Bosonic indistinguishability-dependent contextuality

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Producing contextual correlations in sequences of measurements on single quantum systems faces two major problems. One is the experimental difficulty of performing sequences of ideal measurements on high-dimensional quantum systems, a problem that also affects other forms of quantum temporal correlations. The other is the simulability with classical light of existing contextuality experiments with photons. Here, we introduce a scheme that solves both problems. We show that, by encoding quantum information in $n \geqslant 2$ indistinguishable bosons in $m \geqslant 2$ modes, targeting observables exploiting the bunching-antibunching of bosons, and performing ideal measurements by dispersively coupling these systems with auxiliary qubits, it is possible to realize sequential quantum measurements on high-dimensional quantum systems and produce contextuality which cannot be simulated by classical light, as it relies on indistinguishability and higher-order interference.

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I. INTRODUCTION

Kochen-Specker (KS) contextuality [1–3] or contextuality between ideal measurements is a fundamental property of quantum mechanics. An ideal (or sharp) measurement [4–6] of an observable is one that does not disturb any compatible observable and, in particular, yields the same result when it is repeated. Because of this property, classical intuition suggests that ideal measurements are revealing predetermined results which are independent of which other ideal measurements of compatible observables are performed in the same trial. However, in quantum mechanics, sequences of ideal measurements of compatible observables [7–9] can produce correlations which cannot be explained assuming predefined noncontextual results [10–14], as they violate inequalities, called noncontextuality (NC) inequalities [10–14], which must be satisfied by any noncontextual model. Violations of NC inequalities can be observed by locally measuring spatially separated subsystems, as in Bell tests [15], and in experiments with sequential measurements on noncomposite systems [7–9,16–20]. KS contextuality has multiple applications [21] and plays a fundamental role in quantum computation [22–24] and quantum foundations [5,25].

While sequential measurements on single quantum systems allow, in principle, for quantum correlations with a large degree of contextuality [26,27], interesting temporal quantum correlations [28,29], and practical applications such as KS contextuality-based dimension witnessing [30,31], self-testing [32,33], and sequential measurements-based machine learning [34], these correlations require high-dimensional quantum systems. The problem is that we are far from being able to experimentally test these predictions, as the largest quantum system on which sequential ideal measurements have been carried out has dimension $d = 4$ [7]. Moreover, current platforms do offer a way to circumvent this limitation.

On the other hand, so far, all KS contextual correlations in photonic experiments (e.g., [16,19]) can be simulated with classical light [35,36], as they use first-order coherence measurements and thus probability distributions can be associated to single-mode intensities [35]. This classical simulability contrasts with the fact that both achieving quantum advantage via boson sampling [37,38] and universal quantum computing with linear optics [39] crucially rely on bosonic indistinguishability and higher-order interference. Hence, a fundamental question is whether there is a method to produce KS contextuality with photons relying on indistinguishability and higher-order interference. Although there are previous works on contextuality for bosonic systems [40–43], in the present work we are addressing this question.

In this article, we introduce a method to produce quantum correlations between sequences of ideal measurements on systems of $n \geqslant 2$ identical bosons propagating through $m \geqslant 2$ spatially distinct bosonic modes. The method has two distinguishing features. First, correlations cannot be simulated with classical light, as they require boson indistinguishability and higher-order interference. Due to this property, we will refer to the contextuality produced by this method as bosonic indistinguishability-dependent contextuality (BIC).

Second, the method allows performing sequential measurements (including ideal ones) on high-dimensional quantum systems and can be implemented in actual experiments, as it is based on two recent experimental developments, namely, the ability to encode and manipulate high-dimensional quantum systems in bosonic systems (see, e.g., [38]), and the ability to couple bosonic systems to external qubits [44,45].

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II. THE SIMPLEST EXAMPLE OF BOSONIC INDISTINGUISHABILITY-DEPENDENT CONTEXTUALITY

KS contextuality requires quantum systems of dimension $d \geqslant 3$ [1–3]. The simplest bosonic system producing KS contextuality consists of $n = 2$ indistinguishable photons propagating through $m = 2$ distinguishable modes (called upper and lower), as this defines a three-dimensional Hilbert space spanned by the states $|2,0\rangle$, $|0,2\rangle$, and $|1,1\rangle$, where $|n_a, n_b\rangle$ is the state with n_a photons in the upper mode and n_b photons in the lower mode. In terms of the creation operators for the upper and lower modes, $a^\dagger$ and $b^\dagger$, respectively, these states can be written as

$$|n_a, n_b\rangle = \frac{(a^\dagger)^{n_a}(b^\dagger)^{n_b}}{\sqrt{n_a!}\sqrt{n_b!}}|0,0\rangle. \tag{1}$$

The simplest contextuality witness for $d = 3$ is the Klyachko-Can-Binicioğlu-Shumovsky (KCBS) inequality [10], which can be written as

$$\kappa = -\frac{1}{3}\sum_{j=1}^{5}\langle A_j A_{j+1}\rangle \leqslant 1, \tag{2}$$

where A_j are observables with possible results -1 and $+1$, $\langle A_j A_{j+1}\rangle$ is the mean value of the product of the results of A_j and A_{j+1}, and the sum in the subindex is taken modulo 5. Testing the KCBS inequality in $d = 3$ requires preparing a particular initial state $|v\rangle = (v_x, v_y, v_z)^T$ and then performing two sequential compatible ideal measurements of the type $A_j = 2|v_j\rangle\langle v_j| - \mathbb{1}$.

For preparing arbitrary pure states, we allow the two modes to interact in a beam splitter (BS). This produces the following transformation (see Appendix A):

$$U_{\rm BS}(\theta,\phi)aU^\dagger_{\rm BS}(\theta,\phi) = \cos(\theta/2)a - e^{i\phi}\sin(\theta/2)b, \tag{3a}$$

$$U_{\rm BS}(\theta,\phi)bU^\dagger_{\rm BS}(\theta,\phi) = e^{-i\phi}\sin(\theta/2)a + \cos(\theta/2)b, \tag{3b}$$

where a and b are the annihilation operators for the upper and lower modes, respectively, and θ and φ are the angles accounting for the transmissivity and phase shift introduced by the BS, respectively.

If one begins with state $|1,1\rangle$, which is easy to prepare in various quantum optics devices [46–48], then the BS produces (see Appendix A)

$$U_{\rm BS}(\theta,\phi)|1,1\rangle = \tfrac{\sin\theta e^{i\phi}}{\sqrt{2}}|2,0\rangle - \tfrac{\sin\theta e^{-i\phi}}{\sqrt{2}}|0,2\rangle + \cos\theta|1,1\rangle. \tag{4}$$

By choosing θ and ϕ, one can produce the bosonic analog of $|v\rangle = (v_x, v_y, v_z)^T$, with $v_i \in \mathbb{R}$. This follows from the fact that $|v\rangle$ can be written as $(v_+, v_-, v_0)^T$ in a spherical basis, where $v_\pm = (\mp v_x + iv_y)/\sqrt{2}$ and $v_0 = v_z$. Using that $v_x = \sin\theta\cos\phi$, $v_y = \sin\theta\sin\phi$, and $v_z = \cos\theta$, (v_+, v_-, v_0) correspond to the components in the basis $\{|2,0\rangle, |0,2\rangle, |1,1\rangle\}$, respectively, as shown in Eq. (4). Therefore, for preparing the bosonic analog $|\psi_{\rm in}\rangle$ of a qutrit state $|v\rangle$, one can start with $|1,1\rangle$and then apply a suitably chosen BS. This is what is meant to happen in the block "state preparation" in Fig. 1(a).

The fundamental elements for connecting contextuality to bosonic indistinguishability are the choice of observables and

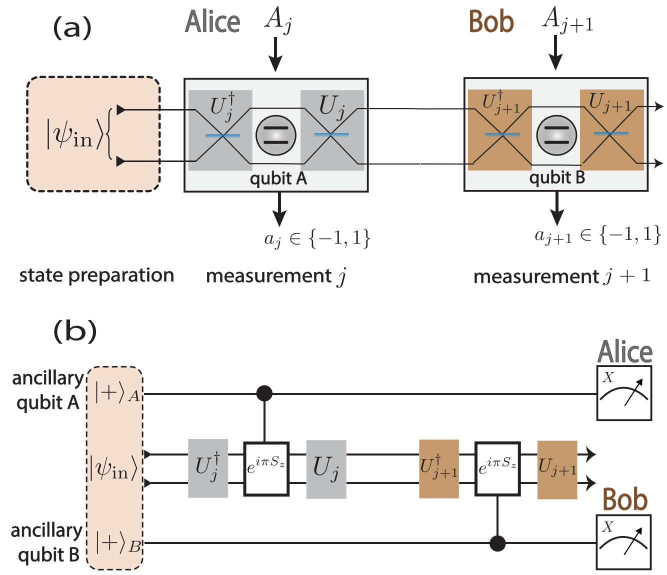

FIG. 1. (a) Schematic setup of two sequential ideal measurements on an initial state of an $n = 2$-photon $m = 2$-mode system. (b) Circuit representation of the operations used for the two sequential ideal measurements. Each of the measurements is finished only when the corresponding party reads out its qubit

the method for performing ideal measurements of them. An ideal measurement is typically implemented by applying first a suitable unitary transformation, then performing an ideal measurement of a suitable observable, and then applying the adjoint unitary transformation. For the unitary transformations, we use BSs: one before the measurement, implementing $U^\dagger_{\rm BS}(\theta_j,\phi_j)$, and one after the measurement, implementing $U_{\rm BS}(\theta_j,\phi_j)$; see Fig. 1, where $U_j \equiv U_{\rm BS}(\theta_j,\phi_j)$.

For the observable, we use that, for state (4), the probability for detecting one photon in each of the output ports of the BS is $p_+ = \cos^2\theta$ and the probability of observing two-photon bunching is $p_- = 1 - p_+$, as it was was demonstrated by Hong, Ou, and Mandel (HOM) [49] for the case $\theta = \pi/2$. The bunching of the two photons is originated by their indistinguishability. Hence, the suitable observable is the one in which outcome $+1$ corresponds to the two photons being detected in coincidence at the two modes, and outcome -1 to the bunching of the two photons. The ideal measurement of this observable is represented by the projector $\Pi^+ = |1,1\rangle\langle 1,1|$, associated to outcome $+1$, and $\Pi^- = \mathbb{1} - |1,1\rangle\langle 1,1|$, associated to outcome -1. Therefore, the measurement corresponding to $|v_j\rangle\langle v_j|$ is represented in our setup by $U_{\rm BS}(\theta_j,\phi_j)|1,1\rangle\langle 1,1|U^\dagger_{\rm BS}(\theta_j,\phi_j)$.

The maximum quantum violation with ideal measurements [50] of the KCBS inequality (2) is achieved for $\langle v_j|\psi_{\rm in}\rangle = \langle v_{j+1}|\psi_{\rm in}\rangle = \cos\gamma$, where $\gamma = \cos^{-1}(1/5^{1/4})$. In this case, one obtains $\langle\psi_{\rm in}|A_jA_{j+1}|\psi_{\rm in}\rangle = -4\cos^2\gamma + 1$, which gives

$$\kappa = \frac{4\sqrt{5}-5}{3} \approx 1.315. \tag{5}$$

This value can be achieved by a simple choice of parameter values, $\theta = 0$ in the BS of the state preparation (thus $|\psi_{\rm in}\rangle = |1,1\rangle$), and choosing the angles of the BSs for the observables A_j as follows: $\theta_j = \cos^{-1}(1/5^{1/4})$ and $\phi_j = 4\pi j/5$.

III. SEQUENTIAL IDEAL MEASUREMENTS OF OBSERVABLES

KS contextuality requires ideal measurements (i.e., not disturbing any compatible observable). A measurement on the $n \geqslant 2$-photon $m \geqslant 2$-mode system that ends by detecting the photons impedes further measurements and, therefore, cannot be ideal. An ideal measurement must be nondestructive. Moreover, it requires an interaction between the $n \geqslant 2$-photon $m \geqslant 2$-mode system and an ancillary system, capable of encoding the result of the measurement in the ancilla while leaving the bosonic system in the postmeasurement state given by Lüders' rule [6].

The dispersive regime [44,48,51,52] provides an efficient method to do it without destroying the photon number state. Let us consider an interaction Hamiltonian of the form

$$H_{\rm disp} = \frac{\lambda(t)}{2} S_z \otimes \sigma_z, \tag{6}$$

with $S_z = (a^\dagger a - b^\dagger b)/2$ and $\sigma_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$, where $\{|\uparrow\rangle, |\downarrow\rangle\}$ is an orthogonal basis of states of the ancillary qubit ($\hbar$ set to 1). The interaction is engineered in such a way that $\int_0^\tau dt\lambda(t) = \pi$. Therefore, $U_{\rm disp}(\pi) = e^{i\pi S_z\otimes\sigma_z/2}$. This has the following effect:

$$U_{\rm disp}(\pi)|2,0\rangle|+\rangle = |2,0\rangle|-\rangle, \tag{7a}$$

$$U_{\rm disp}(\pi)|0,2\rangle|+\rangle = |0,2\rangle|-\rangle, \tag{7b}$$

$$U_{\rm disp}(\pi)|1,1\rangle|+\rangle = |1,1\rangle|+\rangle, \tag{7c}$$

where $|\pm\rangle = (|\downarrow\rangle \pm |\uparrow\rangle)/\sqrt{2}$ are states of the ancillary qubit. Thus, by using this dispersive coupling, we can encode the outcomes of the measurement in the state of the ancillary qubit. The measurement result can be later read out by performing projective measurement on the ancillary qubit. It is worth adding that by virtue of the invariance of the total number of photons, coupling the qubit to a single mode is sufficient to measure the relevant observables (see Appendix B). Therefore, without losing generality $e^{i\frac{\pi}{2}a^\dagger a\otimes\sigma_z}$ generates similar phase shift to the qubit state conditioned on the antibunched photon number state, i.e., $e^{i\frac{\pi}{2}a^\dagger a\otimes\sigma_z}|1,1\rangle|\pm\rangle = |1,1\rangle|\mp\rangle$.

Similarly, the measurement of any of the required observables involves $U_{A_j} = U_{\rm BS}(\theta_j,\phi_j)e^{i\frac{\pi}{2}a^\dagger a\otimes\sigma_z}U^\dagger_{\rm BS}(\theta_j,\phi_j)$ acting on the initial state,

$$\begin{aligned} U_{A_j}|\psi_{\rm in}\rangle|-\rangle &= \cos\gamma|v_j\rangle|+\rangle + \sin\gamma|v_j^\perp\rangle|-\rangle \\ &= \Pi_j^+|\psi_{\rm in}\rangle|+\rangle + \Pi_j^-|\psi_{\rm in}\rangle|-\rangle, \end{aligned} \tag{8}$$

where $\Pi_j^\pm = (\mathbb{1} \pm \mathcal{P}_j)/2$ and $\mathcal{P}_j = U_{\rm BS}(\theta_j,\phi_j)(-e^{i\pi a^\dagger a} \otimes \mathbb{1}_b)U^\dagger_{\rm BS}(\theta_j,\phi_j)$ is the photon-number parity operator acting on the bosonic modes subspace, and measured by measuring the qubit coupled to the upper mode giving outcome probabilities $p_\pm = \langle\psi_{\rm in}|\Pi_j^\pm|\psi_{\rm in}\rangle$ on the $\{|+\rangle, |-\rangle\}$ basis. Therefore,

$$\langle\sigma_x\rangle = p_+ - p_- = \langle\psi_{\rm in}|\mathcal{P}_j|\psi_{\rm in}\rangle, \tag{9}$$

where $p_+ = \cos^2\gamma$ and $p_- = 1 - p_+$. For the two-photon case the observable reduces to $\mathcal{P}_j \equiv 2|v_j\rangle\langle v_j| - \mathbb{1}$, defined in a $d = 3$ space.

The crucial requirement is that the interaction between the photons and the qubit should preserve the indistinguishability of the input photons. The dispersive interaction induces σ_z eigenstate-dependent frequency shift to the both modes, i.e., $\omega_{a(b)} = \omega \pm \lambda$. However, the qubit initialized $|\pm\rangle$ does not induce a frequency shift. The second measurement is similar.

This technique enables sequential ideal measurements on the $n = 2$-photon $m = 2$-mode system. The initial state of the two ancilla qubits $|-\rangle_A|-\rangle_B$ after the action of $U_{A_{j+1}}U_{A_j}$ turns into ρ_{AB} (see Appendix E). The joint probabilities are

$$p_{a_j a_{j+1}} = \langle\psi_{\rm in}|\Pi_j^{a_j}\Pi_{j+1}^{a_{j+1}}\Pi_j^{a_j}|\psi_{\rm in}\rangle, \tag{10}$$

where $a_j, a_{j+1} \in \{-,+\}$. Once $p_{a_j a_{j+1}}$ are inserted into the mean values, we obtain

$$\langle\sigma_x^A \otimes \sigma_x^B\rangle_{\rho_{AB}} \equiv \langle\psi_{\rm in}|\mathcal{P}_j\mathcal{P}_{j+1}|\psi_{\rm in}\rangle, \tag{11}$$

for compatible measurements. Note that $p_{++} = 0$. This is due to the bosonic bunching or HOM-like effect yielding $p_{+|+} = |\langle 1,1|U_j^\dagger U_{j+1}|1,1\rangle|^2 = 0$, as a direct consequence of bosonic indistinguishability. The propagating bosonic system acts as a quantum bus [53], giving rise to correlations which cannot be produced by coupling to (semi)classical fields. This contrasts with the fact that a HOM-like effect can be mimicked by proper phase control of classical fields interfering at the BS [54]. However, we can show that any correlation produced by coupling to such classical fields or coherent states never leads to the violation of the NC inequalities (see Appendix F). In the case of stationary rather than propagating bosonic mode, an alternative scheme involves sequential measurements on a single ancilla qubit (see Appendix B1). This very scheme is already implemented efficiently in Ref. [51] for two-mode two-photon and even for higher photon numbers.

IV. HIGHER-DIMENSIONAL BOSONIC INDISTINGUISHABILITY-DEPENDENT CONTEXTUALITY

So far, we have shown how to use BIC for producing KS contextual correlations between sequential measurements on qutrits. Here, we discuss how to use it for producing correlations between sequential measurements on qudits of arbitrary $d \geqslant 3$. We will focus on the case of sequences of two dichotomic measurements, as this is sufficient for generating any matrix of quantum KS contextual correlations [55]. For other forms of high-dimensional KS contextuality, see Appendix B.

There are several possible ways to achieve high-dimensional BIC combining the Fock space encoding and readout techniques [56,57]. The most compact one consists of using $n \geqslant 2$ photons in $m = 2$ modes [58]. In this case, the most convenient choice for dichotomic observables is that represented by $\Pi^+ = |n_a, n_b\rangle\langle n_a, n_b|$, with $n_a = n_b$, and $\mathbb{1} - \Pi^+$. This allows for producing KS contextual correlations between sequential measurements on quantum systems of any odd dimension [59] (see Appendix B2). In this case, the coupling is engineered such that $H_{\rm disp} = \lambda(t)|n_a\rangle\langle n_a| \otimes \sigma_z$ [60,61].

For experimentally testing BIC, multiple dispersively coupled ancillary qubits (as in Fig. 1) are not necessary. A single one suffices. This follows, on the one hand, from the observation that sequences of two dichotomic measurements are sufficient for producing any matrix of quantum KS contextual correlations [55], and, on the other hand, from the observation that the second measurement does not need to be ideal, since no further measurements will be performed afterward. This holds for any $d \geqslant 3$. For this reason, in this section we will focus on sequences of two dichotomic measurements in which only the first measurement is ideal.

There are several ways to achieve high-dimensional BIC. The most compact way to achieve any odd dimension d consists of using $n = d - 1$ photons in $m = 2$ modes [58]. This is just a particular case of the more general case of using $n \geqslant 2$ indistinguishable bosons in $m \geqslant 2$ modes. If there are no extra constraints (e.g., of the number of allowed bosons per mode), this defines a quantum system of dimension

$$d = \frac{(n+m-1)!}{n!(m-1)!}, \quad (12)$$

assuming that there are no limitations in the number of bosons per mode. In this case, the most convenient choice for dichotomic observables is that represented by $\Pi^+ = \sum_{n_1,\ldots,n_m} |n_1,\ldots,n_m\rangle\langle n_1,\ldots,n_m|$ and $\mathbb{1} - \Pi^+$. The unitaries needed for observables of the form $\Pi_j^+ = U_j \Pi^+ U_j^\dagger$ can be implemented using configurations made of BSs, m-port BSs [62–65], or fiber loops [66]. In principle, any given unitary can be approximated with high fidelity by applying global optimization methods (see, e.g., [67]) to the variables available in these configurations.

For implementing ideal measurements, we engineer the coupling of the bosonic system with an external qubit such that

$$U_{\text{disp}}(\pi) = (\mathbb{1} - \Pi^+) \otimes \mathbb{1} + \Pi^+ \otimes \sigma_z. \quad (13)$$

This enables ideal measurement of dichotomic observables such as $\mathcal{P}^{(n,m)} = 2\Pi^+ - \mathbb{1}$.

Current technology allows both to encode high-dimensional quantum systems using n bosons in m modes, and to make ideal measurements on them using dispersive coupling with external qubits. For example, systems with up to $n = 76$ and $m = 100$ yielding a state space dimension of $d \approx 10^{30}$ have been experimentally demonstrated [38], and sequential measurements via coupling with an external qubit have been demonstrated in bosonic superconducting devices with up to $n = 30$ and $m = 2$ [44] and $m = 11$ [45].

Regarding the measurements necessary for BIC, recall that, in practice, to implement any given unitary on a single photon in m modes, the usual approach is approximating it with a given infidelity by a global optimization method applied to the reconfigurable elements available in the setup. These elements range from multiports made of beam splitters and phase sifters to multicore fiber integrated multiport interferometers and fiber loops. Results for single photons (see, e.g., Ref. [67]) suggest that scalability is not an issue, as relatively compact configurations manage to achieve very high fidelities for relatively high dimension.

All these results point out that BIC may be a reliable way to overcome current limitations and produce KS contextual correlations using higher-dimensional quantum systems.

V. STATE-INDEPENDENT BOSONIC INDISTINGUISHABILITY-DEPENDENT CONTEXTUALITY

One of the interesting possibilities of quantum KS contextuality is that it can be state independent, which means that, for any quantum system of dimension three or larger, there are sets of measurements and contextuality witnesses that have the same value (beyond the corresponding noncontextual bound) for any quantum state [11–14]. The same happens for BIC.

To show this effect, one can consider the bosonic equivalents of the witness of Yu and Oh, YO [13], or its optimal version, opt_3 [14], and apply the method described before. As can be easily checked, while the noncontextual bound for both witnesses is 1, any state in the basis $\{|2,0\rangle, |0,2\rangle, |1,1\rangle\}$ gives the value $25/24 \approx 1.042$ for YO and the value $83/75 \approx 1.107$ for opt_3 (see Appendix C).

VI. MAXIMUM CONTEXTUALITY REQUIRES PERFECT INDISTINGUISHABILITY

Here, we discuss the connection between maximal BIC and bosonic indistinguishability. For that, we study what happens when bosons are not perfectly indistinguishable. This may occur due to, e.g., that they have a different polarization or that there is a time delay between them.

To model the effect of distinguishability, one can replace the initial state $|1,1\rangle$, with which the state preparation was fed, with the state $|1,1_\eta\rangle = a^\dagger b_\eta^\dagger |\text{vac}\rangle$ with $b_\eta^\dagger = \sqrt{1-\eta^2} b^\dagger + \eta b_\perp^\dagger$, where $a^\dagger$ and $b^\dagger$ are creation operators of indistinguishable photons in the upper and lower modes, respectively, while $b_\perp^\dagger$ is a creation operator of photons in the lower mode which are perfectly distinguishable from the former (e.g., $a^\dagger$ and $b^\dagger$ create horizontally polarized photons and $b_\perp^\dagger$ creates vertically polarized ones). $\eta \in [0,1]$ quantifies the degree of distinguishability between the two photons, with $\eta = 0$ representing perfect indistinguishability and $\eta = 1$ perfect distinguishability.

The BS mixes the upper and lower modes regardless of how distinguishable the photons are. Therefore, we are dealing with four distinct modes then, i.e., those corresponding to a, $a_\perp$, b, and $b_\perp$.

Assuming that the dispersive coupling merely depends on the number of photons in each transmission line and not on the degree of freedom with respect to which the photons are distinguishable,

$$H_{\text{disp}} = \lambda(t)\frac{(N_a - N_b)}{2} \otimes \sigma_z = \lambda(t)(S_z + S_z^\perp) \otimes \sigma_z, \quad (14)$$

where $N_a = a^\dagger a + a_\perp^\dagger a_\perp$ denoting the number of photons in the upper transmission line and $N_b = b^\dagger b + b_\perp^\dagger b_\perp$ denoting the number of photons in the lower transmission line. Figure 2(a) shows how the value of κ depends on η: The maximum value is only obtained when the photons are perfectly indistinguishable. Otherwise, contextuality decreases as distinguishability increases. This implies that contextuality can be used to certify boson indistinguishability in a way that cannot be simulated with classical light. This contrasts with the fact that the HOM effect can be simulated with classical

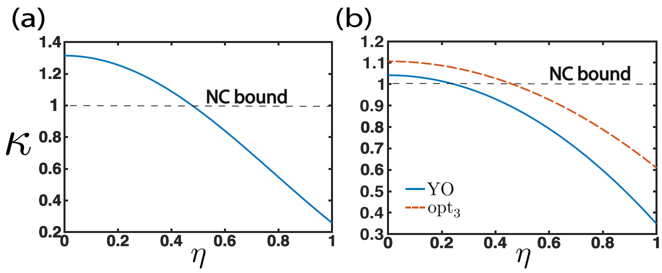

FIG. 2. Value of the contextuality witness [with noncontextual (NC) bound 1] as a function of the degree of bosonic indistinguishability η, defined in the text. (a) For the witness defined in (2), with initial states $U_{\rm BS}(\theta,\phi)|1,1_\eta\rangle$, defined in the text, and the observables A_j used to obtain the quantum maximum of κ. (b) For the Yu-Oh and the optimal state-independent inequality opt_3 witnesses (see Appendix C), for *any* initial state $U_{\rm BS}(\theta,\phi)|1,1_\eta\rangle$ (i.e., no matter the values of θ and ϕ).

light [54]. Moreover, since the maximum quantum violation of the KCBS inequality allows for self-testing [32] (i.e., certification using only the observed correlations), Fig. 2(a) shows that BIC can be used to self-test boson indistinguishability. Therefore, BIC provides an alternative quantitative test of quantum indistinguishability [68,69].

Interestingly, Fig. 2(b) shows exactly the same behavior, but now the degree of contextuality does not depend on θ and ϕ of the initial state, but only on the degree of distinguishability η (see Appendix D for details).

However, just as the maximal violation of a bipartite Bell inequality does not always require maximal entanglement, so the maximal violation of a noncontextuality inequality cannot be expected to always require perfect indistinguishability. Each case has to be studied separately.

VII. DELAYED MEASUREMENTS

The method for performing sequential measurements described before also opens an interesting possibility, namely, deciding at will the order in which the sequential measurements are "finished." This possibility comes from the fact that, in the case of sequences of two measurements, the quantum state after the interaction with the second ancillary qubit is a coherent superposition of the four quantum states corresponding to the four combinations of results for the two measurements (see Appendix E). This superposition can be "collapsed" in three different ways: (i) by first reading out the second qubit and only then reading out the first qubit, (ii) by first reading out the first and then the second, or (iii) by spacelike separating the readouts. This offers a possibility beyond what can be done in standard sequential measurement experiments (e.g., [9]), where the readout of the result of the second measurement cannot be spacelike separated from the readout of the result of the result of the first measurement. This possibility can stimulate a new generation of sequential measurement experiments and tests of collapse models [70] and causality in quantum mechanics.

VIII. CONCLUSIONS

Current contextuality experiments with sequential measurements on quantum systems have a limitation: they are only possible for low-dimensional systems and there is no prospect of overcoming this problem. This prevents testing experimentally some interesting forms of quantum contextuality and other temporal quantum correlations, and progressing toward practical applications. Moreover, existing photonic contextuality experiments can be simulated with classical light, which makes them useless for obtaining quantum advantage. In this article, we have introduced a form of photonic contextuality, dubbed bosonic indistinguishability-dependent contextuality (BIC), which produces contextual correlations which cannot be simulated with classical light, connects maximum contextuality with perfect indistinguishability, and allows sequential measurements on high-dimensional quantum systems. We have also shown that current technology permits preparing the states and performing the measurements necessary to observe BIC on high-dimensional quantum systems. Therefore, we believe that BIC and the methods presented provide a realistic path to unlock experimental progress in sequential measurements on high-dimensional quantum systems and pave the way toward practical applications.

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APPENDIX A: ACTION OF A BEAM SPLITTER ON AN $n = 2$-PHOTON $m = 2$-MODE STATE AND GENERATION OF THE BOSONIC EQUIVALENT OF ANY QUTRIT STATE WITH REAL COMPONENTS

The Hamiltonian of a beam splitter (BS) coupling the two input modes (upper and lower) is

$$H_{\rm BS} = \frac{i\theta}{2}(e^{-i\phi}a^\dagger b - e^{i\phi}b^\dagger a), \tag{A1}$$

where $a^\dagger$ and a are the creation and annihilation operators for the upper mode, $b^\dagger$ and b are the creation and annihilation operators for the lower mode, and θ and φ are the angles accounting for the transmissivity and phase shift introduced by the BS, respectively. Therefore, the BS transformation is

$$U_{\rm BS}(\phi,\theta) = \exp\left[\frac{\theta}{2}(e^{-i\phi}a^\dagger b - e^{i\phi}b^\dagger a)\right]. \tag{A2}$$

In the qutrit subspace spanned by the basis $\{|2,0\rangle, |0,2\rangle, |1,1\rangle\}$, where $|n_a, n_b\rangle$ is the state in which there are n_a photons in the upper mode and n_b photons in the lower mode, the action of $U_{\rm BS}(\phi,\theta)$ is

$$U_{\rm BS}(\phi,\theta)=\begin{pmatrix}\cos^2\frac{\theta}{2} & \frac{1}{\sqrt{2}}e^{i\phi}\sin\theta & e^{i2\phi}\sin^2\frac{\theta}{2}\\ -\frac{1}{\sqrt{2}}e^{-i\phi}\sin\theta & \cos\theta & \frac{1}{\sqrt{2}}e^{i\phi}\sin\theta\\ e^{-i2\phi}\sin^2\frac{\theta}{2} & -\frac{1}{\sqrt{2}}e^{-i\phi}\sin\theta & \cos^2\frac{\theta}{2}\end{pmatrix}, \tag{A3}$$

which, applied to $|1,1\rangle$, produces the state given by Eq. (3) in the main text.

By suitably chosen θ and ϕ of the BS, one can produce the bosonic analog of any $|v\rangle$ with real components. This can be seen as follows. Consider

$$|v\rangle = v_x|e_x\rangle + v_y|e_y\rangle + v_z|e_z\rangle, \tag{A4}$$

where $(v_x, v_y, v_z)\in\mathbb{R}^3$ and $\{|e_x\rangle, |e_y\rangle, |e_z\rangle\}$ is a Cartesian basis. Consider the spherical basis, defined as

$$|e_\pm\rangle = \mp\tfrac{1}{\sqrt{2}}(|e_x\rangle \pm i|e_y\rangle), \tag{A5a}$$

$$|e_0\rangle = |e_z\rangle, \tag{A5b}$$

where i denotes the imaginary unit. Then,

$$|v\rangle = v_-|e_-\rangle + v_+|e_+\rangle + v_0|e_0\rangle. \tag{A6}$$

The components in the spherical basis are related to the components in the Cartesian basis by

$$v_\pm = \tfrac{1}{\sqrt{2}}(\mp v_x + iv_y), \tag{A7a}$$

$$v_0 = v_z. \tag{A7b}$$

Now notice that $|v\rangle$ can be written as

$$|v\rangle = \sin\theta\cos\phi|e_x\rangle + \sin\theta\sin\phi|e_y\rangle + \cos\theta|e_z\rangle, \tag{A8}$$

with $0\leqslant\theta<\pi$ and $0\leqslant\phi<\pi$. Therefore, the components of $|v\rangle$ in the spherical basis are

$$v_+ = \tfrac{1}{\sqrt{2}}(-\sin\theta\cos\phi + i\sin\theta\sin\phi) = -\tfrac{\sin\theta}{\sqrt{2}}e^{-i\phi}, \tag{A9a}$$

$$v_- = \tfrac{1}{\sqrt{2}}(\sin\theta\cos\phi + i\sin\theta\sin\phi) = \tfrac{\sin\theta}{\sqrt{2}}e^{i\phi}, \tag{A9b}$$

$$v_0 = \cos\theta, \tag{A9c}$$

which, as shown in Eq. (3) in the main text, are in one-to-one correspondence with the components of $U_{\rm BS}(\theta,\phi)|1,1\rangle$ in the basis $\{|2,0\rangle, |0,2\rangle, |1,1\rangle\}$, respectively. The correspondence is $|0,2\rangle \Leftrightarrow |e_+\rangle$, $|2,0\rangle \Leftrightarrow |e_-\rangle$, and $|1,1\rangle \Leftrightarrow |e_0\rangle$.

APPENDIX B: HIGH-DIMENSIONAL BIC

In this section we present two different examples of going to higher-dimensional BIC.

1. BIC with Pauli-like observables

In the discussion on high-dimensional BIC in the main text, we have focused on BIC produced by bosonic observables equivalent to Hilbert space observables represented by rank-one projectors. Here, we apply the same ideas to show how to produce BIC equivalent to that generated by Pauli observables on $q\geqslant 2$-qubit systems. For every $q\in\mathbb{N}$, the set of Pauli observables for q qubits is the set of 4^q-1 nontrivial quantum observables represented by q-term tensor products of the 2×2 identity matrix and the Pauli matrices σ_x, σ_y, and σ_z. Pauli observables allow for simple proofs of contextuality [71,72], compact contextuality experiments with sequential measurements [7,11], and for showing that the degree of (state-independent) contextuality can grow with the number N of qubits [26]. The interest of BIC is that it provides a way to experimentally observe this last prediction.

Consider n photons and $m=2$ modes. Using Schwinger's representation of the angular momentum operators [73],

$$S_0 =(a^\dagger a + b^\dagger b)/2, \tag{B1a}$$

$$S_x =(a^\dagger b + b^\dagger a)/2, \tag{B1b}$$

$$S_y =-i(a^\dagger b - b^\dagger a)/2, \tag{B1c}$$

$$S_z =(a^\dagger a - b^\dagger b)/2, \tag{B1d}$$

where $a^\dagger$ (a) and $b^\dagger$ (b) are the creation (annihilation) operators for the upper and lower modes, respectively, we can identify three bosonic observables analogous to the ones represented by the three Pauli matrices. The first one is

$$\mathcal{P}_z = (-1)^{a^\dagger a}\otimes \mathbb{1}_b = e^{i\pi a^\dagger a}\otimes\mathbb{1}_b \tag{B2}$$

$$= \sum_{n_a=0}^{n}(-1)^{n_a}|n_a, n-n_a\rangle\langle n_a, n-n_a|, \tag{B3}$$

which is a parity operator acting on a $d=n+1$-dimensional Hilbert space. $\mathcal{P}_z$ divides the Fock space into even and odd subspace. Notice that $\mathcal{P}_z = e^{i\pi(S_0+S_z)} = e^{i\pi a^\dagger a}\otimes\mathbb{1}_b$. In the basis with fixed total number of photons,

$$\mathcal{P}_z = e^{i\pi n/2}\sum_{n_a,n_b,n_a',n_b'}\langle n_a, n_b|e^{i\pi S_z}|n_a', n_b'\rangle|n_a, n_b\rangle\langle n_a', n_b'|. \tag{B4}$$

Therefore, for example, in the case of $n=n_a+n_b\in$ even we have

$$\langle e^{i\pi a^\dagger a}\rangle = -\langle e^{i\pi S_z}\rangle. \tag{B5}$$

The other two operators are achieved by the following unitary transformation of $\mathcal{P}_z$:

$$\mathcal{P}_x = U_{BS}(\pi/2,0)\mathcal{P}_z U_{BS}^\dagger(\pi/2,0), \tag{B6}$$

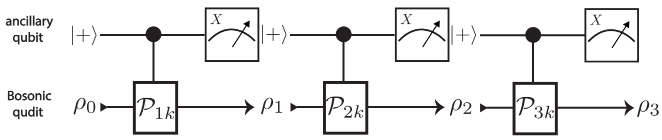

FIG. 3. Scheme for sequential measurements of bosonic parity operators suitable for stationary bosons.

$$\mathcal{P}_y = U_{BS}(\pi/2, \pi/2)\mathcal{P}_z U_{BS}^\dagger(\pi/2, \pi/2). \tag{B7}$$

Assuming that $n = n_a + n_b$, we obtain

$$\mathcal{P}_x = \sum_{n_a=0}^{n} |n_a\rangle|n-n_a\rangle\langle n-n_a|\langle n_a|, \tag{B8}$$

$$\mathcal{P}_y = i^n \sum_{n_a=0}^{n} (-1)^{n_a}|n_a\rangle|n-n_a\rangle\langle n-n_a|\langle n_a|. \tag{B9}$$

$\mathcal{P}_x$, $\mathcal{P}_y$, and $\mathcal{P}_z$ are the bosonic equivalent to σ_x, σ_y, and σ_z, respectively. They satisfy

$$\mathcal{P}_i\mathcal{P}_j = \mathbb{1}_{n+1}\delta_{ij} + i^n\epsilon_{ijk}\mathcal{P}_k, \tag{B10}$$

which is defined in the basis spanned by $|n_a, n_b\rangle$, with $n = n_a + n_b$.

For the Pauli matrices we have that

$$\bigotimes_{l=1}^{m}\sigma_i^{(l)}\bigotimes_{l=1}^{m}\sigma_j^{(l)} = \mathbb{1}_{2^m}\delta_{ij} + i^m\epsilon_{ijk}\otimes_{l=1}^{m}\sigma_k^{(l)}, \tag{B11}$$

where $\bigotimes_{l=1}^{m}\sigma_i^{(l)} = \sigma_i^{(1)}\otimes\cdots\otimes\sigma_i^{(m)}$. The dispersive coupling for realizing ideal measurements of the above bosonic operators is $H_{\rm disp} = \lambda a^\dagger a\otimes\sigma_z$.

For the q-qubit case, we use m distinguishable m modes with $n = n_1 + \cdots + n_m$ photons. In this case, for example,

$$\begin{aligned}\mathcal{P}_{\mathcal{Z}^m} &= \mathcal{P}_{z_1}\otimes\cdots\otimes\mathcal{P}_{z_m} = (-1)^{a_1^\dagger a_1+\cdots+a_m^\dagger a_m} \\ &= \sum_{n_1,\ldots,n_m=0}^{n} (-1)^{n_1+\cdots+n_m}|n_1,\ldots,n_m\rangle\langle n_1,\ldots,n_m|,\end{aligned} \tag{B12}$$

which can be ideally measured by the coupling $H_{\rm disp} = \sum_{i=1}^m \lambda_i a_i^\dagger a_i\otimes\sigma_z$.

For example, to perform an ideal measurement of $\mathcal{P}_{z_1}\otimes\mathbb{1}$, $\mathbb{1}\otimes\mathcal{P}_{z_2}$, and $\mathcal{P}_{\mathcal{Z}^2} = \mathcal{P}_{z_1}\otimes\mathcal{P}_{z_2}$, we engineer the dispersive couplings to be $H_{\rm disp} = \lambda_1 a_1^\dagger a_1\otimes\sigma_z$, $H_{\rm disp} = \lambda_2 a_2^\dagger a_2\otimes\sigma_z$, and $H_{\rm disp} = (\lambda_1 a_1^\dagger a_1 + \lambda_2 a_2^\dagger a_2)\otimes\sigma_z$, respectively. The sequences of three measurements required for showing BIC growing with the dimension [26] are a direct generalization of the method described in the main text. See Fig. 3. For example, measuring the context in the kth column is given by

$$\langle\sigma_x(t_1)\sigma_x(t_2)\sigma_x(t_3)\rangle_{\rm seq} = {\rm tr}(\rho_0\mathcal{P}_{1k}\mathcal{P}_{2k}\mathcal{P}_{3k}). \tag{B13}$$

The efficient implementation of such sequence is reported in several setups. See, e.g., Refs. [56,74]. Interesting generalized photon parity operators and their measurements are introduced in Ref. [74]. This construction allows two distinct possibilities to go to high-dimensional contextuality: one keeps the number of loaded photons fixed and increases the number of modes. The other possibility is to increase the number of photons rather than the number of modes, which is closely related to a technique called bosonic encoding of multiple qubits [75]. The latter is also closer to the spirit of BIC for reaching a compact way of high-dimensional contextuality.

For example,

$$\otimes_{l=1}^m\sigma_x^{(l)} \equiv \mathcal{P}_{\mathcal{X}^m}, \tag{B14}$$

where, $\mathcal{P}_{\mathcal{X}^m} = \mathcal{P}_{x_1}\otimes\cdots\otimes\mathcal{P}_{x_m}$ and $\mathcal{P}_{x_i} = \sum_{n_{a_i}=0}^{N=1}|n_{a_i}\rangle|N-n_{a_i}\rangle\langle N-n_{a_i}|\langle n_{a_i}|$. The alternative realization which takes advantage of the large bosonic Hilbert space of high photon number modes is

$$\otimes_{k=1}^m\sigma_x^{(k)} \equiv \mathcal{P}_x = \sum_{n_a=0}^{N=2^m-1}|n_a\rangle|N-n_a\rangle\langle N-n_a|\langle n_a|, \tag{B15}$$

which are supposed to be the bosonic equivalence of the measurement of the multiqubit Pauli operators, and multiqubit collective rotations $\mathcal{U}_i = U_i^{(1)}\otimes\cdots\otimes U_i^{(m)}$, where $U_i^{(k)} = e^{-i\theta\sigma_i^{(k)}/2}$. From the identity

$$S_i = \frac{1}{2}\sum_{k=1}^m\sigma_i^{(k)}, \quad i = x, y, z, \tag{B16}$$

one can see how the beam splitter realizes $\mathcal{U}_i = e^{-i\theta S_i}$ transforming two-mode bosonic systems, identical to collective spin rotation.

From (B10) we can construct the negative and positive contexts introduced in Ref. [26].

2. Odd-dimensional case

Here, we discuss how to produce forms of contextuality requiring rank-one projectors and quantum systems of odd dimension $d \geqslant 3$ as, e.g., those in Ref. [59]. The bosonic encoding and the implementation of the measurements is straightforward. In the following we present an explicit example. The only challenges are engineering the coupling between the ancilla qubit and the bosonic modes and applying the required unitaries for measuring the observables.

The coupling $H_{\rm disp} = \lambda(t)|n_a\rangle\langle n_a|\otimes\sigma_z$ can generate the following unitary evolution:

$$U_{\rm disp}(\pi) = e^{i\frac{\pi}{2}|n_a\rangle\langle n_a|\otimes\sigma_z}, \tag{B17}$$

causing π-phase shift to the ancillary qubit if and only if the Fock state is $|n_a\rangle$. This number-dependent phase shift technique enables measuring,

$$\mathcal{P}_0 = 2\left|\frac{n}{2},\frac{n}{2}\right\rangle\left\langle\frac{n}{2},\frac{n}{2}\right| - \mathbb{1}, \tag{B18}$$

where in this special case we have $|n_a\rangle = |n/2\rangle$. The observables act on $d = n + 1$-dimensional subspace with $n \in$ even. Let us consider $d = 5$ explicitly as the simplest case beyond the qutrit example.

Using proper unitary operation U_k acting on the two modes we can implement the corresponding observable $A_k = 2|v_k\rangle\langle v_k| - \mathbb{1}$. Thus,

$$|v_k\rangle = |\theta,\phi,\varphi_k\rangle = U_k|2,2\rangle = \cos\theta|2,2\rangle + \sin\theta|\phi,\varphi_k\rangle, \tag{B19}$$

where $|\phi, \varphi_k\rangle = \cos\phi|2\varphi_k\rangle + \sin\phi|\varphi_k\rangle$ and $\varphi_k = \frac{2\pi}{7}k$,

$$|2\varphi_k\rangle = \frac{1}{\sqrt{2}}(e^{i2\varphi_k}|4,0\rangle + e^{-i2\varphi_k}|0,4\rangle), \tag{B20}$$

$$|\varphi_k\rangle = \frac{1}{\sqrt{2}}(e^{i\varphi_k}|3,1\rangle + e^{-i\varphi_k}|1,3\rangle). \tag{B21}$$

The maximum violation, in this example, occurs for the initial state $|n/2, n/2\rangle$, known as the twin Fock state which is shown to be a resource for quantum metrology. As is apparent, the specification of the optimal projectors relies on the generalized HOM quantum interference between four indistinguishable photons.

APPENDIX C: WITNESSES FOR STATE-INDEPENDENT CONTEXTUALITY

The Yu-Oh inequality [13] can be written as

$$\text{YO} = -\frac{1}{8}\left(\sum_{j\in V}\langle A_j\rangle + \frac{1}{2}\sum_{(j,k)\in E}\langle A_jA_k\rangle\right) \leqslant 1, \tag{C1}$$

where A_j are 13 observables with possible results -1 and $+1$, $V = V_1 \cup V_2$, with $V_1 = \{1, 2, \ldots, 9\}$ and $V_2 = \{A, B, C, D\}$, and $E = E_1 \cup E_2$, with $E_1 = \{(1,4), (1,7), (2,5), (2,8), (3,6), (3,9), (4,7), (5,8), (6,9)\}$ and $E_2 = \{(1,2), (1,3), (2,3), (4,A), (4,D), (5,B), (5,D), (6,C), (6,D), (7,B), (7,C), (8,A), (8,C), (9,A), (9,B)\}$.

The optimal version of the Yu-Oh inequality [14] can be written as

$$\text{opt}_3 = -\frac{1}{25}\left(\sum_{j\in V_1}\langle A_j\rangle + 2\sum_{j\in V_2}\langle A_j\rangle + \sum_{(j,k)\in E_1}\langle A_jA_k\rangle + 2\sum_{(j,k)\in E_2}\langle A_jA_k\rangle - 3\sum_{(j,k,l)\in T}\langle A_jA_kA_l\rangle\right) \leqslant 1, \tag{C2}$$

where $T = \{(1,4,7), (2,5,8), (3,6,9)\}$.

By choosing observables of the form $A_k = 2|v_k\rangle\langle v_k| - \mathbb{1}$, with

$$\begin{aligned}
&|v_1\rangle = (1,0,0)^T, && |v_8\rangle = (\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}})^T,\\
&|v_2\rangle = (0,1,0)^T, && |v_9\rangle = (\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0)^T,\\
&|v_3\rangle = (0,0,1)^T, && |v_A\rangle = (-\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}})^T,\\
&|v_4\rangle = (0, \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}})^T, && |v_B\rangle = (\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}})^T,\\
&|v_5\rangle = (\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}})^T, && |v_C\rangle = (\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}})^T,\\
&|v_6\rangle = (\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, 0)^T, && |v_D\rangle = (\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}})^T,\\
&|v_7\rangle = (0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}})^T,
\end{aligned} \tag{C3}$$

and applying the method described in the main text, we obtain the value $25/24 \approx 1.042$ for YO and the value $83/75 \approx 1.107$ for opt_3.

APPENDIX D: EFFECT OF PARTIAL DISTINGUISHABILITY

The BS mixes the two spatial modes regardless of the degree of freedom that makes the photons distinguishable. That is,

$$U_{BS}(\phi,\theta) = \exp\left\{\frac{\theta}{2}[e^{-i\phi}(a^\dagger b + a_\perp^\dagger b_\perp) - e^{i\phi}(b^\dagger a + b_\perp^\dagger a_\perp)]\right\}. \tag{D1}$$

For example, suppose that $a = a_H$ annihilates horizontally polarized photons and $a_\perp = a_V$ annihilates vertically polarized photons. Then, we effectively have four different modes, i.e., $a^\dagger|\text{vac}\rangle \equiv |1,0,0,0\rangle$, $a_\perp^\dagger|\text{vac}\rangle \equiv |0,1,0,0\rangle$, $b^\dagger|\text{vac}\rangle \equiv |0,0,1,0\rangle$, and $b_\perp^\dagger|\text{vac}\rangle \equiv |0,0,0,1\rangle$. Consider the following input state:

$$|1_H, 1_\eta\rangle = a_H^\dagger\left(\sqrt{1-\eta^2}b_H^\dagger + \eta b_V^\dagger\right)|\text{vac}\rangle \tag{D2}$$

$$= \sqrt{1-\eta^2}|1,1,0,0\rangle + \eta|1,0,0,1\rangle, \tag{D3}$$

where $|n_{aH}, n_{aV}, n_{bH}, n_{bV}\rangle$ is the state with n_{aH} horizontally polarized photons and n_{aV} vertically polarized photons in the upper mode and n_{bH} horizontally polarized photons and n_{bV} vertically polarized photons in the lower mode. The action of the BS on this state involves $U(\theta, \phi_j)|1_H, 1_H\rangle = |v_j\rangle$, which is the same as Eq. (3) in the main text, and $U(\theta, \phi_j)|1_H, 1_V\rangle = |\tilde{v}_j\rangle$ which is

$$|\tilde{v}_j\rangle = \cos^2\frac{\theta}{2}|1,0,0,1\rangle - \sin^2\frac{\theta}{2}|0,1,1,0\rangle + \frac{\sin\theta}{2}(e^{-i\phi_j}|1,1,0,0\rangle - e^{i\phi_j}|0,0,1,1\rangle). \tag{D4}$$

Therefore,

$$U(\theta,\phi_j)|1, 1_\eta\rangle = \sqrt{1-\eta^2}|v_j\rangle + \eta|\tilde{v}_j\rangle. \tag{D5}$$

The joint detection probability is then

$$p_+ = (1-\eta^2)\cos^2\theta + \eta^2\frac{1+\cos^2\theta}{2} \tag{D6}$$

and the bunching probability is $p_- = 1 - p_+$.

APPENDIX E: QUANTUM STATE AFTER THE INTERACTION WITH TWO SUCCESSIVE ANCILLARY QUBITS

The state of the bosonic system, the ancillary qubit for the first measurement, and the ancillary qubit for the second measurement, before the readout of the two qubits, is

$$\begin{aligned}
|\psi_{\text{out}}\rangle &= U_{A_{j+1}}U_{A_j}|\psi_{\text{in}}\rangle|-\rangle_A|-\rangle_B\\
&= \sqrt{p_{++}}|v_{j+1}\rangle|+\rangle_A|+\rangle_B + \sqrt{p_{+-}}|v_{j+1}^\perp\rangle|+\rangle_A|-\rangle_B\\
&\quad + \sqrt{p_{-+}}|v_{j+1}\rangle|-\rangle_A|+\rangle_B + \sqrt{p_{--}}|v_{j+1}^\perp\rangle|-\rangle_A|-\rangle_B,
\end{aligned} \tag{E1}$$

where

$$p_{++} = |\langle v_j|v_{j+1}\rangle|^2 \cos^2\gamma, \tag{E2a}$$

$$p_{+-} = (1 - |\langle v_j|v_{j+1}\rangle|^2)\cos^2\gamma, \tag{E2b}$$

$$p_{-+} = |\langle v_j|v_{j+1}^{\perp}\rangle|^2 \sin^2\gamma, \tag{E2c}$$

$$p_{--} = (1 - |\langle v_j|v_{j+1}^{\perp}\rangle|^2)\sin^2\gamma, \tag{E2d}$$

with

$$|v_j^{\perp}\rangle = \sin\theta|1,1\rangle - \cos\theta\frac{e^{i\phi_j}|2,0\rangle - e^{-i\phi_j}|0,2\rangle}{\sqrt{2}}. \tag{E3}$$

State (E1) is a coherent superposition of the four distinct combinations of the measurement results for A_i and A_{j+1}. The order in which the two sequential measurements are finished is determined by the order in which the ancillary qubits are readout. The reduced state of the two qubits after tracing out the bosonic degree of freedom is

$$\rho_{AB} = p_{--}|--\rangle\langle--| + (1-p_{--}) \times (|+-\rangle\langle+-| + |-+\rangle\langle-+|). \tag{E4}$$

APPENDIX F: CORRELATIONS FROM CLASSICAL FIELDS

In this section, we sketch an argument that although classical fields can mimic a HOM-like result, it cannot lead to the violation of NC inequalities.

The phase difference π between the classical fields or coherent states of the BS input ports leads to a total destructive interference at one of the output ports of the 50 : 50 beam splitter. Therefore, for example,

$$\langle\alpha|\langle-\alpha|U_{\rm BS}^{\dagger}a^{\dagger}ab^{\dagger}bU_{\rm BS}|\alpha\rangle|-\alpha\rangle = 0, \tag{F1}$$

very much like the result we get for two identical photons described by $|1,1\rangle$ interfering at the same time at a balanced beam splitter. In the former case the suppression of the coincidence is merely due to classical destructive interference while in the latter the destructive interference is between the states' amplitudes of indistinguishable photons.

Let us reevaluate the correlation measurement for the case when the analogous coupling between ancillary qubit and classical fields, described by the classical c-number function $\mathcal{E}_c$, reads

$$H = \lambda\mathcal{E}_c(t)\sigma_z. \tag{F2}$$

The interaction causes a phase shift, $\varphi_j = 2\lambda\int_{t_j-\tau_j}^{t_j} dt'\mathcal{E}_c(t')$, with respect to the $|\pm\rangle$ basis and therefore the measurement result at time t_j only depends on the accumulated phase during the interaction period τ_j. Thus,

$$\langle\sigma_x^A\rangle = \cos\varphi_j. \tag{F3}$$

The key point is that the measurement result of each qubit is independent of the other one. This is because measuring a qubit has no back action to the classical field affecting the measurement result of the other. Therefore, correlation between the qubit measurements coupled to classical fields, $\mathcal{E}_c$, casts to

$$\langle\sigma_x^A\otimes\sigma_x^B\rangle = \cos\varphi_j\cos\varphi_{j+1}. \tag{F4}$$

In the case of classical random fields the general correlation is a probabilistic (convex) combination of (F4). That is,

$$\langle\sigma_x^A\otimes\sigma_x^B\rangle = \int d\varphi_j d\varphi_{j+1} p(\varphi_j,\varphi_{j+1})\cos\varphi_j\cos\varphi_{j+1}, \tag{F5}$$

where $p(\varphi_j,\varphi_{j+1})$ are the marginals of a single joint probability $p(\varphi_1,\dots,\varphi_5)$. It can be seen that the correlations of the type (F5) cannot lead to the violation of NC inequalities.

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