Converting Contextuality into Nonlocality

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We introduce a general method which converts, in a unified way, any form of quantum contextuality, including any form of state-dependent contextuality, into a quantum violation of a bipartite Bell inequality. As an example, we apply the method to a quantum violation of the Klyachko-Can-Binicioğlu-Shumovsky inequality.

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Introduction.—Nonlocal games [1,2] provide an intuitive understanding of where the advantage of quantum resources lies and a framework, used in computer science [3], to analyze quantum protocols. Contextuality is known to be a crucial resource for some forms of computation with quantum speed up [4–6]. However, although some forms of contextuality can be converted into nonlocal games, there is no universal method for converting any form of contextuality into a nonlocal game. The aim of this Letter is to introduce a unified method that achieves this task.

The research on how contextuality can be converted into nonlocality started with the works of Stairs [7] and Heywood and Redhead [8], extending the proofs of the Kochen-Specker (KS) theorem [9] to bipartite scenarios with entanglement and has evolved in many ways in connection to extensions of the KS theorem [10,11], Bell inequalities [12–14], and nonlocal games [15–20].

So far, the forms of contextuality that can be converted into nonlocality are (i) Those forms of state-dependent contextuality (SD-C) corresponding to scenarios whose measurements can be distributed between two or more parties in such a way that each party has at least two incompatible measurements. (ii) Those that are produced by KS sets [9] (i.e., sets of rank-one projectors which do not admit a "KS assignment," i.e., an assignment of 0 or 1 satisfying that two orthogonal projectors cannot both have assigned 1, and for every set of mutually orthogonal projectors summing the identity, one of them must be assigned 1) or by proofs of the KS theorem (e.g., [21,22]) that can be reduced to KS sets [23,24]. For methods of conversion, see, e.g., [1,14]. (iii) Those produced by some particular state-independent contextuality (SI-C) sets [25] (i.e., sets of projectors which produce noncontextual behaviors for any initial state) that are not KS sets. For methods of conversion, see [26]. (iv) In addition, some constraint satisfaction problems and local no-hidden-variables proofs can be converted into nonlocal games [17–20]. In each case, "convert" may mean a different thing.

Forms of contextuality that we do not know how to convert into nonlocality are those produced by sequentially measuring noncomposite systems initially prepared in specific states in contextuality scenarios which cannot be embedded in Bell scenarios. A particularly relevant example is the quantum violation of the Klyachko-Can-Binicioğlu-Shumovsky (KCBS) inequality [27] with single qutrits. This is arguably the most fundamental form of quantum SD-C produced by noncomposite systems, as the KCBS inequality is the only nontrivial tight noncontextuality inequality [28] in the scenario with the smallest number of measurements in which qutrits produce contextuality (qutrits are the quantum systems of smallest dimension that produce contextuality [9]), and because it plays a crucial role for understanding quantum contextuality [29–31].

The aim of this Letter is to provide a general unified method capable of converting any form of SD-C or SI-C into bipartite nonlocality. The philosophy behind the method is guided by the recognition of the singular role of SI-C, as pointed out in, e.g., [32] ("we argue that a primitive entity of contextuality should embrace state-independence"). The method takes any set of measurements that provides SD-C and identifies the minimal extension of it that provides SI-C and then converts the SI-C into bipartite nonlocality preserving the gap between quantum and noncontextual theories in the SI-C (which becomes the gap between quantum and local theories).

First, we describe the method, which has three steps. Then, we apply the method to a quantum violation of the KCBS inequality. Finally, we provide an intuitive explanation of how it works and discuss its virtues and limitations.

Method.—An ideal measurement of an observable A is a measurement of A that yields the same outcome when repeated and does not disturb any compatible (i.e., jointly measurable) observable. A context is a set of ideal measurements of compatible observables. A scenario is characterized by a number of measurements, their outcomes, and relations of compatibility [31]. In quantum theory, every ideal measurement is represented by the spectral projectors of a self-adjoint operator, and compatible observables correspond to commuting operators.

A behavior for a scenario (i.e., a set of probability distributions for each of its contexts) is contextual if the probability distributions for each context cannot be obtained as the marginals of a global probability distribution on all observables. Otherwise, the behavior is noncontextual. Contextuality is detected by the violation of noncontextuality inequalities whose bounds are derived solely from the assumption of outcome noncontextuality [25,27,33–35]. Any quantum contextual behavior can be produced by a set of rank-one projectors $S = \{\Pi_1, \ldots, \Pi_n\}$ acting on a quantum state $|\psi\rangle$ in a Hilbert space of dimension $d \geq 3$ [29]. Given S, contexts are subsets of S containing mutually commuting projectors.

Step 1: A SI-C set is critical if by removing any of its elements the resulting set is not a SI-C set. A KS set is critical [36] if by removing any of its elements the resulting set is not a KS set. Here we show that every set S producing SD-C can be extended into a critical SI-C set $S'' = S \cup S'$. To prove this, we need the following result [37,38]. In $d \geq 3$, given any two nonorthogonal rank-one projectors Π_A and Π_B, there is a set of projectors E such that, for any KS assignment f, $f(\Pi_A) + f(\Pi_B) \leq 1$. The set $\Pi_A \cup E \cup \Pi_B$ is called a true-implies-false set (TIFS) [39], definite prediction set [37], 01-gadget [38], or Hardy-like proof [40].

The construction of a critical SI-C set containing S is as follows. Let G be the graph of orthogonality of S. Let N be the minimum number of disjoint bases that cover all the vertices of G. If S allows for SD-C, then $N \geq 3$ [29]. If $N < d+1$, then we add disjoint bases until the total of number of disjoint bases is $N+1$. Then, we use the construction shown, for $d=3$, in Fig. 1(a) and, for $d=4$, in Fig. 1(d), and which works similarly for any $d \geq 5$, based on creating TIFSs between some specific nodes. If $N > d+1$, then we use the construction shown, for different combinations of d and N, in Figs. 1(b), 1(c), or 1(e). In all cases, the resulting set is a critical KS set in dimension d for the reasons explained in Fig. 1. If one removes any of the nodes in each of the constructions in Fig. 1, then the resulting set admits a KS noncontextual assignment. Some SI-C sets are not KS sets (e.g., [25,41,42]). Hence, the resulting critical KS set could, in principle, not be a critical SI-C set. However, this problem can be solved by suitably choosing the extra nodes used for the TIFSs in Fig. 1 [43].

A minimal critical SI-C set is a critical SI-C set of minimum cardinality. The previous proof guarantees that critical SI-C sets containing S exist. However, the method used in the proof does not guarantee that the resulting critical SI-C set is minimal. To obtain a minimal critical SI-C set S'' from S, we can use the following results. Let us call $\mathcal{G}$ the graph of orthogonality of S'', and let d be the dimension of the Hilbert space. Necessary conditions for S'' to be a SI-C set are that the chromatic number of $\mathcal{G}$ satisfies $\chi(\mathcal{G}) > d$ [55] and that the fractional chromatic number satisfies $\chi_f(\mathcal{G}) > d$ [56,57]. These conditions allow us to identify candidates to be minimal critical SI-C sets containing any given SD-C set. Then, we can use the necessary and sufficient condition for being a SI-C set [57] to check whether or not they are SI-C sets. This condition states that a set of rank-one projectors $S'' = \{\Pi_i, \ldots, \Pi_n\}$ is a SI-C set if and only if there are nonnegative numbers $w = (w_1, \ldots, w_n)$ and a number $0 \leq y < 1$ such that $\sum_{j \in \mathcal{I}} w_j \leq y$ for all $\mathcal{I}$, where $\mathcal{I}$ is any independent set of $\mathcal{G}$, and $\sum_i w_i \Pi_i \geq \mathbb{1}$.

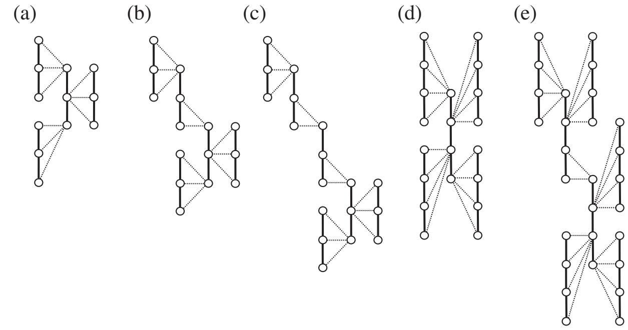

FIG. 1. Every node represents a rank-one projector. A continuous vertical line between $d \geq 3$ nodes indicates that they are mutually orthogonal. Hence, in dimension d, in any KS assignment, one of them has to be assigned 1. A dashed line between two nodes indicates that there is a TIFS between (and including) them. Hence, in any KS assignment, both of them cannot be assigned 1. Construction to obtain a critical KS set in dimension $d \geq 3$ from $N \geq d+1$ disjoint bases: (a) For $d=3$ and $N=d+1$. (b) For $d=3$ and $N=d+2$. (c) For $d=3$ and $N=d+3$. (d) For $d=4$ and $N=d+1$. (e) For $d=4$ and $N=d+2$. The construction works similarly for any $d \geq 3$ and $N \geq d+1$. In all cases, it is impossible to assign to the depicted nodes the values 0 or 1 satisfying that one of the d nodes in each continuous vertical line must be 1, while nodes connected by a dashed line cannot both be 1. However, such an assignment is possible whenever we remove any of the depicted nodes.

In practice, finding a critical SI-C set containing S is not a problem. However, proving that it has minimal cardinality may be difficult [43]. See [40,68] for examples of such proofs [43]. Nevertheless, minimality is only required for elegance; to connect SD-C to nonlocality, what matters is the criticality of the SI-C set.

Step 2: As pointed out in [57], the weights w needed to guarantee that S'' is a SI-C set generate a noncontextuality inequality violated by any quantum state. The results in [29,54] allow us to express this inequality as

$$\sum_{i\in V(\mathcal{G})} w_i P(\Pi_i = 1) - \sum_{(i,j)\in E(\mathcal{G})} \max(w_i, w_j) P(\Pi_i = 1, \Pi_j = 1) \overset{\text{NCHV}}{\leq} \alpha(\mathcal{G}, w), \tag{1}$$

where $P(\Pi_i = 1, \Pi_j = 1)$ is the probability of obtaining outcome 1 in the measurement associated to Π_i (which has possible outcomes 1 and 0) and also in the measurement associated to Π_j, $V(\mathcal{G})$, and $E(\mathcal{G})$ are the sets of vertices and edges of $\mathcal{G}$, respectively, $\alpha(\mathcal{G}, w)$ is the independence number of $(\mathcal{G}, w)$ [i.e., the graph in which weight w_i is assigned to each $i \in V(\mathcal{G})$], and NCHV stands for noncontextual hidden-variable theories. The independence number of a (weighted) graph is the cardinality of its largest set of vertices (taking their weights into account) such that no two are adjacent.

Step 3: This step has two ingredients. One is the following method, introduced in [7,8] and used extensively since then to embed a KS set in a bipartite Bell scenario. In each run of the experiment, we prepare a pair of particles in the two-qudit maximally entangled state

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |kk\rangle, \tag{2}$$

distribute one particle to Alice and the other to Bob, and allow Alice (Bob) to freely and independently choose and perform one measurement from S'' (from the set obtained by taking the complex conjugate of the elements in S''). Here, we apply this embedding not only to KS sets but to any SI-C set.

The second ingredient is the observation that the behavior produced by this state and these measurements violate the following Bell inequality:

$$\sum_{i\in V(\mathcal{G})} w_i P(\Pi_i^A = 1, \Pi_i^B = 1) - \sum_{(i,j)\in E(\mathcal{G})} \frac{\max(w_i, w_j)}{2} [P(\Pi_i^A = 1, \Pi_j^B = 1) + P(\Pi_j^A = 1, \Pi_i^B = 1)] \overset{\text{LHV}}{\leq} \alpha(\mathcal{G}, w), \tag{3}$$

where $P(\Pi_i^A = 1, \Pi_j^B = 1)$ is the probability that Alice obtains outcome 1 for measurement Π_i on her particle and Bob obtains outcome 1 for measurement Π_j on his particle. LHV stands for local hidden-variable theories.

That (3) is a Bell inequality follows from the fact that, for LHV theories, the maximum of the left-hand side of (3) is always attained by a deterministic assignment for the outcomes of the elements of S'' in Alice's particle and a deterministic assignment for the outcomes of the elements of the complex conjugate of S'' in Bob's particle. To maximize the left-hand side of (3), we need to maximize (taking into account the weights) the number of projectors Π_i to which outcome 1 is assigned both in Alice's and Bob's particles, while minimizing the number of adjacent Π_j to which outcome 1 is assigned, which is exactly the definition of independence number of a (weighted) graph $(\mathcal{G}, w)$. We can translate this violation into a nonlocal game with quantum advantage following the method in [2] (Sec. II.B4).

The interest of Bell inequality (3) comes from the following observations. Noncontextuality inequalities of the form (1) are in one-to-one correspondence with Bell inequalities of the form (3). The noncontextual bound in (1) is equal to the local bound in (3). The quantum violation of (1) for the maximally mixed state using S'' is equal to the quantum violation of the Bell inequality (3) for the maximally entangled state (2) and using S'' in Alice's side and the complex conjugate of S'' in Bob's side. Moreover, if S'' admits a weight w for which the left-hand side of (1) is represented in quantum theory by $\lambda\mathbb{1}$ with $\lambda > \alpha(\mathcal{G}, w)$ (as is the case in many critical SI-C sets, e.g., [25,58,78]), then, all quantum states violate inequality (1) by the same value, and this violation coincides with that of the Bell inequality (3) for state (2).

Overall, step 3 is an interesting result by itself, as it applies to any SI-C set (and not only to sets that can be reduced to KS sets, as [1,14]) and preserves the gap between quantum and noncontextual theories (while previous methods [1,14,26] do not).

Converting KCBS contextuality into nonlocality.—Here, we apply the method described above to a set of projectors [74,75] leading to a violation of the KCBS inequality [27]. The method works for any form of contextuality. The example has been chosen for its relevance and simplicity, as we can use a previous result [68] to identify S''.

Consider $S = \{\Pi_1, \ldots, \Pi_5\}$, where $\Pi_i = |v_i\rangle\langle v_i|$, with

$$|v_1\rangle = (1, 0, 0)^T, \tag{4a}$$

$$|v_2\rangle = \frac{1}{\sqrt{2}}(0, 1, 1)^T, \tag{4b}$$

$$|v_3\rangle = \frac{1}{\sqrt{3}}(1, -1, 1)^T, \tag{4c}$$

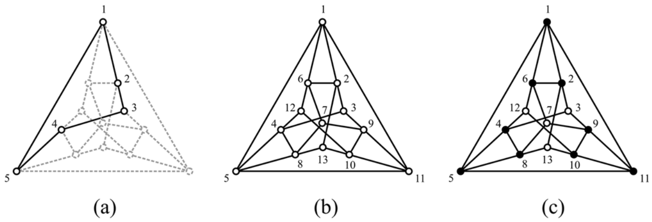

FIG. 2. (a) Five-vertex graph G that represents the relations of orthogonality between the projectors $S = \{\Pi_1, \ldots, \Pi_5\}$ needed to violate the KCBS inequality (5). Projector Π_i is represented by vertex i, mutually orthogonal projectors are represented by adjacent vertices. (b) Extended 13-vertex graph $\mathcal{G}$ representing the relations of orthogonality between elements of the smallest SI-C set $S'' = \{\Pi_1, \ldots, \Pi_{13}\}$ that contains S. (c) Vertex-weighted graph $(\mathcal{G}, w)$ with the weights that produce the largest SI-C. Vertices in white have weight 2 and vertices in black have weight 3. These are the weights used in the SI-C inequality (1) and the Bell inequality (3).

$$|v_4\rangle = \frac{1}{\sqrt{2}}(1,1,0)^T, \tag{4d}$$

$$|v_5\rangle = (0,0,1)^T. \tag{4e}$$

These measurements violate the KCBS inequality [27], which can be written [54] as

$$\sum_{i\in V(G)} P(\Pi_i = 1) - \sum_{(i,j)\in E(G)} P(\Pi_i = 1, \Pi_j = 1) \leq \alpha(G), \tag{5}$$

where G is the graph in Fig. 2(a), for which $\alpha(G) = 2$. For example [74,75], the state $|\psi\rangle = (1/\sqrt{3})(1,1,1)^T$ gives $2 + \frac{1}{9}$, which violates inequality (5).

Step 1: The smallest critical SI-C set S'' that contains S is the Yu-Oh set [25]. This follows from the proof in [57] that the Yu-Oh set is the SI-C set of rank-1 projectors with minimum cardinality. Therefore, $S' = \{\Pi_6, \ldots, \Pi_{13}\}$, where $\Pi_i = |v_i\rangle\langle v_i|$, with

$$|v_6\rangle = \frac{1}{\sqrt{2}}(0,1,-1)^T, \tag{6a}$$

$$|v_7\rangle = \frac{1}{\sqrt{3}}(1,1,1)^T, \tag{6b}$$

$$|v_8\rangle = \frac{1}{\sqrt{2}}(1,-1,0)^T, \tag{6c}$$

$$|v_9\rangle = \frac{1}{\sqrt{2}}(1,0,-1)^T, \tag{6d}$$

$$|v_{10}\rangle = \frac{1}{\sqrt{2}}(1,0,1)^T, \tag{6e}$$

$$|v_{11}\rangle = (0,1,0)^T, \tag{6f}$$

$$|v_{12}\rangle = \frac{1}{\sqrt{3}}(-1,1,1)^T, \tag{6g}$$

$$|v_{13}\rangle = \frac{1}{\sqrt{3}}(1,1,-1)^T. \tag{6h}$$

The graph $\mathcal{G}$ that represents the relations of orthogonality between the projectors $S'' = \{\Pi_1, \ldots, \Pi_{13}\}$ is shown in Fig. 2(b).

Step 2: The set of weights $\{w_1, \ldots, w_{13}\}$ leading to the largest gap between quantum and noncontextual theories for inequality (1) for S'' is $w_i = 2$ for $i = 3, 7, 12, 13$, and $w_i = 3$, otherwise. See Fig. 2(c). This follows from the observation that, in this case, the noncontextuality inequality (1) has $\alpha(\mathcal{G}, w) = 11$, while it is violated by any quantum state of dimension $d = 3$, since, for any initial state (including the maximally mixed state), the left-hand side of (1) is $\frac{1}{3}(2\times 4 + 3\times 9) = 11 + \frac{2}{3}$.

Step 3: Distributing pairs of particles in the maximally entangled state (2), with $d = 3$, between Alice and Bob and allowing each of them to perform a randomly chosen spacelike separated measurement from S'' (in this case, S'' and its complex conjugate are equal), we obtain a nonlocal behavior as the local bound of the Bell inequality (3) is $\alpha(\mathcal{G}, w) = 11$, while the value for the left-hand side of (3) is, again, $11 + \frac{2}{3}$.

Explanation, virtues, and limitations.—Here, we give some intuition of how the method works. The set of states (in dimension $d \geq 3$) that yield contextual behaviors grows as the set of measurements grows from S to S''. For example, while the state $|\psi'\rangle = (1/\sqrt{3})(1,-1,1)^T$ does not violate inequality (5), it violates a similar noncontextuality inequality replacing S by $\{\Pi_1, \ldots, \Pi_9\}$ [79]. When all the measurements in S'' are used, then even the maximally mixed state produces contextuality and weights can be adjusted [57] to produce equal state-independent violation of a noncontextuality inequality for all states [25,33–35].

The Bell inequality (3) follows from the SI-C inequality (1) by noticing that (1) can be tested in experiments consisting of two sequential measurements on a maximally mixed state. We can assume that each of these measurements is performed by a different party. Sometimes Alice is the first to measure and Bob the second, and sometimes vice versa. Sometimes both parties measure the same Π_i, sometimes they measure different but compatible projectors. This view leads to the Bell inequality (3) which shares the classical bound and it is also violated by the same amount when preparing pairs in state (2) and giving one particle to Alice and the other to Bob, as, in this case, Alice's and Bob's outcomes are perfectly correlated, and Alice's and Bob's local states are maximally mixed states.

Virtues: (I) While inequalities (1) and (5) are noncontextuality inequalities that might only be testable by performing sequential nondemolition measurements on single systems [80–82], inequality (3) is a Bell inequality that can be tested by performing local measurements on spatially separated systems and can be converted into a nonlocal game. (II) The "compatibility" or "sharpness" loophole [83] in contextuality experiments with sequential measurements vanishes in the Bell test, as, there, measurements do not need to be ideal (or sharp) [31] and observables on different particles are automatically compatible. (III) The gap between quantum and local theories for the Bell inequality (3) is the same as the gap between quantum and noncontextual theories for the SI-C inequality (1), and both are produced using the same measurements. (IV) The violation of the Bell inequality (3) by the measurements in S'' and state (2) vanishes whenever we remove from S'' any element of S. This follows from the fact that, in that case, inequality (1) is not violated by the maximally mixed state. Therefore, the maximally entangled state (2) fails to violate the Bell inequality (3), as the local states of Alice and Bob are maximally mixed. This property follows from the fact that S'' is a critical SI-C set. (V) There is no "contextuality-nonlocality tradeoff" [84,85]. The quantum violations of the SI-C inequality (1) and the Bell inequality (3) can be tested simultaneously in the same experiment. According to quantum theory, the experiment would give (equal) violations of both inequalities. The violation of (1) can be observed by allowing one of the parties, e.g., Alice, to perform sequential measurements. The violation of (3) can be observed by considering the first (or second) measurements of Alice and the (only) measurements of Bob. It would be interesting to observe these simultaneous violations in an actual experiment.

Limitations: Except for the case $S = S''$, the nonlocal behavior resulting from the application of this method does not have the same gap between quantum and local theories than the gap between quantum and noncontextual theories of the original SD-C behavior. Arguably, no method exists that preserves this gap for all forms of state-dependent contextuality.

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