Since $J \ll I$, there exists $\delta > 0$ such that

$$(\forall g \in +B)\ g \le h,\ I(g) < \delta \Rightarrow J(g) < \varepsilon/2. \tag{1}$$

Put $k := \frac{J(h)}{\delta}$ and $n \ge k$. Then

$$|J(h) - (J \wedge nI)(h)| = J(h) - (J \wedge nI)(h) \le J(h) - (J \wedge kI)(h).$$

Moreover, by the formula

$$(J \wedge kI)(h) = \inf\{J(h_1) + kI(h_2) : h_1, h_2 \in +B, h_1 + h_2 = h\}$$

there exist h_1 and h_2 such that $(J \wedge kI)(h) + \frac{\varepsilon}{2} \ge J(h_1) + kI(h_2)$.

So,

$$\begin{aligned}|J(h) - (J \wedge nI)(h)| &\le J(h) - (J \wedge kI)(h) \le J(h) - (J(h_1) + kI(h_2) - \varepsilon/2)\\ &= J(h_2) - kI(h_2) + \varepsilon/2.\end{aligned}$$

Therefore, it is enough to show that $g \le h$ implies $J(g) - kI(g) \le \varepsilon/2$ to complete the proof.

If $I(g) < \delta$, by (1), then

$$J(g) - kI(g) = J(g) - \frac{J(h)}{\delta} I(g) < \frac{\varepsilon}{2} - \frac{J(h)}{\delta}\delta < \frac{\varepsilon}{2} - J(h) \le \frac{\varepsilon}{2}.$$

If $I(g) \ge \delta$ then

$$J(g) - kI(g) = J(g) - \frac{J(h)}{\delta} I(g) = J(g) - J\left(\frac{I(g)}{\delta} h\right) < J\left(g - \frac{I(g)}{\delta} h\right)$$

and $I(g) \ge \delta$ and $h \ge g$ lead to

$$J(g) - kI(g) \le J(g - g) = 0 \le \varepsilon/2. \qquad \square$$

Theorem 4.1 says that for each $J \in +AC(I)$, there exists a sequence $J_m := mI \wedge J$, $m \in \mathbb{N}$, in $+C_1(I)$ such that $J_m \to J$ pointwise. (Note that $J_m \in C_1(I)$ because $J_m \le mI$ and $J \ge 0$.) Therefore, $C_1(I)$ is sequentially dense in $AC(I)$:

Corollary 4.2. $\forall J \in AC(I)\ \exists \{J_m\} \subseteq C_1(I)\ \forall h \in B\ J_m(h) \to J(h)$.

The sequential density property is the most important fact for us rather than the explicit form of the sequence $\{J_m\}$. This viewpoint clarifies our techniques and enables us to solve the problem of approximate representation in a natural context.

5. Representability of Absolutely Continuous Functionals

We are now in a position to deduce the approximate Radon–Nikodým Theorem 5.2 from Theorem 3.1 and Corollary 4.2. We call attention upon the completeness property of $C_2(I)$ which is, merely, the dual space of B equipped with the I_2-norm.

For these (a priori independent) results to be properly combined, it seems natural to impose the relation $C_1(I) \subseteq C_2(I)$ which is a weakening of the condition $1 \in B$ as showed by the next lemma.

Lemma 5.1. *Let (X, B, I) be a Loomis system with $1 \in B$. Then $C_1(I) \subseteq C_2(I)$.*

Proof. Let J be in $C_1(I)$. Given $h \in B$, there exists $M > 0$ such that

$$|J(h)| \leq MI_1(h) = MI(|h|) = MI(1\ |h|) \leq MI_2(1)\ I_2(|h|) = M_0 I_2(h),$$

where $M_0 := MI_2(1)$ and we have used the Cauchy–Schwarz inequality. Thus, $J \in C_2(I)$, and then $C_1(I) \subseteq C_2(I)$. □

The converse of Lemma 5.1 is false as shown by Example 6.3. As a consequence, the following theorem strictly generalizes that in [3, p. 446].

Theorem 5.2 (An Approximate Radon–Nikodým Theorem). *Let (X, B, I) be a Loomis system verifying $C_1(I) \subseteq C_2(I)$. Then $AC(I) \subseteq R(I)$; i.e.,*

$$J \ll I \ \Rightarrow\ \exists \{v_m\} \subseteq B\ \forall h \in B\ J(h) = \lim_{m\to\infty} I(hv_m).$$

Proof. Consider J in $AC(I)$. By Corollary 4.2 there exists a sequence $\{J_m\}$ in $C_1(I)$ such that $J_m \to J$ pointwise. Using the hypothesis, the fact that $J_m \in C_1(I) \subseteq C_2(I)$ for each $m \in \mathbb{N}$, and applying Theorem 3.1, we see that there exists an I_2-Cauchy sequence $\{v_m(n)\}_{n\in\mathbb{N}}$ in B such that $I_{v_m(n)} \to J_m$ pointwise. By $\|\cdot\|$ we denote the dual norm for I_2 in $C_2(I)$, i.e., the canonical norm of operators on $C_2(I)$. In fact, $I_{v_m(n)}$ converges to J_m in this norm. (Note that J_m and $I_{v_m(n)}$ are in $C_2(I)$ for all $m \in \mathbb{N}$.) Take $\varepsilon > 0$. There exists n_0 such that $\forall p, q \geq n_0\ I_2(v_m(p) - v_m(q)) < \varepsilon$. Now, if $p, q \geq n_0$ and $h \in B$ with $I_2(h) \leq 1$, it follows that

$$\begin{aligned}|(I_{v_m(p)} - I_{v_m(q)})(h)| &= |I_{v_m(p)}(h) - I_{v_m(q)}(h)| = |I(h\ v_m(p)) - I(h\ v_m(q))| \\ &= |I(h\ [v_m(p) - v_m(q)])| \leq I_2(h)\ I_2(v_m(p) - v_m(q)) \leq I_2(v_m(p) - v_m(q)) < \varepsilon\end{aligned}$$

so that $\|I_{v_m(p)} - I_{v_m(q)}\| < \varepsilon$, i.e., for each m fixed in $\mathbb{N}$, $\{I_{v_m(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach (normed complete) space $(C_2(I), \|\cdot\|)$ (which is the dual space of (B, I_2)). Hence $\{I_{v_m(n)}\}$ converges to some element in $(C_2(I), \|\cdot\|)$. Moreover, $I_{v_m(n)} \to J_m$ pointwise. Therefore, $I_{v_m(n)} \to J_m$ in the norm $\|\cdot\|$ of $C_2(I)$.

Now, for each $m \in \mathbb{N}$, we consider $n_m \in \mathbb{N}$ such that $\|I_{v_m(n_m)} - J_m\| < 1/m$ and put $u_m := v_m(n_m)\ \forall m \in \mathbb{N}$. So that $\|I_{u_m} - J_m\| < 1/m$.

Given $h \in B$ and $\varepsilon > 0$, let $m_1 \in \mathbb{N}$ such that $\forall m \geq m_1\ \|I_{u_m} - J_m\|\ I_2(h) < \varepsilon/2$, and $J_m(h) \to J(h)$ implies that there exists too $m_2 \in \mathbb{N}$ such that $\forall m \geq m_2\ |J_m(h) - J(h)| < \varepsilon/2$.

Putting $m \geq \max\{m_1, m_2\}$, we deduce finally that

$$\begin{aligned}&|J(h) - I(u_m h)| \leq |J(h) - J_m(h)| + |J_m(h) - I_{u_m}(h)| \\ &\leq |J(h) - J_m(h)| + \|J_m - I_{u_m}\| I_2(h) < \varepsilon/2 + \varepsilon/2 = \varepsilon\end{aligned}$$

i.e., $I(u_m h) \to J(h)\ \forall h \in B$. □

6. Comments and Examples

This section is devoted to clarify, both at a theoretical and a practical level, the comprehension of the conditions in Theorem 5.2, by establishing the relation between $C_1(I) \subseteq C_2(I)$ and the existence of I_2-approximate I-units. For the benefit of the reader, some examples, mostly given in [4], are included.

A sequence $\{u_n\}$ in B is an *I_2-approximate I-unit* if $I(u_n h) \to I(h)$ $\forall h \in B$ and $\sup_{n\in\mathbb{N}} I_2(u_n) < \infty$. If we put

$$R_2(I) := \{J \in B' : \exists v_n \in B;\ \sup_{n\in\mathbb{N}} I_2(v_n) < \infty, I_{v_n}(h) \to J(h)\ \forall h \in B\}$$

then the existence of I_2-approximate I-units is equivalent to $I \in R_2(I)$.

Lemma 6.1. *Let (X, B, I) be a Loomis system. Then the following are equivalent:*
(i) $C_1(I) \subseteq C_2(I)$,
(ii) $I \in C_2(I)$,
(iii) $I_1 \le MI_2$ *for some* $M > 0$.

PROOF. (i) $\Rightarrow$ (ii) $I \in C_1(I)$ because $|I(h)| \le I(|h|)$ $\forall h \in B$, and then, by (i), $I \in C_2(I)$.

(ii) $\Rightarrow$ (iii) There exists $M > 0$ such that $|I(h)| \le MI_2(h)$ $\forall h \in B$. Therefore, if $h \in +B$ then $I_1(h) = I(h) = |I(h)| \le MI_2(h)$ and for arbitrary $h \in B$

$$\begin{aligned} I_1(h) &= I_1(|h|) = I_1(h^+ + h^-) \le I_1(h^+) + I_1(h^-) \\ &\le MI_2(h^+) + MI_2(h^-) \le 2MI_2(|h|) = 2MI_2(h). \end{aligned}$$

(iii) $\Rightarrow$ (i) Let $J \in C_1(I)$. There exists $M' > 0$ such that $|J(h)| \le M'I_1(h) \le M'MI_2(h)$ $\forall h \in B$, and this gives $J \in C_2(I)$. $\square$

Proposition 6.2. *Let (X, B, I) be a Loomis system with $I \in R_2(I)$. Then $C_1(I) \subseteq C_2(I)$.*

PROOF. If $I \in R_2(I)$ then there exists $\{u_n\}$ in B with $I_2(u_n) \le M$ $\forall n \in \mathbb{N}$ such that $I(u_n h) \to I(h)$. By the Cauchy–Schwarz inequality,

$$|I(u_n h)| \le I_2(u_n) I_2(h) \le MI_2(h) \quad \forall h \in B$$

so that, by taking limits, it follows that $|I(h)| \le MI_2(h)$ $\forall h \in B$ and hence $I \in C_2(I)$. Now, Lemma 6.1 says that $C_1(I) \subseteq C_2(I)$. $\square$

This result makes easier to check the hypotheses of Theorem 5.2, as showed by the following

EXAMPLE 6.3. Let $B := C_0([0,1])$ be the space of continuous functions $f : [0,1] \to \mathbb{R}$ such that $f(0) = f(1) = 0$ and $I(f) := \int_0^1 f(t)\,dt$ $\forall f \in B$.

Note that, putting $L^p := L^p([0,1])$, with the Lebesgue measure, $B \subseteq L^\infty \subseteq L^2 \subseteq L^1$ and $I_1 = \|\cdot\|_1$, $I_2 = \|\cdot\|_2$ restricted to B. Let $u \in B$ be piecewise-linear and equal to 1 on $[\frac{1}{4}, \frac{3}{4}]$, and $u_n := (nu) \wedge 1 \in B$, $n \in \mathbb{N}$.

Since u_n is an I_2-approximate I-unit, Proposition 6.2 gives $C_1(I) \subseteq C_2(I)$. Therefore, we can apply Theorem 5.2 and so $AC(I) \subseteq R(I)$.

In fact, if we define $\pi : L^1 \to B'$ mapping g to $\pi(g) := I_g$, where $I_g : B \to \mathbb{R}$ is given by $\pi(g)(f) = I_g(f) = I(gf)$ $\forall f \in B$, then it can be proved that

$$SR(I) = \pi(B) \subset C_1(I) = \pi(L^\infty) \subset C_2(I) = \pi(L^2) \subset AC(I) = \pi(L^1)$$

and all inclusions are strict. In particular, the inclusion $C_1(I) \subset C_2(I)$ is strict.

However, the existence of I_2-approximate I-units is not equivalent to the inclusion $C_1(I) \subseteq C_2(I)$, as the following example shows:

EXAMPLE 6.4. Take the basic function system $B = l^2(\mathbb{N}) = \{x : \mathbb{N} \to \mathbb{R},\ \sum_{k=1}^\infty x_k^2 \le \infty\}$, and the linear functional $I(x) = \sum_{j=1}^\infty \beta_j x_j,\ x \in B,$ and fixed $\beta \in l^1$ with $\beta_j > 0$. Here $SR(I) = \pi(\beta l^2) \subset C_1(I) = \pi(\beta l^\infty) \subset C_2(I) = \pi(\sqrt{\beta} l^2)$, and there is no I_2-approximate I-units.

Finally, we give an example showing that $C_1(I) \subseteq C_2(I)$ is not true in general.

EXAMPLE 6.5. We now consider $B = l^1(\mathbb{N}) = \{x : \mathbb{N} \to \mathbb{R},\ \sum_{k=1}^{\infty} |x_k| \leq \infty\}$ and the integral $I(x) = \sum_{j=1}^{\infty} x_j\ \forall x \in B$. In this case $SR(I) = \pi(l^1) \subset C_2(I) = \pi(l^2) \subset C_1(I) = \pi(l^{\infty})$, and all inclusions are strict.

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