Note

On partitions with k corners not containing the staircase with one more corner

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ABSTRACT

We give three proofs of the following result conjectured by Carriegos, De Castro-García and Muñoz Castañeda in their work on enumeration of control systems: when $\binom{k+1}{2} \leq n < \binom{k+2}{2}$, there are as many partitions of n with k corners as pairs of partitions (α, β) such that $\binom{k+1}{2} + |\alpha| + |\beta| = n$.

1. Introduction

Integer partitions (finite weakly decreasing sequences of positive integers) are fundamental objects in enumerative combinatorics. The terms of a partition are called its *parts*. Several parameters are attached to a partition, such as: its length, its weight (the sum of its parts), its largest part, the size of its Durfee square (i.e. the maximum k such that the partition has k parts $\geq k$) …Another parameter is the number of distinct sizes of its parts. It is also the number of corners in the diagram of the partition. For instance, the partition $(7, 4, 4, 2, 2, 2, 1)$ has parts of 4 different sizes (7,4,2, and 1). Accordingly, its diagram has 4 corners, as shown in Fig. 1(a). We will call for short *partitions with k corners* the partitions with parts of k different sizes.

Recently, the problem of counting partitions with k corners has arisen in the context of the enumeration of linear control systems with coefficients in a commutative rings, in papers by Carriegos, De Castro-García and Muñoz Castañeda [4,5]. The present note is devoted to proving the following result that they conjectured.

Theorem 1 ([4, Conjecture 30]). *When $\binom{k+1}{2} \leq n < \binom{k+2}{2}$, there are as many partitions of n with k corners as pairs of partitions (α, β) such that $|\alpha| + |\beta| = n - \binom{k+1}{2}$.*

Note that $\binom{k+1}{2}$ is the size of the *staircase partition of length k*, which is $(k, k-1, k-2, \ldots, 1)$; see Fig. 2.

We actually give three proofs of Theorem 1. The first one (Section 3) is based on generating series. The second one (Section 4) is based on an identity due to Fine on some statistics on partitions. The last one (Section 5) is a bijective proof of the following more general result.

Theorem 2. *For any $k \geq 0$ and $m \geq 0$, there are as many pairs of partitions (α, β) with $|\alpha| + |\beta| = m$ whose lengths fulfill $\ell(\alpha) + \ell(\beta) \leq k$, as partitions of $m + \binom{k+1}{2}$ with k corners whose diagrams do not contain the diagram of the staircase partition with length $k + 1$.*

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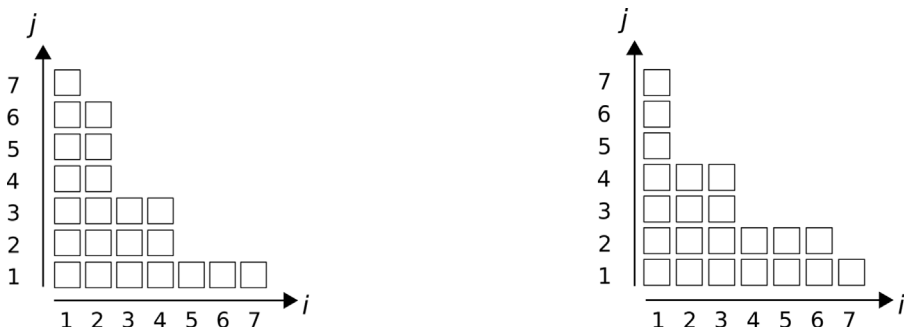
(a) The diagram of the partition $\lambda = (7, 4, 4, 2, 2, 2, 1)$. The lengths of its rows are the parts 7, 4, 4, 2, 2, 2, 1 of λ. This diagram has 4 corners, whose positions (i, j) are $(1, 7)$, $(2, 6)$, $(4, 3)$ and $(7, 1)$.

(b) The diagram of the partition $\lambda' = (7, 6, 3, 3, 1, 1, 1)$. The parts of the λ are the lengths of the columns in the diagram of λ'.

Fig. 1. The diagrams of the partitions $\lambda = (7, 4, 4, 2, 2, 2, 1)$ and $\lambda' = (7, 6, 3, 3, 1, 1, 1)$, which are conjugate of each other. Each point (i, j) of a diagram is represented as a square box centered at (i, j).
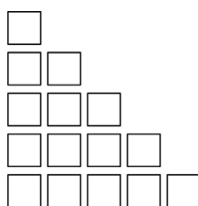

Fig. 2. The diagram of $(5, 4, 3, 2, 1)$, which is the staircase partition of length 5.

Proof of Theorem 1 from Theorem 2. Consider $n < \binom{k+2}{2}$ and set $m = n - \binom{k+1}{2}$. Then $m \leq k$. Observe now that all pairs of partitions (α, β) with $|\alpha| + |\beta| = m$ fulfill $\ell(\alpha) + \ell(\beta) \leq |\alpha| + |\beta| = m \leq k$. Also, the diagram of a partition of n cannot contain the diagram of the staircase partition of length $k + 1$, since n is smaller than $\binom{k+2}{2} = 1 + 2 + \cdots + (k + 1)$. Theorem 1 follows. $\square$

2. Basic facts and notations

2.1. Partitions and their diagrams

In this section, we recall classical operations and notations for integer partitions. See [11], [9, I.1], [12, p. 58] or [2, Ch. I] for further details, basic definitions and general presentations.

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, its *diagram* is the set of integer points (i, j) such that $1 \leq j \leq \ell(\lambda)$ and $1 \leq i \leq \lambda_j$. In graphical representations, the points of the diagram of λ are often drawn as square boxes centered at that point. The *conjugate* of λ is the partition, denoted λ', whose diagram is obtained from the diagram of λ by applying the reflection that swaps the coordinates. The *parts* of λ are the nonzero terms $\lambda_1, \lambda_2 \ldots$ and the *length* of λ, denoted with $\ell(\lambda)$, is the number of nonzero terms (non-necessarily distinct). By $\lambda \vdash n$ we mean that λ is a partition of n. We call n the *weight* of λ, and denote it with $|\lambda|$.

For instance, the partition $\lambda = (7, 4, 4, 2, 2, 2, 1)$ is a partition of $7 + 4 + 4 + 2 + 2 + 2 + 1 = 22$ (so $|\lambda| = 22$ and $\lambda \vdash 22$), with length 7 (denoted $\ell(\lambda) = 7$) and parts 7, 4, 4, 2, 2, 2 and 1. Its conjugate partition is $\lambda' = (7, 6, 3, 3, 1, 1, 1)$. The diagrams of λ and λ' are shown in Fig. 1.

In sequences and partitions, we will make use of the notation p^m for "m occurrences of p". For instance, the partition $(7, 4, 4, 2, 2, 2, 1)$ will be also denoted $(7, 4^2, 2^3, 1)$.

Weakly decreasing sequences of non-negative integers with trailing zeros will be identified with the partition obtained by deleting the trailing zeros. For instance, $(7, 4, 4, 2, 2, 2, 1, 0, 0, 0)$ will be identified to the partition $(7, 4, 4, 2, 2, 2, 1)$.

Consider two partitions $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$. Their *sum* is the partition $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots)$. For each i, let a_i (resp. b_i) be the multiplicity of i in α (resp. in β). The *union* of the partitions α and β is the partition (denoted $\alpha \cup \beta$) in which the multiplicity of i is $a_i + b_i$. For instance, if $\alpha = (7, 4, 4, 2, 2, 2, 1)$ and $\beta = (4, 2, 1)$ then $\alpha + \beta = (7 + 4, 4 + 2, 4 + 1, 2 + 0, 2 + 0, 2 + 0, 1 + 0) = (11, 6, 5, 2, 2, 2, 1)$ and $\alpha \cup \beta = (7, 4, 4, 4, 2, 2, 2, 1, 1)$. The two operations are related by the identity $(\alpha + \beta)' = \alpha' \cup \beta'$.

2.2. Generating series

Given any nonnegative integer k, the product

$$\prod_{i=1}^{k} \frac{1}{1-q^i}$$

expands as

$$\sum_{a_1, a_2, \ldots, a_k} q^{1a_1 + 2a_2 + \cdots + ka_k},$$

where the sum is carried over all k–tuples $(a_1, a_2, \ldots, a_k)$ of nonnegative integers. Interpreting each integer a_i as the multiplicity of i as a part of a partition with weight $1a_1 + 2a_2 + \cdots + ka_k$, we see that the series also writes $\sum_\lambda q^{|\lambda|}$, where the sum is over all partitions λ whose parts are all $\leq k$. It is thus the generating series for these partitions, according to their weight (i.e. for each n, the coefficient of q^n in this series is the number of partitions of n whose parts are all $\leq k$).

Similarly, the infinite product

$$\prod_{i=1}^{\infty} \frac{1}{1-q^i}$$

is the generating series of all partitions, according to their weight,.

2.3. Corners of partitions

A *corner* of (the diagram of) λ is a point (i, j) in the diagram of λ, such that neither $(i + 1, j)$ nor $(i, j + 1)$ is in the diagram of λ. The partitions with parts of k distinct sizes are exactly the partitions whose diagram has k corners.

We will denote with $\nu(n; k)$ the number of partitions of n with k corners.

Let $\rho_k = (k, k - 1, \ldots, 1)$. This partition is called the *staircase partition of length k*. Fig. 2 shows the diagram of the staircase partition of length 5. The staircase partition of length k is the smallest partition with k corners, in a sense made precise by the following lemma, that we will use implicitly in the sequel.

Lemma 3. *For any $k \geq 0$, the diagram of any partition with k corners contains the diagram of ρ_k.*

Proof. Let $k \geq 0$ and let λ be a partition with k corners. Then λ is of the form $(p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k})$ with $p_1 > p_2 > \cdots > p_k > 0$ and all $m_i > 0$.

For each i, set $\mu_i = p_i - (k - i + 1)$. Therefore, for all $i < k$, we have $\mu_i - \mu_{i+1} = p_i - p_{i+1} - 1 \geq 0$. We have also $\mu_k = p_k - 1 \geq 0$. This shows that $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ is a partition. We have:

$$\lambda = (\rho_k + \mu) \cup (p_1^{m_1-1} p_2^{m_2-1} \cdots p_k^{m_k-1}).$$

This shows that the diagram of λ contains the diagram of ρ_k. $\quad\square$

We will also make repeated use of the following converse of Lemma 3.

Lemma 4. *Let $k \geq 0$, and let λ be a partition. If the diagram of λ does not contain the diagram of ρ_{k+1}, then λ has at most k corners. This is the case in particular for any partition λ of weight less than $\binom{k+2}{2}$.*

2.4. References on partitions with k corners

It has been remarked in [1] that the number of corners is a statistic on partitions that has been seldom considered. A notable and early exception is the work of MacMahon [10] relating the enumeration of partitions according to their number of corners with number theory. MacMahon's work was expanded by Andrews [3] with a focus on asymptotics. MacMahon provided a formula for the generating series of partitions according to their weight and number of corners (identity (1) below in Section 3, where we make use of it). Refinements for this generating series were studied by Alladi [1]. Let us mention also that part of MacMahon's results on this topic have been rediscovered by some authors unaware of his work (for instance [8]).

The sequence counting the partitions of n with k corners is number A116608 in The On-Line Encyclopedia of Integer Sequences [6].

Note finally that another possible name for the partitions with k corners could have been "partitions with k distinct parts", but this name is already taken and widely used for the partitions with k distinct parts *all of multiplicity* 1, famous for having the same generating function as the partitions in odd parts.

3. Proof with generating series

In this section, we prove Theorem 1 using the generating series of the numbers $v(n; k)$, of partitions of n with k corners, which is defined as

$$F(x, q) = \sum_{k,n} v(n; k)x^k q^n.$$

Our starting point will be the following expression for F, due to MacMahon.

Theorem 5 ([10, p. 90]; or see (3.2) in Andrews' account [3]). *The generating series F is given by*

$$F(x, q) = \sum_{j=0}^{\infty} \frac{(x-1)^j q^{\binom{j+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^j)} \cdot \prod_{i=1}^{\infty} \frac{1}{1-q^i}. \tag{1}$$

Proof of Theorem 1 from Theorem 5. Consider n and k as in Theorem 1. The number $v(n; k)$ is the coefficient of $x^n q^k$ in $F(x, q)$.

In the right-hand side of (1), the summands with indices $j < k$ have degree in x at most $k - 1$, and thus do not contribute to the coefficient of $x^k q^n$. The expansions of the summands with indices $j > k$ involve only monomials $x^m q^d$ with $d \geq \binom{k+2}{2} > n$, and thus these summands do not contribute to the coefficient of $x^k q^n$.

Therefore, $v(n; k)$ is the coefficient of $x^k q^n$ in the summand with index $j = k$, which is

$$\frac{(x-1)^k q^{\binom{k+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^k)} \cdot \prod_{i=1}^{\infty} \frac{1}{1-q^i}.$$

After expanding $(x-1)^k$ as $x^k +$ terms of smaller degree in x, we get that $v(n; k)$ is simply the coefficient of q^n in

$$\frac{q^{\binom{k+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^k)} \cdot \prod_{i=1}^{\infty} \frac{1}{1-q^i}$$

which is the coefficient of $q^{n-\binom{k+1}{2}}$ in

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^k)} \cdot \prod_{i=1}^{\infty} \frac{1}{1-q^i}. \tag{2}$$

The left factor in (2) is the generating series for the partitions λ with parts $\leq k$; the right factor is the generating series for all partitions μ (see Section 2.2). Therefore, for any h, the coefficient of q^h in (2) is the number of pairs of partitions (λ, μ) such that $|\lambda| + |\mu| = h$, and λ has all parts $\leq k$. For $h \leq k$, the condition on the sizes of the parts of λ can be dropped. This is the case in particular for $h = n - \binom{k+1}{2}$, since $n < \binom{k+2}{2} = \binom{k+1}{2} + k + 1$. $\square$

4. Proof from statistics on partitions

We now give another proof of Theorem 1, based on the following result due to Fine.

Theorem 6 ([7, Theorem 4 in Chapter 2]). *For any $r \geq 0$,*

$$\sum_{\lambda \vdash n} \binom{Q(\lambda)}{r} = \sum_{\lambda \vdash n} m_1(\lambda)m_2(\lambda)\cdots m_r(\lambda), \tag{3}$$

where $Q(\lambda)$ is the number of corners of λ, and $m_i(\lambda)$ stands for the multiplicity of i as a part of λ.

Proof of Theorem 1 from Theorem 6. Consider n and k such that

$$\binom{k+1}{2} \leq n < \binom{k+2}{2}.$$

After Lemma 4, the diagram of any partition of n has at most k corners. Apply Theorem 6 with $r = k$. For any $\lambda \vdash n$, either λ has less than k corners, and then $\binom{Q(\lambda)}{k} = 0$, or λ has exactly k corners, and then $\binom{Q(\lambda)}{k} = 1$. The left-hand side in (3) is thus $v(n; k)$.

The right-hand side is

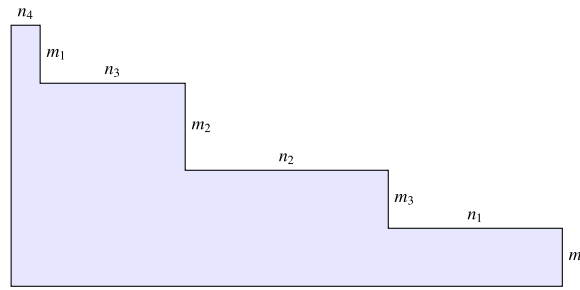$$\sum_{\lambda \vdash n} m_1(\lambda)m_2(\lambda)\cdots m_k(\lambda). \tag{4}$$

Fig. 3. The diagram of a partition with 4 corners, with its border coordinates $(m_1, m_2, m_3, m_4; n_1, n_2, n_3, n_4)$..

Note that if a partition $\lambda \vdash n$ has some part $j > k$, then at least one of its parts $i \leq k$ must have multiplicity 0 in λ (otherwise $n \leq \binom{k+2}{2}$ cannot be fulfilled). Such a partition does not contribute to the sum (4).

Consider the map $(\alpha, \beta) \mapsto \alpha \cup \beta \cup \rho_k$ from the pairs of partitions (α, β) with parts $\leq k$. If $\alpha = (k^{a_k} \cdots 2^{a_2} 1^{a_1})$ and $\beta = (k^{b_k} \cdots 2^{b_2} 1^{b_1})$, the image of (α, β) is the partition $(k^{a_k+b_k+1} \cdots 2^{a_2+b_2+1} 1^{a_1+b_1+1})$. This shows that the number of preimages of a partition λ, under this map, is $m_1(\lambda) m_2(\lambda) \cdots m_k(\lambda)$. Indeed, each pair mapped to λ is determined by the choice of the multiplicities a_i, that must fulfill $0 \leq a_i \leq m_i(\lambda) - 1$ for all i, whence there are $m_i(\lambda)$ choices for a_i. The sum (4) now interprets as the cardinality of the inverse image of the set of all partitions $\lambda \leq n$. This inverse image is the set of all pairs of partitions (α, β) with parts $\leq k$ such that $|\alpha| + |\beta| + |\rho_k| = n$. This last condition simplifies as $|\alpha| + |\beta| = n - \binom{k+1}{2}$. Finally, since $n < \binom{k+2}{2}$, we have that $n - \binom{k+1}{2} < k + 1$. Therefore, the condition that the parts of α and β are all at most k can be dropped, as it is already implied by the condition on their weights. The statement of Theorem 1 is obtained. $\square$

5. Bijective proof

It has been shown in the introduction that Theorem 1 follows from the more general Theorem 2. In this section, a bijective proof of Theorem 2 is provided.

We will use the *border coordinates* for partitions, that we introduce now.

Let λ be a partition with k corners. Let $p_1, p_2, \ldots, p_k$ be the distinct parts of λ, ordered increasingly; $p_1 < p_2 < \cdots < p_k$. Let $q_1 < q_2 < \cdots < q_k$ be the distinct parts of the conjugate partition λ'. For each i, let m_i (resp. n_i) be the multiplicity of p_i (resp. q_i) in λ (resp. λ'). We call the pair of sequences $(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_k)$ the *border coordinates* of λ, since they are the lengths of the vertical and horizontal segments in the border of the diagram of λ (see Fig. 3).

The border coordinates n_i are directly obtained as differences of consecutive parts. Precisely,

$$n_k = p_1, \quad \text{and for all } i > 1, \ n_{k+1-i} = p_i - p_{i-1}. \tag{5}$$

Indeed, $p_1, p_2, \ldots, p_k$ are the first coordinates of the corners listed from left to right, while $n_{k-1}, n_{k-2}, \ldots, n_1$ are the differences between the first coordinates of consecutive corners. The same relation holds between the m_i and the q_i.

Lemma 7. *Let λ be a partition with k corners and border coordinates*

$$(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_k).$$

Let $p_1 < p_2 < \cdots < p_k$ be the parts of λ, and let $q_1 < q_2 < \cdots < q_k$ be the parts of λ'.

1. *Let γ be a partition whose parts are all among the parts of λ. For each i, let c_i be the multiplicity of p_i in γ. Then the border coordinates of $\lambda \cup \gamma$ are*

$$(m_1 + c_1, m_2 + c_2, \ldots, m_k + c_k; n_1, n_2, \ldots, n_k).$$

2. *Let γ be a partition such that all parts of γ' are among the parts of λ'. For each i, let d_i be the multiplicity of q_i in γ'. Then the border coordinates of $\lambda + \gamma$ are*

$$(m_1, m_2, \ldots, m_k; n_1 + d_1, n_2 + d_2, \ldots, n_k + d_k).$$

Proof. We prove part 1. We have $\lambda = (p_k^{m_k} \cdots p_2^{m_2} p_1^{m_1})$ and $\gamma = (p_k^{c_k} \cdots p_2^{c_2} p_1^{c_1})$. Therefore

$$(\lambda \cup \gamma) = (p_k^{m_k+c_k} \cdots p_2^{m_2+c_2} p_1^{m_k+c_1}).$$

This gives the first half of the border coordinates of $\lambda \cup \gamma$. For the second half, observe that, after (5), the n_i do not depend on the multiplicities of the p_i, and thus remain unaffected by the union with γ.

Part 2 is straightforwardly deduced from part 1 by means of the identity $\lambda + \gamma = (\lambda' \cup \gamma')'$, and noting that conjugation swaps the m_i's and the n_i's in the border coordinates. $\square$
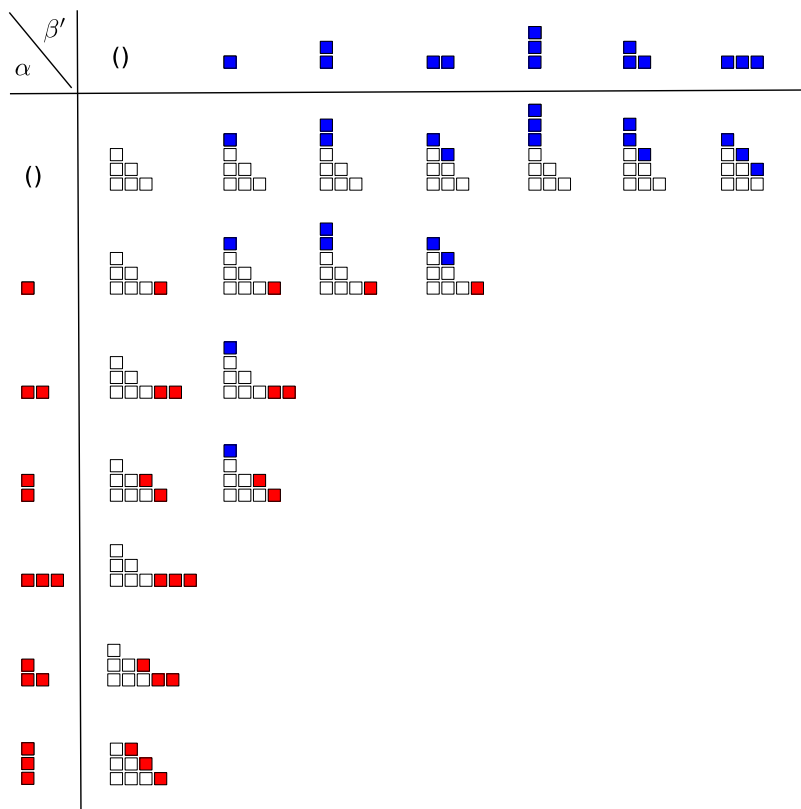
Fig. 4. The bijection $(\alpha, \beta) \mapsto (\rho_3 \cup \beta') + \alpha$ described in Lemma 8. Its puts in correspondence the pairs of partitions (α, β) with $|\alpha| + |\beta| \leq 3$, with the partitions λ with 3 corners and weight $< \binom{5}{2}$, whose diagrams do not contain the diagram of ρ_4. This illustrates Theorem 1 for $k = 3$. Rows correspond to the partitions α, and columns to the partitions β. The conjugate partition β' is drawn. The entry in row α and column β is the diagram of the partition $(\rho_3 \cup \beta') + \alpha$. The staircase partition ρ_3 is left in white, and the new points of the diagram coming from α and β' are in color.

Theorem 2 follows straightforwardly from the more precise lemma below.

Lemma 8. *The map* $(\alpha, \beta) \mapsto (\rho_k \cup \beta') + \alpha$ *establishes a bijection between the pairs of partitions* (α, β) *such that* $\ell(\alpha) + \ell(\beta) \leq k$, *and the partitions* λ *with k corners whose diagrams do not contain the diagram of* ρ_{k+1}.

The case $k = 3$ of the bijection defined in Lemma 8 is shown in Fig. 4.

Proof. Let α and β be two partitions whose lengths have sum at most k. There exist $p \geq \ell(\alpha)$ and $q \geq \ell(\beta)$ with $p+q = k$. Set b_i (resp. a_i) for the multiplicity of i in β' (resp. α').

Since β has length at most k, all parts of β' are smaller than or equal to k, and thus all parts of β' are among those of ρ_k. Lemma 7 applies: the border coordinates of $\rho_k \cup \beta'$ are $(1 + b_1, 1 + b_2, \ldots, 1 + b_k; 1^k)$.

The corners of the diagram of ρ_k are the pairs $(i, k + 1 - i)$ for i from 1 to k. Since β has length at most q, i.e. β' has all its parts smaller than or equal to q, performing the union with β' does not affect the columns of the diagram whose indices i fulfill $i > q$. Therefore, all pairs $(i, k + 1 - i)$ for $i > q$ are still corners of $\rho_k \cup \beta'$. As a consequence, the numbers $k + 1 - i$ for $i > q$, which are the numbers $j \leq p$, are still parts of $(\rho_k \cup \beta')'$. This shows that all parts of α' are among the parts of $(\rho_k \cup \beta')'$, allowing to apply again Lemma 7. The conclusion now is that the border coordinates of $(\rho_k \cup \beta') + \alpha$ are

$$(1 + b_1, 1 + b_2, \ldots, 1 + b_k; 1 + a_1, 1 + a_2, \ldots, 1 + a_k).$$

This shows clearly that the map Φ that sends each pair of partitions (α, β) whose lengths have sum at most k to $(\rho_k + \beta') + \alpha$ is injective. Indeed, the multiplicities a_i and b_i of α' and β' can be read from the border coordinates of $\Phi(\alpha, \beta)$.

Besides, since the union with β' affects only the first q columns of the diagram, and the sum with α affects only the first p rows, the point $(q + 1, p + 1)$, that is not in the diagram of ρ_k, is still not in the diagram of $(\rho_k \cup \beta') + \alpha$. Since $q + 1 + p + 1 = k + 2$, this point $(q + 1, p + 1)$ lies in the diagram of ρ_{k+1}. This shows that the diagram of $\Phi(\alpha, \beta)$ does not contain the diagram of ρ_{k+1}. On the other hand, $\Phi(\alpha, \beta)$ has k corners exactly (since it has $2k$ border coordinates). We

conclude that Φ takes its values in the set $\mathcal{S}_k$ of all partitions with k corners whose diagrams do not contain the diagram of ρ_{k+1}.

Let us now show that Φ has image $\mathcal{S}_k$ exactly. Let λ be a partition with k corners, whose diagram does not contain the diagram of ρ_{k+1}. There exists (i_0, j_0), lying in the diagram of ρ_{k+1}, but not in the diagram of λ. Since λ has k corners, its diagram contains the diagram of ρ_k. Therefore (i_0, j_0) is not in the diagram of ρ_k, and belongs to the set difference of the diagram of ρ_{k+1} and the diagram of ρ_k. Thus $i_0 + j_0 = k + 2$. Moreover, any point (i, j) of the diagram, and in particular any corner, must fulfill $i < i_0$ or $j < j_0$. Let p be the number of corners (i, j) with $i < i_0$, and let q be the number of corners (i, j) with $j < j_0$. Then $k \leq p + q$. We have $p \leq i_0 - 1$ since there is at most one corner in each column. Similarly, $q \leq j_0 - 1$, because there is at most one corner in each row. Altogether, we get $k \leq p + q \leq (i_0 - 1) + (j_0 - 1) = k$. As a consequence, $k = p + q$ and $p = i_0 - 1$, $q = j_0 - 1$. There is one corner in each of the first p columns and one corner in each of the first q rows, and no corner belongs at the same time to some of the first p columns and to some of the first q rows. We conclude that λ has border coordinates of the form $(m_1, \ldots, m_q, 1^p; n_1, \ldots, n_p, 1^q)$ for some positive numbers m_i and n_i. This coincides with the border coordinates of $(\rho_k \cup \beta') + \alpha$ for $\beta' = (q^{m_q - 1} \cdots 2^{m_2 - 1} 1^{m_1 - 1})$ and $\alpha' = (p^{n_p - 1} \cdots 2^{n_2 - 1} 1^{n_1 - 1})$. Clearly, partitions are determined by their border coordinates. Therefore, λ is equal to $(\rho_k \cup \beta') + \alpha$ for the partitions α' and β' defined above (and $\alpha = (\alpha')'$). This shows that all $\lambda \in \mathcal{S}_k$ are in the image of Φ, which terminates the proof. $\square$

Acknowledgments

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