BI-OBJECTIVE OPTIMAL CONTROL OF SOME PDES: NASH EQUILIBRIA AND QUASI-EQUILIBRIA*

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Abstract. This paper deals with the solution of some multi-objective optimal control problems for several PDEs: linear and semilinear elliptic equations and stationary Navier-Stokes systems. Specifically, we look for Nash equilibria associated with standard cost functionals. For linear and semilinear elliptic equations, we prove the existence of equilibria and we deduce related optimality systems. For stationary Navier-Stokes equations, we prove the existence of Nash quasi-equilibria, *i.e.* solutions to the optimality system. In all cases, we present some iterative algorithms and, in some of them, we establish convergence results. For the existence and characterization of Nash quasi-equilibria in the Navier-Stokes case, we use the formalism of Dubovitskii and Milyutin. In this context, we also present a finite element approximation and we illustrate the techniques with numerical experiments.

1. Introduction

We consider bi-objective optimal control problems for various PDEs and systems. First, an introductory problem corresponding to a linear elliptic PDE is analyzed with detail. Then, we deal with a similar semilinear elliptic PDE. Finally, we deal with the stationary Navier-Stokes system, that is, the equations satisfied by the time-independent velocity field and pressure of an incompressible viscous fluid flow.

Our aims are to prove existence, characterize efficiently the equilibria and, also, compute numerical solutions to these multi-objective control problems. They are very important from the mathematical viewpoint and appear frequently in the applications; for some previous works on the subject, see for instance [2].

In classical control theory, we usually find a state equation or system and one control, with the task of achieving a predetermined goal. Frequently (but not always), the goal is to minimize a cost functional within a prescribed family of admissible controls; see for instance [10, 16]. A different and interesting situation arises when several (in general, conflictive or contradictory) objectives are considered. This may happen, for example, if the cost function is the sum of several terms and it is not clear that an average provides a reasonable criterion. Also, it can be expectable to have more than one control acting on the equation. In these cases, we are led to consider multi-objective control problems. In contrast with the mono-objective case, various strategies for the

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choice of good or the best controls can appear, depending on many circumstances. Moreover, these strategies can be cooperative or noncooperative (depending on whether or not several controls mutually cooperate in order to achieve prescribed goals).

There exist several concepts of equilibrium for multi-objective problems, with origin in game theory. Each of them can be identified with a strategy. Thus, let us mention the non-cooperative optimization strategy proposed by Nash [18], the Pareto cooperative strategy [19] and the Stackelberg hierarchical-cooperative strategy [24]. In the context of the control of PDEs, up to date, there have been some works on the subject like the seminal papers by Lions [15, 17] and other more recent contributions, like [6, 21, 22].

In this paper, we will be concerned with Nash equilibria and quasi-equilibria associated with standard cost functionals. To be more precise, let us give some details in the case of a linear elliptic equation.

Thus, let the fluid domain be a bounded open set $\Omega \subset \mathbb{R}^N$, with $N \geq 1$. Let us introduce four nonempty open subsets, $\mathcal{O}_1$, $\mathcal{O}_2$, ω_1 and ω_2 and let us assume that a function u_{id} defined on $\mathcal{O}_i$ is given for $i = 1, 2$. We would like to find a couple $(f_1, f_2) \in L^2(\omega_1) \times L^2(\omega_2)$ (the control pair) with the following property: the associated state u, that is, the solution to the Dirirchlet problem

$$\begin{cases} -\nu \Delta u = f_1 1_{\omega_1} + f_2 1_{\omega_2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

is such that (f_1, f_2) is a Nash equilibrium for the functionals

$$J_i(f_1, f_2) := \frac{a}{2} \int_{\mathcal{O}_i} |u - u_{id}|^2 + \frac{\mu}{2} \int_{\omega_i} |f_i|^2, \quad i = 1, 2, \tag{1.2}$$

where $a, \mu > 0$ (see Sect. 2.1).

Our first main goal will be to find conditions under which at least one Nash equilibrium exists. The second one will be to characterize these equilibria in terms of first order optimality conditions, *i.e.* to derive a system that the optimal solution, together with some associated adjoint states must satisfy. The third one will be to show how Nash equilibria can be computed and present related convergent algorithms. These tasks will be successfully accomplished below. Then, we will deal with some more complex equations and systems.

In particular, in what concerns optimality conditions, we will try to extend to this context the techniques usually employed for distributed control problems (see for instance [1, 10, 16]).

At some point, we will apply the so called formalism of Dubovitskii and Milyutin. This approach was introduced in the context of mathematical programming and has been succesfully applied to the solution of many optimal control problems for ODEs since the 70's. A good presentation of its applications to these areas can be found in Girsanov [11]; see also Flett [9]. Later, these techniques have been applied successfully to some distributed control problems; see [3, 4]. In the present context this method is appropriate; note that this would not be so clear if state constraints were imposed.

The plan of this paper is the following.

In Section 2, we consider the relatively simple problem given by (1.1)–(1.2). We prove the existence of Nash equilibria, we provide an optimality system and we present some iterative algorithms.

In Section 3, a more complicated problem is analyzed: a semilinear elliptic PDE together with functionals of the same kind. Here, in view of the nonlinearity, we must work with Nash equilibria and Nash quasi-equilibria, that is solutions to the optimality system. Again, existence, characterization and computation-oriented results are established.

Section 4 deals with the stationary Navier-Stokes system. New difficulties are found: nonlinearity, lack of uniqueness, lack of regularity of the functionals, etc. We also discuss the existence and optimality characterization of Nash equilibria and quasi-equilibria. Additionally, we present some iterative algorithms and the results of several numerical experiments.

Finally, some additional comments and open questions are presented in Section 5.

2. Introductory problem: a linear elliptic PDE

In the sequel, we denote by $\| \cdot \|$ and $(\cdot, \cdot)$ the usual L^2 norm and inner product, respectively. The symbol 1_D will be used to denote the characteristic function of the set D. Also, C will stand for a generic positive constant. For simplicity, we will assume that only two controls act on the system and two functionals are taken into account but very similar considerations hold for systems with a larger number of controls and functionals.

2.1. Definition of Nash equilibria

Let $\Omega \subset \mathbb{R}^N$ be a nonempty bounded connected open set with regular boundary $\partial\Omega$ and let us assume that ω_1 and ω_2 are nonempty disjoint open subsets of Ω.

We will consider the problems

$$\begin{cases} -\Delta u = f_1 1_{\omega_1} + f_2 1_{\omega_2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where the $f_i \in L^2(\omega_i)$ are the controls and u is the state.

Let $\mathcal{O}_1$ an $\mathcal{O}_2$ be open sets, representing prescribed observation domains and let the J_i be given by

$$J_i(f_1, f_2) := \frac{a}{2} \int_{\mathcal{O}_i} |u - u_{id}|^2 + \frac{\mu}{2} \int_{\omega_i} |f_i|^2, \quad i = 1, 2, \tag{2.2}$$

where the $u_{id} \in L^2(\mathcal{O}_i)$ are given functions and a and μ are positive constants.

The first bi-objective control problem considered in this paper is the following:

Find a *Nash equilibria* associated with (2.1) and (2.2), that is, a pair $(\hat{f}_1, \hat{f}_2) \in L^2(\omega_1) \times L^2(\omega_2)$ such that

$$\begin{cases} J_1(\hat{f}_1, \hat{f}_2) \le J_1(f_1, \hat{f}_2) & \forall f_1 \in L^2(\omega_1), \\ J_2(\hat{f}_1, \hat{f}_2) \le J_2(\hat{f}_1, f_2) & \forall f_2 \in L^2(\omega_2). \end{cases} \tag{2.3}$$

In this case, since the control-to-state mapping is well-defined, linear and continuous and the cost functionals J_i are quadratic, strictly convex and C^1, it is immediate to deduce that $(\hat{f}_1, \hat{f}_2)$ is a Nash equilibria if and only if

$$\frac{\partial J_1}{\partial f_1}(\hat{f}_1, \hat{f}_2) = 0, \quad \frac{\partial J_2}{\partial f_2}(\hat{f}_1, \hat{f}_2) = 0. \tag{2.4}$$

2.2. Existence and characterization of Nash equilibria

For future purposes, note that

$$(\frac{\partial J_i}{\partial f_i}(f_1, f_2), g_i) = \int_{\omega_i} (a\varphi_i + \mu f_i) \, g_i \quad \forall f_i, g_i \in L^2(\omega_i), \tag{2.5a}$$

where φ_i is the i-th adjoint state associated with (f_1, f_2), *i.e.* the solution to

$$\begin{cases} -\Delta\varphi_i = (u - u_{id})1_{\mathcal{O}_i} & \text{in } \Omega, \\ \varphi_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5b}$$

where u is the state associated with (f_1, f_2).

We can now present the main result of the section. It deals with the existence and characterization of Nash equilibria.

Theorem 2.1. Let us assume that $(\hat{f}_1, \hat{f}_2) \in L^2(\omega_1) \times L^2(\omega_2)$. Then

1. $(\hat{f}_1, \hat{f}_2)$ is a Nash equilibrium if only if there exist $\hat{u}$, $\hat{\varphi}_1$ and $\hat{\varphi}_2$ such that

$$
\begin{cases}
-\Delta \hat{u} = \hat{f}_1 1_{\omega_1} + \hat{f}_2 1_{\omega_2} & \text{in } \Omega, \\
\hat{u} = 0, & \text{on } \partial\Omega, \\
-\Delta \hat{\varphi}_i = (\hat{u} - u_{id}) 1_{\mathcal{O}_i} & \text{in } \Omega, \\
\hat{\varphi}_i = 0 & \text{on } \partial\Omega, \\
\hat{f}_i = -\dfrac{a}{\mu} \left. \hat{\varphi}_i \right|_{\omega_i}, & i = 1, 2.
\end{cases}
\tag{2.6}
$$

2. There exists $\chi = \chi(\Omega, \mathcal{O}_1, \mathcal{O}_2, \omega_1, \omega_2)$ such that, if $a/\mu \leq \chi$, (2.6) possesses exactly one solution. Consequently, if $a > 0$, $\mu > 0$ and a/μ is sufficiently small, there exists a unique Nash equilibrium for J_1 and J_2.
3. Let us assume that $\mathcal{O}_1 = \mathcal{O}_2$. Then, for any $a > 0$ and $\mu > 0$, there exists exactly one solution to (2.6), that is, a unique Nash equilibrium associated with (2.1) and (2.2).

Proof. **1.** Let us first assume that $(\hat{f}_1, \hat{f}_2)$ is a Nash equilibrium. Then, (2.4) holds. In view of (2.5a)–(2.5b), one must have

$$
\hat{f}_i = -\frac{a}{\mu} \left. \hat{\varphi}_i \right|_{\omega_i} \quad \text{for } i = 1, 2
$$

(here, $\hat{\varphi}_i$ solves (2.5b) for $u = \hat{u}$). Hence, (2.6) is satisfied.

Conversely, if $(\hat{f}_1, \hat{f}_2), \hat{u}$ and the $\hat{\varphi}_i$ satisfy (2.6), then we see from (2.5a) that (2.4) holds and $(\hat{f}_1, \hat{f}_2)$ is a Nash equilibrium. This proves part 1.

2. Note that (2.6) can be written in the form

$$
\left(Id + \frac{a}{\mu}\Lambda_0\right)(\hat{f}_1, \hat{f}_2) = \frac{a}{\mu}(z_1, z_2), \quad (\hat{f}_1, \hat{f}_2) \in L^2(\omega_1) \times L^2(\omega_2),
\tag{2.7}
$$

where $\Lambda_0 : L^2(\omega_1) \times L^2(\omega_2) \mapsto L^2(\omega_1) \times L^2(\omega_2)$ is given by

$$
\Lambda_0(f_1, f_2) := (\left. \psi_1 \right|_{\omega_1}, \left. \psi_2 \right|_{\omega_2}), \quad \psi_i = (-\Delta)^{-1}((-\Delta)^{-1}(f_1 1_{\omega_1} + f_2 1_{\omega_2})) 1_{\mathcal{O}_\rangle})
$$

and

$$
(z_1, z_2) := ((-\Delta)^{-1} \left. (u_{1d} 1_{\mathcal{O}}) \right|_{\omega_1}, (-\Delta)^{-1} \left. (u_{2d} 1_{\mathcal{O}}) \right|_{\omega_2}).
$$

It is easy to check that Λ_0 is a linear compact operator. Consequently, if a/μ is sufficiently small, $Id + \frac{a}{\mu}\Lambda_0$ is an isomorphism and there exists exactly one solution to (2.7). This proves part 2.

3. Suppose now that $\mathcal{O}_1 = \mathcal{O}_2$, Then Λ_0 is self-adjoint and positive, that is,

$$
(\Lambda_0(f_1, f_2), (f_1, f_2)) \geq 0 \quad \forall (f_1, f_2) \in L^2(\omega_1) \times L^2(\omega_2).
$$

Therefore, for any $a, \mu > 0$, (2.7) is uniquely solvable and the solution is moreover the unique minimizer of the functional $K : L^2(\omega_1) \times L^2(\omega_2) \mapsto \mathbb{R}$, given by

$$
K(f_1, f_2) := \frac{1}{2}((Id + \frac{a}{\mu}\Lambda_0)(f_1, f_2), (f_1, f_2)) + \frac{a}{\mu}((z_1, z_2), (f_1, f_2)).
\tag{2.8}
$$

Thus, the proof of part 3 is also done. $\qquad\square$

Note that, when $\mathcal{O}_1 \neq \mathcal{O}_2$, the operator Λ_0 is not necessarily self-adjoint. As a consequence, for large a/μ there can be no solution to (2.7). However, since Λ_0 is compact, we can make use of *Fredholm's Alternative* and affirm that there exist at most $\beta_1, \beta_2, \dots$ with $\beta_n \to +\infty$ such that, if $a/\mu \neq \beta_n$ for all $n \geq 1$, a unique Nash equilibrium exists.

2.3. Algorithms and convergence

We present in this section several algorithms that can be used for the computation of Nash equilibria. The first one relies on a fixed-point reformulation of (2.6):

ALG 1:

(a) Choose $f_i^0 \in L^2(\omega_i)$, $i = 1, 2$.
(b) Then, for given $n \geq 0$ and $f_i^n \in L^2(\omega_i)$, compute the solution u^n to

$$\begin{cases} -\Delta u^n = f_1^n 1_{\omega_1} + f_2^n 1_{\omega_2} & \text{in } \Omega, \\ u^n = 0 & \text{on } \partial\Omega \end{cases} \tag{2.9}$$

and the solutions φ_i^n to the systems

$$\begin{cases} -\Delta \varphi_i^n = (u^n - u_{id}) 1_{\mathcal{O}_i} & \text{in } \Omega, \\ \varphi_i^n = 0 & \text{on } \partial\Omega \end{cases} \tag{2.10}$$

and, finally, take

$$f_i^{n+1} = -\frac{a}{\mu} \, \varphi_i^n|_{\omega_i}, \quad i = 1, 2. \tag{2.11}$$

The following convergence result holds:

Theorem 2.2. There exists $\chi_0 \in (0, \chi]$, such that, if $a/\mu \leq \chi_0$, the controls provided by **ALG 1** satisfy $(f_1^n, f_2^n) \to (\hat{f}_1, \hat{f}_2)$ strongly in $L^2(\omega_1) \times L^2(\omega_2)$, where $(\hat{f}_1, \hat{f}_2)$ is the unique Nash equilibrium associated with J_1 and J_2. Furthermore, the speed of convergence is at least linear.

The proof is immediate: it suffices to rewrite (2.6) in the form (2.7) and note that **ALG 1** is the usual fixed-point iteration method for the mapping $\Lambda : L^2(\omega_1) \times L^2(\omega_2) \mapsto L^2(\omega_1) \times L^2(\omega_2)$, where

$$\Lambda(f_1, f_2) := -\frac{a}{\mu} \Lambda_0(f_1, f_2) + \frac{a}{\mu}(z_1, z_2).$$

The following two algorithms are inspired by the classical optimal step gradient and conjugate gradient methods.

ALG 2:

(a) Choose $f_i^0 \in L^2(\omega_i)$, $i = 1, 2$.
(b) Then, for given $n \geq 0$ and $f_i^n \in L^2(\omega_i)$, compute the solution u^n to (2.9) and the solution φ_i^n to (2.10) and take

$$f_i^{n+1} = f_i^n - \rho_i^n g_i^n, \quad i = 1, 2, \tag{2.12}$$

where

$$g_i^n = a\,\varphi_i^n|_{\omega_i} + \mu f_i^n \tag{2.13}$$

and

$$\rho_1^n = \operatorname*{argmin}_{\rho \geq 0} J_1(f_1^n - \rho g_1^n, f_2^n), \qquad \rho_2^n = \operatorname*{argmin}_{\rho \geq 0} J_2(f_1^n, f_2^n - \rho g_2^n). \tag{2.14}$$

ALG 3:

(a) Choose $f_i^0 \in L^2(\omega_i)$, $i = 1, 2$.
(b) For $n = 0$, perform one step of **ALG 2** and take $d_i^0 = g_i^0$, $i = 1, 2$.
(c) Then, for given $n \geq 1$ and $f_i^n \in L^2(\omega_i)$, compute the solution u^n to to (2.9) and the solution φ_i^n to (2.10) and take

$$f_i^{n+1} = f_i^n - \rho_i^n d_i^n, \tag{2.15}$$

where

$$\begin{cases} d_i^n = g_i^n + \gamma_i^n d_i^{n-1}, \quad \gamma_i^n = \dfrac{\|g_i^n\|_{L^2(\omega_i)}^2}{\|g_i^{n-1}\|_{L^2(\omega_i)}^2}, \\[2mm] g_i^n = a\,\varphi_i^n|_{\omega_i} + \mu f_i^n \end{cases} \tag{2.16}$$

and

$$\rho_1^n = \operatorname*{argmin}_{\rho \geq 0} J_1(f_1^n - \rho d_1^n, f_2^n), \qquad \rho_2^n = \operatorname*{argmin}_{\rho \geq 0} J_2(f_1^n, f_2^n - \rho d_2^n). \tag{2.17}$$

Note that, in these iterates, the functions $\rho \mapsto J_1(f_1^n - \rho g_1^n, f_2^n)$, $\rho \mapsto J_2(f_1^n, f_2^n - \rho g_2^n)$, etc. are quadratic and strictly convex. Consequently, the ρ_i^n can be computed explicitly.

To the best of our knowledge, the convergence of **ALG 2** and **ALG 3** cannot be guaranteed. Note however that we advance in the directions indicated by the partial derivatives of the J_i and, consequently, it is reasonable to expect that they do converge.

Now, let us assume as before that $\mathcal{O}_1 = O_2 = \mathcal{O}$. Then, the Nash equilibrium is the unique minimizer of the functional K in (2.8) and it makes sense to consider the following algorithms:

ALG 2': Optimal Step Gradient Method for K.

(a) Solve the systems:

$$\begin{cases} -\Delta Z_i = u_{id} 1_{\mathcal{O}} & \text{in } \Omega, \\ Z_i = 0 & \text{on } \partial\Omega \end{cases}$$

for $i = 1$ and 2.
(b) Choose $f_i^0 \in L^2(\omega_i)$, $i = 1, 2$.
(c) Then, for given $n \geq 0$ and $f_i^n \in L^2(\omega_i)$, compute the solution u^n to (2.9) and the solution ψ^n to

$$\begin{cases} -\Delta \psi^n = u^n 1_{\mathcal{O}}, & \text{in } \Omega, \\ \psi^n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.18}$$

and take

$$f_i^{n+1} = f_i^n - \rho^n d_i^n \quad i = 1, 2,$$ (2.19)

where

$$d_i^n = f_i^n + \frac{a}{\mu} \psi^n|_{\omega_i} + \frac{a}{\mu} Z_i$$ (2.20)

and

$$\rho^n = \operatorname*{argmin}_{\rho \geq 0} K(f_1^n - \rho d_1^n, f_2^n - \rho d_2^n),$$ (2.21)

ALG 3': Optimal Step Conjugate Gradient Method for K.

(a) Solve the systems:

$$\begin{cases} -\Delta Z_i = u_{id} 1_{\mathcal{O}} & \text{in } \Omega, \\ Z_i = 0 & \text{on } \partial\Omega \end{cases}$$

for $i = 1$ and 2.
(b) Choose $f_i^0 \in L^2(\omega_i)$, $i = 1, 2$.
(c) For $n = 0$, perform one step of **ALG 2'** and take d_i^0, $i = 1, 2$.
(d) Then, for given $n \geq 1$ and $f_i^n \in L^2(\omega_i)$, compute the solution u^n to to (2.9) and the solution ψ^n to (2.18) and take

$$f_i^{n+1} = f_i^n - \rho^n d_i^n,$$ (2.22)

where

$$\begin{cases} d_i^n = g_i^n + \gamma^n d_i^{n-1}, \quad \gamma^n = \dfrac{\|(g_1^n, g_2^n)\|_{L^2(\omega_1) \times L^2(\omega_2)}^2}{\|(g_1^{n-1}, g_2^{n-1})\|_{L^2(\omega_1) \times L^2(\omega_2)}^2}, \\ g_i^n = f_i^n + \dfrac{a}{\mu} \psi^n|_{\omega_i} + \dfrac{a}{\mu} Z_i \end{cases}$$ (2.23)

and

$$\rho^n = \operatorname*{argmin}_{\rho \geq 0} K(f_1^n - \rho d_1^n, f_2^n - \rho d_2^n).$$ (2.24)

Again, the functions $\rho \mapsto K(f_n^1 - \rho d_n^1, f_n^2 - \rho d_n^2)$ are quadratic and strictly convex, whence the ρ^n can be easily found. The following result holds:

Theorem 2.3. Assume that $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}$. Then, the controls furnished by **ALG 2'** and **ALG 3'** satisfy $(f_1^n, f_2^n) \to (\hat{f}_1, \hat{f}_2)$ strongly in $L^2(\omega_1) \times L^2(\omega_2)$, where $(\hat{f}_1, \hat{f}_2)$ is the unique Nash equilibrium associated with J_1 and J_2.

The proof is immediate in view of the properties of the linear operator Λ_0. It suffices to directly apply well known classical results; see for instance [7, 20]. It is remarkable that, in this particular case, **ALG 2** and **ALG 2'** and **ALG 3** and **ALG 3'** only differ in the definitions of the step parameters ρ_i^n and ρ^n.

3. The case of a semilinear elliptic PDE

3.1. Nash equilibria and quasi-equilibria

This section is devoted to introduce Nash optima in the semilinear case. Now, we must distinguish equilibria from quasi-equilibria and take into account the particularities of each of them.

Let us assume that

$$\begin{cases} \phi : \mathbb{R} \mapsto \mathbb{R} \text{ is of class } C^1, \\ 0 \le \phi'(s) \le C \quad \forall s \in \mathbb{R}. \end{cases} \tag{3.1}$$

The state equation reads as follows:

$$\begin{cases} -\Delta u + \phi(u) = f_1 1_{\omega_1} + f_2 1_{\omega_2} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

It is then well known that, for each $(f_1, f_2) \in L^2(\omega_1) \times L^2(\omega_2)$, there exists exactly one (strong) solution u to (3.2). As in Section 2, we will consider the cost functionals J_i in (2.2), where the $u_{id} \in L^2(\mathcal{O}_i)$ and $a, \mu > 0$. Again, it will be said that $(\hat{f}_1, \hat{f}_2)$ is a Nash equilibrium for (3.2) and (2.2) if (2.3) is satisfied.

From standard optimal control theory, it is known that, under the previous assumptions on ϕ, the cost functionals J_i are again C^1 and satisfy

$$(\frac{\partial J_i}{\partial f_i}(f_1, f_2), g_i) = \int_{\omega_i} (a\varphi_i + \mu f_i)g_i \quad \forall f_i, g_i \in L^2(\omega_i),$$

where φ_i is the unique solution to

$$\begin{cases} -\Delta\varphi_i + \phi'(u)\varphi_i = (u - u_{id})1_{\mathcal{O}_i} \quad \text{in } \Omega, \\ \varphi_i = 0, \quad \text{on } \partial\Omega \end{cases} \tag{3.3}$$

(that is, φ_i is the i-th adjoint state corresponding to f_1 and f_2) and u is the solution to (3.2); see for instance [10, 16].

Definition 3.1. It will be said that $(\hat{f}_1, \hat{f}_2)$ is a Nash quasi-equilibrium for (3.2) and (2.2) if $(\hat{f}_1, \hat{f}_2)$ satisfies (2.4), that is, $(\hat{f}_1, \hat{f}_2)$ solves, together with $\hat{u}$ and the $\hat{\varphi}_i$, the optimality system

$$\begin{cases} -\Delta\hat{u} + \phi(\hat{u}) = \hat{f}_1 1_{\omega_1} + \hat{f}_2 1_{\omega_2} \quad \text{in } \Omega, \\ \hat{u} = 0 \quad \text{on } \partial\Omega, \\ -\Delta\hat{\varphi}_i + \phi'(\hat{u})\hat{\varphi}_i = (\hat{u} - u_{id})1_{\mathcal{O}_i} \quad \text{in } \Omega, \\ \hat{\varphi}_i = 0 \quad \text{on } \partial\Omega, \\ \hat{f}_i = -\frac{a}{\mu} \left. \hat{\varphi}_i \right|_{\omega_i}, \quad i = 1, 2. \end{cases} \tag{3.4}$$

Note that, if $(\hat{f}_1, \hat{f}_2)$ is a Nash equilibrium, then $(\hat{f}_1, \hat{f}_2)$ is also a Nash quasi-equilibrium. However, the converse is not necessarily true.

Theorem 3.2. Let us assume that $N \le 8$ and, besides (3.1), ϕ is of class C^2 and $|\phi'| + |\phi''| \le M$ for some $M > 0$. Then, there exists $\eta > 0$, only depending on Ω, u_{1d}, u_{2d} and M, such that, if $a/\mu \le \eta$, the following assertions are equivalent:

(a) $(\hat{f}_1, \hat{f}_2)$ is a Nash equilibrium for (3.2) and (2.2).
(b) $(\hat{f}_1, \hat{f}_2)$ is a Nash quasi-equilibrium for (3.2) and (2.2).

Proof. We have to prove that, under the previous conditions, any quasi-equilibrium satisfies (2.3). Thus, let us assume that (2.4) holds and let the functionals $\tilde{J}_i$ be given as follows:

$$\tilde{J}_1(f_1) := J_1(f_1, \hat{f}_2) \quad \text{and} \quad \tilde{J}_2(f_2) := J_2(\hat{f}_1, f_2).$$

Let us first prove that there exists $\eta_0 > 0$ and $C_0 > 0$, only depending on Ω, the u_{id} and M, such that, if $a/\mu \le \eta_0$, any quasi-equilibrium $(\hat{f}_1, \hat{f}_2)$ satisfies

$$\|\hat{f}_i\|_{L^2(\omega_i)} \le C_0, \quad i = 1, 2. \tag{3.5}$$

Indeed, if we take the PDEs for u and φ_i, we respectively multiply by u and φ_i and we integrate in Ω, we get

$$\|\nabla u\|^2 + \int_\Omega \phi(u)u + \frac{a}{\mu}\left(\int_{\omega_1} u\varphi_1 + \int_{\omega_2} u\varphi_2\right) = 0$$

and

$$\|\nabla \varphi_i\|^2 + \int_\Omega \phi'(u)|\varphi_i|^2 - \int_{\mathcal{O}_i} u\varphi_i = -\int_{\mathcal{O}_i} u_{id}\varphi_i.$$

Now, from the properties of ϕ and Hölder and Young inequalities, we see that

$$\|\nabla u\|^2 \le C\frac{a}{\mu}\left(\|\nabla \varphi_1\|^2 + \|\nabla \varphi_2\|^2\right)$$

and

$$\|\nabla \varphi_i\|^2 \le C + C\frac{a}{\mu}\left(\|\nabla \varphi_1\|^2 + \|\nabla \varphi_2\|^2\right).$$

Consequently, if a/μ is sufficiently small one has (3.5) for some C_0.

Observe that, in view of the properties of ϕ, the $\tilde{J}_i$ are twice continuosly differentiable. Let us see that, if a/μ is sufficiently small, $\tilde{J}_i''(f_i; g_i, g_i) > 0$ for all $f_i \in L^2(\omega_i)$, all nonzero $g_i \in L^2(\omega_i)$ and $i = 1, 2$.

For instance, let us take $i = 1$. Then,

$$(\tilde{J}_1'(f_1), g_1)_{L^2(\omega_1)} = \int_{\omega_1}\left(a\varphi_1 + \mu f_1\right)g_1\,dx \quad \forall f_1, g_1 \in L^2(\omega_1), \tag{3.6}$$

where φ_1 is, together with u, the solution to

$$\begin{cases} -\Delta u + \phi(u) = f_1 1_{\omega_1} + \hat{f}_2 1_{\omega_2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ -\Delta \varphi_1 + \phi'(u)\varphi_1 = (u - u_{1d})1_{\mathcal{O}_1} & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7a}$$

For any small $\varepsilon > 0$ and any $g_1 \in L^2(\omega_1)^N$, let us introduce u^ε and φ_1^ε, with

$$\begin{cases} -\Delta u^\varepsilon + \phi(u^\varepsilon) = (f_1 + \varepsilon g_1)1_{\omega_1} + \hat{f}_2 1_{\omega_2} & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \\ -\Delta \varphi_1^\varepsilon + \phi'(u^\varepsilon)\varphi_1^\varepsilon = (u^\varepsilon - u_{1d})1_{\mathcal{O}_1} & \text{in } \Omega, \\ \varphi_1^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7b}$$

Let us put

$$z := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(u^\varepsilon - u) \quad \text{and} \quad \psi_1 := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\varphi_1^\varepsilon - \varphi_1).$$

Note that these limits exist in $H_0^1(\Omega)$. This is easy to see by substracting (3.7a) from (3.7b), dividing by ε and letting $\varepsilon \to 0$ (here, we must use that $\phi \in C^2(\mathbb{R})$ and ϕ' and ϕ'' are uniformly bounded). Furthermore, one has

$$\begin{cases} -\Delta z + \phi'(u)z = g_1 1_{\omega_1} & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \\ -\Delta \psi_1 + \phi'(u)\psi_1 + \phi''(u)z\varphi_1 = z1_{\mathcal{O}_1} & \text{in } \Omega, \\ \psi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Therefore,

$$\tilde{J}_1''(f_1; g_1, g_1) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\tilde{J}_1'(f_1 + \varepsilon g_1) - \tilde{J}_1'(f_1), g_1) = \int_{\omega_1} (a\psi_1 + \mu g_1)\, g_1. \tag{3.9}$$

Observe that, from classical elliptic regularity results, one has

$$\|\varphi_1\|_{H^2} \leq C_1(1 + \|u\|) \quad \text{and} \quad \|\nabla u\| \leq C_2(\|f_1\| + \|\hat{f}_2\|) \tag{3.10}$$

for some constants $C_1 = C_1(\Omega, u_{1d}, u_{2d})$ and $C_2 = C_2(\Omega)$. On the other hand, from the PDE satisfied by ψ_1 and z, one also has

$$\|\nabla \psi_1\| \leq C_3(1 + \|\varphi_1\|_{L^{N/2}})\|\nabla z\| \quad \text{and} \quad \|\nabla z\| \leq C_2\|g_1\|, \tag{3.11}$$

where $C_3 = C_3(\Omega, M)$. Therefore, taking into account that, for $N \leq 8$, $H^2(\Omega) \hookrightarrow L^{N/2}(\Omega)$ with continuous embedding, we see from (3.10) and (3.11) that

$$\tilde{J}_1''(f_1; g_1, g_1) \geq \mu\|g_1\|^2 - aC_4(1 + \|f_1\|)\|g_1\|^2$$

for some $C_4 = C_4(\Omega, u_{1d}, u_{2d}, M)$.

Clearly, this proves that, if $N \leq 8$ and a/μ is sufficiently small, $\tilde{J}_1''(f_1, g_1, g_1) > 0$ for all $f_1 \in L^2(\omega_1)$ and all nonzero $g_1 \neq 0$ and, consequently, $\tilde{J}_i$ is strictly convex and possesses a unique global minimum at $\hat{f}_i$. A similar fact holds for $\tilde{J}_2$. That is, $(\hat{f}_1, \hat{f}_2)$ is a Nash equilibrium for (3.2) and (2.2). □

3.2. Existence of Nash equilibria and quasi-equilibria

We can now prove the existence of Nash equilibria for (3.2) and (2.2):

Theorem 3.3. Let the assumptions in Theorem 3.2 be satisfied. The following assertions hold:

1. There exists $\eta_0 \leq \eta$, only depending on u_{1d}, u_{2d} and M such that, if $a/\mu \leq \eta_0$, there exists at least one Nash equilibrium $(\hat{f}_1, \hat{f}_2)$ for (3.2) and (2.2).
2. There exists $\eta_1 \leq \eta_0$ such that, whenever $a/\mu \leq \eta_1$, the Nash equilibrium is unique.

Proof. To prove the existence of a Nash equilibrium, it suffices to check that, if a/μ is sufficiently small, the optimality system (3.4) possesses at least one solution. To this purpose, we can use *Schauder's Fixed-Point Theorem*.

Thus, let us consider the mapping $\Psi : L^2(\omega_1) \times L^2(\omega_2) \mapsto L^2(\omega_1) \times L^2(\omega_2)$ defined as follows: $(f_1, f_2) = \Psi(\tilde{f}_1, \tilde{f}_2)$ if and only if

$$f_i = -\frac{a}{\mu}\, \varphi_i|_{\omega_i}, \quad i = 1, 2,$$

where φ_i is the solution to (3.3) and u is the state associated with $\tilde{f}_1$ and $\tilde{f}_2$. It is not difficult to see that Ψ is well-defined, continuous and compact. Furthermore, if a/μ is small enough, Ψ maps the whole space $L^2(\omega_1) \times L^2(\omega_2)$ into a ball. This can be seen from the following estimates:

(a) First, from (3.2) and the properties satisfied by ϕ, one has

$$\|\nabla u\|^2 + \int_\Omega \left(\phi(u) - \phi(0)\right) u = -\frac{a}{\mu} \int_\Omega \left(\varphi_1 1_{\omega_1} + \varphi_2 1_{\omega_2}\right) u - \phi(0) \int_\Omega u,$$

whence

$$\|\nabla u\|^2 \leq C\left(\left(\frac{a}{\mu}\right)^2 \left(\|\nabla \varphi_1\|^2 + \|\nabla \varphi_2\|^2\right) + 1\right). \tag{3.12a}$$

(b) Then, taking into account (3.3), we deduce that

$$\|\nabla \varphi_i\|^2 + \int_\Omega \phi'(u)|\varphi_i|^2 = \int_{\mathcal{O}_i} (u - u_{id})\varphi_i$$

and consequently

$$\|\nabla \varphi_i\|^2 \leq C(\|\nabla u\|^2 + 1). \tag{3.12b}$$

From (3.12a) and (3.12b), our assertion follows.

Hence, we can apply Schauder's Theorem to Ψ and the existence of a Nash equilibrium is ensured. This ends the proof of existence.

Let us now see that, if a/μ is small enough, the solution to (3.4) is unique.

In order to fix ideas, let us assume that $3 \leq N \leq 8$ (the case $N = 2$ is similar and even easier).

Let us assume that there exist two Nash equilibria (f_1^1, f_2^1) and (f_1^2, f_2^2) in $L^2(\omega_1) \times L^2(\omega_2)$. Then they solve the following systems for $j = 1$ and 2:

$$\begin{cases} -\Delta u^j + \phi(u^j) = f_1^j 1_{\omega_1} + f_2^j 1_{\omega_2} & \text{in } \Omega, \\ u^j = 0 & \text{on } \partial\Omega, \\ -\Delta \varphi_i^j + \phi'(u^j)\varphi_i^j = (u^j - u_{id})1_{\mathcal{O}_i} & \text{in } \Omega, \\ \varphi_i^j = 0 & \text{on } \partial\Omega, \\ f_i^j = -\dfrac{a}{\mu}\varphi_i^j 1_{\omega_i}, \quad i = 1, 2. \end{cases}$$

Let us set $f_i := f_i^1 - f_i^2$, $u := u^1 - u^2$ and $\varphi_i := \varphi_i^1 - \varphi_i^2$ for $i = 1, 2$. Then

$$
\begin{cases}
-\Delta u + \phi'(\tilde{u})u = f_1 1_{\omega_1} + f_2 1_{\omega_2} & \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega, \\
-\Delta \varphi_i + \phi'(u^1)\varphi_i + \phi''(\overline{u})\varphi_i^2 u = u 1_{\mathcal{O}_i} & \text{in } \Omega, \\
\varphi_i = 0 \quad \text{on } \partial\Omega, \\
f_i = -\dfrac{a}{\mu}\varphi_i 1_{\omega_i}, \quad i = 1, 2.
\end{cases}
\tag{3.13}
$$

Here, it is assumed that one has $\tilde{u}(x) = \beta(x)u^1(x) + (1 - \beta(x))u^2(x)$ and $\overline{u}(x) = \lambda(x)u^1(x) + (1 - \lambda(x))u^2(x)$ for some $\beta(x), \lambda(x) \in (0,1)$ and all $x \in \Omega$. We deduce that

$$
\int_\Omega \Big(|\nabla u|^2 + \phi'(\tilde{u})|u|^2 \Big) dx = (f_1 1_{\omega_1}, u) + (f_2 1_{\omega_2}, u)
$$

and, therefore,

$$
\|\nabla u\|^2 \leq C\frac{a^2}{\mu^2}(\|\varphi_1\|^2 + \|\varphi_2\|^2).
\tag{3.14}
$$

With a similar argument, denoting by 2^* the maximal exponent r such that $H^1(\Omega) \hookrightarrow L^r(\Omega)$ and $(2^*)'$ its conjugate, i.e. $2^* = (2N)/(N-2)$ and $(2^*)' = 2N/(N+2)$, we get:

$$
\begin{aligned}
\|\nabla \varphi_i\|^2 &\leq C\|\varphi_i^2 u\|_{L^{(2^*)'}}\|\varphi_i\|_{L^{2^*}} + C\|u\|\|\varphi_i\| \\
&\leq \frac{1}{2}\|\nabla \varphi_i\|^2 + C\|u\|^2 + C\|\varphi_i^2\|_{L^{N/2}}^2\|u\|_{L^{2^*}}^2 \\
&\leq \frac{1}{2}\|\nabla \varphi_i\|_{L^{N/2}}^2 + C\Big(1 + \|\varphi_i^2\|^2\Big)\|\nabla u\|^2,
\end{aligned}
$$

whence

$$
\|\nabla u\|^2 \leq C\frac{a^2}{\mu^2}(1 + \|\varphi_1^2\|_{L^{N/2}}^2 + \|\varphi_2^2\|_{L^{N/2}}^2)\|\nabla u\|^2.
\tag{3.15}
$$

For $N \leq 8$, one has $N/2 \leq 2N/(N-4)$. Accordingly, $H^2(\Omega) \hookrightarrow L^{N/2}(\Omega)$ and, from the usual elliptic estimates, the following is found:

$$
\|\varphi_i^2\|_{L^{N/2}}^2 \leq C\|(u^2 - u_{id})1_{\mathcal{O}_i}\|^2 \leq C(1 + \|u^2\|^2) \leq C\Big(1 + \frac{a^2}{\mu^2}(\|\varphi_1^2\|^2 + \|\varphi_2^2\|^2)\Big), \quad i = 1, 2.
\tag{3.16}
$$

This indicates that, if a/μ is sufficiently small, $\|\varphi_1^2\|_{L^{N/2}}^2 + \|\varphi_2^2\|_{L^{N/2}}^2 \leq C$ and then, using (3.15), we see that $u = 0$. Thus, in this case, we necessarily have $f_i = 0$ for $i = 1, 2$ and the proof is done.

$\square$

In view of what can be said in the linear case, it is reasonable to expect that, when $\mathcal{O}_1 = \mathcal{O}_2$, there exist quasi-equilibria for any $a > 0$ and any $\mu > 0$. However, to our knowledge, this is unknown.

3.3. Algorithms and convergence

In this section, we present some iterative algorithms, similar to those considered in the linear case.

ALG 4:

(a) Choose $(f_1^0, f_2^0) \in L^2(\omega_1) \times L^2(\omega_2)$.

(b) Then, for given $n \geq 0$ and $(f_1^n, f_2^n) \in L^2(\omega_1) \times L^2(\omega_2)$, compute the solution u^n to

$$\begin{cases} -\Delta u^n + \phi(u^n) = f_1^n 1_{\omega_1} + f_2^n 1_{\omega_2} & \text{in } \Omega, \\ u^n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.17}$$

the solution φ_i^n to the system

$$\begin{cases} -\Delta \varphi_i^n + \phi'(u^n)\varphi_i^n = (u^n - u_{id})1_{\mathcal{O}_i} & \text{in } \Omega, \\ \varphi_i^n = 0 & \text{on } \partial\Omega \end{cases} \tag{3.18}$$

and, finally, take

$$f_i^{n+1} = -\frac{a}{\mu} \left. \varphi_i^n \right|_{\omega_i} \tag{3.19}$$

for $i = 1, 2$.

Theorem 3.4. Let the assumptions in Theorem 3.3 be satisfied. There exists $\eta_2 \leq \eta_1$ such that, if $a/\mu \leq \eta_2$, the couples provided by **ALG 4** satisfy $(f_1^n, f_2^n) \to (\hat{f}_1, \hat{f}_2)$ strongly in $L^2(\omega_1) \times L^2(\omega_2)$, where $(\hat{f}_1, \hat{f}_2)$ is the unique Nash equilibrium.

This proof is easy. Indeed, note that, under the assumptions in Theorem 3.4, the couple (f_1, f_2) is a Nash equilibrium if and only if satisfies a fixed-point equation $(f_1, f_2) = \Psi(f_1, f_2)$, where Ψ is a contraction in $L^2(\omega_1) \times L^2(\omega_2)$. Therefore, the iterates $(f_1^{n+1}, f_2^{n+1}) = \Psi(f_1^n, f_2^n)$ converge in this space independently of the choice of (f_1^0, f_2^0).

ALG 5:

(a) Choose $(f_1^0, f_2^0) \in L^2(\omega_1) \times L^2(\omega_2)$.

(b) Then, for given $n \geq 0$ and $(f_1^n, f_2^n) \in L^2(\omega_1) \times L^2(\omega_2)$, compute the solution u^n to (3.17) and the solution φ_i^n to (3.18) and take

$$f_i^{n+1} = f_i^n - \rho_i^n g_i^n, \tag{3.20}$$

where

$$g_i^n = a \left. \varphi_i^n \right|_{\omega_i} + \mu f_i^n \tag{3.21}$$

and

$$\rho_1^n = \underset{\rho \geq 0}{\text{argmin}} \, J_1(f_1^n - \rho g_1^n, f_2^n), \qquad \rho_2^n = \underset{\rho \geq 0}{\text{argmin}} \, J_2(f_1^n, f_2^n - \rho g_2^n). \tag{3.22}$$

ALG 6:

(a) Choose $(f_1^0, f_2^0) \in L^2(\omega_1) \times L^2(\omega_2)$.

(b) For $n = 0$, perform one step of **ALG 5** and take $d^0 = g^0$.

(c) Then, for given $n \geq 1$, $(f_1^n, f_2^n) \in L^2(\omega_1) \times L^2(\omega_2)$, $g_i^{n-1} \in L^2(\omega_i)$ and $d_i^{n-1} \in L^2(\omega_i)$, compute the solution u^n to (3.17) and the solution φ_i^n to (3.18) and take

$$f_i^{n+1} = f_i^n - \rho_i^n d_i^n, \tag{3.23}$$

where

$$\begin{cases} d_i^n = g_i^n + \gamma_i^n d_i^{n-1}, \quad \gamma_i^n = \dfrac{(g_i^n - g_i^{n-1}, g_i^n)_{L^2(\omega_i) \times L^2(\omega_i)}}{\|g_i^{n-1}\|_{L^2(\omega_i)}^2}, \\ g_i^n = a \, \varphi_i^n|_{\omega_i} + \mu f_i^n \end{cases} \tag{3.24}$$

and

$$\rho_1^n = \operatorname*{argmin}_{\rho \geq 0} J_1(f_1^n - \rho g_1^n, f_2^n), \qquad \rho_2^n = \operatorname*{argmin}_{\rho \geq 0} J_2(f_1^n, f_2^n - \rho g_2^n). \tag{3.25}$$

Note that in **ALG 3** and **ALG 6**, the coefficient γ_i^n is given by different expressions. The reason is that, now, the system is nonlinear and, as usual, it seems better to impose Polak condition to ensure convergence.

In the iterates in **ALG 5** and **ALG 6**, we have to compute the steps ρ_1^n and ρ_2^n. Now, the functions to minimize are not quadratic and these numbers cannot be computed exactly. Thus, in practice, we must apply a one-dimensional minimization method to these purposes. More details are given below.

4. THE STATIONARY NAVIER-STOKES SYSTEM

This section is devoted to analyze Nash equilibria and quasi-equilibria for the stationary Navier-Stokes equations. In view of the properties of the state system and, in particular, nonlinearity and possible lack of uniqueness, this task will be much more complicated than in Sections 2 and 3. In fact, we will only be able to prove the existence of quasi-equilibria; that Nash equilibria exists is an open question.

The stationary Navier-Stokes equations are the following:

$$\begin{cases} -\nu \Delta u + (u \cdot \nabla)u + \nabla p = f_1 1_{\omega_1} + f_2 1_{\omega_2} \quad \text{in } \Omega, \\ \nabla \cdot u = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

The state variables are $u = (u_1, \ldots, u_N)$ and p. They can be interpreted as the velocity field and the pressure of a steady viscous Newtonian fluid flow. The controls are $f_1 1_{\omega_1}$ and $f_2 1_{\omega_2}$ and can viewed as external fields of forces respectively applied at the points in ω_1 and ω_2.

4.1. Nash equilibria and quasi-equilibria

The positive constant ν is the kinematic viscosity of the fluid. It must be regarded as a measure of the fluid internal friction.

As in the previous sections, let us introduce the functionals J_i with

$$J_i(f_1, f_2, u) := \frac{a}{2} \int_{\mathcal{O}_i} |u - u_{id}|^2 + \frac{\mu}{2} \int_{\omega_i} |f_i|^2, \quad i = 1, 2, \tag{4.2}$$

where the $u_{id} \in L^2(\mathcal{O}_i)^N$ and $a, \mu > 0$.

Note that, here, the J_i depend not only on the controls f_i but also on the associated state u. This is due to the possible non-uniqueness of solution to (4.1), that can be true when ν is not sufficiently large with respect to Ω and the $\|f_i\|_{L^2(\omega_i)}$.

Definition 4.1. It will be said that $(\hat{f}_1, \hat{f}_2) \in L^2(\omega_1)^N \times L^2(\omega_2)^N$ is a Nash equilibrium for (4.1) and (4.2) if there exists an associated state $(\hat{u}, \hat{p})$ satisfying

$$\begin{cases} J_1(\hat{f}_1, \hat{f}_2, \hat{u}) \leq J_1(f_1, \hat{f}_2, u) & \forall f_1 \in L^2(\omega_1)^N \text{ and any associated state } u \text{ to } (f_1, \hat{f}_2), \\ J_2(\hat{f}_1, \hat{f}_2, \hat{u}) \leq J_2(\hat{f}_1, f_2, u) & \forall f_2 \in L^2(\omega_2)^N \text{ and any associated state } u \text{ to } (\hat{f}_1, f_2). \end{cases} \tag{4.3}$$

Definition 4.2. It will be said that $(\hat{f}_1, \hat{f}_2) \in L^2(\omega_1)^N \times L^2(\omega_2)^N$ is a Nash quasi-equilibrium for (4.1) and (4.2) if there exists a solution $(\hat{u}, \hat{p}, \hat{\varphi}_1, \hat{q}_1, \hat{\varphi}_2, \hat{q}_2)$ to the following coupled system

$$\begin{cases} -\nu \Delta \hat{u} + (\hat{u} \cdot \nabla)\hat{u} + \nabla \hat{p} = \hat{f}_1 1_{\omega_1} + \hat{f}_2 1_{\omega_2} & \text{in } \Omega, \\ \nabla \cdot \hat{u} = 0 & \text{in } \Omega, \\ \hat{u} = 0 & \text{on } \partial\Omega, \\ -\nu \Delta \hat{\varphi}_i + (\hat{u} \cdot \nabla)\hat{\varphi}_i + (\nabla \hat{u})^t \hat{\varphi}_i + \nabla \hat{q}_i = (\hat{u} - u_{id})1_{\mathcal{O}_i} & \text{in } \Omega, \\ \nabla \cdot \hat{\varphi}_i = 0 & \text{in } \Omega, \\ \hat{\varphi}_i = 0 & \text{on } \partial\Omega, \\ \hat{f}_i = -\dfrac{a}{\mu}\hat{\varphi}_i 1_{\omega_i}, \quad i = 1, 2. \end{cases} \tag{4.4}$$

4.2. Existence of Nash quasi-equilibria

Let us recall some classical spaces, usual for the analysis of the Navier Stokes equations:

$$H := \{v \in L^2(\Omega)^N : \ \nabla \cdot v = 0 \text{ in } \Omega, \ v \cdot n = 0 \text{ on } \partial\Omega\},$$

$$V := \{v \in H_0^1(\Omega)^N : \ \nabla \cdot v = 0 \text{ in } \Omega\}.$$

They are closed subspaces of $L^2(\Omega)^N$ and $H_0^1(\Omega)^N$, respectively; accordingly, they are Hilbert spaces for the inner products $(\cdot, \cdot)$ and $(\cdot, \cdot)_{H_0^1}$. Also, we have the compact embeddings $V \hookrightarrow H \equiv H' \hookrightarrow V'$ where X' denotes the dual space of X.

In the sequel, we will have to use the Stokes operator $A : D(A) \subset H \mapsto H$, with $D(A) = H^2(\Omega)^N \cap V$ and

$$Av := P(-\Delta v) \quad \forall v \in D(A),$$

where $P : L^2(\Omega)^N \mapsto H$ is the usual orthogonal projector. It is known that A can be uniquely extended to a bounded linear operator in $\mathcal{L}(V; V')$, again denoted by A.

Then, A is self-adjoint and one has

$$\langle Av, w \rangle = (v, w)_{H_0^1} \quad \forall v, w \in V.$$

Let us consider the trilinear continuous forms $b(\cdot, \cdot, \cdot)$ and $\hat{b}(\cdot, \cdot, \cdot)$, with

$$b(u, v, w) := \sum_{i,j=1}^{N} \int_{\Omega} u_i \, \partial_i v_j \, w_j \quad \forall u, v, w \in V$$

and

$$\hat{b}(u, v, w) := -b(v, u, w) + b(w, u, v) \quad \forall u, v, w \in V.$$

Note that there exist bilinear continuous mappings B and $\hat{B}$ with

$$\langle B(u,v), w \rangle = b(u,v,w) \quad \text{and} \quad \langle \hat{B}(u,v), w \rangle = \hat{b}(u,v,w) \quad \forall u, v, w \in V.$$

The existence of a quasi-equilibrium for (4.1) and (4.2) is guaranteed by the following result:

Theorem 4.3. There exists $\beta > 0$, only depending on Ω, ν, the ω_i, the $\mathcal{O}_i$ and the u_{id}, such that, if $a/\mu \leq \beta$, the optimality system (4.4) possesses at least one solution.

Proof. The proof is not difficult. Thus, let us rewrite (4.4) in the form

$$\begin{pmatrix} u \\ \varphi_1 \\ \varphi_2 \end{pmatrix} = Z \begin{pmatrix} u \\ \varphi_1 \\ \varphi_2 \end{pmatrix},$$

where $Z : V^3 \mapsto V^3$ is the following continuous and compact mapping:

$$Z \begin{pmatrix} u \\ \varphi_1 \\ \varphi_2 \end{pmatrix} := \frac{1}{\nu} \begin{pmatrix} A^{-1}\big(-(u \cdot \nabla)u - \frac{a}{\mu}(\varphi_1 1_{\omega_1} + \varphi_2 1_{\omega_2}) \big) \\ A^{-1}\big(-(u \cdot \nabla)\varphi_1 - (\nabla u)^t \varphi_1 + (u - u_{1d})1_{\mathcal{O}_1} \big) \\ A^{-1}\big(-(u \cdot \nabla)\varphi_2 - (\nabla u)^t \varphi_2 + (u - u_{2d})1_{\mathcal{O}_2} \big) \end{pmatrix}.$$

Now, we want to find a fixed-point of Z. It is easy to see that, if a/μ is sufficiently small, the fixed-points of Z are uniformly bounded. Consequently, form the Leray-Schauder Principle, we deduce that, for small a/μ, (4.4) is solvable. $\qquad\square$

In this case, we cannot ensure the existence of Nash equilibria. This is due to the lack of uniqueness of the control-to-state mapping and the lack of convexity of the related functionals. However, the following holds:

Theorem 4.4. Let $(\hat{f}_1, \hat{f}_2)$ be a Nash equilibrium for (4.1) and (4.2). Then, $(\hat{f}_1, \hat{f}_2)$ is a Nash quasi-equilibrium.

We will give a proof of this result relying on the Dubovitsky-Milyutin formalism (see [11]). To this purpose, we must recall some technical results.

Lemma 4.5. Let $K_1, \ldots, K_n$ be convex cones in a Banach space X with apex at 0. For each i, we assume that either K_i is open or it is a closed subspace. Then the following conditions are equivalent:

- $\displaystyle\bigcap_{i=1}^n K_i = \emptyset.$
- There exist linear functionals $f_i \in K_i^*$ with $i = 1, \ldots, n$, not all zero, such that $f_1 + f_2 + \cdots + f_n = 0$.

Here, for any i, we have denoted by K_i^* the dual cone to K_i, that is, $K_i^* := \{f \in X' : f(e) \geq 0 \;\; \forall e \in K_i\}$. For the proof, see for instance Lemma 5.11 of [11].

Note that the adjoint system

$$\begin{cases} -\nu \Delta \varphi_i + (u \cdot \nabla)\varphi_i + (\nabla u)^t \varphi_i + \nabla q_i = (u - u_{id})1_{\mathcal{O}_i} & \text{in } \Omega, \\ \nabla \cdot \varphi_i = 0 & \text{in } \Omega, \\ \varphi_i = 0 & \text{on } \partial\Omega, \end{cases}$$

can be equivalently rewritten in the form

$$\begin{cases} -\nu \Delta \varphi_i - D\varphi_i \cdot u + \nabla \tilde{q}_i = (u - u_{id})1_{\mathcal{O}_i} & \text{in } \Omega, \\ \nabla \cdot \varphi_i = 0 & \text{in } \Omega, \\ \varphi_i = 0 & \text{on } \partial\Omega, \end{cases}$$

where $D\varphi_i = \frac{1}{2}(\nabla\varphi_i + (\nabla\varphi_i)^t)$ and the pressure has been redefined. This observation will be used in the following lemma and the following section.

Lemma 4.6. Let $(f_1, f_2) \in L^2(\omega_1)^N \times L^2(\omega_2)^N$ be given and let $u \in H^2(\Omega)^N \cap V$ be (together with p) an associated solution to (4.1). Let $R : V \mapsto V$ be the linear mapping defined by $R\varphi := (\nu A)^{-1}(-\hat{B}(u, \varphi))$, where A is the (extended) Stokes operator in V. Then, for any non-empty open set $\omega \subset \Omega$,

$$[\varphi]_\omega := \| \, \varphi|_\omega \, \|_{L^2(\omega)} \tag{4.5}$$

is a norm in $N(Id + R)$.

Proof. From the usual alliptic estimates for the solutions to stationary Navier-Stokes systems, one has $u \in H^2(\Omega)^N$, $\nabla u \in L^6(\Omega)^{N\times N}$ and $u \in L^\infty(\Omega)^N$, see for instance [23]. Consequently, R is well defined and compact. We only have to prove that, for every $\varphi \in N(Id + R)$ with $\varphi|_\omega = 0$, one has $\varphi \equiv 0$.

Thus, let us assume that $\varphi \in V$ and $\varphi + (\nu A)^{-1}(-\hat{B}(u, \varphi)) = 0$, that is,

$$\begin{cases} -\nu\Delta\varphi - D\varphi \cdot u + \nabla\tilde{q} = 0 & \text{in } \Omega, \\ \nabla \cdot \varphi = 0 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume that $\varphi = 0$ a.e. in ω. Then, we can use the unique continuation property of the Stokes system with coefficients in L^∞ (see [8]) and deduce that, certainly, φ vanishes identically. $\square$

Lemma 4.7. Let (f_1, f_2), (u, p) and ω be as in Lemma 4.6. Let the (φ_n, ψ_n) be given in $V \times V$ with

$$\varphi_n|_\omega \to \varphi|_\omega \text{ in } L^2(\omega), \quad \psi_n := \varphi_n + R\varphi_n \quad \text{and} \quad \psi_n \to \psi \in V.$$

Then φ_n is uniformly bounded in V.

Proof. First, note that, since R is a compact operator, $\dim(N(Id + R)) < +\infty$ and $R(Id + R)$ is closed, in view of Fredholm's Alternative Theorem (see for instance [5]).

Now, let $\tilde{\varphi}_n$ be, for each n, the unique function in $N(Id + R)$ satisfying

$$\|\varphi_n - \tilde{\varphi}_n\|_{H_0^1} = \inf_{\tilde{\varphi} \in N(Id+R)} \|\varphi_n - \tilde{\varphi}\|_{H_0^1}.$$

Then, $\psi_n = (\varphi_n - \tilde{\varphi}_n) + R(\varphi_n - \tilde{\varphi}_n)$. Also, $\|\varphi_n - \tilde{\varphi}_n\|_{H_0^1}$ is bounded by a constant C. Indeed, if this were not the case, we could assume that

$$\|\varphi_n - \tilde{\varphi}_n\|_V = \text{dist}(\varphi_n, N(Id + R)) \to +\infty.$$

Let us introduce

$$\zeta_n := \frac{\varphi_n - \tilde{\varphi}_n}{\|\varphi_n - \tilde{\varphi}_n\|_{H_0^1}}.$$

Then

$$\zeta_n + R\zeta_n = \frac{\psi_n}{\|\varphi_n - \tilde{\varphi}_n\|_{H_0^1}}. \tag{4.6}$$

Since the $\|\zeta_n\|_V = 1$ and $\psi_n \to \psi$ in V, we see from (4.6) that $\zeta_n \to \zeta$ for some $\zeta \in N(Id + R)$. So,

$$\left\|\varphi_n - \left(\tilde{\varphi}_n + \|\varphi_n - \tilde{\varphi}_n\|_{H_0^1}\right)\right\|_{H_0^1} = \|\varphi_n - \tilde{\varphi}_n\|_{H_0^1} \|\zeta_n - \zeta\|_{H_0^1} \geq \|\varphi_n - \tilde{\varphi}_n\|_{H_0^1}$$

for all n. But this is an absurd, since $\zeta_n \to \zeta$ in V.

Finally, let us deduce that $\|\varphi_n\|_{H_0^1} \leq C$. We can write that $\varphi_n = \tilde{\varphi}_n + \eta_n$, with

- $\|\eta_n\|_{H_0^1} = \|\varphi_n - \tilde{\varphi}_n\|_{H_0^1} \leq C.$
- $\|\tilde{\varphi}_n\|_{H_0^1} \leq C[\tilde{\varphi}_n]_\omega$ (note that the $\tilde{\varphi}_n$ belong to a finite dimensional space) and

$$[\tilde{\varphi}_n]_\omega \leq [\varphi_n]_\omega + [\eta_n]_\omega \leq [\varphi_n]_\omega + C\|\eta_n\|_{H_0^1} \leq C.$$

This ends the proof.   □

Now, we can prove the main result in this section.

Thus, let $(\hat{f}_1, \hat{f}_2)$ be a Nash equilibrium for (4.1) and (4.2). Let us introduce the sets

$$\mathcal{F}_1 = \{(f_1, u) \in L^2(\omega_1)^N \times D(A) : (f_1, \hat{f}_2, u) \text{ solves (4.1) (together with some } p \text{ for } f_2 = \hat{f}_2\}$$

and

$$\mathcal{F}_2 = \{(f_2, u) \in L^2(\omega_2)^N \times D(A) : (\hat{f}_1, f_2, u) \text{ solves (4.1) (together with some } p \text{ for } f_1 = \hat{f}_1\}.$$

By assumption, there exists a strong solution $(\hat{u}, \hat{p})$ to (4.1) such that $(\hat{f}_1, \hat{u})$ and $(\hat{f}_2, \hat{u})$ respectively solve the following extremal problems:

$$\begin{cases} \text{Minimize } \hat{J}_1(f_1, u) = J_1(f_1, \hat{f}_2, u) \\ \text{Subject to } (f_1, u) \in \mathcal{F}_1 \end{cases} \tag{4.7a}$$

and

$$\begin{cases} \text{Minimize } \hat{J}_2(f_2, u) = J_1(\hat{f}_1, f_2, u) \\ \text{Subject to } (f_2, u) \in \mathcal{F}_2. \end{cases} \tag{4.7b}$$

Let us introduce some cones associated with (4.7a) and (4.7b). To this end, we set

$$M_1(f_1, u) := -\nu A u + B(u, u) - f_1 1_{\omega_1} - \hat{f}_2 1_{\omega_2} \quad \forall (f_1, u) \in L^2(\omega_1)^N \times V$$

and

$$M_2(f_2, u) := -\nu A u + B(u, u) - \hat{f}_1 1_{\omega_1} - f_2 1_{\omega_2} \quad \forall (f_2, u) \in L^2(\omega_2)^N \times V.$$

It is then clear that M_1 and M_2 are continuosly differentiable mappings, with

$$M_i'(f_i, u)(h_i, w) = \nu A w + B(u, w) + B(w, u) - h_i 1_{\omega_i} \quad \forall (h_i, w) \in L^2(\omega_i)^N \times V.$$

We also have

$$M_i'(\hat{f}_i, \hat{u})^* \varphi_i = (-\varphi_i|_{\omega_i}, \nu A \varphi_i - \hat{B}(\hat{u}, \varphi_i)) \quad \forall \varphi_i \in V.$$

Then the cones of descent directions associated with (4.7a) and (4.7b) are the sets

$$D_i := \{(h_i, v) \in L^2(\omega_i)^N \times V : \langle \hat{J}'_i(\hat{f}_i, \hat{u}), (h_i, v) \rangle < 0\}$$

and the spaces of tangent directions are the

$$T_i := \{(h_i, v) \in L^2(\omega_i)^N \times V : M'_i(\hat{f}_i, \hat{u})(h_i, v) = 0\} = N(M'_i(\hat{f}_i, \hat{u})).$$

Let us prove that $R(M'_i(f, u)^*)$ is closed in $L^2(\omega_i)^N \times V$ for each $i = 1, 2$. Indeed, if $(h, w) \in \overline{R(M'_i(f_i, u)^*)}$, there exists fields $\varphi_n \in V$ such that

$$- \varphi_n|_{\omega_i} \to h \quad \text{strongly in } L^2(\omega_i)^N \quad \text{and} \quad \nu A \varphi_n - \hat{B}(\hat{u}, \varphi_n) \to w \quad \text{strongly in } V'.$$

Let us introduce the linear mapping $S = \nu A(Id + R)$, where R is the linear operator in Lemma 4.6. From Fredholm's Alternative, we have that $R(S)$ is closed in V' and, from Lemma 4.7, we find that the φ_n are uniformly bounded in V. As an inmediate consequence, we see that, at least for a subsequence, $\varphi_n \to \varphi$ weakly in V, with $- \varphi|_{\omega_i} = h$ and $S\varphi = w$. In other words, $(h, w) \in R(M'_i(\hat{f}_i, \hat{u})^*)$.

At this moment, we can apply the Dubovitskiy-Milyutin formalism to (4.7a) and (4.7b) and deduce that the descent cone D_i and the tangent space T_i are disjoint for $i = 1, 2$ (see [11]):

$$D_i \cap T_i = \emptyset \quad \text{for } i = 1, 2.$$

In view of Lemma 4.5, there exist $(h'_i, w'_i) \in D_i^*$ and $(h'_i, w'_i) \in T_i^*$, not both zero, such that

$$(h'_i, w'_i) + (h'_i, w'_i) = (0, 0). \tag{4.8}$$

Taking into account the definitions of D_i and T_i and the fact that $R(M'_i(\hat{f}_i, \hat{u})^*)$ is closed, we see at once that

$$D_i^* = \{-\lambda \hat{J}'_i(\hat{f}_i, \hat{u}) : \lambda \geq 0\} \quad \text{and} \quad T_i^* = (N(M'_i(\hat{f}_i \hat{u})))^\perp = R(M'_i(f_i, u)^*).$$

Hence, there exist $\varphi_1, \varphi_2 \in V$ such that

$$h'_i = - \varphi_i|_{\omega_i}, \quad w'_i = \nu A \varphi_i - \hat{B}(\hat{u}, \varphi_i) \tag{4.9}$$

and (4.8) can be rewritten in the form

$$\int_{\mathcal{O}_i} (\hat{u} - u_{id}) w_i + \mu \int_{\omega_i} \hat{f}_i h_i = \int_{\omega_i} h'_i h_i + \langle w'_i, w_i \rangle \quad \forall (h_i, w_i) \in L^2(\omega_i)^N \times V. \tag{4.10}$$

Taking $w_i = 0$, we find that $h'_i = \mu f_i$.

On the other hand, taking $h_i = 0$ and recalling (4.9), we see that $\varphi_i|_{\omega_i} = -\mu \hat{f}_i$ for each $i = 1, 2$ and

$$\int_\Omega \left(\nu \nabla \varphi_i \cdot \nabla w_i - (\hat{u} \cdot \nabla) \varphi_i \cdot w_i + (\nabla \hat{u})^t \varphi_i \cdot w_i \right) = a \int_{\mathcal{O}_i} (\hat{u} - u_{id}) w_i \quad \forall w_i \in V.$$

Therefore, φ_i solves, together with some $q_i \in L^2(\Omega)$, the following linear system for each $i = 1, 2$

$$\begin{cases} -\nu \Delta \varphi_i - (\hat{u} \cdot \nabla) \varphi_i + (\nabla \hat{u})^t \varphi_i + \nabla q_i = a(\hat{u} - u_{id}) 1_{\mathcal{O}_i} & \text{in } \Omega, \\ \nabla \cdot \varphi_i = 0 & \text{in } \Omega, \\ \varphi_i = 0 & \text{on } \partial \Omega \end{cases} \tag{4.11a}$$
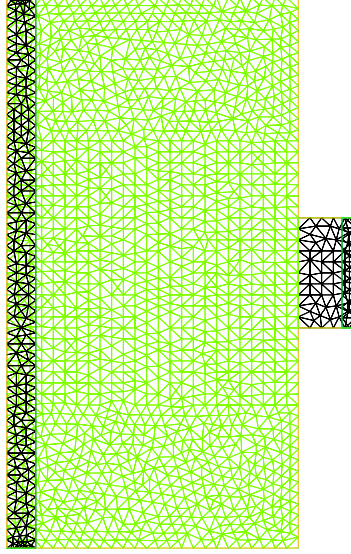
FIGURE 1. Test 1 – The domain and a "rough" mesh; Ω is composed of the bands ω_1 and ω_2, the large rectangle $\mathcal{O}_1$ and the small rectangle $\mathcal{O}_2$. Number of nodes: 1519. Number of triangles: 2876.

and, furthermore,

$$\hat{f}_i = -\frac{1}{\mu} \left. \varphi_i \right|_{\omega_i} . \tag{4.11b}$$

From (4.1), (4.11a) and (4.11b), we deduce that $(\hat{f}_1, \hat{f}_2)$ is a Nash quasi-equilibrium and the proof is done.

4.3. Algorithms

This section is dedicated to the presentation of four iterative algorithms for the computation of a Nash quasi-equilibrium for (4.1) and (4.2). Three of them are similar to those in Sections 2.3 and 3.3; the fourth one is of the Newton kind.

ALG 7: Fixed-Point-like Method

(a) Choose $(f_1^0, f_2^0) \in L^2(\omega_1)^N \times L^2(\omega_2)^N$ and $u^0 \in V$.
(b) Then, for given $n \geq 0$, $(f_1^n, f_2^n) \in L^2(\omega_1)^N \times L^2(\omega_2)^N$ and $u^n \in V$, compute a solution (u^{n+1}, p^{n+1}) to

$$\begin{cases} -\nu \Delta u^{n+1} + (u^n \cdot \nabla) u^{n+1} + \nabla p^{n+1} = f_1^n 1_{\omega_1} + f_2^n 1_{\omega_2} & \text{in } \Omega, \\ \nabla \cdot u^{n+1} = 0 \quad \text{in } \Omega, \\ u^{n+1} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.12}$$

a solution $(\varphi_i^{n+1}, q_i^{n+1})$ to the system

$$\begin{cases} -\nu \Delta \varphi_i^{n+1} - D\varphi_i^{n+1} \cdot u^{n+1} + \nabla q_i^{n+1} = (u^{n+1} - u_{id}) 1_{\mathcal{O}_i} & \text{in } \Omega, \\ \nabla \cdot \varphi_i^{n+1} = 0 \quad \text{in } \Omega, \\ \varphi_i^{n+1} = 0 \quad \text{on } \partial\Omega \end{cases} \tag{4.13}$$

and, finally, set

$$f_i^{n+1} = -\frac{a}{\mu} \left. \varphi_i^{n+1} \right|_{\omega_i} \tag{4.14}$$

for $i = 1, 2$.

ALG 8: Optimal-Step-Gradient-like Method

(a) Choose $(f_1^0, f_2^0) \in L^2(\omega_1)^N \times L^2(\omega_2)^N$ and $u^0 \in V$.

(b) Then, for given $n \geq 0$, $(f_1^n, f_2^n) \in L^2(\omega_1)^N \times L^2(\omega_2)^N$ and $u^n \in V$, compute the solution (u^{n+1}, p^{n+1}) to (4.12) and the solution $(\varphi_i^{n+1}, q_i^{n+1})$ to (4.13) and set

$$f_i^{n+1} = f_i^n - \rho_i^n g_i^{n+1}, \tag{4.15}$$

where

$$g_i^{n+1} = a \left. \varphi_i^{n+1} \right|_{\omega_i} + \mu f_i^n \tag{4.16}$$

and

$$\rho_1^n = \underset{\rho \geq 0}{\operatorname{argmin}} \, J_1^n(f_1^n - \rho g_1^n, f_2^n), \quad \rho_2^n = \underset{\rho \geq 0}{\operatorname{argmin}} \, J_2^n(f_1^n, f_2^n - \rho g_2^n). \tag{4.17}$$

Here, J_1^n and J_2^n are the quadratic functionals respectively similar to J_1 and J_1 obtained when we replace (4.1) by the n-th linear system

$$\begin{cases} -\nu \Delta u + (u^n \cdot \nabla) u + \nabla p = f_1 1_{\omega_1} + f_2 1_{\omega_2} & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega. \end{cases}$$

ALG 9: Optimal-Step-Conjugate-Gradient-like Method

(a) Choose $(f_1^0, f_2^0) \in L^2(\omega_1)^N \times L^2(\omega_2)^N$ and $u^0 \in V$.

(b) For $n = 0$, perform one step of **ALG 8** and set $d_i^0 = g_i^0$.

(c) Then, for given $n \geq 1$, $(f_1^n, f_2^n) \in L^2(\omega_1)^N \times L^2(\omega_2)^N$ and $u^n \in V$, compute the solution (u^{n+1}, p^{n+1}) to (4.12), the solution $(\varphi_i^{n+1}, q_i^{n+1})$ to (4.13) and then set

$$f_i^{n+1} = f_i^n - \rho_i^n d_i^n, \tag{4.18}$$

where

$$\begin{cases} d_i^n = g_i^n + \gamma_i^n d_i^{n-1}, \quad \gamma_i^n = \dfrac{(g_i^n - g_i^{n-1}, g_i^n)}{\|g_i^{n-1}\|^2}, \\ g_i^n = a \left. \varphi_i^n \right|_{\omega_i} + \mu f_i^n \end{cases} \tag{4.19}$$

and

$$\rho_1^n = \underset{\rho \geq 0}{\operatorname{argmin}} \, J_1^n(f_1^n - \rho d_1^n, f_2^n), \quad \rho_2^n = \underset{\rho \geq 0}{\operatorname{argmin}} \, J_2^n(f_1^n, f_2^n - \rho d_2^n). \tag{4.20}$$

As in Section 3, we find in **ALG 8** and **ALG 9** some 1D extremal problems. Again, the functions to minimize are not quadratic and a numerical method is needed. This will be explained below.

Before presenting the results of some simulations, let us describe another algorithm. It is based on Newton's iterates and, like **ALG 7**, aims to compute directly a solution to the optimality system (4.4). In practice, this method is much faster but, as is usual for Newton methods and variants, needs a nontrivial starting process (see below).

ALG 10: Newton Method

We want to solve (4.4) with $\nu = \tilde{\nu}$. We fix a decreasing factor $\tilde{a} \in (0,1)$ and we do as follows.

(a) Choose $(f_1^0, f_2^0) \in L^2(\omega_1)^N \times L^2(\omega_2)^N$ and $\nu^0 \in \mathbb{R}^+$ and compute a solution (u^0, p^0) to the Stokes problem

$$\begin{cases} -\nu^0 \Delta u^0 + \nabla p^0 = f_1^0 1_{\omega_1} + f_2^0 1_{\omega_2} & \text{in } \Omega, \\ \nabla \cdot u^0 = 0 & \text{in } \Omega, \\ u^0 = 0 & \text{on } \partial\Omega \end{cases} \tag{4.21}$$

and a solution (φ_i^0, q_i^0) to

$$\begin{cases} -\nu^0 \Delta \varphi_i^0 + \nabla q_i^0 = (u^0 - u_{id})1_{\mathcal{O}_i} & \text{in } \Omega, \\ \nabla \cdot \varphi_i^0 = 0 & \text{in } \Omega, \\ \varphi_i^0 = 0 & \text{on } \partial\Omega \end{cases} \tag{4.22}$$

and take

$$f_i^0 = -\frac{a}{\mu} \, \varphi_i^0\big|_{\omega_i} \quad \text{for } i = 1,2 \text{ and then } \nu^1 = \max\{\tilde{\nu}, \tilde{a}\nu^0\}.$$

(b) Then, for given $n \geq 0$, ν^n, $(f_1^n, f_2^n) \in L^2(\omega_1)^N \times L^2(\omega_2)^N$, (u^n, p^n) and (φ_i^n, q_i^n), do the following:

(b.1) Take $f_i^{n,0} = -\frac{a}{\mu}\varphi_i^n\,|_{\omega_i}$, $u^{n,0} = u^n$, $\varphi_i^{n,0} = \varphi_i^n$ and $\nu^{n+1} = \max(\tilde{a}\nu^n, \tilde{\nu})$.

(b.2) For given $k \geq 0$, $f_i^{n,k}, u^{n,k}$ and $\varphi_i^{n,k}$, set until convergence

$$F^{n,k} := -\nu^{n+1}\Delta u^{n,k} + (u^{n,k} \cdot \nabla)u^{n,k} - f_1^{n,k}1_{\omega_1} - f_2^{n,k}1_{\omega_2}$$
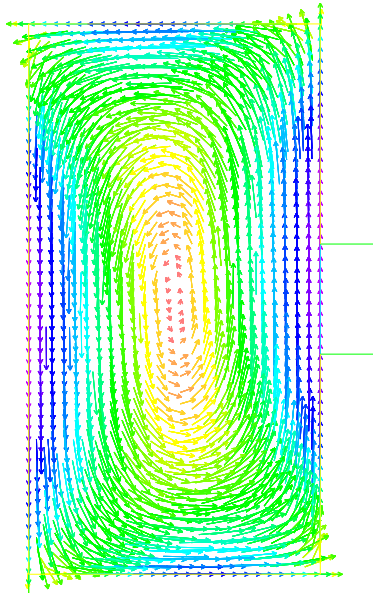
and

$$G_i^{n,k} := -\nu^{n+1}\Delta\varphi_i^{n,k} - (u^{n,k} \cdot \nabla)\varphi_i^{n,k} + (\nabla u^{n,k})^t\varphi_i^{n,k} - (u^{n,k} - u_{id})1_{\mathcal{O}_i},$$

compute the solution $(v^k, h^k, \psi_i^k, \eta_i^k)$ to

$$\begin{cases} -\nu^{n+1}\Delta v^k + (u^{n,k} \cdot \nabla)v^k + (v^k \cdot \nabla)u^{n,k} + \nabla h^k = F^{n,k}, & \text{in } \Omega, \\ \nabla \cdot v^k = 0 & \text{in } \Omega, \\ v^k = 0 & \text{on } \partial\Omega, \\ -\nu^{n+1}\Delta\psi_i^k - (u^{n,k}\cdot\nabla)\psi_i^k - (v^k\cdot\nabla)\varphi_i^{n,k} \\ \qquad\qquad + (\nabla u^{n,k})^t\psi_i^k + (\nabla v^k)^t\varphi_i^{n,k} + \nabla\eta_i^k = G_i^{n,k} & \text{in } \Omega, \\ \nabla \cdot \psi_i^k = 0 & \text{in } \Omega, \\ \psi_i^k = 0 & \text{on } \partial\Omega \end{cases} \tag{4.23}$$

and take

$$u^{n,k+1} = u^{n,k} - v^k, \quad \varphi_i^{n,k+1} = \varphi_i^{n,k} - \psi_i^k, \quad \nu^{n+1} = \max(\tilde{\nu}, \tilde{a}\nu^n). \tag{4.24}$$

FIGURE 2. Test 1 – The function u_{1d}.

Note that **ALG 7** and **ALG 10** are designed to compute a solution to the optimality system (4.4) (a quasi-equilibrium) that, possibly, is not be a Nash equilibrium. Thus, from the viewpoint of the Calculus of Variations, **ALG 7** and **ALG 10** can be seen as "indirect methods". Contrarily, **ALG 8** and **ALG 9** can be viewed as realizations of the "direct method" of Calculus of Variations.

4.4. Numerical experiments

In order to illustrate the behavior of the previous algorithms, we will present in this section the results of some numerical experiments. The computations have been performed with the FreeFem++ package (see [14]).

In our experiments, we try to solve numerically the optimality system (4.4), where a is chosen with $0 < a < 2$ and $\mu = 2 - a$. In the tests, several values of a and the Reynolds number $Re := UL/\nu$ have been chosen. Here, the characteristic speed and length are respectively given by $U = \max(\|u_{1d}\|, \|u_{2d}\|)$ and $L = \max_{x,x' \in \overline{\Omega}} |x - x'|$.

4.4.1. Test 1

In this first test, our domain is composed of two rectangles $\mathcal{O}_1$ and $\mathcal{O}_2$ and we assume that the controls act on the narrow bands ω_1 and ω_2. In order to solve numerically the systems (4.12), (4.13), (4.21), (4.22) and (4.23), we have used meshes like the one in Figure 1 and a mixed finite element formulation with continuous piecewise $\mathbb{P}_1$-bubble and $\mathbb{P}_1$ functions respectively for the velocity field and the pressure; for details, see [12, 13].

The data u_{id} are the following: $u_{1d} = \nabla \times \psi_{1d}$, where ψ_{1d} is the solution to the problem

$$\begin{cases} -\Delta \psi_{1d} = 1 & \text{in } \mathcal{O}_1, \\ \psi_{1d} = 0 & \text{on } \partial \mathcal{O}_1 \end{cases} \tag{4.25}$$

and $u_{2d} \equiv 0$. That means that the "desired" (ideal) configurations correspond to a uniformly rotating flow in $\mathcal{O}_1$ and a fluid at rest in $\mathcal{O}_2$ (see Fig. 2).

For **ALG 8** and **ALG 9**, the computation of the steps ρ_i^n has been performed by applying a standard dichotomy process. This provides acceptable values after a reduced amount of iterates.
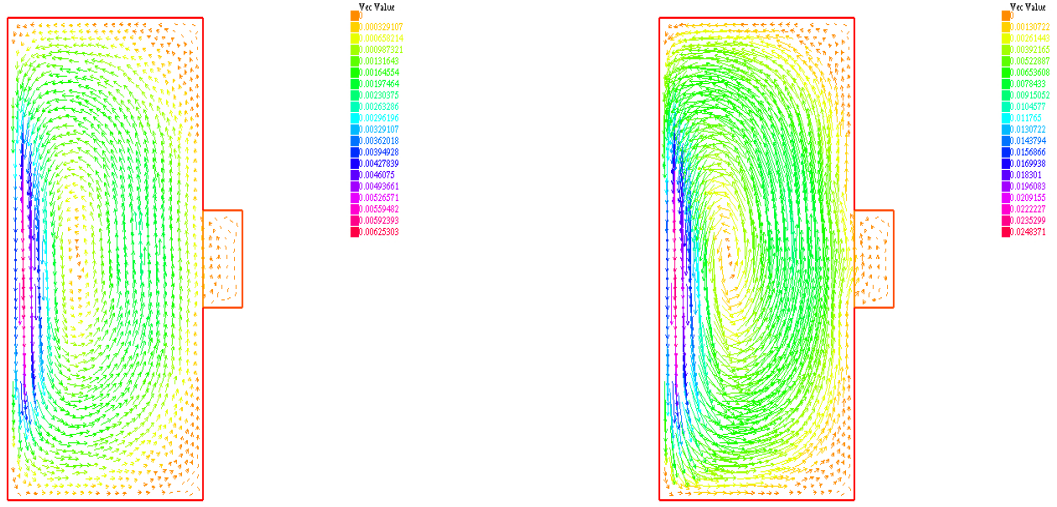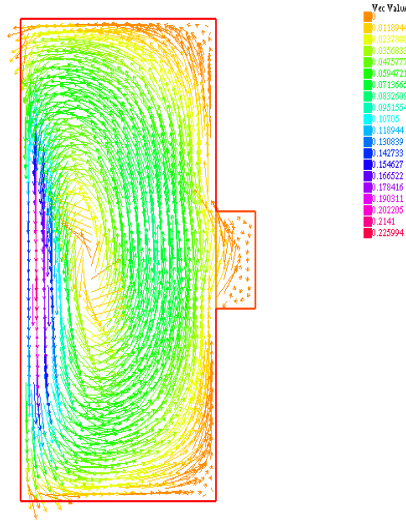
(A) $Re = 7$, number iterates: 5, $\max_{\overline{\Omega}} |u| = 0.077$.

(B) $Re = 14$, number iterates: 5, $\max_{\overline{\Omega}} |u| = 0.21$.

(C) $Re = 44$, number iterates: 14, $\max_{\overline{\Omega}} |u| = 1.54$.

FIGURE 3. Test 1 – The computed velocity fields computed with **ALG 8** for various Re. Here, $a = 1.2$ and $\mu = 2 - a = 0.8$.

In the case of **ALG 7**, **ALG 8** and **ALG 9**, the stopping test has been

$$\|u^{n+1} - u^n\|_{L^\infty} + \|p^{n+1} - p^n\|_{L^\infty} + \|q^{n+1} - q^n\|_{L^\infty} \leq \varepsilon,$$

with $\varepsilon = 10^{-6} Re$. This choice establishes a criterion with uniform severity. This has also been the stopping criterion for the external iterates in **ALG 10**. For the internal loops (indexed by k), the stopping test has been

$$\|u^{n,k+1} - u^{n,k}\|_{L^\infty} + \|\varphi^{n,k+1} - \varphi^{n,k}\|_{L^\infty} \leq \varepsilon.$$
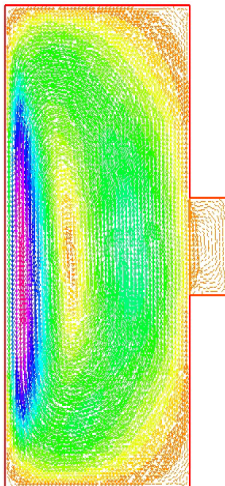
FIGURE 4. Test 1 – The computed velocity field with **ALG 10** for $Re = 141$. Now, $a = 1.8$ and $\mu = 2 - a = 0.2$. The mesh is composed of 6003 vertices and 11684 triangles; $\max_{\overline{\Omega}} |u| = 3.17$.

		a				
		0.1	0.5	1	1.5	1.8
	7.	3	5	5	5	5
Re	14.	3	5	5	9	9
	44.	7	7	11	51	129
	100.	8	18	41	107	1838
	141.	12	31	127	2206	4997

(A) Grad. (**ALG 8**)

		a				
		0.1	0.5	1	1.5	1.8
	7.	2	2	3	3	4
Re	14.	2	3	4	4	5
	44.	4	6	11	13	17
	100.	6	9	13	16	23
	141.	8	14	17	21	37

(B) Conjugate Grad. (**ALG 9**)

		a				
		0.1	0.5	1	1.5	1.8
	7	2	3	3	3	3
Re	14.	3	3	3	3	4
	44.	7	7	9	33	35
	100.	129	130	137	139	161
	141.	234	267	318	359	--

(C) Newton (**ALG 10**)

FIGURE 5. Test 1 – The numbers of iterates for various values of Re, a and $\mu = 2 - a$.

In all cases, the starting control pair has been $(f_1^0, f_2^0) = (0, 0)$. On the other hand, for all choices of a and Re, all the algorithms provide the same solution.

Some results are depicted in Figure 3. There, we can see the computed velocity fields corresponding to some Nash quasi-equilibria, for $a = 1.2$ and various Re. As expected, the appearance of the velocity field in $\mathcal{O}_1$ (resp. $\mathcal{O}_2$) reminds us of u_{1d} (resp. u_{2d}).

We also present in Figure 4 the velocity field computed with **ALG 10**, a finer mesh and a higher Re.

In order to compare the behavior of the Gradient-like, Conjugate Gradient-like and Newton methods, we present some related numerical values in the Tables in Figure 5. Specifically, we have indicated there the numbers of iterates needed in each case to fulfill the stopping test and produce an approximation. In the case of **ALG 10**, we have included the total numbers of iterates (that is, the iterates needed for initialization have been taken into account).

Also, we have included in Figure 6 a Table with a comparison of the computation times and required numbers of iterates for each method.

The convergence of the numerical aproximations toward a quasi-equilibrium has been checked in the following way. We have considered several meshes corresponding to triangles of typical sizes $0.2, 0.13, 0.1, \ldots$, as indicated in Table 1. For $h = 0.05$, with 23327 vertices and 46012 triangles, we have solved (4.1)–(4.2) for $a = 1.8$

	CPU TIME	ITERATES
GRADIENT	28305.9	2206
CONJUGATE GRADIENT	3920.53	21
NEWTON	848.034	259

FIGURE 6. Test 1 – Computation times (in seconds) and numbers of iterates to reach an error $\leq \varepsilon = 10^{-6} Re$ for $Re = 141.$, $a = 1.5$ and $\mu = 2 - a = 0.5$.

TABLE 1. Test 1 – L^2 relative errors associated with various numerical approximations.

h	Number of vertices	Number of triangles	L^2 relative error
0.2	1519	2876	0.024683
0.1333	3419	6658	0.010674
0.1	6003	11684	0.0074805
0.08	9489	18578	0.0066609
0.0667	13176	25870	0.002432
0.0571	18431	36302	0.002237
0.05	23327	46012	0.
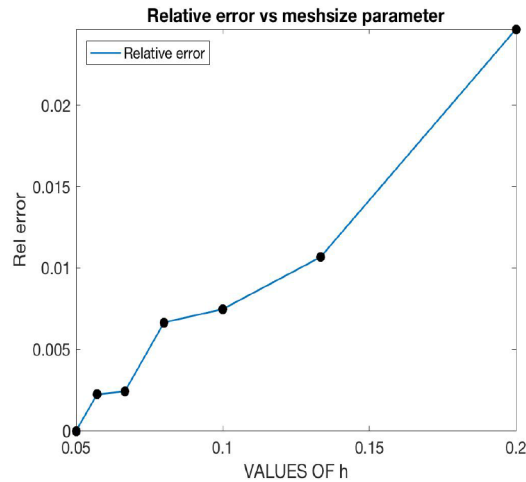
FIGURE 7. Test 1 – Relative errors of the numerical approximations for various meshsize parameters h.

and $Re = 141$ and we have accepted that the computed solution is, for practical purposes, our Nash quasi-equilibrium. Then, we have measured the relative L^2 errors associated with the computations with the other grids. See Table 1 and Figure 7 for quantitative information.

4.4.2. Test 2

We consider the flow of a fluid in a pipe. The fluid enters with a parabolic horizontal profile on the left. It is allowed to exit vertically upwards and downwards before it reaches the observation domains $\mathcal{O}_1$ and $\mathcal{O}_2$ (see Fig. 10). Our objective is to know how we have to act in two specific control regions (ω_1 and ω_2) to make the fluid velocity reach a Nash quasi-equilibrium associated with u_{1d} and u_{2d} (see Fig. 9). The domain and a mesh are depicted in Figure 8 (see the legend for explanations). What we do now is equivalent to solve a
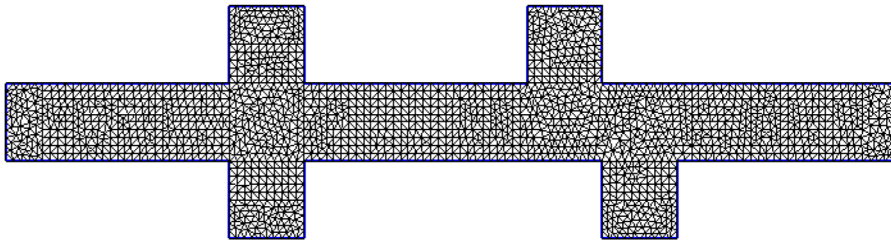
FIGURE 8. Test 2- The domain and a "rough" mesh; Ω is composed of the main pipe, two first rectangles (ω_1 and ω_2), a second upper rectangle $\mathcal{O}_1$ and a second lower rectangle $\mathcal{O}_2$. Number of nodes: 1541. Number of triangles: 2774.
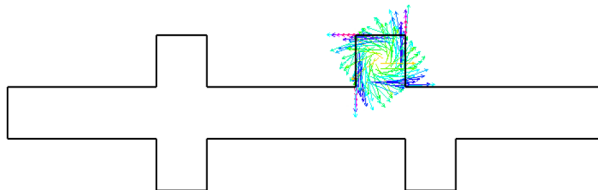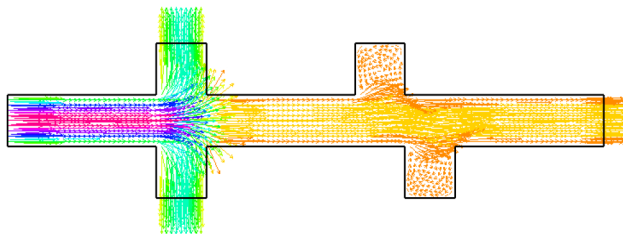

FIGURE 9. Test 2 – The function u_{1d}.

FIGURE 10. Test 2 – The "free" parabolic profile entering the pipe.

bi-objective boundary control problem, where the controls act on the parts of $\partial\omega_1$ and $\partial\omega_2$ that "touch" the pipe.

The finite element method has been the same as before.

The data u_{id} are similar to those for Test 1: $u_{1d} = \nabla \times \psi_{1d}$, where ψ_{1d} is the solution to (4.25) and $u_{2d} \equiv 0$. Thus, the "desired" (ideal) configurations correspond to a uniformly rotating flow in $\mathcal{O}_1$ and a fluid at rest in $\mathcal{O}_2$ (see Fig. 9). The stopping test and the initial controls (f_1^0, f_2^0) have been as in the previous section. Again, all algorithms have provided the same solution in all cases.

We present in Figure 10 the velocity field corresponding to a "free" state, that is, an uncontrolled solution.

Some results are depicted in Figure 11 for large a and various Re.

We have also checked the convergence of the numerical aproximations with results similar to those in Section 4.4.1. We have solved (4.1)–(4.2) for $a = 1.99$, $Re = 240$ and $h = 0.025$, which corresponds to a mesh with 30405 vertices and 59448 triangles. We have assumed that this computed solution coincides with the Nash quasi-equilibrium and we have measured the relative L^2 errors associated with the other approximations. For details, see Table 2 and Figure 12.

5. CONCLUSIONS, FURTHER COMMENTS AND OPEN QUESTIONS

In this paper, we have been able to analyze and sometimes solve several that bi-objective control problems for stationary PDEs. The existence of Nash equilibria (or at least quasi-equilibria) has been established. We
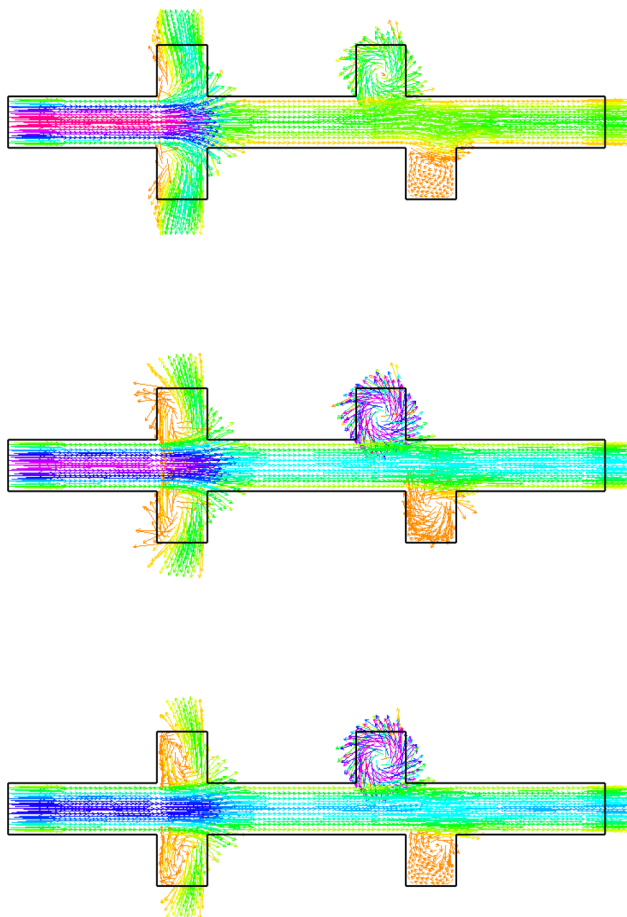
FIGURE 11. Test 2 – The final velocity fields computed with **ALG 10** for various Reynolds numbers and $a = 1.99$. From top to bottom, $Re = 240$, $Re = 720$ and $Re = 1200$.

TABLE 2. Test 2 – L^2 relative errors associated with various numerical approximations.

h	Number of vertices	Number of triangles	L^2 relative error
0.1	1958	3574	0.014601
0.0667	4251	7998	0.0076592
0.05	7706	14770	0.0031519
0.04	11248	21644	0.0024084
0.0333	16420	31818	0.0010536
0.0286	21867	42542	0.0010438
0.025	30405	59448	0.

have also deduced related characterizations (optimality systems) and several iterative algorithms have been introduced.

Our numerical experiments, that concern Navier-Stokes control, provide useful illustrations and can be viewed as a confirmation of the interest of the subject.
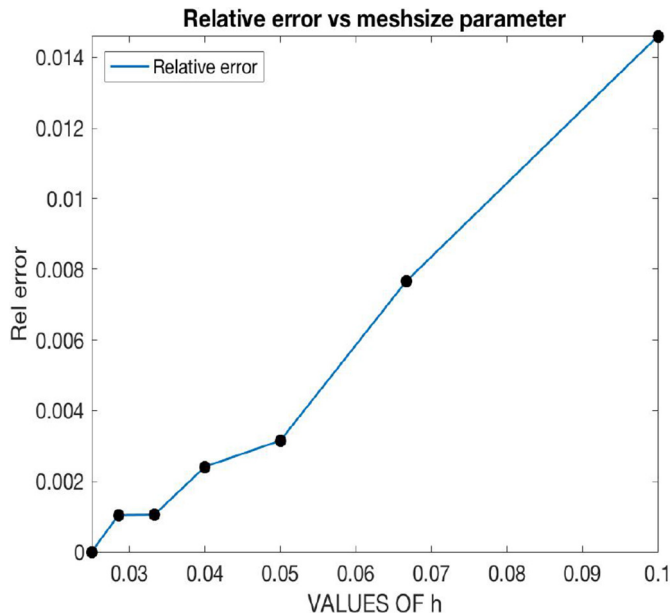
FIGURE 12. Test 2 – Relative errors of the numerical approximations for various meshsize parameters h.

The problems considered in the previous sections lead to a set of interesting open questions. Let us indicate some of them:

- It is unknown whether convergence results can be established for **ALG 2** and **ALG 3** (see Sect. 2.3). It is reasonable to expect this at least for small a/μ. The same can be said for **ALG 5** and **ALG 6** (Sect. 3.3) and **ALG 8** and **ALG 9** (Sect. 4.3).

  The reason is that, in these iterates, we respect the gradient or conjugate gradient main idea and, to go from f_i^n to f_i^{n+1}, we use either the direction $\partial J_i/\partial f_i$ or a suitable approximation.

- It is also unknown whether Nash equilibria exist for the Navier-Stokes system (4.1) and the functionals given in (4.2). As can be deduced from the arguments in Section 4.2, the lack of uniqueness of the system and the lack of convexity of the functionals are nontrivial difficulties in this context.

  Indeed, it does not seem easy to adapt the arguments in the proof of Theorem 3.2. One could try to find an equilibrium directly, forgetting the optimality system and minimizing J_1 and J_2 in an iterative process. But, to our knowledge, no argument has been given up to now.

- Finally, the extension of the results in Section 4 to the time-dependent Navier-Stokes system is also a challenge. It is not difficult to prove some related assertions in the 2D case. However, once more, not many things can be said when the spatial dimension is three.

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