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Gevrey Expansions of Hypergeometric Integrals II

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We study integral representations of the Gevrey series solutions of irregular hypergeometric systems under certain assumptions. We prove that, for such systems, any Gevrey series solution, along a coordinate hyperplane of its singular support, is the asymptotic expansion of a holomorphic solution given by a carefully chosen integral representation.

1 Introduction

In [10] (see also [11, 12]) the authors introduce and study A-hypergeometric systems and their solutions, generalizing many classical hypergeometric differential equations. General A-hypergeometric systems, also known as GKZ systems, are finitely generated D-modules, where $D := \mathbb{C}[x]\langle\partial\rangle = \mathbb{C}[x_1,\dots,x_n]\langle\partial_1,\dots,\partial_n\rangle$ stands for the complex n-th Weyl algebra.

Let us first recall some preliminary notions and results in D–module theory. Given a left D–ideal $J \subseteq D$, we consider the cyclic D–module $M := D/J$. A solution f of M is an element of a left D–module $\mathscr{F}$ such that $P \cdot f = 0,\ \forall P \in J$. In this paper we only consider the cases when $\mathscr{F}$ is either the space of holomorphic functions or the space

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of *Gevrey series* (of order $s \in \mathbb{R}$) along $Y = \{x_n = 0\}$ at $p \in Y$. We recall that such a Gevrey series is an expression of the form $f = \sum_{m=0}^{\infty} f_m x_n^m$ where $f_m = f_m(x_1, \ldots, x_{n-1})$ is holomorphic at p and $\sum_{m=0}^{\infty} f_m x_n^m/(m!)^{s-1}$ is convergent at p. The smallest possible s (if any) so that this latter condition holds is called the *Gevrey index* of f.

On the other hand, if $u, v \in \mathbb{R}^n$ satisfy $u + v \in \mathbb{R}^n_{>0}$ one can consider the *graded ideal* (or *initial ideal*) of J with respect to $L = (u, v)$, denoted by $\mathrm{in}_L(J)$, which is an ideal in the polynomial ring $\mathbb{C}[x, \xi] = \mathbb{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$, see, for example, [3, page 28]. Its zero set $V(\mathrm{in}_L(J)) \subseteq \mathbb{C}^{2n}$ is the *L–characteristic variety* of the cyclic D–module $M = D/J$, see, for example, [28, Definition 3.1]. If $F = (u, v)$ with $u = (0, \ldots, 0), v = (1, \ldots, 1)$ then $\mathrm{Ch}(M) := V(\mathrm{in}_F(J))$ is simply called the characteristic variety of M. The D–module M is said to be *holonomic* if the dimension of $\mathrm{Ch}(M)$ is n. The *singular locus* of M is the Zariski closure of the image of $\mathrm{Ch}(M) \setminus \{\xi_1 = \cdots = \xi_n = 0\} \subseteq \mathbb{C}^{2n}$ by the projection $\mathbb{C}^{2n} \longrightarrow \mathbb{C}^n$, $(x, \xi) \mapsto x$. On the other hand, set $V := (-e_n, e_n)$, where $e_n = (0, \ldots, 0, 1)$, and denote $L_s := F + (s-1)V$ for $s > 1$. The L_s-characteristic variety is known to be locally constant with respect to $s > 1$ except at a finite set of values called the *slopes* of M along Y, see [17]. If M is holonomic and it has a Gevrey solution with Gevrey index $s > 1$ along Y then s is a slope of M along Y, see [18, Théorème 2.4.2] and [23] for a more general and stronger statement.

The input data for a GKZ system is a pair (A, β) where β is a vector in $\mathbb{C}^d$ and $A = (a_{k\ell}) = (a(1), \ldots, a(n)) \in (\mathbb{Z}^d)^n$ is a $d \times n$ matrix whose ℓ-th column is $a(\ell)$ and $\mathbb{Z}A := \sum_{k=1}^{d} \mathbb{Z}a(k) = \mathbb{Z}^d$. The *toric ideal* $I_A \subset \mathbb{C}[\partial] := \mathbb{C}[\partial_1, \ldots, \partial_n]$ is the ideal generated by the family of binomials $\partial^u - \partial^v$, where $u, v \in \mathbb{N}^n$ and $Au = Av$ (we assume $0 \in \mathbb{N}$). Following [10, 11], the hypergeometric ideal associated with the pair (A, β) is

$$H_A(\beta) := DI_A + D(E_1 - \beta_1, \ldots, E_d - \beta_d),$$

where $E_k = \sum_{\ell=1}^{n} a_{k\ell} x_\ell \partial_\ell$ is the $k-$th Euler operator associated with the k-th row of A. The corresponding hypergeometric D-module (or A-hypergeometric system) is $M_A(\beta) := \frac{D}{H_A(\beta)}$.

In [11] and [1, Thm. 3.9] the authors prove that any hypergeometric system $M_A(\beta)$ is holonomic. Moreover, a characterization of the regularity of $M_A(\beta)$, in the sense of D–module theory [18, 23], is provided in the series of papers [16, 27, 28]. The holonomic D-module $M_A(\beta)$ is *regular* if and only if the toric ideal I_A is homogeneous for the standard grading in the polynomial ring $\mathbb{C}[\partial]$. In particular the condition to be regular for $M_A(\beta)$ is independent of the parameter vector β.

The dimension of the space of germs of holomorphic solutions of $M_A(\beta)$ around a generic point in $\mathbb{C}^n$ equals $d!\,\mathrm{Vol}(\Delta_A)$ if β is generic (see [11], [1, Cor. 5.20], and [22]). Here Δ_A is the convex hull in $\mathbb{R}^d$ of the points $\mathbf{0}, a(1), \ldots, a(n)$, where $\mathbf{0} \in \mathbb{R}^d$ is the origin, and $\mathrm{Vol}(\Delta_A)$ is its Euclidean volume. These holomorphic solutions are represented as Γ–series in [11] (see also [24] and [7]) when β is generic enough.

A. Adolphson considers in [1, Sec. 2] integral representations of solutions of $M_A(\beta)$ that involve exponentials of polynomial functions and appropriate integration cycles. In [6], A. Esterov and K. Takeuchi prove that the generic holomorphic solution spaces are in fact completely described by Adolphson's integral representations along rapid decay cycles as introduced by M. Hien in [14] and [15]. Such type of integrals are also used in [20] and generalized in [21], where they are called Laplace integrals.

The slopes, see [18], of $M_A(\beta)$ along coordinate subspaces are described in [28]. Their corresponding irregularity sheaves and Gevrey series solutions, defined in [23], are studied and described for generic parameters β in [7] (see also [8, 9]). Moreover, in [4, Proposition 5.3 and Remark 5.4] these Gevrey series solutions of $M_A(\beta)$ are interpreted as asymptotic expansions of certain of its holomorphic solutions under some assumption on the Gevrey index of the series, via the so-called *modified A-hypergeometric systems* introduced in [29].

In [5], and when A is a row matrix with positive integer entries, the authors develop a link between Gevrey series solutions of $M_A(\beta)$ and holomorphic solutions in sectors following Adolphson's approach. They prove that any Gevrey series solution, along the singular support of the system $M_A(\beta)$, is the asymptotic expansion of a holomorphic solution given by a carefully chosen integral representation.

In this paper we further develop this link when the matrix $A = (a(1), \ldots, a(n)) \in (\mathbb{Z}^d)^n$ satisfies two conditions. Since the rank of A is assumed to be d, we may also assume, after a possible reordering of the columns, that the 1st d columns of A determine a $(d-1)$-simplex σ. We further assume that A satisfies the following two conditions (see Assumption 4.1): (1) the points $a(d+1), \ldots, a(n-1)$ belong to the interior of the convex hull Δ_σ of σ and the origin; and (2) the point $a(n)$ is not in Δ_σ and belongs to the open positive cone of σ. Figure 1 shows an example of an allowed column set configuration for a 2×5 matrix A, where Δ_σ is the triangle.

Under these two conditions we have that $Y = \{x_n = 0\}$ is an irreducible component of the singular locus of $M_A(\beta)$ [1, Sec. 3], there is only one slope of $M_A(\beta)$ along Y [28] and, if β is generic enough, the dimension of the space of Gevrey series solutions of $M_A(\beta)$ along Y is $d!\,\mathrm{Vol}(\Delta_\sigma)$ [7].

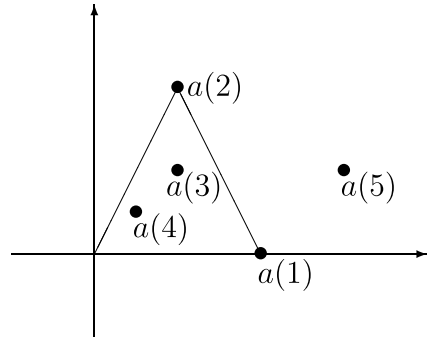

Fig. 1.

We prove in Theorem 4.3 that for generic $\beta \in \mathbb{C}^d$, the space of Gevrey series solutions of $M_A(\beta)$, along the hyperplane Y, has a basis given by asymptotic expansions of holomorphic solutions of $M_A(\beta)$ described by Adolphon's integral representations. These integrals are solutions of type

$$I_C(\beta; x) = I_C(\beta; x_1, \ldots, x_n) := \int_C t^{-\beta-1} \exp\left(\sum_{\ell=1}^{n} x_\ell t^{a(\ell)}\right) \mathrm{d}t,$$

where $t = (t_1, \ldots, t_d)$, $dt = dt_1 \cdots dt_d$ and C runs over a finite set of cycles on the universal covering of $(\mathbb{C}^*)^d$. These are Borel–Moore cycles or cycles with closed support on the universal covering of $(\mathbb{C}^*)^d$, a notion for which we refer to [25, II,5.3]). Moreover, we prove in Theorem 5.8 that these cycles can be replaced by a set of rapid decay homology cycles in the sense of [15].

Here is a summary of the content of this paper. In Section 2 we consider a general matrix A as before but not necessarily satisfying previous conditions (1) and (2) (see Assumption 4.1). Following a construction in [13, Sec. 4.4], we describe cycles $C_{p,\delta}$ in the universal covering of $(\mathbb{C}^*)^d$, depending on a given point $x \in \mathbb{C}^n$. We fix a maximal simplex $\sigma \subset \{1, \ldots, n\}$, that is, the set $\{a(k) \mid k \in \sigma\}$ is a basis of $\mathbb{R}^d$. Then this cycle depends only on $x_\sigma := (x_k)_{k\in\sigma}$, and on vectors $p \in \mathbb{Z}^\sigma$ and $\delta \in \mathbb{R}^\sigma$ with components δ_k satisfying $|\delta_k| < 1/2$. In Section 2.3 we give a sufficient condition for the integrand of $I_{p,\delta}(\beta; x) := I_{C_{p,\delta}}(\beta; x)$ to have moderate growth along $C_{p,\delta}$. This is a step towards sufficient conditions of convergence for $I_{p,\delta}(\beta; x)$ that are developed in Section 3.

In Section 3, we perform the appropriate toric change of variables in the universal covering of $(\mathbb{C}^*)^d$, like in [13], which reduces the description of asymptotic

expansions for the integrals $I_{p,\delta}(\beta; x)$ to the study of integrals of type

$$F_{p,\delta}(\beta; y) := \int_{D_{p,\delta}} t^{-\beta-1} \exp\left(t_1 + \cdots + t_d + \sum_{j=d+1}^{n} y_j t^{a(j)}\right) \mathrm{d}t,$$

where the cycle $D_{p,\delta}$ is the image of $C_{p,\delta}$ under the change of variables. The new integral $F_{p,\delta}(\beta; y)$ looks like a particular case of $I_{p,\delta}(\beta; x)$, with the 1st $d \times d$ submatrix $(a(1), \ldots, a(d))$ equal to the identity matrix. However, the matrix $A = (a(1), \ldots, a(n))$ is now allowed to have rational non integer coefficients. The crucial point for convergence statements is a condition of rapid decay at infinity, see inequality (3.10). We prove that, under some conditions, the integral $F_{p,\delta}(\beta; y)$ is absolutely convergent when $\Re\beta_k < 0$ for $k \in \sigma$ and $y \in (\mathbb{C}^*)^{n-d}$; see Lemmata 3.1 and 3.2.

Section 4 contains some of the main results of this paper. We assume that the matrix A defined in Section 3 satisfies more conditions in Assumption 4.4, deduced from conditions (1) and (2) in Assumption 4.1 already considered for the original matrix. First we prove that the conditions for convergence in Lemma 4.5, can be obtained in practice for every $y \in \mathbb{C}^{n-d}$ with $y_n \neq 0$.

We fix $p \in \mathbb{Z}^d$ and $\delta \in \mathbb{R}^d$ once for all and we omit these subindexes in our formulas. As a step towards previously mentioned Theorem 4.3, we prove in Theorem 4.7 that if $\Re\beta < 0$, there is an asymptotic expansion with respect to the variable y_n in some sector in $\mathbb{C}^*$:

$$F(\beta; y) \underset{y_n \to 0}{\sim} \sum_{m \in \mathbb{N}} A(\beta; m, y') \frac{y_n^m}{m!}, \tag{1.1}$$

where $y' = (y_{d+1}, \ldots, y_{n-1})$ and

$$A(\beta; m, y') := \int_{D_{p,\delta}} t^{-\beta-1+ma(n)} \exp\left(t_1 + \cdots + t_d + \sum_{j=d+1}^{n-1} y_j t^{a(j)}\right) \mathrm{d}t.$$

Assumption 4.4 plays an essential role in the proof of this result. Without assumption (1), we might need to impose further conditions on the arguments of y, e.g. conditions $(3.5)_j$ for all j, in order to guarantee the convergence of $F(\beta; y)$. Without condition (2), the vertex $a(n)$ could have negative components and the integrals defining the coefficients $A(\beta; m, y')$ would fail to be convergent for m large enough.

Then we prove in Lemma 4.9 that $F(\beta; y)$ admits a meromorphic continuation $\widetilde{F}(\beta; y)$, with respect to the variable β, with poles at most in a countable locally finite union of hyperplanes $\mathcal{P}$ in $\mathbb{C}^d$. The proof of this lemma uses that the points

$a(d+1),\dots,a(n)$ belong to $\sum_{k=1}^{d}\mathbb{R}_{>0}a(k)=\mathbb{R}^{d}_{>0}$, which follows from conditions (1) and (2). The set $\mathcal{P}$ is contained in the set of so-called *resonant parameters* of A [12, 2.9] and it is explicitly described in terms of the columns of A. We also prove in Lemma 4.10 that, for any fixed parameter $\beta\notin\mathcal{P}$, the meromorphic continuation $\widetilde{F}(\beta;y)$ admits an asymptotic expansion along $y_n=0$ and that the coefficients $\widetilde{A}(\beta;m,y')$ of this expansion are the analytic continuation of the previously introduced $A(\beta;m,y')$.

In Section 5 we prove that when $\Re\beta<0$ and β is sufficiently general, the integrals $F(\beta;y)$ are in fact equal to integrals over rapid decay cycles in the sense of [15] (see Theorem 5.3). The statements involving Borel–Moore cycles are weaker because the analytic continuations are not expressed by integral along cycles when $\Re\beta_k>0$ for some k. Another reason is that they are not cycles in the suitable homology adapted to the problem, like for Hien's rapid decay homology. The notion of rapid decay cycles is explained in Section 5.1. Section 5.2 is devoted to the construction of rapid decay cycles. We start from a product of Hankel contours, along which the hypergeometric integrals are grossly divergent, but then we build a refined towards infinity version of this product along which convergent integrals are obtained. These integrals in Section 5 are also defined when $\Re\beta_k\geq 0$ for some k and they are still solutions of $M_A(\beta)$. In Section 5.2 we prove, by using Section 4, that these integrals admit asymptotic expansions as Gevrey series solutions of $M_A(\beta)$ for non resonant β in $\mathbb{C}^d$.

2 Products of Lines for Rapid Decay

2.1 Notations

Let us slightly change our notation used in the introduction and let us start with a pair (B,γ), where $B:=(b(1),\dots,b(n))\in(\mathbb{Z}^d)^n$ is a $d\times n$ matrix, described as a list of columns such that $\mathbb{Z}B:=\mathbb{Z}b(1)+\cdots+\mathbb{Z}b(n)=\mathbb{Z}^d$ and where γ is a parameter vector in $\mathbb{C}^d$. We are concerned with integrals:

$$I_C(\gamma;x)=I_C(\gamma;x_1,\dots,x_n):=\int_C t^{-\gamma-\mathbf{1}}\exp\left(\sum_{\ell=1}^{n}x_\ell t^{b(\ell)}\right)\mathrm{d}t,$$

where $\mathbf{1}=(1,\dots,1)\in\mathbb{N}^d$ and C is a suitable cycle.

To make precise this definition let us specify some conventions and notations. As already mentioned, C is a cycle on the universal covering $(\widetilde{\mathbb{C}^*})^d$ of $(\mathbb{C}^*)^d$. We identify $(\widetilde{\mathbb{C}^*})^d$ with $\mathbb{C}^d$ or with $\mathbb{R}^d_{>0}\times\mathbb{R}^d$ and write $z=(\log r+\sqrt{-1}\,\theta)$ or (r,θ), respectively, for the coordinates on $(\widetilde{\mathbb{C}^*})^d$ with θ_k a branch of $\arg t_k$ $t_k=\exp(z_k)$, and $r_k=|t_k|$. We set, for

any vector $v \in \mathbb{C}^d$, $t^v = \prod_{k=1}^d t_k^{v_k}$. This is a multivalued monomial, namely the function on the universal covering:

$$\exp\langle z, v\rangle = \exp\left(\sum_{k=1}^d v_k(\log r_k + \sqrt{-1}\,\theta_k)\right),$$

where we set, given two vectors $u, v \in \mathbb{C}^d$, $\langle u, v\rangle = \sum_{k=1}^d u_k v_k$.

2.2 Description of cycles of rapid decay at infinity

If $\tau \subset \{1, \dots, n\}$, we denote by B_τ the matrix whose columns are $b(j)$ with $j \in \tau$ and by $\overline{\tau}$ the complement of τ in $\{1, \dots, n\}$.

Recall that a subset $\sigma \subset \{1, \dots, n\}$ is called a maximal simplex for B if the columns $\{b(k), k \in \sigma\}$ form a basis of $\mathbb{R}^d$. Such a maximal simplex σ is also called a base in [11, Sec. 1.1]. We often identify the set σ with the set of columns $\{b(k), k \in \sigma\}$.

We fix a maximal simplex σ for B and take $x \in \mathbb{C}^n$ such that $x_k \neq 0$ for all $k \in \sigma$. We also fix $p = (p_k)_{k\in\sigma} \in \mathbb{Z}^\sigma \simeq \mathbb{Z}^d$, $\delta = (\delta_k)_{k\in\sigma} \in \mathbb{R}^\sigma \simeq \mathbb{R}^d$ such that $|\delta_k| < \frac{1}{2}$ for all $k \in \sigma$. We denote by $C_{p,\delta}$ the cycle in the space $(\widetilde{\mathbb{C}^*})^d$ described by the following condition on the argument $\theta := \arg t$ of $t \in (\mathbb{C}^*)^d$ (i.e. ${}^t\theta := (\arg t_1, \dots, \arg t_d)$):

$$\arg(x_k t^{b(k)}) = \arg x_k + \langle b(k), \theta\rangle = (1 + \delta_k + 2p_k)\pi \quad \text{for all} k \in \sigma. \tag{2.1}$$

Remark 2.1. The cycle $C_{p,\delta}$ depends on $x_\sigma := (x_k)_{k\in\sigma} \in (\mathbb{C}^*)^\sigma \simeq (\mathbb{C}^*)^d$ and also on a choice of its argument. However, a change in this choice yields only a reindexation by p of the unchanged set of these cycles. For that reason in all our statements we stick on $x_\sigma \in (\mathbb{C}^*)^\sigma$ without passing to the universal covering of $(\mathbb{C}^*)^\sigma$.

From now on we will denote $I_{p,\delta}(\gamma; x) = I_{C_{p,\delta}}(\gamma; x)$. The cycles $C_{p,\delta}$ are a slightly modified version of cycles considered in [13, Sec. 4.4].

Let us set $\Theta := \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[+ 2\pi\mathbb{Z}$. The equality (2.1) can be globally rewritten using matrix notation:

$$\arg x_\sigma + {}^tB_\sigma\theta = (\mathbf{1} + \delta + 2p)\pi \in \Theta^\sigma. \tag{2.2}$$

There is a unique solution θ of the previous equation

$$\theta = ({}^tB_\sigma)^{-1}\left(-\arg x_\sigma + (\mathbf{1} + \delta + 2p)\pi\right) \tag{2.3}$$

so that $C_{p,\delta}$ is the cartesian product of d open half–lines.

Given $p, p' \in \mathbb{Z}^d$, let $\theta = \arg t,\ \theta' = \arg t'$ be the corresponding unique solutions for equation (2.2).

If $({}^tB_\sigma)^{-1}(p-p') \in \mathbb{Z}^d$ then $\theta - \theta' \in 2\pi\mathbb{Z}^d$ and the projections of the two cycles $C_{p,\delta}$ and $C_{p',\delta}$ on $(\mathbb{C}^*)^d$ are the same. We check that the convergence of the two integrals along the cycles $C_{p,\delta}$ and $C_{p',\delta}$ are then equivalent to each other and, moreover, the integral solutions differ only by a constant factor:

$$I_{p',\delta}(\gamma; x) = \int_{C_{p',\delta}} t^{-\gamma-1} \exp\left(\sum_{\ell=1}^{n} x_\ell t^{b(\ell)}\right) \mathrm{d}t = e^{-2\pi\sqrt{-1}(\sum_{k=1}^{d} m_k \gamma_k)} I_{p,\delta}(\gamma; x)$$

for some $m_k \in \mathbb{Z},\ k = 1, \ldots, d$.

When p varies in a set of representatives of $\frac{\mathbb{Z}^d}{\mathbb{Z}\,{}^tB_\sigma}$, we will see that the convergence of the integral $I_{p,\delta}(\gamma; x)$ depends on δ (see Remark 2.2 and Lemma 4.5). However, choosing in each such class an appropriate δ, we can find, as a consequence of our main result and under some conditions (see Assumption 4.1), $[\mathbb{Z}^d : \mathbb{Z}\,{}^tB_\sigma] = |\det B_\sigma|$ many integral solutions $I_{p,\delta}(\gamma; x)$ which are linearly independent (see Theorem 4.3).

We will see in the proof of Lemma 3.2, after the change of variables defined in Section 3, that the cycles $C_{p,\delta}$ are of rapid decay at infinity.

2.3 Sufficient conditions for moderate growth

Sufficient conditions for the convergence of the integral $I_{p,\delta}(\gamma; x)$ are detailed in the next section (see Lemma 3.2 and Remark 3.6). As a preliminary step let us look here at a condition for bounding the exponential term in that integral; let us notice that condition (2.1) implies that $\Re(x_k t^{b(k)}) < 0$ along $C_{p,\delta}$ for any $x_k \in \mathbb{C}^*,\ k \in \sigma$. If we additionally could ensure that

$$\arg(x_j t^{b(j)}) \in \Theta \quad \text{for all } j \in \overline{\sigma} \text{ such that } x_j \neq 0 \tag{2.4}$$

(see Remark 2.2 below) then the argument of the exponential has negative real part along $C_{p,\delta}$; hence, the absolute value of the exponential term in the integral $I_{p,\delta}(\gamma; x)$ is bounded by 1. Then if we take into account the term $t^{-\gamma-1}$, the integrand of $I_{p,\delta}(\gamma; x)$ has moderate growth along $C_{p,\delta}$.

Remark 2.2. Let us notice that condition (2.3) determines a unique cycle $C_{p,\delta}$ for a given p and δ. It is not clear that for given $x \in (\mathbb{C}^*)^\sigma \times \mathbb{C}^{\overline{\sigma}}$ and $p \in \mathbb{Z}^\sigma$ one can always choose $\delta \in \mathbb{R}^\sigma$ for this cycle to satisfy conditions (2.2) and (2.4). It is therefore interesting to weaken these conditions by keeping only the significant ones. In Lemma 3.2 and

Remark 3.6, completed by Remark 3.3, we do this with a reduced version of the variables x renamed y.

We will see in the next section that conditions (2.2) and (2.4) are sufficient convergence conditions for the integrals $I_{p,\delta}(\gamma; x)$ when combined with a condition on the parameter γ. After an appropriate change of variables we can interpret them as a condition of rapid decay at infinity, see the proof of Lemma 3.2.

We notice that $C_{p,\delta}$ is a Borel–Moore cycle in $(\widetilde{\mathbb{C}^*})^d$ but not in general a rapid decay cycle in the sense of [14], see Remark 3.5.

However, we shall prove in Section 5 that the integral along $C_{p,\delta}$ is equal to an integral along a rapid decay cycle (see Theorem 5.3) under Assumption 4.4, and for values of γ that guarantee convergence. Our result can be then interpreted in the frame of [6, Th. 4.5].

3 A Change of Variables and Explicit Calculations

We will assume for simplicity, after a possible reordering of the variables, that the maximal simplex σ is $\{1, \dots, d\}$. Let us fix $x \in (\mathbb{C}^*)^d \times \mathbb{C}^{n-d}$, and an argument of all x_k with $k \in \sigma$. We consider the finite to one covering $(\mathbb{C}^*)^d \to (\mathbb{C}^*)^d$ of degree $\det B_\sigma$, given by the formula:

$$s_k = x_k t^{b(k)} \text{ for } k \in \sigma.$$

We think of it as a (ramified) toric change of variables. We fix a branch of $\log x_\sigma$ and we consider the bijective change of variables on the universal covering $(\widetilde{\mathbb{C}^*})^d \simeq \mathbb{C}^d$, given by

$$\log s_k - \log x_k = \log t \cdot b(k).$$

Fractional powers like x_σ^v with $v \in \mathbb{Q}^d$ have the natural meaning $x_\sigma^v = \exp(\log x_\sigma \cdot v)$, and the inverse mapping on $(\widetilde{\mathbb{C}^*})^d$ can be read as follows using these fractional powers:

$$t_k = \left(\frac{s}{x_\sigma}\right)^{B_\sigma^{-1} e(k)} \text{ for } k \in \sigma,$$

where $\left(\frac{s}{x_\sigma}\right)$ is the vector with coordinates s_k/x_k and $(e(k))_{k\in\sigma}$ is the standard basis of $\mathbb{Z}^d$. The image $D_{p,\delta}$ of the cycle $C_{p,\delta}$ described in Section 2, is determined by the conditions:

$$\arg s_k = (1 + \delta_k + 2p_k)\pi \quad \text{for all } k \in \sigma. \tag{3.1}$$

The d-form $\frac{dt}{t} = \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_d}{t_d}$ is well defined on $(\widetilde{\mathbb{C}^*})^d$, and we have

$$\frac{dt_k}{t_k} = \frac{ds^{B_\sigma^{-1}e(k)}}{s^{B_\sigma^{-1}e(k)}} = \sum_{i=1}^{d} \left(B_\sigma^{-1}\right)_{i,k} \frac{ds_k}{s_k},$$

so $\frac{dt}{t} = \det B_\sigma^{-1} \cdot \frac{ds}{s}$. The integral $I_{p,\delta}$ is transformed as follows:

$$\begin{aligned} I_{p,\delta}(\gamma; x) &= \int_{C_{p,\delta}} t^{-\gamma-\mathbf{1}} \exp\left(\sum x_\ell t^{b(\ell)}\right) \mathrm{d}t = \int_{C_{p,\delta}} t^{-\gamma} \exp\left(\sum x_\ell t^{b(\ell)}\right) \frac{\mathrm{d}t}{t} \\ &= \int_{D_{p,\delta}} \left(\frac{s}{x_\sigma}\right)^{-B_\sigma^{-1}\gamma} \exp\left(\sum_{k\in\sigma} s_k + \sum_{j\notin\sigma} x_j x_\sigma^{-B_\sigma^{-1}b(j)} \cdot s^{B_\sigma^{-1}b(j)}\right) \det(B_\sigma^{-1}) \frac{\mathrm{d}s}{s}. \end{aligned}$$

The final result is

$$I_{p,\delta}(\gamma; x) = \det(B_\sigma^{-1}) x_\sigma^{\,B_\sigma^{-1}\gamma} \int_{D_{p,\delta}} s^{-B_\sigma^{-1}\gamma-\mathbf{1}} \exp\left(\sum_{k\in\sigma} s_k + \sum_{j\notin\sigma} x_j x_\sigma^{-B_\sigma^{-1}b(j)} \cdot s^{B_\sigma^{-1}b(j)}\right) \mathrm{d}s.$$

Lemma 3.1. Sufficient conditions for the absolute convergence of $I_{p,\delta}(\gamma; x)$ are

$$\Re B_\sigma^{-1}\gamma < 0 \text{ and } \Re(x_j x_\sigma^{-B_\sigma^{-1}b(j)} s^{B_\sigma^{-1}b(j)}) < 0 \quad \forall j \in \overline{\sigma} \text{ such that } x_j \neq 0,\ \forall s \in D_{p,\delta}.$$

Proof. The 2nd condition is a direct translation of (2.4). It is sensitive to the choice modulo $2\pi\mathbb{Z}$ of $\arg s_k$, for $k \in \sigma$, since the matrix $B_\sigma^{-1}b(j)$ may have coefficients in $\mathbb{Q} \setminus \mathbb{Z}$.

We have $s_k = -|s_k| e^{\sqrt{-1}\,\pi\delta_k}$ and $(|s_1|, \ldots, |s_d|) \in \mathbb{R}^d_{>0}$ parametrizes the cycle. Since all the terms in the argument of the exponential term in $I_{p,\delta}(\gamma; x)$ have negative real part, we have

$$\Re\left(\sum_{k\in\sigma} s_k + \sum_{j\notin\sigma} x_j x_\sigma^{-B_\sigma^{-1}b(j)} s^{B_\sigma^{-1}b(j)}\right) \leq \Re\left(\sum_{k\in\sigma} s_k\right) \leq -c\left(\sum_{k\in\sigma} |s_k|\right)$$

with $c := \min_k \cos(\pi\delta_k) > 0$.

Therefore, the integral $I_{p,\delta}(\gamma; x)$, without the prefactor $\det(B_\sigma^{-1}) x_\sigma^{\,B_\sigma^{-1}\gamma}$, is dominated by the following convergent integral with $\alpha_k = \Re(-(B_\sigma^{-1}\gamma)_k) - 1 > -1$:

$$\int_{\mathbb{R}^d_{>0}} r^\alpha \exp\left(-c \sum_{k\in\sigma} r_k\right) \mathrm{d}r = c^{-|\alpha|-d} \Gamma(\alpha + \mathbf{1}),$$

where $r = (r_1, \ldots, r_d)$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and $\Gamma(\alpha + \mathbf{1}) = \prod_{k=1}^d \Gamma(\alpha_k + 1)$. ∎

We now define a reduction of the integral $I_{p,\delta}(\gamma;x)$, which contains all the essential information. We put aside the initial monomial ${x_\sigma}^{B_\sigma^{-1}\gamma}$, and the constant $\det(B_\sigma^{-1})$, and the remaining integral can be expressed via a function of $n-d$ variables y indexed by $\overline{\sigma}$:

$$G_{p,\delta}(\gamma;y)=G_{p,\delta}(\gamma;y_{d+1},\dots,y_n):=\int_{D_{p,\delta}} s^{-B_\sigma^{-1}\gamma-\mathbf{1}}\exp\left(\sum_{k\in\sigma}s_k+\sum_{j\notin\sigma}y_j s^{B_\sigma^{-1}b(j)}\right)\mathrm{d}s. \tag{3.2}$$

The formula relating $I_{p,\delta}(\gamma;x)$ to the previous integral is

$$I_{p,\delta}(\gamma;x)=\det(B_\sigma^{-1}){x_\sigma}^{B_\sigma^{-1}\gamma}G_{p,\delta}(\gamma;y), \tag{3.3}$$

where $y_j=x_j x_\sigma^{-B_\sigma^{-1}b(j)}$ for $j\in\overline{\sigma}$.

3.1 Reduced version of hypergeometric integrals

In order to simplify subsequent calculations we shall use a more handy version of the integral $G_{p,\delta}(\gamma;y)$ by renaming the exponents.

We consider $\beta\in\mathbb{C}^d$ and a $d\times n$ matrix $A=(a(1),\dots,a(n))$ with rational coefficients and with $(a(1),\dots,a(d))=(e(1),\dots,e(d))$ the unit matrix. Writing t instead of s, we define

$$F_{p,\delta}(\beta;y):=\int_{D_{p,\delta}} t^{-\beta-\mathbf{1}}\exp\left(t_1+\cdots+t_d+\sum_{j=d+1}^{n}y_j t^{a(j)}\right)\mathrm{d}t \tag{3.4}$$

and we recover $G_{p,\delta}$ by setting $A=B_\sigma^{-1}B$ and $\beta=B_\sigma^{-1}\gamma$ in $F_{p,\delta}$.

Now we transpose to $F_{p,\delta}(\beta;y)$ the two sufficient conditions in Lemma 3.1. The 1st one simply becomes $\Re\beta_k<0$, for all $k\in\sigma$, and we shall assume it until the end of this section. Then we focus on the 2nd condition in Lemma 3.1. This condition transposed to $F_{p,\delta}(\beta;y)$ is

$$\arg(y_j t^{a(j)})=\arg\, y_j+\langle\mathbf{1}+\delta+2p,\, a(j)\rangle\pi\in\Theta\quad\text{for } j\in\overline{\sigma}\text{ such that } y_j\neq 0. \tag{3.5}$$

We shall prove in the next two lemmas, that a part of conditions $(3.5)_\ell$ is already sufficient to guarantee the convergence of the integral $F_{p,\delta}(\beta;y)$.

We set $y_k = 1$ for $k = 1, \dots, d$. Notice that condition $(3.5)_k$ is satisfied for all $k \in \sigma$ because $t_k = y_k t^{a(k)}$ and $|\delta_k| < 1/2$. So, condition (3.5) is equivalent to

$$\arg(y_\ell t^{a(\ell)}) = \arg y_\ell + \langle \mathbf{1} + \delta + 2p,\, a(\ell)\rangle \pi \in \Theta \quad \text{for } \ell \in \{1, \dots, n\} \text{ such that } y_\ell \neq 0. \tag{3.6}$$

Recall that Δ_A denotes the convex hull of $\{\mathbf{0}, a(1), \dots, a(n)\}$ in $\mathbb{R}^d$. Let $\partial\Delta_A$ be the union of the facets of Δ_A not containing the origin and let τ_A be the set of indices $\ell \in \{1, \dots, n\}$ such that $a(\ell) \in \partial\Delta_A$. We denote by $\eta \subset \tau_A$ the set of indices for the vertices of Δ_A.

Lemma 3.2. Assume that $y_\ell \neq 0$ if $\ell \in \tau_A$. The set of conditions $(3.6)_\ell$ for ℓ in τ_A is sufficient for the integral $F_{p,\delta}(\beta; y)$ to be absolutely convergent, when $\Re\beta < 0$.

Proof. Let $M(t, y)$ denote the argument in the exponential term in the integral (3.4). We need to provide a bound for $\Re M(t, y)$. We set $\tau := \tau_A \cup \sigma$. We can write, for $j \notin \tau$,

$$a(j) = \sum_{\ell \in \eta} \nu_{j\ell}\, a(\ell) \tag{3.7}$$

with $0 \leq \sum_{\ell\in\eta} \nu_{j\ell} < 1$ and $\nu_{j\ell} \geq 0$ for all $\ell \in \eta$ and $j \notin \tau$.

We set

$$\xi_\ell := |y_\ell t^{a(\ell)}| \text{ for } \ell \in \tau.$$

By the condition $(3.6)_\ell$, there exists $\vartheta \in]0, \frac{\pi}{2}[$ such that for all $\ell \in \tau$ one has

$$\Re(y_\ell t^{a(\ell)}) \leq -\xi_\ell \cos\vartheta.$$

Recall that

$$M(t,y) = \sum_{\ell\in\tau} y_\ell t^{a(\ell)} + \sum_{j\notin\tau} y_j t^{a(j)} = \sum_{\ell\in\tau} y_\ell t^{a(\ell)} + \sum_{j\notin\tau} y_j \prod_{\ell\in\eta} (t^{a(\ell)})^{\nu_{j\ell}} =$$

$$= \sum_{\ell\in\tau} y_\ell t^{a(\ell)} + \sum_{j\notin\tau} \frac{y_j}{\prod_{\ell\in\eta} y_\ell^{\nu_{j\ell}}} \prod_{\ell\in\eta} (y_\ell t^{a(\ell)})^{\nu_{j\ell}}.$$

Therefore, we get

$$\Re M(t,y) \leq -\left(\sum_{\ell\in\tau} \xi_\ell\right)\cos\vartheta + \sum_{j\notin\tau} \left|\frac{y_j}{\prod_{\ell\in\eta} y_\ell^{\nu_{j\ell}}}\right| (\max_{\ell\in\eta} \xi_\ell)^{\sum_{\ell\in\eta}\nu_{j\ell}} \leq \tag{3.8}$$

$$\leq -\left(\sum_{\ell\in\tau} \xi_\ell\right)\cos\vartheta + K \max\left(\left(\sum_{\ell\in\tau} \xi_\ell\right)^\kappa, 1\right), \tag{3.9}$$

where $\kappa := \max_{j\notin\tau}(\sum_{\ell\in\eta} v_{j\ell}) < 1$, and $K = \sum_{j\notin\tau}\left|\frac{y_j}{\prod_{\ell\in\eta} y_\ell^{v_{j\ell}}}\right|$ is a constant for a fixed value of $y \in (\mathbb{C}^*)^n$. Set $\xi := \sum_{\ell\in\tau}\xi_\ell$. We see that $\Re M(t,y)$ is bounded by $-\xi\cos\vartheta + K\max(\xi^\kappa, 1)$ that tends to $-\infty$ when $\xi \to +\infty$.

It also follows that $|\exp(M(t,y))|$ is bounded:

$$|\exp(M(t,y))| \le \exp(-\xi\cos\vartheta + K\max\left(\xi^\kappa, 1\right)) \le e^L,$$

where $L := \sup_{\xi\in\mathbb{R}_{>0}}(-\xi\cos\vartheta + K\max(\xi^\kappa, 1))$.

Since $\sigma \subseteq \tau$, we have $\xi = \left(\sum_{\ell\in\tau}\xi_\ell\right) \ge |t_1| + \cdots + |t_d|$ and we get a rapid decay condition when $|t_1| + \cdots + |t_d| \to +\infty$. There are positive constants $c > 0$ and $R > 0$ such that

$$|t_1| + \cdots + |t_d| > R \Longrightarrow |\exp(M(t,y))| < \exp(-c(|t_1| + \cdots + |t_d|)).$$

It is convenient for further calculation to incorporate the upper bound e^L in a global inequality. For $C = L + cR > 0$ we have that

$$\forall t \in D_{p,\delta}, \quad |\exp(M(t,y))| < \exp(C - c(|t_1| + \cdots + |t_d|)). \tag{3.10}$$

The absolute convergence of the integral $F_{p,\delta}(\beta; y)$ follows now exactly as in the proof of Lemma 3.1 by the assumption $\Re\beta_k < 0$ for all $k \in \{1, \ldots, d\}$. ■

Remark 3.3. Notice that for fixed $p \in \mathbb{Z}^d$ and $\delta \in \mathbb{R}^d$ with $|\delta_k| < 1/2$, the set of conditions $(3.6)_\ell$ on $y = (y_{d+1}, \ldots, y_n)$, for $\ell \in \tau_A$, defines an open set $\mathcal{W} \subset \mathbb{C}^{\{d+1,\ldots,n\}\setminus\tau_A} \times (\mathbb{C}^*)^{\tau_A\setminus\sigma}$. On the factor $(\mathbb{C}^*)^{\tau_A\setminus\sigma}$ this open set is a product of open sectors. In the more specific situation of Section 4, since $\tau_A = \sigma \cup \{n\}$, there will be only one sector for the variable y_n, see Remark 4.6.

Remark 3.4. If y varies in a compact neighborhood of a given point $y_0 \in \mathcal{W}$ we can replace in (3.10) the constants c, C by constants independent from $y = (y_{d+1}, \ldots, y_n)$. We can also take a uniform bound for $|t^{-\beta-1}|$, when $\Re\beta$ is bounded from below. This implies that $F_{p,\delta}(\beta; y)$ is analytic with respect to (β, y) by Lebesgue's theorem on dominated convergence for integrals.

Remark 3.5. Notice that, in general, we don't have rapid decay at the origin. For example, if the matrix A has only positive entries, the exponential term in the integral

$F_{p,\delta}(\beta; y)$ tends to 1, when $|t_1| + \cdots + |t_d| \to 0$, and the integrand behaves at the origin as the factor $t^{-\beta-\mathbf{1}}$.

Recall that η is the set of vertices of Δ_A different from the origin. We may weaken the hypothesis in Lemma 3.2 as follows: we have an analogous formula to (3.7) for all $j \notin \eta \cup \sigma$, namely $a(j) = \sum_{\ell\in\eta} \nu_{j\ell} a(\ell)$, with $\kappa_j := \sum_{\ell\in\eta} \nu_{j\ell} \leq 1$. Precisely, $\kappa_j = 1$ for $j \in \tau \setminus (\eta \cup \sigma)$ and $\kappa_j < 1$ for $j \notin \tau = \tau_A \cup \sigma$. We set $K_j = \left| \frac{y_j}{\prod_{\ell\in\eta} y_\ell^{\nu_{j\ell}}} \right|$ for $j \notin \eta \cup \sigma$ and we obtain the following.

Remark 3.6. The set of conditions $(3.6)_\ell$ for $\ell \in \eta$, is sufficient for the convergence of $F_{p,\delta}(\beta; y)$ when y varies in the nonempty open set in $(\mathbb{C}^*)^{n-d}$ defined by $\sum_{j\in\tau\setminus(\eta\cup\sigma)} K_j < \cos\vartheta$.

More precisely, we can write down a refined upper bound of the real part of the exponent

$$\Re M(t, y) \leq -\left(\sum_{\ell\in\eta\cup\sigma} \xi_\ell\right) \cos\vartheta + \sum_{j\notin\tau} K_j \left(\sum_{\ell\in\eta\cup\sigma} \xi_\ell\right)^{\kappa_j} + \sum_{j\in\tau\setminus(\eta\cup\sigma)} K_j \left(\sum_{\ell\in\eta\cup\sigma} \xi_\ell\right)^{1}.$$

Thus, the conclusion follows by an argument similar to the one in Lemma 3.2.

4 Obtaining the Gevrey Series

Given a matrix $B = (b(1), \ldots, b(n))$ as in Section 2 and $\tau \subset \{1, \ldots, n\}$ we denote by Δ_τ the convex hull of $b(k)$, $k \in \tau$, and the origin. We assume, after a possible reordering of the variables, that $\sigma := \{1, \ldots, d\}$ is a maximal simplex for B, i.e. $\{b(1), \ldots, b(d)\}$ is a basis of $\mathbb{R}^d$.

Assumption 4.1. We assume that the matrix B satisfies

(1) The points $b(d+1), \ldots, b(n-1)$ belong to the interior of Δ_σ and
(2) $b(n)$ is not in Δ_σ and belongs to the open positive cone of σ.

Remark 4.2. We notice that, under the above assumption, it follows from [1] that, for any $\gamma \in \mathbb{C}^d$, the singular locus of the hypergeometric system $M_B(\gamma)$ is equal to $\bigcup_{k\in\sigma}\{x_k = 0\} \cup \{x_n = 0\}$. Furthermore, by [28], $M_B(\gamma)$ has a unique slope along the coordinate hyperplane $\{x_n = 0\}$.

The main result in this section is the following theorem announced in the Introduction.

Theorem 4.3. In the above situation, let us assume that $\gamma \in \mathbb{C}^d$ and $\Re(B_\sigma^{-1}\gamma) < 0$. Then,

(1) There exists a finite number of cycles $C_{p,\delta}$ such that all the Gevrey solutions of $M_B(\gamma)$ along the hyperplane $x_n = 0$ can be described as linear combinations of the asymptotic expansions of the integral solutions $I_{C_{p,\delta}}(\gamma; x)$.
(2) For each cycle $C = C_{p,\delta}$, there are meromorphic continuations with respect to γ in $\mathbb{C}^d$, of both $I_C(\gamma; x)$ and of the coefficients of the asymptotic expansion to the whole $\mathbb{C}^d$.
(3) For any $\gamma \in \mathbb{C}^d$ that is not a pole, the meromorphic continuation of $I_C(\gamma; x)$ has an asymptotic expansion whose coefficients are precisely the values at γ of the meromorphic continuations of the coefficients of the asymptotic expansion of $I_C(\gamma; x)$.

In the next three subsections we are proving the analogous result for the reduced version $F_{p,\delta}$ of the hypergeometric integrals $I_{C_{p,\delta}}$; see Section 3.1. The transfer of the results to the integrals $I_{C_{p,\delta}}$ in the form of Theorem 4.3 is immediate.

4.1 Existence of asymptotic expansions for the integrals

As we did in Section 3.1, let us consider $\beta \in \mathbb{C}^d$ and $A = (a(1), \dots, a(n))$ is a $d \times n$ matrix with rational coefficients and with $(a(1), \dots, a(d)) = (e(1), \dots, e(d))$ the unit matrix. Let us denote by $|a| = a_1 + \cdots + a_d$ the sum of the coordinates of any vector $a \in \mathbb{Q}^d$. Assumption 4.1 takes the following form in this reduced presentation.

Assumption 4.4. The matrix A satisfies

(1) For $j = d+1, \dots, n-1$, the rational vector $a(j)$ is in the open positive orthant in $\mathbb{Q}^d$ and $|a(j)| < 1$.
(2) The rational vector $a(n)$ belongs to the open positive orthant in $\mathbb{Q}^d$ and $|a(n)| > 1$.

Lemma 4.5. For $\Re\beta < 0$ and under Assumption 4.4 one can find for each $y_{n,0} \in \mathbb{C}^*$, and each $p \in \mathbb{Z}^d$ a value of the parameter $\delta \in]-\frac{1}{2}, \frac{1}{2}[^d$ such that the integral $F_{p,\delta}(\beta; y)$ is absolutely convergent.

Proof. Indeed by Lemma 3.2 it is sufficient to choose δ such that equation $(3.5)_n$ is satisfied. Such a δ exists because $|a(n)| > 1$ and the image of $]\frac{-1}{2}, \frac{1}{2}[^d$ by the map $\delta \to \langle \delta, a(n)\rangle$ is an interval $]-|a(n)|/2, |a(n)|/2[$ of length greater than 1. ∎

Remark 4.6. Let us fix p, δ and a branch α_n of $\arg y_{n,0}$ satisfying condition $(??)_n$. This defines the subset $\mathfrak{R} \subset \mathbb{R}$ of allowed arguments $\arg y_n$. Let $S_{p,\delta} \subset \mathbb{C}^*$ be the sector defined by $\arg y_n \in \mathfrak{R}_0$ the connected component of α_n in $\mathfrak{R}$. Like in Remark 3.3, $S_{p,\delta}$ is independent of the choice of α_n.

Theorem 4.7. If $\Re\beta_k < 0$ for all $k = 1, \ldots, d$, then for any given $y_{n,0} \in \mathbb{C}^*$ there is an asymptotic expansion with respect to the variable y_n in the open sector $S_{p,\delta}$:

$$F_{p,\delta}(\beta; y) \underset{y_n \to 0}{\sim} \sum_{m \in \mathbb{N}} A_{p,\delta}(\beta; m, y') \frac{y_n^m}{m!},$$

where $y' = (y_{d+1}, \ldots, y_{n-1})$ and

$$A_{p,\delta}(\beta; m, y') := \int_{D_{p,\delta}} t^{-\beta - \mathbf{1} + ma(n)} \exp\left(t_1 + \cdots + t_d + \sum_{j=d+1}^{n-1} y_j t^{a(j)}\right) \mathrm{d}t.$$

Proof. We have to prove that for any integer $N > 0$ there exists $K_N = K_N(\beta, y') > 0$ such that

$$\left| F_{p,\delta}(\beta; y) - \sum_{m=0}^{N-1} A_{p,\delta}(\beta; m, y') \frac{y_n^m}{m!} \right| \leq K_N |y_n|^N$$

holds for every $y_n \in S_{p,\delta}$.

Let

$$\Phi_N(z) := e^z - \sum_{m=0}^{N-1} \frac{z^m}{m!}$$

for $z \in \mathbb{C}$. Then we have

$$|\Phi_N(z)| \leq \frac{|z|^N}{N!} \text{ for all } z \text{ such that } \Re z < 0.$$

Recall that by the assumption on δ we have $\Re(y_n t^{a(n)}) < 0$ when $t \in D_{p,\delta}$ since $y_n \in S_{p,\delta}$. Thus, we have

$$\left| F_{p,\delta}(\beta; y) - \sum_{m=0}^{N-1} A_{p,\delta}(\beta; m, y') \frac{y_n^m}{m!} \right| = \left| y_n^N \right| \left| Q_{p,\delta}(\beta; y, N) \right|,$$

where

$$Q_{p,\delta}(\beta; y, N) = \int_{D_{p,\delta}} t^{a(n)N - \beta - \mathbf{1}} \exp\left(t_1 + \cdots + t_d + \sum_{j=d+1}^{n-1} y_j t^{a(j)}\right) \frac{\Phi_N(y_n t^{a(n)})}{(y_n t^{a(n)})^N} \mathrm{d}t.$$

The absolute value of the integrand in $Q_{p,\delta}(\beta; y, N)$ is bounded by the function

$$\frac{1}{N!}\left| t^{a(n)N-\beta-\mathbf{1}} \exp\left(t_1 + \cdots + t_d + \sum_{j=d+1}^{n-1} y_j t^{a(j)}\right)\right|,$$

which is independent of y_n and integrable over $D_{p,\delta}$ by Lemma 3.2 (that can be applied to the submatrix of A defined by its 1st $n-1$ columns because of Assumption 4.4). Notice that we use here that $\Re(\beta - a(n)N) < 0$ for all $N > 0$ since $a(n)$ does not have negative coordinates.

Thus, there exists $K_N = K_N(\beta, y') > 0$ such that $\left|Q_{p,\delta}(\beta; y, N)\right| \leq K_N$. This finishes the proof. ■

Remark 4.8. Notice that $A_{p,\delta}(\beta; m, y') = F_{p,\delta}(\beta - ma(n); y')$ for the submatrix of A defined by its 1st $n-1$ columns. In particular it is analytic with respect to (β, y') by Remark 3.4.

We extend Theorem 4.7 to nonnegative values of $\Re\beta_k$ in Section 4.2.

4.2 Analytic continuation with respect to β

In this section we focus on the analytic dependency of $F(\beta; y) = F_{p,\delta}(\beta; y)$ on β. Let us take $y_{n,0} \in \mathbb{C}^*$ and $p \in \mathbb{Z}^d$. We choose δ as in Lemma 4.5 and we omit p, δ in the remainder of this subsection. We assume now that y belongs to $\mathbb{C}^{n-d-1} \times S_{p,\delta}$, where the sector $S_{p,\delta}$ is defined in Remark 4.6.

The integral $F(\beta; y)$ is a solution of the reduced GG-system (see [13]):

$$\beta_k F(\beta; y) = \sum_{\ell=d+1}^{n} a(\ell)_k\, y_\ell\, F(\beta - a(\ell); y) + F(\beta - e(k); y) \quad \text{for} k = 1, \ldots, d \tag{4.1}$$

$$F(\beta - a(\ell); y) = \frac{\partial F}{\partial y_\ell}(\beta; y) \quad \text{for} \ell = d+1, \ldots, n. \tag{4.2}$$

Lemma 4.9. The function $F(\beta; y)$ admits a meromorphic continuation with respect to β, denoted by $\widetilde{F}(\beta; y)$, with poles at most along the countable locally finite union of hyperplanes

$$\mathcal{P} := \bigcup_{k=1}^{d} \{\beta \in \mathbb{C}^d \mid \beta_k \in \pi_k(\mathbb{N}A)\},$$

where $\mathbb{N}A = \mathbb{N}a(1) + \cdots + \mathbb{N}a(n)$ and $\pi_k : \mathbb{Q}^d \to \mathbb{Q}$ denotes the projection to the k-th coordinate.

Proof. The initial domain of analyticity of $F(\beta; y)$ is defined by $\Re\beta_k < 0$ for all $k = 1, \ldots, d$. Let us fix conditions $\Re\beta_k < 0$ for $k = 2, \ldots, d$ and extend the domain of analyticity in the coordinate β_1 using equation $(4.1)_1$ as follows. The functions $F(\beta - a(\ell); y)$ for $\ell = d+1, \ldots, n$ and $F(\beta - e(1); y)$ are analytic for $\Re\beta_1 < \widetilde{a}_1 := \min_\ell\{a(\ell)_1, 1\}$ and hence it follows from equation $(4.1)_1$ that $F(\beta; y)$ is meromorphic in $\Re\beta_1 < \widetilde{a}_1$ with at most a pole in $\beta_1 = 0$.

In the general inductive step for the variable β_1, we assume that $F(\beta; y)$ is meromorphic in the half-space $\Re\beta_1 < (q-1)\widetilde{a}_1$. Then, on the domain defined by $\Re\beta_1 < q\widetilde{a}_1$, the right-hand side of $(4.1)_1$ is meromorphic, with poles of type $\beta_1 = c+1$, or $\beta_1 = c + a(\ell)_1$, where $\beta_1 = c$ runs over all the poles of $F(\beta; y)$. We obtain that $F(\beta; y)$ is also meromorphic in the same domain adding these new poles to those already found. Thus, by induction, we get that $F(\beta; y)$ is also meromorphic for $\beta_1 \in \mathbb{C}$ and $\Re\beta_k < 0$ for $k = 2, \ldots, d$ with poles at most along $\beta_1 = \sum_{\ell=d+1}^{n} m_\ell a(\ell)_1 + m'$, for all $m_{d+1}, \ldots, m_n, m' \in \mathbb{N}$. By an analogous argument in $k = 2, \ldots, d$ we get the result. ∎

Notice that the equations (4.2) are then satisfied by $\widetilde{F}(\beta; y)$ by analytic continuation on $U := \mathbb{C}^d \setminus \mathcal{P}$.

Lemma 4.10. For any fixed $\beta \in$, $\widetilde{F}(\beta; y)$ admits an asymptotic expansion along $y_n = 0$ in $S_{p,\delta}$. Furthermore, the coefficients $\widetilde{A}(\beta; m, y')$ of this expansion are analytic with respect to $\beta \in U$. Hence, they are analytic continuations of the coefficients $A(\beta; m, y')$ described in Theorem 4.7.

Proof. It follows from an induction starting from Theorem 4.7 and parallel to the one used in the proof of Lemma 4.9 that for any fixed $\beta \in$, $\widetilde{F}(\beta; y)$ admits asymptotic expansions along $y_n = 0$ in $S_{p,\delta}$. By construction, these analytic continuations satisfy equation (4.1), for any $\beta \in U$. This implies that the coefficients $\widetilde{A}(\beta; m, y')$ of these expansions satisfy the following equations for $k = 1, \ldots, d$:

$$\begin{aligned}\beta_k \widetilde{A}(\beta; m, y') = \widetilde{A}(\beta - e(k); m, y') + \sum_{\ell=d+1}^{n-1} a(\ell)_k\, y_\ell\, \widetilde{A}(\beta - a(\ell); m, y') \\ + m a(n)_k \widetilde{A}(\beta - a(n); m-1, y').\end{aligned} \tag{4.3}$$

Again by an induction like in Lemma 4.9, using (4.3) and Remark 4.8, $\widetilde{A}(\beta; m, y')$ is analytic with respect to β and y', hence as a function of β it is an analytic continuation to U of $A(\beta; m, y')$. ∎

We have proved the following theorem that implies the last sentence in Theorem 4.3 when we return to the integrals $I_C(\beta; y)$.

Theorem 4.11. There is an asymptotic expansion along $y_n = 0$, in an appropriate open sector $S_{p,\delta}$ around any half-line $\mathbb{R}_{>0} \cdot y_{n,0} \subset \mathbb{C}^*$:

$$\widetilde{F}_{p,\delta}(\beta; y) \underset{y_n \to 0}{\sim} \sum_{m \in \mathbb{N}} \widetilde{A}_{p,\delta}(\beta; m, y') \frac{y_n^m}{m!},$$

where $\widetilde{A}_{p,\delta}(\beta; m, y')$ is the analytic continuation of $A_{p,\delta}(\beta; m, y')$ to $\beta \in U$.

4.3 Parametrizations

We go on working with the reduced form of the integral described in (3.2–3.4), and we study integrals of the form

$$F_{p,\delta}(\beta; y) = \int_{D_{p,\delta}} t^{-\beta-\mathbf{1}} \exp\left(t_1 + \cdots + t_d + \sum_{j=d+1}^{n} y_j t^{a(j)}\right) \mathrm{d}t.$$

Lemma 4.12. If $\Re\beta < 0$, then

$$A^0_{p,\delta}(\beta) := \int_{D_{p,\delta}} t^{-\beta-\mathbf{1}} \exp(t_1 + \cdots + t_d)\mathrm{d}t = e^{\sqrt{-1}\,\pi\langle 2p+\mathbf{1}, -\beta\rangle} \Gamma(-\beta),$$

where $\Gamma(-\beta) := \prod_{k=1}^{d} \Gamma(-\beta_k)$.

Proof. The integrand $t^{-\beta-\mathbf{1}} \exp(t_1 + \cdots + t_d)\mathrm{d}t$ is of rapid decay at infinity in the product of d sectors defined by the condition:

$$\arg(t_k) \in [(1 + \min\{0, \delta_k\} + 2p_k)\pi,\ (1 + \max\{0, \delta_k\} + 2p_k)\pi],\ k \in \sigma.$$

Thus, since this product of sectors contain $D_{p,0}$ and $D_{p,\delta}$, we know by elementary considerations in one complex variable, that $A^0_{p,\delta}(\beta)$ does not depend on $\delta_k \in]-\frac{1}{2}, \frac{1}{2}[$ and so $A^0_{p,0}(\beta) = A^0_{p,\delta}(\beta)$.

We parametrize $D_{p,0}$ by $t_k = \rho_k e^{\sqrt{-1}\,\pi(2p_k+1)} = -\rho_k$ with $\rho_k \in]0, +\infty)$, and the result follows directly from the expression that we obtain

$$A^0_{p,0}(\beta) = \int_{]0,+\infty)^d} \exp(\sqrt{-1}\,\pi\langle 2p + \mathbf{1}, -\beta\rangle) \rho^{-\beta-\mathbf{1}} \exp(-\rho_1 - \cdots - \rho_d)\mathrm{d}\rho.$$

■

Since $A^0_{p,\delta}(\beta)$ does not depend on δ, from now on we drop δ and set $A^0_p(\beta) := A^0_{p,0}(\beta) = A^0_{p,\delta}(\beta)$.

We notice that $F_{p,\delta}(\beta; y)$ is locally constant with respect to δ by a similar homotopy argument. However, the dependency on δ of $F_{p,\delta}(\beta; y)$ must be kept because the argument by homotopy works only for small perturbations of δ. This does not allow a reduction of δ to zero. Let us now make the analytic continuation of the coefficients of the asymptotic expansion described in Theorem 4.7 more precise by developing them with respect to y'.

Lemma 4.13. The coefficients of the asymptotic expansion described in Theorem 4.7 are analytic functions of the variables y' with the following power series development:

$$A_{p,\delta}(\beta; m, y') = \sum_{\mathbf{m}' \in \mathbb{N}^{n-d-1}} A^0_p\left(\beta - ma(n) - \sum_{j=d+1}^{n-1} m_j a(j)\right) \frac{y'^{\mathbf{m}'}}{\mathbf{m}'!}. \tag{4.4}$$

Furthermore, this expansion is still valid for the meromorphic continuation of $A_{p,\delta}(\beta; m, y')$ found in Lemma 4.10 and the meromorphic continuation of $A^0_p(\beta)$ deduced from Lemma 4.12.

Proof. Recall that, when $\Re\beta < 0$ the coefficient we consider has the form

$$A_{p,\delta}(\beta; m, y') = \int_{D_{p,\delta}} \varphi(\beta; y'; t)\mathrm{d}t$$

with

$$\varphi(\beta; y'; t) = t^{-\beta-\mathbf{1}+ma(n)} \exp\left(t_1 + \cdots + t_d + \sum_{j=d+1}^{n-1} y_j t^{a(j)}\right).$$

We set $|t_k| = \rho_k$ for $k = 1, \ldots, d$ and we parametrize $D_{p,\delta}$ by $\rho \in \mathbb{R}^d_{>0}$. We fix a polydisc $Q = \{y' \mid |y_j| < R_j, j = d+1, \ldots, n-1\} \subset \mathbb{C}^{n-1-d}$.

The function $\varphi(\beta; y'; t)$ is holomorphic with respect to $y' \in \mathbb{C}^{n-d-1}$. By the same argument as in the proof of Lemma 3.2 and inequality (3.10), the integrand $\varphi(\beta; y'; t)\mathrm{d}t$ is dominated, via the parametrization $t_k = e^{(1+\delta_k+2p_k)\sqrt{-1}\,\pi}\rho_k$ and up to a constant factor, by

$$\rho^{-\Re\beta+ma(n)-\mathbf{1}} \exp(C - c(\rho_1 + \cdots + \rho_d))\mathrm{d}\rho$$

for some constants $C, c \in \mathbb{R}_{>0}$. These constants depend only on Q but not on $y' \in Q$ by Remark 3.4 applied to $A_{p,\delta}(\beta; m, y')$ instead of $F_{p,\delta}(\beta; y)$.

For each $j = d+1, \dots, n-1$, the integral

$$\int_{D_{p,\delta}} \frac{\partial \varphi(\beta; y'; t)}{\partial y_j} \mathrm{d}t$$

has an expression similar to the one for $A_{p,\delta}(\beta; m, y')$, with β replaced by $\beta - a(j)$. By the same argument as for φ, the integrand $\frac{\partial \varphi(\beta;y';t)}{\partial y_j}\mathrm{d}t$ is dominated, up to a constant factor, by

$$\rho^{-\Re\beta + ma(n) + a(j) - \mathbf{1}} \exp(C_j - c_j(\rho_1 + \cdots + \rho_d))\mathrm{d}\rho$$

for some constants $C_j, c_j \in \mathbb{R}_{>0}$, independent of y' in the polydisk Q.

By Lebesgue's theorem on dominated convergence for integrals, these considerations prove that $A_{p,\delta}(\beta; m, y')$ is holomorphic with respect to y' and that

$$\frac{\partial A_{p,\delta}(\beta; m, y')}{\partial y_j} = \int_{D_{p,\delta}} \frac{\partial \varphi(\beta; y'; t)}{\partial y_j} \mathrm{d}t$$

for all $j = d+1, \dots, n-1$. If we iterate the argument we obtain an expression of the partial derivatives of $A_{p,\delta}$, up to any order $\mathbf{m}' = (m_{d+1}, \dots, m_{n-1})$:

$$\frac{\partial^{|\mathbf{m}'|} A_{p,\delta}(\beta; m, y')}{\partial^{m_{d+1}} y_{d+1} \cdots \partial^{m_{n-1}} y_{n-1}} = \int_{D_{p,\delta}} \frac{\partial^{|\mathbf{m}'|} \varphi(\beta; y'; t)}{\partial^{m_{d+1}} y_{d+1} \cdots \partial^{m_{n-1}} y_{n-1}} \mathrm{d}t.$$

Setting $y' = 0$ in this last expression gives the coefficients of the Taylor expansion of $A_{p,\delta}(\beta; m, y')$ with respect to y' at the origin. This proves the equality (4.4) when $\Re\beta < 0$.

The last claim of this lemma follows from the explicit calculation in Lemma 4.12 from which we see that the coefficient of $\frac{y'^{\mathbf{m}'}}{\mathbf{m}'!}$ is equal to

$$A_p^0\left(\beta - ma(n) - \sum_{j=d+1}^{n-1} m_j a(j)\right) =$$
$$e^{\sqrt{-1}\,\pi\left(2p+\mathbf{1}, -\beta + ma(n) + \sum_{j=d+1}^{n-1} m_j a(j)\right)} \Gamma\left(-\beta + ma(n) + \sum_{j=d+1}^{n-1} m_j a(j)\right).$$

By the standard properties of the Γ-function, this coefficient admits a meromorphic continuation with respect to β, with poles along a subset of $\mathcal{P}$ defined in Lemma 4.9.

When $\beta \in \mathbb{C}^d \setminus \mathcal{P}$ the right-hand side of (4.4) is still defined and yields a convergent power series defined for all $y' \in \mathbb{C}^{n-d-1}$ because of the conditions $|a(j)| < 1$ for $j = d+1, \ldots, n-1$, and standard estimates on Γ-functions (see, e.g., [7, Lemma 3.8]).

Therefore, it is an analytic continuation of the power series defined for $\Re\beta < 0$. The equality (4.4) follows everywhere in $\mathbb{C}^d \setminus \mathcal{P}$ with the meromorphic continuation of $A_{p,\delta}(\beta; m, y')$ on the left-hand side, defined in Lemma 4.10. ■

Remark 4.14. Notice that as a consequence of Lemma 4.13 the function $A_{p,\delta}(\beta; m, y')$ does not depend on δ.

4.4 Space of asymptotic expansions and Gevrey series

In this section we finish the proof of Theorem 4.3. For any $\mathbf{k} \in \mathbb{N}^{n-d}$, let us set

$$\Lambda_{\mathbf{k}} := \{\mathbf{k} + \mathbf{m} = (k_{d+1} + m_{d+1}, \ldots, k_n + m_n) \in \mathbb{N}^{n-d} : \ A_{\overline{\sigma}}\mathbf{m} \in \mathbb{Z}^d\}$$

and define

$$S_{\mathbf{k}}(\beta; y) := \sum_{\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k}}} e^{|A_{\overline{\sigma}}(\mathbf{k}+\mathbf{m})|\pi\sqrt{-1}} \Gamma\left(-\beta + A_{\overline{\sigma}}(\mathbf{k}+\mathbf{m})\right) \frac{y^{\mathbf{k}+\mathbf{m}}}{(\mathbf{k}+\mathbf{m})!}.$$

Notice that the coefficients of the series $S_{\mathbf{k}}$ are meromorphic with respect to $\beta \in \mathbb{C}^d$ with at most simple poles along each hyperplane in $\mathcal{P}$. In particular, if $\beta \notin \mathcal{P}$ all these series are well-defined nonzero power series with support equal to $\Lambda_{\mathbf{k}}$ since the Gamma function does not have any zero. It can be proved by using standard estimates of Gamma functions that these series are Gevrey along $y_n = 0$ with Gevrey index $|a(n)| > 1$.

Let $\Omega \subseteq \mathbb{N}^{n-d}$ be a set of cardinality $[\mathbb{Z}A : \mathbb{Z}A_\sigma]$ such that

$$\{A_{\overline{\sigma}}\mathbf{k} + \mathbb{Z}A_\sigma : \ \mathbf{k} \in \Omega\} = \mathbb{Z}A/\mathbb{Z}A_\sigma = \mathbb{Z}A/\mathbb{Z}^d.$$

We notice that the existence of such $\Omega \subseteq \mathbb{N}^{n-d}$ follows from [7, Lemma 3.2]. It is clear that $\mathcal{G} = \{S_{\mathbf{k}}(\beta; y) : \ \mathbf{k} \in \Omega\}$ is a linearly independent set because the series $S_{\mathbf{k}}$ have pairwise disjoint supports $\Lambda_{\mathbf{k}}$. Using Theorem 4.7, Lemma 4.12, and Lemma 4.13, we have

$$\begin{aligned} F_{p,\delta}(\beta; y) &\underset{y_n \to 0}{\sim} \sum_{q_n \in \mathbb{N}} A_{p,\delta}(\beta; q_n, y') \frac{y_n^{q_n}}{q_n!} \\ &= \sum_{\mathbf{q} \in \mathbb{N}^{n-d}} A_p^0(\beta - A_{\overline{\sigma}}\mathbf{q}) \frac{y^{\mathbf{q}}}{\mathbf{q}!} = \sum_{\mathbf{q} \in \mathbb{N}^{n-d}} e^{\sqrt{-1}\,\pi\langle \mathbf{1}+2p, -\beta + A_{\overline{\sigma}}\mathbf{q}\rangle} \Gamma\left(-\beta + A_{\overline{\sigma}}\mathbf{q}\right) \frac{y^{\mathbf{q}}}{\mathbf{q}!} \end{aligned}$$

$$
\begin{aligned}
&= e^{\sqrt{-1}\,\pi\langle \mathbf{1}+2p,-\beta\rangle} \sum_{\mathbf{k}\in\Omega} \sum_{\mathbf{k}+\mathbf{m}\in\Lambda_{\mathbf{k}}} e^{\sqrt{-1}\,\pi\langle \mathbf{1}+2p,A_{\overline{\sigma}}(\mathbf{k}+\mathbf{m})\rangle} \Gamma\left(-\beta + A_{\overline{\sigma}}(\mathbf{k}+\mathbf{m})\right) \frac{y^{\mathbf{k}+\mathbf{m}}}{(\mathbf{k}+\mathbf{m})!} \\
&= e^{\sqrt{-1}\,\pi\langle \mathbf{1}+2p,-\beta\rangle} \sum_{\mathbf{k}\in\Omega} e^{\sqrt{-1}\,\pi\langle \mathbf{1}+2p,A_{\overline{\sigma}}\mathbf{k}\rangle} \sum_{\mathbf{k}+\mathbf{m}\in\Lambda_{\mathbf{k}}} e^{\sqrt{-1}\,\pi\langle \mathbf{1},A_{\overline{\sigma}}\mathbf{m}\rangle} \Gamma \\
&\qquad \left(-\beta + A_{\overline{\sigma}}(\mathbf{k}+\mathbf{m})\right) \frac{y^{\mathbf{k}+\mathbf{m}}}{(\mathbf{k}+\mathbf{m})!} \\
&= e^{\sqrt{-1}\,\pi\langle \mathbf{1}+2p,-\beta\rangle} \sum_{\mathbf{k}\in\Omega} e^{\sqrt{-1}\,\pi\langle 2p,A_{\overline{\sigma}}\mathbf{k}\rangle} S_{\mathbf{k}}(\beta;y).
\end{aligned}
$$

Notice that previous power series is formal with respect to y_n, with convergent coefficients. More precisely, it is a Gevrey series along $y_n = 0$ with Gevrey index $|a(n)| > 1$. We notice also that $\Re\beta < 0$ implies that $\Re(\beta - A_{\overline{\sigma}}\mathbf{q}) < 0$ for all $\mathbf{q} \in \mathbb{N}^{n-d}$, by using Assumption (4.4), which guarantees the convergence of all the integrals involved in Section 4.3. By the last claim in Lemma 4.13, this calculation is valid everywhere in the domain of analytic continuation $\mathbb{C}^d \setminus \mathcal{P}$, since the argument applies also to the coefficients of the series $S_{\mathbf{k}}(\beta;y)$.

The matrix of coefficients of the series $S_{\mathbf{k}}(\beta;y)$ in the asymptotic expansions of the functions

$$
e^{\sqrt{-1}\,\pi\langle 2p+\mathbf{1},\beta\rangle} F_{p,\delta}(\beta;y)
$$

is $(e^{\sqrt{-1}\,\pi\langle 2p,A_{\overline{\sigma}}\mathbf{k}\rangle})_{\mathbf{k},p}$, where $\mathbf{k}$ varies in Ω. If p varies in an appropriate set of $[\mathbb{Z}A : \mathbb{Z}^d]$ elements, this matrix is square and invertible. Indeed, we have $\mathbb{Z}A/\mathbb{Z}A_\sigma = \mathbb{Z}A/\mathbb{Z}^d \simeq \mathbb{Z}^d/\mathbb{Z}M$, where M is the matrix of coordinates of the canonical basis of $\mathbb{Z}^d$ with respect to a basis of $\mathbb{Z}A/\mathbb{Z}^d$. Thus, the matrix $(e^{\sqrt{-1}\,\pi\langle 2p,A_{\overline{\sigma}}\mathbf{k}\rangle})_{\mathbf{k},p}$ is invertible by [19, Proposition 6.3], if p runs in a set of representatives of the quotient $\mathbb{Z}^d/\mathbb{Z}\,{}^tM$. In particular, if $\beta \notin \mathcal{P}$ the set of holomorphic functions $F_{p,\delta}(\beta;y)$, where p varies in this set of representatives, is a linearly independent set and any Gevrey series along $y_n = 0$ in the space generated by the series $\{S_{\mathbf{k}}(\beta;y) : \mathbf{k} \in \Omega\}$ is an asymptotic expansion of a linear combination of the integrals $F_{p,\delta}(\beta;y)$.

Now if we start from the matrix B in Section 2 and we apply the above results with the matrix $A = B_\sigma^{-1}B = (I, B_\sigma^{-1}B_{\overline{\sigma}})$ and the parameter $\beta = B_\sigma^{-1}\gamma$, we obtain a similar statement for the integrals $I_C(\gamma;x)$ using (3.3) and (3.4) if we set $y_j = x_j x_\sigma^{-a(j)}$ for all $j = d+1,\dots,n$, or $y = x_{\overline{\sigma}} x_\sigma^{-B_\sigma^{-1}B_{\overline{\sigma}}}$. Precisely, M can be chosen to be B_σ. We get that $x_\sigma^{B_\sigma^{-1}\gamma} \cdot \mathcal{G}$ is a linearly independent set of Gevrey series solutions of $M_B(\gamma)$ along $x_n = 0$ with Gevrey index $|a(n)| = |B_\sigma^{-1}b(n)| > 1$ if $\beta \notin \mathcal{P}$. Again this transformation involves a

choice of $\arg x_\sigma$, but by Remark 2.1 and (3.3), a change in this choice does not modify the basis $\mathcal{G}$ except for constant factors.

It is enough to prove that the dimension of the space of Gevrey series solutions of $M_B(\gamma)$ along $x_n = 0$ is at most equal to $|\Omega| = [\mathbb{Z}^d : \mathbb{Z}B_\sigma]$ when $\Re\beta < 0$. To this end, notice first that, if

$$f = \sum_{m=0}^{\infty} f_m(x_1, \ldots, x_{n-1}) x_n^m$$

is a Gevrey series belonging to this space, then the initial part of f with respect to the weight vector $w = (0, \ldots, 0, 1) \in \mathbb{R}^n$ has the form $\mathrm{in}_w(f) = f_m(x_1, \ldots, x_{n-1}) x_n^m$ for some $m \geq 0$ and it is hence a holomorphic function. Thus, by the same argument as in the proof of [27, Th. 2.5.5], it is a (holomorphic) solution of $\mathrm{in}_{(-w,w)}(H_B(\gamma))$. This last ideal is the initial ideal with respect to w of the hypergeometric ideal associated with (B, γ) (see [27, p. 4]). In particular, the dimension of the space of Gevrey solutions is at most equal to the rank of $\mathrm{in}_{(-w,w)}(H_B(\gamma))$, because one can choose a basis of Gevrey solutions of $M_B(\gamma)$ such that their initial parts are also linearly independent (see [27, Proposition 2.5.7]).

On the other hand, by using [27, Lemma 2.1.6] for $(u, v) = (\mathbf{0}, \mathbf{1})$ and $(u', v') = (-w, w)$, we have that the initial ideal of $\mathrm{in}_{(-w,w)}(H_B(\gamma))$ with respect to $(\mathbf{0}, \mathbf{1})$ is

$$\mathrm{in}_{(\mathbf{0},\mathbf{1})}(\mathrm{in}_{(-w,w)}(H_B(\gamma))) = \mathrm{in}_L(H_B(\gamma))$$

for $L = (-w, w) + \epsilon(\mathbf{0}, \mathbf{1})$ with $\epsilon > 0$ small enough.

Thus, by [28, Th. 4.21, Rk. 4.23, and Th. 4.28] for $L = (-w, w) + \epsilon(\mathbf{0}, \mathbf{1})$ and Assumption 4.1, we have that the holonomic rank of $\mathrm{in}_{(-w,w)}(H_B(\gamma))$ equals $|\Omega|$ if γ is not *rank–jumping* for B (i.e., if $\mathrm{rank}(M_B(\gamma)) = d! \,\mathrm{Vol}(\Delta_B)$), a condition that is weaker than $\Re\beta = \Re(B_\sigma^{-1}\gamma) < 0$ by [1, Th. 5.15] (see also [27, Cor. 4.5.3]). This finishes the proof of Theorem 4.3.

Remark 4.15. Notice that the proof of Theorem 4.3 shows that the constructed set of Gevrey series solutions $x_\sigma^{B_\sigma^{-1}\gamma} \cdot \mathcal{G}$ is still a basis of the space of Gevrey solutions of $M_B(\gamma)$ along $x_n = 0$ when γ is not rank-jumping and $\beta = B_\sigma^{-1}\gamma \notin \mathcal{P}$, where $\mathcal{P}$ is defined in Lemma 4.9. We do not know if under Assumption 4.1 the condition of γ being rank-jumping implies $\beta \in \mathcal{P}$. However, it is true that if γ is rank-jumping then it is *semi-resonant* [1]. In particular, under Assumption 4.1, γ is semi-resonant for B if and only if $\beta \in \mathcal{P}' := \cup_{k=1}^d \{\beta \in \mathbb{C}^d |\ \beta_k \in \pi_k(\mathbb{Z}A \cap \mathbb{R}^d_{\geq 0})\}$, where π_k is the projection to the k-th coordinate. Notice also that $\mathcal{P} \subseteq \mathcal{P}'$.

Remark 4.16. In [7, Sec. 3] the author constructs certain Gevrey series solutions $\varphi_{v^{\mathbf{k}}}$ for the hypergeometric system $M_B(\gamma)$. Using Euler's reflection formula, $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ for $z \notin \mathbb{Z}$, it can be easily shown that, for all $\mathbf{k} \in \Omega$ and when β is generic enough,

$$S_{\mathbf{k}}(B_\sigma^{-1}\gamma;\, x_{\overline{\sigma}}x_\sigma^{-B_\sigma^{-1}B_{\overline{\sigma}}}) = \frac{\pi^d e^{\sqrt{-1}\,\pi|A_{\overline{\sigma}}\mathbf{k}|}}{\sin(\pi(-\beta + A_{\overline{\sigma}}\mathbf{k}))} \cdot \varphi_{v^{\mathbf{k}}}.$$

The genericity condition here means that $\beta \notin \mathcal{P}$ and that $\beta - A_{\overline{\sigma}}\mathbf{k}$ does not have integer coordinates for all $\mathbf{k} \in \Omega$.

5 Integrals Over Rapid Decay Cycles

The goal of this section is to prove that when $\Re\beta < 0$ is sufficiently general, the integrals studied in Theorem 4.3, are in fact integrals over rapid decay cycles in the sense of [15]. These integrals are defined without the condition $\Re\beta < 0$ and are still solutions of our GKZ system when $\Re\beta_k \geq 0$ for some k. By meromorphic continuation proved in Theorem 4.11 they admit asymptotic expansions as Gevrey series solution for all β sufficiently general in $\mathbb{C}^d$.

5.1 Description of rapid decay cycles

In this section we first briefly recall the theory of rapid decay homology by M. Hien in [15, Sec. 5.1] and give a sufficient condition to detect a cycle for this homology.

Let U be a complex quasi-projective variety over $\mathbb{C}$ of dimension d. Let $h \in \mathcal{O}(U)$ and let X be a smooth projective compactification of U, such that $D = X \setminus U$ is a normal crossing divisor, and h extends to a map $h : X \longrightarrow \mathbb{P}^1$.

Let us denote by $\pi : \widetilde{X}(D) \longrightarrow X^{an}$ the real oriented blow-up along D as defined in [26, 8.2]. The space $\widetilde{X} := \widetilde{X}(D)$ can be embedded into a real Euclidian space as a semi-analytic subset, and h induces a map $\widetilde{h} : \widetilde{X} \longrightarrow \widetilde{\mathbb{P}^1}$, where $\widetilde{\mathbb{P}^1} \to \mathbb{P}^1$ is the real blow-up of infinity.

Let us describe the morphism π, locally at $p \in D$ with local coordinates $t_1, \ldots, t_d$ such that $p = 0$ and $D = \{t_1 \cdots t_k = 0\}$,

$$\begin{array}{rcl} \pi : ([0,\epsilon) \times S^1)^k \times B(0,\epsilon)^{d-k} & \longrightarrow & \mathbb{C}^d \\ ((r_j, e^{\sqrt{-1}\,\theta_j})_{j=1}^k, t') & \mapsto & (r_1 \cdot e^{\sqrt{-1}\,\theta_1}, \ldots, r_k \cdot e^{\sqrt{-1}\,\theta_k}, t'), \end{array}$$

where $t' = (t_{k+1}, \ldots, t_d)$ and $\epsilon > 0$ is a small real number.

Consider a regular flat algebraic connection $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega_U$, restriction to U of a regular meromorphic connection $(\mathcal{E}_X(*D), \nabla)$ on X, with $\mathcal{E}_X$ a lattice of this connection.

On the oriented blow-up $\widetilde{X}$, we consider the sheaf $\mathcal{A}_{\widetilde{X}}^{\leq D}$ of holomorphic functions that are flat along $\widetilde{D} := \pi^{-1}(D)$.

A section of $\mathcal{A}_{\widetilde{X}}^{\leq D}$ on an open set $\widetilde{V} \subseteq \widetilde{X}$ is a holomorphic function u on $\widetilde{V} \setminus \widetilde{D}$ such that, for any compact set $K \subset \widetilde{V}$, and all $N = (N_1, \ldots, N_k) \in \mathbb{N}^k$, there exists a constant $C_{K,N} > 0$ satisfying

$$|u(t)| \leq C_{K,N} |t_1|^{N_1} \cdots |t_k|^{N_k}, \quad \forall t = (t_1, \ldots, t_d) \in K \setminus \widetilde{D} \tag{5.1}$$

in terms of local coordinates as above such that locally $D = \{t_1 \cdots t_k = 0\}$.

The twisted connection $\nabla_h = \nabla - dh\wedge = \exp(h) \circ \nabla \circ \exp(-h)$ on U extends to a morphism of sheaves over $\widetilde{X}$

$$\pi^{-1}\mathcal{E}_X \otimes \mathcal{A}_{\widetilde{X}}^{\leq D} \longrightarrow \pi^{-1}\mathcal{E}_X \otimes \mathcal{A}_{\widetilde{X}}^{\leq D} \otimes_{\pi^{-1}\mathcal{O}_{X^{an}}} \pi^{-1}\Omega^1_{X^{an}}(*D).$$

The kernel of this extension is denoted by $\mathcal{S}^{<D}$. The restriction of this kernel to U is the set of horizontal sections of ∇_h and it is equal to $\mathcal{S} \otimes \exp(h)$, where $\mathcal{S}$ is the local system of horizontal sections of ∇. Since the coefficients of a section of $\mathcal{S}$, on a basis of $\mathcal{E}$ at a point $P \in D$, have at most a polynomial growth, the germ of $\mathcal{S}^{<D}$ at a point $\widetilde{P} \in \widetilde{D}$ is nonzero if and only if $\exp(\widetilde{h})$ satisfies condition (5.1) at $\widetilde{P}$. At such a point it is equal to the germ of $\pi^{-1}(\mathcal{S}) \otimes \exp(\widetilde{h})$.

The sheaf of rapid decay chains [15, Sec. 5.1] is obtained from the *sheaf* $\mathcal{C}^{-p}_{\widetilde{X},\widetilde{D}}$ *of relative chains mod* $\widetilde{D}$ by tensoring it with $\mathcal{S}^{<D}$:

$$\mathcal{C}^{r.d.,-p}_{\widetilde{X}} := \mathcal{C}^{-p}_{\widetilde{X},\widetilde{D}} \otimes_{\mathbb{C}} \mathcal{S}^{<D}.$$

Let $j : U \hookrightarrow \widetilde{X}$ be the inclusion map. The sheaf $\mathcal{S}^{<D}$ is a subsheaf of $j_*(\mathcal{S} \otimes \exp(h))$, the latter being isomorphic to $j_*(\mathcal{S})$ through the multiplication by $\exp(-h)$. Therefore, $\mathcal{C}^{r.d.,-p}_{\widetilde{X}}$ is a subsheaf of $\mathcal{C}^{-p}_{\widetilde{X},\widetilde{D}} \otimes_{\mathbb{C}} j_*(\mathcal{S} \otimes \exp(h))$, and in the next lemma we determine its image in $\mathcal{C}^{-p}_{\widetilde{X},\widetilde{D}} \otimes_{\mathbb{C}} j_*(\mathcal{S})$. In all what follows we identify $\mathcal{C}^{r.d.,-p}_{\widetilde{X}}$ with this image. This convention is the most appropriate for the expression of integrals.

Lemma 5.1. Let Σ be a semi-algebraic set in U such that $\Re h$ tends to $-\infty$ on Σ with a controlled argument for h, that is, there exists $\delta \in \left]0, \frac{\pi}{2}\right[$ such that for all $R > 0$ there

exists a compact set $K \subset U$, such that, for all $t \in \Sigma \setminus K$ we have

$$\Re(h(t)) < -R \text{ and } \arg(h(t)) \in]\pi - \delta, \pi + \delta[.$$

Then the closure $\overline{\Sigma}$ in $\widetilde{X}$ is a compact semi-algebraic set. Moreover, let $\mathfrak{T}$ be any finite triangulation of $\overline{\Sigma}$, and let $\Upsilon := \sum_\Delta \Delta \otimes \varsigma_\Delta$ be a section of $\mathcal{C}^{-d}_{\widetilde{X},\widetilde{D}} \otimes_{\mathbb{C}} j_*(\mathcal{S})$. This is a finite sum, where Δ runs, possibly with repetitions, over all the d-simplices of $\mathfrak{T}$ that are not included in $\widetilde{D}$ and ς_Δ is a section of $\mathcal{S}$ over $j^{-1}(\Delta) \cap U$. Then Υ is a rapid decay chain, whose support is contained in $\overline{\Sigma}$.

Proof. The divisor D is the union of the components $(D_i)_{i\in I}$ of $h^{-1}(\infty)$ and of other components $(D'_j)_{j\in J}$, such that on each D'_j the restriction of h is surjective on $\mathbb{P}^1$ or takes a finite constant value. From the fact that $|h(t)|$ tends to $+\infty$ on the support $\operatorname{supp}(\Upsilon) := \bigcup\{\Delta \mid \varsigma_\Delta \neq 0\}$ of Υ, we deduce that $\overline{\operatorname{supp}(\Upsilon)} \cap D \subset h^{-1}(\infty) = \bigcup_{i\in I} D_i$, where $\overline{\operatorname{supp}(\Upsilon)}$ denotes the closure of $\pi(\operatorname{supp}(\Upsilon))$ in X. Furthermore, the closure of $\operatorname{upp}(\Upsilon)$ in $\widetilde{X}$ meets $\widetilde{D}$ only at points such that $\arg \tilde{h} \in [\pi - \delta, \pi + \delta]$.

Let $\tilde{P} \in \widetilde{D}$ be such a point and let $P = \pi(\tilde{P})$. We choose local coordinates $(v_1, \dots, v_d)$ centered at P such that a local equation of D is $v_1 \dots v_k = 0$. The local expression of h is

$$h(v) = \frac{w(v_1, \dots, v_d)}{v_1^{m_1} \dots v_k^{m_k}}$$

with all $m_k > 0$ and $w(v)$ a unit since there are no points of indeterminacy. In a small enough neighbourhood of , $h(v) = -\exp(i\delta(v))|h(v)|$, with $\cos\delta(v) \geq \cos\delta > 0$ and $|w(v_1, \dots, v_d)| \geq R'$, for some $R' > 0$. Finally, around P we obtain the expected rapid decay condition because

$$|\exp(h(v))| = \exp(\Re h(v)) \leq \exp\left(-\frac{R'\cos\delta}{|v_1|^{m_1} \dots |v_k|^{m_k}}\right).$$

■

In order to treat integrals $I_C(\beta; x)$ as in the introduction, we consider the connection $(\mathcal{O}_U, \nabla_\beta)$ on $U = (\mathbb{C}^*)^d$ with the differential $\nabla_\beta = d + \beta\frac{dt}{t}\wedge$ and its meromorphic extension $(\mathcal{O}_X(*D), \nabla_\beta)$ to X. It contains a lattice isomorphic to $\mathcal{O}_X$, and the local system of horizontal sections over U is $\mathcal{S}_\beta = \mathbb{C}\cdot t^{-\beta-1}$. We set $h(t) = \sum_{\ell=1}^n x_\ell t^{a(\ell)}$ and we intend to apply Lemma 5.1 for a fixed value of x. For that purpose, we have to use a cycle different from the cycles $C_{p,\delta}$, considered in Section 2, since the support of $C_{p,\delta}$ always have the origin of $\mathbb{C}^d$ in its closure, and when t tends to 0 along $C_{p,\delta}$, h

does not tend to $+\infty$. This cycle is described in detail in Section 5.2. It is is still a Borel–Moore cycle on the universal covering $(\widetilde{\mathbb{C}^*})^d$, whose projection Σ on $(\mathbb{C}^*)^d$ is semi-algebraic. There is a triangulation $\mathfrak{T}$ of its closure $\overline{\Sigma}$ in $\widetilde{X}$ and a set of d-simplices $\Delta \in \mathfrak{T}$ not contained in $\widetilde{D}$, such that C is obtained by taking their restriction to $(\mathbb{C}^*)^d$ and an appropriate lifting to the universal covering $(\widetilde{\mathbb{C}^*})^d$. These liftings induce branches $\varsigma_\Delta = \left(t^{-\beta}\right)_\Delta$ of $t^{-\beta}$ and we identify C with the twisted chain: $\sum \Delta \otimes \varsigma_\Delta$. The formula in [2, page 23], can be directly adapted to the irregular case:

$$\int_{\Delta \otimes \varsigma_\Delta} t^{-\beta} e^{h(x,t)} \frac{\mathrm{d}t}{t} = \int_\Delta \left(t^{-\beta}\right)_\Delta e^{h(x,t)} \frac{\mathrm{d}t}{t} \tag{5.2}$$

and the construction above shows that the integral $I_C(\beta; x)$ along C is the integral along this twisted cycle.

Corollary 5.2. Let us assume that in the above situation $h(x,t)$ satisfies the condition of rapid decay and controlled argument in Lemma 5.1. Then the cycle $\Upsilon = \sum \Delta \otimes \varsigma_\Delta$ associated with C is a rapid decay cycle, and the integral $I_C(\beta; x)$ along this cycle is convergent.

Proof. Only the last assertion requires a proof. Consider again a point $P \in D$, with coordinates $(v_1, \ldots, v_d)$ as in the last argument for Lemma 5.1. Since t is algebraic, $t^{-\beta-1}$ has at most a polynomial growth around P, with respect to $\frac{1}{v_1 \cdots v_k}$. Therefore, $t^{-\beta-1} e^{h(x,t)}$ is locally bounded by an expression of the form $\frac{1}{|v_1 \cdots v_k|^m} \exp\left(-\frac{R' \cos\delta}{|v_1|^{m_1} \cdots |v_k|^{m_k}}\right)$ for some integer $m > 0$. This yields a convergent integral on $\mathcal{U}_P \cap (\mathbb{C}^*)^d$ for some closed neighbourhood $\mathcal{U}_P$ of P. Since $\overline{\Sigma} \cap D$ can be covered by a finite number of such $\mathcal{U}_P$, the integral is indeed convergent. ∎

From now on we will identify a cycle on the universal covering and the corresponding twisted cycle and denote it by the same symbol.

5.2 Realization of solutions by integrals over rapid decay cycles

We state and prove here the main result of this section. We define q_k, for $k = 1, \ldots, d$, as the smallest common denominator of the k-th row of A, that is as the smallest integer such that $q_k a(\ell)_k \in \mathbb{N}$, for $\ell = d+1, \ldots, n$. Recall that, for each $p \in \mathbb{Z}^d$ and $y_{n,0} \in \mathbb{C}^*$ we choose $\delta \in]-\frac{1}{2}, \frac{1}{2}[^d$ such that $(3.6)_n$ is satisfied (see Lemma 4.5).

Theorem 5.3. There is a rapid decay cycle $\widetilde{D}_{p,\delta}$ such that the integral

$$\int_{\widetilde{D}_{p,\delta}} t^{-\beta-\mathbf{1}} \exp\left(t_1 + \cdots + t_d + \sum_{j=d+1}^{n} y_j t^{a(j)}\right) \mathrm{d}t$$

is equal to $F_{p,\delta}(\beta; y)$ up to a nonzero constant factor if $\Re\beta < 0$ and $q_k\beta_k \notin \mathbb{Z}$, for $k = 1, \ldots, d$.

Proof. The proof starts with a preliminary reduction and then has three steps. First, we build cycles depending on a parameter $\epsilon > 0$ for which Corollary 5.2 can be applied. We then show that $F_{p,\delta}(\beta; y)$ is the limit when $\epsilon \longrightarrow 0$ of the integrals over these cycles and finally we prove that these integrals are in fact independent of ϵ.

If we perform the change of coordinates $t_k = e^{\sqrt{-1}\,\pi(1+2p_k+\delta_k)} r_k$ for $k = 1, \ldots, d$, the image of the cycle $D_{p,\delta}$, defined in (3.1) is just the positive orthant $\mathbb{R}^d_{>0}$, and we find

$$F(\beta; y) = \int_{\mathbb{R}^d_{>0}} e^{-\sqrt{-1}\,\pi\langle \mathbf{1}+2p+\delta, \beta\rangle} r^{-\beta} \exp\left(-\sum_{k=1}^{d} e^{\sqrt{-1}\,\pi\delta_k} r_k + \sum_{j=d+1}^{n} z_j r^{a(j)}\right) \frac{\mathrm{d}r}{r} \tag{5.3}$$

with $z_j = e^{\sqrt{-1}\,\pi\langle \mathbf{1}+2p+\delta, a(j)\rangle} y_j$, and $\Re z_n < 0$, since $(3.6)_n$ is satisfied.

For the sake of simplicity we skip the constant $e^{-\sqrt{-1}\,\pi\langle \mathbf{1}+2p+\delta, \beta\rangle}$ and consider only the case $p = \delta =$, $z_j = e^{\sqrt{-1}\pi|a(j)|} y_j$ hence reduce to the integral:

$$H(\beta; y) = \int_{\mathbb{R}^d_{>0}} r^{-\beta} \exp\left(-r_1 - \cdots - r_d + \sum_{j=d+1}^{n} y_j r^{a(j)}\right) \frac{\mathrm{d}r}{r}, \tag{5.4}$$

related to F by the relation: $F_{0,0}(\beta; y) = e^{-\sqrt{-1}\,\pi|\beta|} H(\beta; e^{\sqrt{-1}\pi|a(d+1)|} y_{d+1}, \ldots, e^{\sqrt{-1}\pi|a(n)|} y_n)$. By Lemma 3.2, this integral is convergent since $\Re\beta < 0$ and the condition $(3.5)_n$ transfered to H is $\Re(y_n r^{a(n)}) = r^{a(n)} \Re y_n < 0$. Finally, we are looking for cycles $\Upsilon(\epsilon)$ such that the integral

$$H_{\Upsilon(\epsilon)}(\beta; y) := \int_{\Upsilon(\epsilon)} u^{-\beta} \exp\left(-u_1 - \cdots - u_d + \sum_{j=d+1}^{n} y_j u^{a(j)}\right) \frac{\mathrm{d}u}{u} \tag{5.5}$$

tends to $H(\beta; y)$ when $\epsilon \longrightarrow 0$. The cycle $\widetilde{D}_{0,0}$ in the statement of Theorem 5.3, is the image of $\Upsilon(\epsilon)$ by $t_k = -u_k$ for some $\epsilon > 0$, with the choice of arguments $\arg t_k = \arg u_k + \pi$. The reason is that the same operation changes $H(\beta; y)$ into $F_{0,0}(\beta; y)$.

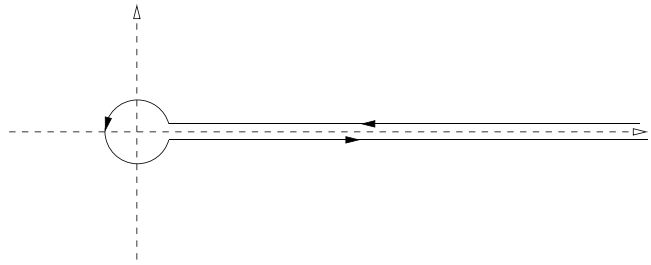

Fig. 2.

Inspecting the proof, the case of a general value of p, δ is a straightforward adaptation.

Recall that by Assumption 4.4 we have that $a(j) \in \mathbb{Q}^d_{>0}$ for all $j = d+1, \ldots, n$, $|a(j)| < 1$, for $j = d+1, \ldots, n-1$ and also $|a| = a_1 + \cdots + a_d > 1$, setting $a := a(n) = {}^t(a_1, \ldots, a_d)$. ■

Notation 5.4. Let $\varpi : (\mathbb{C}^*)^d_v \longrightarrow (\mathbb{C}^*)^d_u$ be the finite covering of multidegree $(q_1, \ldots, q_d)$, given by the formulas $v_k^{q_k} = u_k$ between two samples of the torus $(\mathbb{C}^*)^d$. The interest of this convering is that the exponent $M(u, y)$ in the integrand of $H_{\Upsilon(\epsilon)}(\beta; y)$ is a univalent function of the variable v.

Let us first describe a product of cycles $C := \gamma_1 \times \cdots \times \gamma_d$ on the universal covering $(\widetilde{\mathbb{C}^*})^d$. In Figure 2 we draw the projection of γ_k on $\mathbb{C}^*_{v_k}$, and for the projection on the space $\mathbb{C}^*_{u_k}$ we turn q_k times on the circle of radius ϵ in the k-th component. In Figure 2 the radius is $\epsilon^{\frac{1}{q_k}}$ on the k-th component. We choose the argument of u_k to be 0 or $2q_k\pi$, on the two half-lines of Figure 2, drawn in $\mathbb{C}_{v_k}$. The integrand is the same up to a constant factor on the 2^d different products of the d half-lines in $(\widetilde{\mathbb{C}^*})^d$. With these choices we can think of C in two different ways: as a cycle on $(\widetilde{\mathbb{C}^*})^d$, or as a twisted cycle on either $(\mathbb{C}^*)^d_u$ or $(\mathbb{C}^*)^d_v$.

However, there is a problem of convergence for the integral $H_C(\beta; y)$. The cycle C is a union of products of the type $\left(S_\epsilon\right)^\eta \times ([\epsilon, +\infty))^\tau$. Here S_ϵ is a circle of radius $\epsilon > 0$ and $\eta \cup \tau = \{1, \ldots, d\}$ is a partition of $\{1, \ldots, d\}$. On each piece with $\tau \neq \varnothing \neq \eta$ the integral is not convergent. Indeed, when $k \in \eta$ and u_k varies in S_ϵ, the argument of each monomial $y_\ell u^{a(\ell)}$ takes all values mod 2π. Therefore, the monomial itself takes arbitrarily large positive values, as well as $|u_\ell|$ for $\ell \in \tau$.

We shall build the cycle $\Upsilon(\epsilon)$ as a deformed version of C. We identify $(\widetilde{\mathbb{C}^*})^d$ with $\mathbb{C}^d = \mathbb{R}^d + \sqrt{-1}\,\mathbb{R}^d$ and the covering map $(\widetilde{\mathbb{C}^*})^d \to (\mathbb{C}^*)^d$ with the map $\log r +$

$\sqrt{-1}\,\theta \longrightarrow (r_1 e^{\sqrt{-1}\,\theta_1}, \ldots, r_d e^{\sqrt{-1}\,\theta_d})$. We consider $(\widetilde{\mathbb{C}^*})^d$ as fibered over $\mathbb{R}^d_{>0}$, by the map $\log r + \sqrt{-1}\theta \longrightarrow (r_1, \ldots, r_d)$, with fiber isomorphic to $\sqrt{-1}\,\mathbb{R}^d$. The image of the restriction of this map to $\gamma_1 \times \cdots \times \gamma_d$, is $[\epsilon, +\infty)^d$, with semi–algebraic fibers. The fiber over $(r_1, \ldots, r_d)$ is $\log r + \sqrt{-1}\,F_r$, where $F_r \subset \mathbb{R}^d$ is a subset of arguments $\arg u := (\arg u_1, \ldots, \arg u_d)$, which depends on r in the following way:

(1) Above each point $(r_1, \ldots, r_d)$ in the open orthant $]\epsilon, +\infty)^d$, there are 2^d points with $\arg u \in F_r = \prod_{k=1}^d \{0, 2q_k\pi\}$.
(2) Above the point $\{(\epsilon, \ldots, \epsilon)\}$, the argument $\arg u$ is in $F_r = \prod_{k=1}^d [0, 2q_k\pi]$.
(3) In general, above the product $\{\epsilon\}^\eta \times (]\epsilon, +\infty))^\tau$, the fiber has dimension $|\eta|$, the cardinality of η, with $2^{|\tau|}$ connected components. It is described in the universal covering $(\widetilde{\mathbb{C}^*})^d$, by

$$\begin{aligned} u_k &\in \quad \log\epsilon + [0, 2q_k\pi]\sqrt{-1} \quad \text{if } k \in \eta, \\ u_\ell &\in \quad \log r_\ell + \{0, 2q_\ell\pi\}\sqrt{-1} \quad \text{if } \ell \in \tau. \end{aligned}$$

We choose instead of $\gamma_1 \times \cdots \times \gamma_d$ a cycle $\Upsilon(\epsilon)$ fibered over the subset of $\mathbb{R}^d_{>0}$ described by the equation $r^a \geq \epsilon^{|a|}$, which is the union of 2^d semi-algebraic strata:

(1) $\mathcal{S}_\varnothing = \{r \in \mathbb{R}^d_{>0} \mid r^a > \epsilon^{|a|}\}$
(2) $\mathcal{S}_\eta = \{r \in \mathbb{R}^d_{>0} \mid r^a = \epsilon^{|a|}$, and $\eta = \{p \mid r_p = \min_{1\leq k\leq d} r_k\}\}$, if $\eta \neq \varnothing$.

We shall sometimes write $re^{\sqrt{-1}\,\theta}$ instead of $\log r + \sqrt{-1}\,\theta$, since the abuse of notation fits better with the expression of the integral and it is clear from the context that when the target space is $\widetilde{\mathbb{C}}$, arguments θ are to be considered in $\mathbb{R}$.

Definition 5.5. Description of the cycle $\Upsilon(\epsilon)$:

(1) The fiber of the support of $\Upsilon(\epsilon)$ over the point $r \in \mathcal{S}_\eta$ is

$$\Upsilon(\epsilon)_r := \prod_{k\in\eta} (\log r_k + [0, 2q_k\pi]\sqrt{-1}) \times \prod_{\ell\in\tau} (\log r_\ell + \{0, 2q_\ell\pi\}\sqrt{-1}).$$

(2) Let us take some $r \in \mathcal{S}_\eta$. We have $r = (\rho^\eta; (r_k)_{k\in\tau})$, with $\rho^\eta \in \mathbb{R}^\eta$ the point with all coordinates equal to $\rho > 0$. When $\eta \neq \varnothing$, these data are subject to the conditions:

$$\rho < \min((r_k)_{k\in\tau}), \quad r^a = \rho^{\sum_{j\in\eta} a_j} \prod_{k\in\tau} r_k^{a_k} = \epsilon^{|a|}. \tag{5.6}$$

(3) The projection of $\mathcal{S}_\eta$ on the space $\mathbb{R}^\tau_{>0}$ is a bijection $\mathcal{S}_\eta \longrightarrow U_\eta$ to the open subset described by the inequalities:

$$\epsilon^{|a|} < r_k^{\sum_{j\in\eta} a_j} \prod_{\ell\in\tau} r_\ell^{a_\ell} \text{ for any } k \in \tau,$$

and the value of the η-coordinate ρ of a point $r \in \mathcal{S}_\eta$ is a function $\rho((r_k)_{k\in\tau})$ by implicit equation (5.6).

(4) Let $s \in \{0,\dots,d\}$ be the number of elements in η. Let us denote $\Upsilon(\epsilon)_\eta$ the union of the pieces of the cycle $\Upsilon(\epsilon)$ above the stratum $\mathcal{S}_\eta$. Then $\Upsilon(\epsilon)_\eta$ is the union of 2^{d-s} pieces. A typical piece is indexed by some $(\xi_k)_{k\in\tau} \in (\{0,1\})^\tau$, and parametrized by $\prod_{k\in\eta}[0,2q_k\pi] \times U_\eta \subset \prod_{k\in\eta}[0,2q_k\pi] \times \mathbb{R}^\tau_{>0}$, in the following way:

$$\left((\theta_j)_{j\in\eta}; (r_k)_{k\in\tau}\right) \longrightarrow \left((\rho e^{\sqrt{-1}\,\theta_j})_{j\in\eta}; (r_k e^{2\sqrt{-1}\,\pi\xi_k q_k})_{k\in\tau}\right). \tag{5.7}$$

(5) We choose the coherent system of orientations inspired by the product of cycles γ_k, with the circles positively oriented: we orient $\prod_{k\in\eta}[0,2q_k\pi] \times \mathbb{R}^\tau_{>0}$, by its canonical orientation multiplied by the signature of the permutation (η,τ) of $\{1,\dots,d\}$, and by $(-1)^{d-\sum\xi_k}$.

In fact one can easily check that there is a radial isotopy from $\gamma_1 \times \cdots \times \gamma_d$ to $\Upsilon(\epsilon)$, which yields an oriented stratified isomorphism. Indeed, for $r := (r_1,\dots,r_d) \in (\mathbb{R}^d_{>0})$, with $r_k \geq \epsilon$ for all k, define $r_0 = \min\{r_k\}$. On the half-line $\mathbb{R}_{>0}r$ there is a unique point $r' = (r'_1,\dots,r'_d)$ with $\min\{r'_k\} = \epsilon$, and a unique point $\rho := (\rho_1,\dots,\rho_d)$, such that $\rho^a = \epsilon^{|a|}$ (see Figure 3 for $d = 2$). Let us consider $\rho_0 = \min\{\rho_k\}$ and the linear multiplication on $\mathbb{R}_{>0}r$ by the ratio $\rho_0/\epsilon = r_0 r^{-a/|a|}$, which depends continuously on r. Then the map $\log r + \sqrt{-1}\theta \mapsto \log((\rho_0/\epsilon)r) + \sqrt{-1}\,\theta$ from $\gamma_1 \times \cdots \times \gamma_d$ to $\Upsilon(\epsilon)$ is the mentioned radial isotopy.

Proposition 5.6. The twisted cycle $\Upsilon(\epsilon)$ is a rapid decay cycle. In particular, the integrals $H_{\Upsilon(\epsilon)}(\beta; y)$ are convergent.

Remark 5.7. Again we think of $\Upsilon(\epsilon)$ as well as a cycle on $(\widetilde{\mathbb{C}^*})^d$, or as a twisted cycle on either $(\mathbb{C}^*)^d_u$ or $(\mathbb{C}^*)^d_v$. It can be written as a sum $\Sigma\Delta_u \otimes \varsigma_u$ or $\Sigma\Delta_v \otimes \varsigma_v$, with terms in one-to-one correspondence by $\Delta_u = \varpi\Delta_v$, and the branches ς_u and ς_v of u^β are compatible with the maps $(\widetilde{\mathbb{C}^*})^d \longrightarrow (\mathbb{C}^*)^d_v \longrightarrow (\mathbb{C}^*)^d_u$ and yields the change of variables

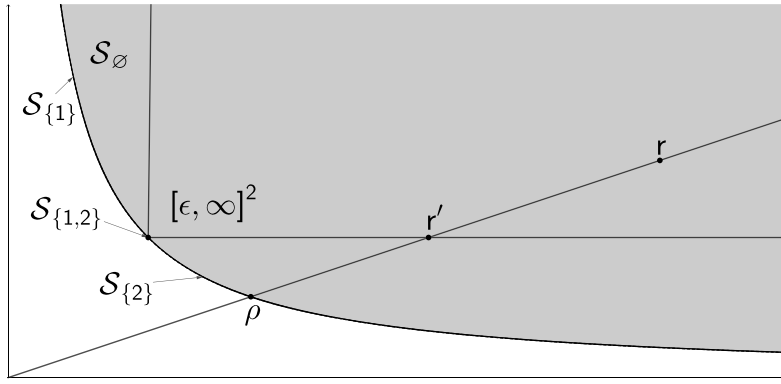

Fig. 3.

formula deduced from (5.2):

$$\int_{\Delta_u} u^{-\beta} e^{M(u,y)} \frac{\mathrm{d}u}{u} = \int_{\Delta_v} v^{-\langle q,\beta\rangle} e^{M(v^q,y)} \prod_{k=1}^{d} q_k \frac{\mathrm{d}v}{v}.$$

Since the exponent $M(v^q, y)$ is univalent our Corollary 5.2 can be applied to $\Upsilon(\epsilon)$ seen as a twisted cycle on $(\mathbb{C}^*)^d_v$ endowed with the pullback $\mathbb{C}\cdot v^{-\langle q,\beta\rangle}$ of the local system $\mathbb{C}\cdot u^{-\beta}$. However, all our calculations can be done with the variable u. Indeed both variables u and v are equivalent for the control of rapid decay at infinity, since $\|u\|_1 := \sum |u_k| = \sum |v_k|^{q_k}$.

Proof of Proposition 5.6. Let us consider a stratum with $\eta \neq \varnothing$. The last monomial of the argument of the exponential in $F(\beta; y)$ satisfies

$$|y_n u^{a(n)}| = |y_n| \epsilon^{|a(n)|}. \tag{5.8}$$

For $\eta = \{1, \dots, d\}$ the fiber over $\mathcal{S}_\eta$ is a compact subset of $(\widetilde{\mathbb{C}^*})^d$ and the integrand of $H_{\Upsilon(\epsilon)}(\beta; y)$ is holomorphic over it, so there is nothing to prove. Let us assume for simplicity that $\eta = \{1, \dots, s\}$ with $1 \leq s < d$. On the stratum $\mathcal{S}_\eta$ we have $r_1 = \cdots = r_s = \rho < \epsilon$. Thus, we imitate the proof of Lemma 3.2 (recall that $\tau_A = \sigma \cup \{n\}$ in our case) to get an upper bound for the real part of $M(u, y) = -u_1 - \cdots - u_d + \sum_{j=d+1}^{n} y_j u^{a(j)}$. Recall that by Assumption 4.4, $a(j) = \sum_{k=1}^{d} \nu_{jk} e(k)$ for $j = d+1, \dots, n-1$ where $\nu_{jk} \geq 0$ and

$|a(j)| = \sum_{k=1}^{d} \nu_{jk} < 1$. Hence,

$$\begin{aligned} \Re\left(\sum_{j=d+1}^{n-1} y_j r^{a(j)}\right) = \Re\left(\sum_{j=d+1}^{n-1} y_j \prod_{k=1}^{d} r^{\nu_{jk} e(k)}\right) &\leq \sum_{j=d+1}^{n-1} |y_j| (\max_{k=1}^{d} r^{e(k)})^{\sum_{k=1}^{d} \nu_{jk}} \\ &\leq K \max\left(1, \left(\sum_{k=1}^{d} r^{e(k)}\right)^{\kappa}\right) \leq K \max(1, (\epsilon s + r_{s+1} + \cdots + r_d)^{\kappa}), \end{aligned} \tag{5.9}$$

where $K = \sum_{j=d+1}^{n-1} |y_j|$ and $\kappa = \max_{j=d+1}^{n-1} \sum_{k=1}^{d} \nu_{jk}$. Finally, using that $\Re(-u_j) \leq \rho < \epsilon$ for $j = 1, \ldots, ,\ \Re(-u_j) = -r_j$ for $j = s+1, \ldots, d$ and (5.8), we obtain

$$\Re M(u, y) \leq \epsilon s - r_{s+1} - \cdots - r_d + |y_n| \epsilon^{|a|} + K \max(1, (\epsilon s + r_{s+1} + \cdots + r_d)^{\kappa}). \tag{5.10}$$

Since $r_{s+1} + \cdots + r_d \leq \|u\|_1 = \rho s + r_{s+1} + \cdots + r_d \leq \epsilon s + r_{s+1} + \cdots + r_d$ for $u \in \Upsilon(\epsilon)_\eta$, we still get an inequality of type (3.10). There are constants $C_\eta, c_\eta > 0$ (depending also on y but independent of ϵ) such that $\Re M(y, u) \leq C_\eta - c_\eta \|u\|_1$ for all $u \in \Upsilon(\epsilon)_\eta$.

A closer look at the argument that proves (5.10) shows that we can write the following upper bound for $|\Im M(u, y)|$:

$$|\Im M(u, y)| \leq d\epsilon + |y_n| \epsilon^{|a(n)|} + K(d\epsilon + \|u\|_1)^{\kappa}.$$

This upper bound, the relation (5.10) in the form of inequality $\Re M(y, u) \leq C_\eta - c_\eta \|u\|_1$ and the fact that $0 < \kappa < 1$ prove that $\Im M(u, y)/\Re M(u, y)$ tends to zero as $\|u\|_1$ tends to infinity. In particular the argument of $M(u, y)$ tends to π along $\Upsilon(\epsilon)_\eta$.

We need a similar result when $\eta = \varnothing$. We notice that $M(u, y)$ is univalent on the 2^d branches of $\Upsilon(\epsilon)_\varnothing$. Following the proof of inequality (5.10) we obtain the inequality:

$$\Re M(u, y) \leq -r_1 - \cdots - r_d - |\Re y_n| r^{a(n)} + K \max(1, (r_1 + \cdots + r_d)^{\kappa}).$$

Since for the imaginary part we have the inequality $|\Im M(u, y)| \leq |\Im y_n| r^{a(n)} + K(\|u\|_1)^{\kappa}$, we deduce that if we set $\alpha_n = |\pi - \arg y_n| \in [0, \pi/2[$, we have for any $\delta > 0$ and any $\|u\|_1$ large enough $\arg M(t, y) \in]\pi - \alpha_n - \delta, \pi + \alpha_n + \delta]$.

Let us use a good compactification X of $(\mathbb{C}^*)^d$, a real blow-up $\pi : \widetilde{X} \longrightarrow X$ of X along D, and apply Corollary 5.2. The behaviour of $\arg M(t, y)$ when $\|u\|_1 \to +\infty$ and the fact that for any $R >$, $\Upsilon(\epsilon) \cap \{u \mid \|u\|_1 \leq R\}$ is compact imply that $\Upsilon(\epsilon)$ is a rapid decay cycle. □ ■

Let us prove that when $\Re\beta < 0$, the integral $H_{\Upsilon(\epsilon)}(\beta; y)$ tends, when $\epsilon \to 0$, to the integral (5.4) multiplied by the factor

$$T := \sum_{\xi\in\{0,1\}^d} (-1)^{d-|\xi|} \exp\left(2\sqrt{-1}\,\pi \sum \beta_k q_k \xi_k\right) = \prod_{k=1}^{d} (\exp(2\sqrt{-1}\,\pi q_k \beta_k) - 1) \tag{5.11}$$

that comes from the parametrization (5.7) for $\eta = \varnothing$. Since (5.4) is clearly the limit of the piece of the integral $H_{\Upsilon(\epsilon)}(\beta; y)$ over $\Upsilon(\epsilon)_{\varnothing}$, it suffices to show that the integrals over $\mathcal{S}_\eta$ for $\eta \neq \varnothing$ tend to zero.

Let us assume again for simplicity that $\eta = \{1, \dots, s\}$ with $1 \le s \le d$. On each piece of $\Upsilon(\epsilon)_\eta$ the parameters are

$$(\theta_1, \dots, \theta_s, r_{s+1}, \dots, r_d) \in \prod_{k\in\eta} [0, 2q_k\pi] \times U_\eta.$$

and the change of variables from the parametrization (5.7) induces the following results in the different factors of the integrand:

$$\bigwedge_{k=1}^{d} \frac{du_k}{u_k} = (\sqrt{-1}\,d\theta_1) \wedge \cdots \wedge (\sqrt{-1}\,d\theta_s) \wedge \frac{dr_{s+1}}{r_{s+1}} \wedge \cdots \wedge \frac{dr_d}{r_d},$$

$$u^{-\beta} = \rho^{-\beta_1-\cdots-\beta_s} r_{s+1}^{-\beta_{s+1}} \cdots r_d^{-\beta_d} \exp\left(\sqrt{-1}\,\left(-\sum_{j=1}^{s} \beta_j\theta_j - \sum_{k=s+1}^{d} 2\pi\beta_k q_k \xi_k\right)\right),$$

$$\begin{aligned} |u^{-\beta}| &= \rho^{-\Re(\beta_1+\cdots+\beta_s)} \prod_{\ell=s+1}^{d} r_\ell^{-\Re\beta_\ell} \exp\left(\sum_{j=1}^{s} \Im\beta_j\theta_j + \sum_{k=s+1}^{d} 2\pi\,\Im\beta_k q_k \xi_k\right) \\ &\le \epsilon^{-\Re(\beta_1+\cdots+\beta_s)} \prod_{\ell=s+1}^{d} r_\ell^{-\Re\beta_\ell} \exp\left(\sum_{k=1}^{d} 2\pi\,|\Im\beta_k| q_k\right). \end{aligned}$$

From these inequalities and the fact that the real part of the argument of the exponential function is bounded from above by

$$C_\eta - c_\eta(r_{s+1} + \cdots + r_d)$$

with $C_\eta, c_\eta \in \mathbb{R}_{>0}$ independent of ϵ, for $\epsilon \in [0, \epsilon_0]$, we see that the integral over $\Upsilon(\epsilon)_\eta$ tends to zero when $\epsilon \to 0$ as expected, because $-\Re(\beta_1 + \cdots + \beta_s) > 0$.

Finally, let us prove that the integral $H_{\Upsilon(\epsilon)}$ does not depend on ϵ: take $0 < \epsilon_1 < \epsilon_2$. We consider $\Upsilon([\epsilon_1, \epsilon_2])$, the noncompact $(d+1)$-cycle

$$\bigcup_{\epsilon \in [\epsilon_1, \epsilon_2]} \{\epsilon\} \times \Upsilon(\epsilon)$$

with oriented boundary $\{\epsilon_1\} \times \Upsilon(\epsilon_1) - \{\epsilon_2\} \times \Upsilon(\epsilon_2)$. Consider then for $R > \epsilon_2$ the compact cycle $\Upsilon_R = \Upsilon([\epsilon_1, \epsilon_2]) \cap ([\epsilon_1, \epsilon_2] \times P_R)$, where P_R is the polydisk

$$P_R = \{u \in \mathbb{C}^d \mid |u_1| \leq R, \ldots, |u_d| \leq R\}.$$

Integrals $H_{\Upsilon(\epsilon)}$ are of the form $H_{\Upsilon(\epsilon)} = \int_{\Upsilon(\epsilon)} \omega$, where ω is a holomorphic d–form independent of ϵ and hence it is a closed form. We have

$$0 = \int_{\Upsilon_R} \mathrm{d}\omega = \int_{\partial \Upsilon_R} \omega.$$

The boundary $\partial \Upsilon_R$ is equal to

$$(\{\epsilon_1\} \times \Upsilon(\epsilon_1)) \cap ([\epsilon_1, \epsilon_2] \times P_R) - (\{\epsilon_2\} \times \Upsilon(\epsilon_2)) \cap ([\epsilon_1, \epsilon_2] \times P_R) + \partial_R.$$

Since by examining the parametrization (5.7) we see that each d-dimensional piece of ∂_R is included in an hyperplane $u_j = R$, hence, the restriction to it of ω is zero. We deduce that the integral of ω on $\Upsilon(\epsilon_j) \cap P_R$ (which can replace $(\{\epsilon_j\} \times \Upsilon(\epsilon_j)) \cap ([\epsilon_1, \epsilon_2] \times P_R)$ because ω does not depend on ϵ) for $j = 1, 2$ are equal. Taking the limit when $R \longrightarrow \infty$ we obtain the result

$$H_{\Upsilon(\epsilon_1)} = H_{\Upsilon(\epsilon_2)}.$$

In the case of general p, δ, we keep the same cycle and work with the integral

$$H_{\Upsilon(\epsilon)}(\beta; y) := \int_{\Upsilon(\epsilon)} u^{-\beta-\mathbf{1}} \exp\left(-\sum_{k=1}^{d} e^{\sqrt{-1}\,\pi\delta_k} u_k + \sum_{j=d+1}^{n} z_j u^{a(j)} \right) \mathrm{d}u,$$

where $z_j = e^{\sqrt{-1}\,\pi\langle \mathbf{1}+2p+\delta, a(j)\rangle} y_j$ and the proof is essentially the same with only an easy modification of inequality (5.10).

In particular, the cycle $\widetilde{D}_{p,\delta}$ in the statement of Theorem 5.3, is the image of $\Upsilon(\epsilon)$ by $t_k = u_k \cdot \exp(\sqrt{-1}\,\pi\,\langle \mathbf{1} + 2p + \delta,\, a(k)\rangle)$.

Conclusion: The integral $H_{\Upsilon(\epsilon)}(\beta; y)$ is analytic as a function of $\beta \in \mathbb{C}^d$. Moreover, when $\Re\beta < 0$, $e^{-\sqrt{-1}\,\pi\langle \mathbf{1}+2p+\delta, \beta\rangle} H_{\Upsilon(\epsilon)}(\beta; y) = T \cdot F_{p,\delta}(\beta; y)$, see (5.11). Hence, it equals the

meromorphic continuation $T \cdot \widetilde{F}_{p,\delta}(\beta; y)$ outside the union of hyperplanes $\mathcal{P}$ described in Lemma 4.9.

When $q_k\beta_k \notin \mathbb{Z}$ for all $k \in \{1, \dots, d\}$, the factor T is non zero and we obtain a Gevrey series expansion for the integral along rapid decay cycles $H_{\gamma(\epsilon)}(\beta; y)$. To check this last claim we have to remark that the set of poles of the analytic continuation $\widetilde{F}_{p,\delta}(\beta; y)$ is contained in $\mathcal{P}$ which is contained in the set defined by $T = \prod_{k=1}^{d}(\exp(2\sqrt{-1}\,\pi q_k\beta_k) - 1) = 0$. This latter set is, under Assumption 4.1, the set of parameters $\beta = B_\sigma^{-1}\gamma$ such that γ is called *resonant* for B (see [12, 2.9]).

Coming back to the general situation of Theorem 4.3, the result of this theorem and the above considerations prove the following theorem:

Theorem 5.8. If Assumption 4.1 is satisfied and $\gamma \in \mathbb{C}^d$ is non resonant for B, then all the Gevrey solutions of $M_B(\gamma)$ along the hyperplane $x_n = 0$ can be described as linear combinations of a fixed set of asymptotic expansions of integral solutions of type $I_C(\gamma; x)$ along rapid decay cycles.

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