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Quasilinear approximation for interval-valued functions via generalized Hukuhara differentiability

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Abstract

In this paper, a new generalized Hukuhara differentiability concept for interval-valued functions defined on $\mathbb{R}^n$ is proposed, which extends the classical Fréchet differentiability notion and provides an interval quasilinear approximation for an interval-valued function in a neighborhood of a point at which such function is gH-differentiable. Moreover, it overcomes the shortcomings generated by the use of the gH-differentiability concept previously presented in the literature, and this presents a good perspective on interval and fuzzy environments. Several properties of this new concept are investigated and compared with the previous concept properties. Furthermore, the gH-differentiability concept is extended for a fuzzy function, and its introduction is argued and illustrated with examples.

Keywords Interval-valued functions · Quasilinear functions · Generalized Hukuhara differentiability

Mathematics Subject Classification 90C30 · 90C70 · 90C99

1 Introduction

It is known that real-valued function differentiability is one of the most important concepts of real analysis, which has been allowing the rigorous formalizations of important physical

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theories as well as many phenomena of real-world description through mathematical models. Moreover, the differential calculus development has provided valuable mathematical tools to several areas as computer science, biology, engineering, economics, among others.

The interval analysis is a mathematics area introduced in the works of Moore (1959), Sunaga (1958), and Warmus (1956), which have been received great highlight for providing mathematical tools that allow modelling and dealing with problems under interval type uncertainties. However, there exist some shortcomings in the algebraic operation “difference” given in these works, which do not allow developing a consistent differential theory in interval spaces, and it has motived the introduction of some different kinds of algebraic arithmetic in interval space (Lodwick 2015; Markov 1979; Plotnikova 2005 and its references).

In Markov (1979), a “difference" on interval spaces is defined, with which was introduced an interval version of the Gâteaux differentiability and was demonstrated that calculus for interval-valued functions of a real variable could be developed. Also in this paper, it was stated that such interval version of the Gâteaux differentiability is equivalent to a specific interval version of the Frechet differentiability. That statement does not hold as it is shown in that paper. Basically, the shortcoming in such statement arises from the fact of that interval spaces do not have a vector structure.

Years later, Stefanini (2008) proposed a generalization of the Hukuhara difference called the generalized Hukuhara difference (gH-difference for short) that coincides with the difference defined in Markov (1979). Using the gH-difference, Stefanini and Bede (2009) introduced a generalization of the Hukuhara differentiability for interval-valued functions of a real variable called the generalized Hukuhara differentiability (gH-differentiability for short), which is also an interval version of the Gâteaux differentiability and which coincides with the differentiability concept given in Markov (1979) (see Chalco-Cano et al. 2011 for details). Also, in Stefanini and Bede (2009), the authors compared differentiability concepts for interval-valued functions previously proposed in the literature and showed that the gH-differentiability is more general than these others. Since then, the gH-differentiability concept has been applied to different fields (Armand et al. 2016; Bede and Stefanini 2013; Chalco-Cano et al. 2012, 2013a, b; Long et al. 2015; Majumder et al. 2016; Villamizar-Roa et al. 2015; Wang et al. 2019), and has been proved to be a powerful tool with many applications in interval and fuzzy-valued functions spaces.

In Ahmad et al. (2016), Chalco-Cano et al. (2013a, b), and Luhandjula and Rangoaga (2014), the gH-differentiability was extended for interval-valued functions of several real variables. However, we understand that these definitions have some important drawbacks which are exposed herein.

Recently, Stefanini and Arana-Jiménez (2019) introduced a new gH-differentiability concept for interval-valued functions of several real variables and they showed that such concept is equivalent to an interval version of the Gâteaux differentiability.

From the approximation theory viewpoint, it is very interesting to obtain a gH-differentiability definition for interval-valued functions of several real variables that extend the classical Fréchet differentiability preserving a linear approximation for an interval-valued function in a neighborhood of a point at which such function is gH-differentiable. Nevertheless, due to the nonexistent vector structure on interval spaces, such linear approximation, in general, it is not possible. In this work, we present a new gH-differentiability definition for interval-valued functions of several real variables which extends the classical Fréchet differentiability and which provides an interval quasilinear approximation for an interval-valued function in a neighborhood of a point at which such function is gH-differentiable. Also, it is shown that this definition is equivalent to the one given (Stefanini and Arana-Jiménez 2019), and for the one-dimensional case, it also coincides with definitions given by Markov (1979) and Stefanini in Stefanini (2008), and gives a correct meaning for the Markov's statement above cited.

This work is organized as follows: Sect. 2 recalls some known results about interval analysis. Section 3 shows that the condition for gH-differentiability given by Markov (1979) is only sufficient, and then, a necessary and sufficient condition for such gH-differentiability, which is the starting point for the introduction of gH-differentiability of interval-valued functions of several real variables, is introduced. Section 4 shows that some of the gH-differentiability previous concepts for interval-valued functions of several real variables introduced in the literature have shortcomings, and it provides a new gH-differentiability definition for interval-valued functions of several real variables that extends the classical version of Fréchet differentiability, generating a gH-differential that is a quasilinear interval-valued function. Section 4 also shows the gH-differentiability concept for interval-valued functions of several real variables given by Stefanini and Arana-Jiménez (2019) which is equivalent to the gH-differentiability one herein introduced. The extension of gH-differentiability to fuzzy environment is presented in Sect. 5. Finally, Sect. 6 presents our last considerations.

2 Preliminaries

Consider the space $I(\mathbb{R})$ of all closed and bounded intervals of real numbers, that is, $I(\mathbb{R}) = \left\{\left[\underline{a}, \overline{a}\right] \ / \ \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \underline{a} \leq \overline{a}\right\}$. Given $A, B \in I(\mathbb{R})$ and $\lambda \in \mathbb{R}$, the interval arithmetic operations are defined by

$$A + B = \left[\underline{a}, \overline{a}\right] + \left[\underline{b}, \overline{b}\right] = \left[\underline{a} + \underline{b}, \overline{a} + \overline{b}\right], \quad \lambda A = \begin{cases} \left[\lambda\underline{a}, \lambda\overline{a}\right] & \text{if } \lambda \geq 0 \\ \left[\lambda\overline{a}, \lambda\underline{a}\right] & \text{if } \lambda < 0, \end{cases} \tag{1}$$

$$\text{and } A \ominus_{gH} B = C \Leftrightarrow \begin{cases} (a) \ A = B + C & \text{if } \mu(B) \leq \mu(A) \\ (b) \ B = A + (-1)C & \text{if } \mu(B) > \mu(A), \end{cases} \tag{2}$$

where $\mu(A)$ denotes the length of an interval $A = [\underline{a}, \overline{a}]$, i.e., $\mu(A) = \overline{a} - \underline{a}$.

We write $A = [\alpha \vee \beta]$ if α and β are the end-points of the interval $A \in I(\mathbb{R})$, but $\alpha \leq \beta$ it is not necessarily satisfied. The gH-difference of two intervals always exists and it is equal (Stefanini and Bede 2009) to

$$A \ominus_{gH} B = [\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}, \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}] = [\underline{a} - \underline{b} \vee \overline{a} - \overline{b}]. \tag{3}$$

The space $(I(\mathbb{R}), H)$ is a complete and separable metric space (Diamond and Kloeden 1994), where $H(A, B) = \max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\}$ is the distance between $A, B \in I(\mathbb{R})$. The absolute value of $A \in I(\mathbb{R})$ is the real number $\|A\|$ given by $H(A, [0, 0]) = \|A\|$.
Next, some known properties related to the interval arithmetic operations and to intervals length are recalled.

Proposition 2.1 (Markov 1979; Stefanini and Bede 2009) *Given $A, B, C \in I(\mathbb{R})$ and $\lambda \in \mathbb{R}$, it follows that:*

1. $\mu(A + B) = \mu(A) + \mu(B)$,
2. $\mu(\lambda A) = |\lambda|\mu(A)$,
3. $\mu(A \ominus_{gH} B) = |\mu(A) - \mu(B)|$,
4. $\lambda(A \ominus_{gH} B) = \lambda A \ominus_{gH} \lambda B$,
5. $A \ominus_{gH} B = -(B \ominus_{gH} A)$,
6. $[0, 0] \ominus_{gH} A = -A$,
7. $A \ominus_{gH} A = [0, 0]$,
8. $A \ominus_{gH} B = [0, 0] \Leftrightarrow A = B$,
9. $(A + B) \ominus_{gH} B = A$,
10. $A \ominus_{gH} (A + B) = -B$.

Given $n \in \mathbb{N}$, an application $F : S \subseteq \mathbb{R}^n \to I(\mathbb{R})$, such that $F(x) = [\underline{f}(x), \overline{f}(x)]$ is called an interval-valued function of several variables, where $\underline{f}, \overline{f} : S \to \mathbb{R}$ are real-valued functions, such that $\underline{f}(x) \leq \overline{f}(x)$, for all $x \in S$, and $\underline{f}$ and $\overline{f}$ are its end-point functions. If $n = 1$, then F is called an interval-valued function of a real variable.

Based on the limit concept of set-valued function (Aubin and Cellina 1984), and on the gH-difference given by Stefanini (2008), the following differentiability concept for interval-valued functions was introduced.

Definition 2.1 (Stefanini and Bede 2009) Let $S \subseteq \mathbb{R}$ be an open and nonempty set and let $F : S \to I(\mathbb{R})$ be an interval-valued function, and then, the generalized Hukuhara derivative (gH-derivative, for short) of F at $x^* \in S$ is defined by

$$F'_{gH}(x^*) = \lim_{h \to 0} \frac{F(x^* + h) \ominus_{gH} F(x^*)}{h}. \tag{4}$$

If $F'_{gH}(x^*) \in I(\mathbb{R})$ satisfying (4) exists, we say that F is generalized Hukuhara differentiable (gH-differentiable, for short) at x^*.

The relationships between the gH-differentiability of an interval-valued function $F(x) = [\underline{f}(x), \overline{f}(x)]$ and the differentiability of its end-point functions, $\underline{f}(x)$ and $\overline{f}(x)$, have been completely studied in Qiu (2020).

3 New necessary and sufficient conditions for gH-differentiability on $\mathbb{R}$

Markov (1979) states that a necessary and sufficient condition for the gH-differentiability of F at x^* is the existence of an interval $A \in I(\mathbb{R})$ and an interval-valued function $P : (-\epsilon, \epsilon) \to I(\mathbb{R})$ with $\lim_{h \to 0} P(h) = [0, 0]$, such that

$$F(x^* + h) \ominus_{gH} F(x^*) = hA + hP(h). \tag{5}$$

However, this condition, in general, it is not necessary as it is shown in the following example.

Example 3.1 Let $F : (-1, 1) \to I(\mathbb{R})$ be the interval-valued function given by $F(x) = (1 - x^2)[1, 2]$. Then, $\underline{f}$ and $\overline{f}$ are differentiable functions, and consequently (see Chalco-Cano et al. 2011), F is gH-differentiable with $F'_{gH}(x) = -2x[1, 2]$. Since

$$F(x + h) \ominus_{gH} F(x) = \left[\min\left\{-h^2 - 2xh, 2(-h^2 - 2xh)\right\}, \max\left\{-h^2 - 2xh, 2(-h^2 - 2xh)\right\}\right]$$

for all $x \in (-1, 1)$ and $h \in \mathbb{R}$, such that $x + h \in (-1, 1)$, then given $x < 0$, there exists $\epsilon_1 > 0$, such that $h + 2x < 0$ for all $h \in (0, \epsilon_1)$, and consequently

$$F(x + h) \ominus_{gH} F(x) = [-h^2 - 2xh, 2(-h^2 - 2xh)].$$

However, does not exist any interval-valued function $P : B(-\epsilon, \epsilon) \to I(\mathbb{R})$, $P(h) = [\underline{p}(h), \overline{p}(h)]$ for all $h \in (-\epsilon, \epsilon)$, such that

$$[-h^2 - 2xh, -2(-h^2 - 2xh)] = h(-2x)[1, 2] + h[\underline{p}(h), \overline{p}(h)]. \tag{6}$$

Indeed, let us suppose that there exists an interval-valued function $P : (-\epsilon, \epsilon) \to I(\mathbb{R})$, $P(h) = [\underline{p}(h), \overline{p}(h)]$ for all $h \in (-\epsilon, \epsilon)$ such that (6) holds, then setting $\tilde{\epsilon} = \min\{\epsilon, \epsilon_1\}$ and given $h \in (0, \tilde{\epsilon})$, it follows that $\underline{p}(h) = -h$ and $\overline{p}(h) = -2h$, that is, $\overline{p}(h) < \underline{p}(h)$, which contradicts the fact that P is an interval-valued function. Therefore, F is gH-differentiable, but does not exist an interval $A \in I(\mathbb{R})$ and an interval-valued function $P : (-\epsilon, \epsilon) \to I(\mathbb{R})$, such that (5) holds.

Although the condition given by Markov is not necessary for gH-differentiability of F at $x^* \in S \subseteq \mathbb{R}$, the next result shows that the condition is sufficient. Moreover, such result presents a relation between the left and right sides of (5) when F it is gH-differentiable at $x^* \in S \subseteq \mathbb{R}$. Before to present such result, a technical proposition is necessary.

Proposition 3.1 *Let $F, P : S \subseteq \mathbb{R} \to I(\mathbb{R})$ be interval-valued functions, such that* $\lim_{x\to x_0} P(x) = [0, 0]$. *Given $x_0 \in S$, it follows that:*

(i) if $F(x) = A + P(x)$, then $\lim_{x\to x_0} F(x) = A$;
(ii) if $\lim_{x\to x_0} F(x) = A$, then $F(x) \subseteq A + P(x)$.

Proof (i) This result follows directly from the metric H properties.
(ii) If $\lim_{x\to x_0} F(x) = A$, from Proposition 7 in Stefanini and Bede (2009), it follows that $\lim_{x\to x_0}(F(x) \ominus_{gH} A) = [0, 0]$.

Let $P : S \to I(\mathbb{R})$ be the interval-valued function given by $P(x) = F(x) \ominus_{gH} A$ for all $x \in S$. Thus, $\lim_{x\to x_0} P(x) = [0, 0]$, and from definition of $\ominus_{gH}$, it follows that:

$$(F(x) \ominus_{gH} A) = P(x) \Leftrightarrow \begin{cases} (a)\ F(x) = A + P(x), \text{ or} \\ (b)\ A = F(x) + (-1)P(x) \end{cases}$$
$$\Leftrightarrow \begin{cases} (a)\ \underline{f}(x) - \underline{a} = \underline{p}(x) \text{ and } \overline{f}(x) - \overline{a} = \overline{p}(x), \text{ or} \\ (b)\ \underline{f}(x) - \underline{a} = \overline{p}(x) \text{ and } \overline{f}(x) - \overline{a} = \underline{p}(x) \end{cases}.$$

If (a) holds, then $F(x) = A + P(x)$. On the other hand, if (b) holds, then $\underline{f}(x) - \underline{a} = \overline{p}(x) \geq \underline{p}(x)$ and $\overline{f}(x) - \overline{a} = \underline{p}(x) \leq \overline{p}(x)$, that is, $\underline{a} + \underline{p}(x) \leq \underline{f}(x)$ and $\overline{f}(x) \leq \overline{a} + \overline{p}(x)$. Therefore, from (a) and (b), it follows that $F(x) \subseteq A + P(x)$. □

Next results follow directly from Proposition 3.1.

Theorem 3.1 *Let $S \subseteq \mathbb{R}$ be an open and nonempty set and let $F : S \subseteq \mathbb{R} \to I(\mathbb{R})$ be an interval-valued function. Given $x^* \in S$, it follows that:*

(i) *If there exist an interval $A \in I(\mathbb{R})$ and an interval-valued function $P : (-\epsilon, \epsilon) \to I(\mathbb{R})$ with* $\lim_{h\to 0} P(h) = [0, 0]$*, such that* (5) *holds, then F is gH-differentiable at x^*.*
(ii) *If F is gH-differentiable at x^*, then there exists an interval-valued function P : $(-\epsilon, \epsilon) \to I(\mathbb{R})$ with* $\lim_{h\to 0} P(h) = [0, 0]$*, such that*

$$F(x^* + h) \ominus_{gH} F(x^*) \subseteq hF'_{gH}(x^*) + hP(h). \tag{7}$$

The following result provides a necessary and sufficient gH-differentiability condition for interval-valued functions of a real variable providing a correct sense for the Markov's statement. This important result is the starting point to define the gH-differentiability for interval-valued functions of several variables.

Theorem 3.2 *Let $S \subseteq \mathbb{R}$ be an open and nonempty set. An interval-valued function $F : S \to I(\mathbb{R})$ is gH-differentiable at $x^* \in S$ if and only if there exist an interval $A \in I(\mathbb{R})$ and an interval-valued function $P : (-\epsilon, \epsilon) \to I(\mathbb{R})$ with* $\lim_{h\to 0} P(h) = [0, 0]$*, such that*

$$(F(x^* + h) \ominus_{gH} F(x^*)) \ominus_{gH} hA = hP(h). \tag{8}$$

Proof If F is gH-differentiable at x^*, then there exists $F'_{gH}(x^*) \in I(\mathbb{R})$, such that $\lim_{h\to 0}\left(\frac{F(x^*+h)\ominus_{gH}F(x^*)}{h} \ominus_{gH} F'_{gH}(x^*)\right) = [0, 0]$. Given $\epsilon > 0$, let $P : (-\epsilon, \epsilon) \to I(\mathbb{R})$ be the interval-valued function given by

$$P(h) = \begin{cases} \frac{F(x^*+h)\ominus_{gH}F(x^*)}{h} \ominus_{gH} F'_{gH}(x^*) & \text{if } h \neq 0, \\ 0 & \text{if } h = 0. \end{cases}$$

Then, $\lim_{h\to 0} P(h) = [0, 0]$. Moreover, given $h \in (-\epsilon, \epsilon)\setminus\{0\}$, it follows that:

$$hP(h) = (F(x^* + h) \ominus_{gH} F(x^*)) \ominus_{gH} hF'_{gH}(x^*).$$

Thus, (8) holds with $A = F'_{gH}(x^*)$. On the other hand, it is easy to see that (8) holds with $A = F'_{gH}(x^*)$ whenever $h = 0$.

Reciprocally, if there exist an interval $A \in I(\mathbb{R})$ and an interval-valued function $P : (-\epsilon, \epsilon) \to I(\mathbb{R})$ with $\lim_{h\to 0} P(h) = [0, 0]$, such that (8) holds, then from Proposition 2.1 and Proposition 7 in Stefanini and Bede (2009), it follows that:

$$\lim_{h\to 0} \left(\frac{F(x^* + h) \ominus_{gH} F(x^*)}{h} \right) = A.$$

Therefore, F is gH-differentiable at x^*. □

Since the gH-differentiability condition given in Theorem 3.2 is an interval version of the Fréchet differentiability, this result shows that the gH-differentiability concept given in Markov (1979) and Stefanini and Bede (2009), which is an interval version of Gatêax differentiability, is equivalent to an interval version of the Fréchet differentiability for interval-valued functions of a real variable.

4 gH-differentiability for interval-valued functions on $\mathbb{R}^n$

This section recalls the gH-differentiability concepts for interval-valued functions of several variables that were introduced in the literature, and it presents a short discussion about the shortcomings generated by such concepts. Moreover, this section presents a new gH-differentiability concept which overcomes the shortcomings above mentioned and it shows that this concept is equivalent to the gH-differentiability one for interval-valued functions of several variables given by Stefanini and Arana-Jiménez (2019).

In Ahmad et al. (2016), Chalco-Cano et al. (2013a), and Luhandjula and Rangoaga (2014), the following interval-valued gH-differentiability definitions for interval-valued-functions of several variables are considered.

Definition 4.1 Let F be an interval-valued function defined on an open and non-empty set $S \subseteq \mathbb{R}^n$ and let $x^* = \left(x_1^*, \ldots, x_n^*\right) \in S$ be fixed. Given the interval-valued function $H_i : \mathbb{R} \to I(\mathbb{R})$ defined by $H_i(x_i) = F(x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_n^*)$, if H_i is gH-differentiable at x_i^*, then we say that F has the ith *partial* gH-derivative at x^* that is defined and denoted by

$$\frac{\partial F}{\partial x_i}(x^*) = H_i'\left(x_i^*\right) \in I(\mathbb{R}).$$

If there exists $\frac{\partial F}{\partial x_i}(x^*)$ for all $i \in \{1, \ldots, n\}$, then the n-tuple $\left(\frac{\partial F}{\partial x_1}(x^*), \ldots \frac{\partial F}{\partial x_n}(x^*)\right)$ of intervals is called the gradient of F and it is denoted by $\nabla_{gH} F(x^*)$.

Definition 4.2 Given $S \subseteq \mathbb{R}^n$ an open and nonempty set, let $F : S \to I(\mathbb{R})$ be an interval-valued function and let $x^* \in S$ be fixed. F is said to be gH-differentiable at x^* if there exist all the partial gH-derivatives $\frac{\partial F}{\partial x_1}(x), \ldots, \frac{\partial F}{\partial x_n}(x)$ on some neighborhood of x^* and are continuous at x^*.

Remark 4.1 Definition 4.2 says that, for $n = 1$, an interval-valued function $F : S \subseteq \mathbb{R} \to I(\mathbb{R})$ is gH-differentiable at $x_0 \in S$ if and only if there exist $F'_{gH}(x)$ in a neighborhood of x_0

and F'_{gH} is continuous at x_0. Therefore, Definition 4.2 does not coincide with Definition 2.1. Moreover, if $F : S \subseteq \mathbb{R} \to I(\mathbb{R})$ is a real-valued function, then Definition 4.2 is more restrictive than the classical real-valued differentiability concept.

Besides Theorem 3.2, our proposal of gH-differentiability concept for interval-valued functions of several variables is presented based on the following remark.

Remark 4.2 It is known that in the classical differential calculus theory, where S is an open set and V is a normed vector space, a function $f : S \subseteq \mathbb{R} \to V$ is differentiable at $x^* \in S$ if and only if there exist a linear function $L_{x^*} : \mathbb{R} \to V$ given by $L_{x^*}(h) = f'(x^*)h$ and a function $p : (-\epsilon, \epsilon) \to V$ with $\lim_{h\to 0} p(h) = 0$, such that

$$f(x^* + h) - f(x^*) = L_{x^*}(h) + hp(h) \text{ for all } h \in \mathbb{R} \text{ with } x^* + h \in S.$$

From Theorem 3.2, we have a similar situation for the gH-differentiability on $\mathbb{R}$. Indeed, given $F : S \to I(\mathbb{R})$, from Theorem 3.2, F is gH-differentiable at $x^* \in S$ if there exist the interval-valued function $T_{x^*} : \mathbb{R} \to I(\mathbb{R})$ given by $T_{x^*}(h) = hF'_{gH}(x^*)$ and an interval-valued function $P : (-\epsilon, \epsilon) \to I(\mathbb{R})$ with $\lim_{h\to 0} P(h) = [0, 0]$, such that

$$(F(x_0 + h) \ominus_{gH} F(x_0)) \ominus_{gH} T_{x^*}(h) = hP(h).$$

However, the interval-valued function $T_{x^*}(h)$, satisfying (8), may not be linear, since $I(\mathbb{R})$ is not a vector space. For example, let us consider $F : (-1, 1) \to I(\mathbb{R})$ given by $F(x) = [-1 + x, x^2]$. Considering $x^* = 0$ and given $P : (-\epsilon, \epsilon) \to I(\mathbb{R})$ defined by

$$P(h) = \begin{cases} \frac{(F(h)\ominus_{gH} F(0))\ominus_{gh} h[-1,0]}{|h|} & \text{if } h \neq 0 \\ 0 & \text{if } h = 0, \end{cases}$$

it follows that $\lim_{h\to 0} P(h) = [0, 0]$ and $(F(0 + h) \ominus_{gH} F(0)) \ominus_{gH} h[-1, 0] = |h|P(h)$. Therefore, $A = [-1, 0] = F'_{gH}(0)$. However, given $h_1 = -\frac{1}{2}$ and $h_2 = \frac{1}{2}$, it follows that:

$$\begin{aligned} T_{x^*}(h_1 + h_2) = T_{x^*}(0) = 0[-1, 0] = [0, 0] \neq \left[-\frac{1}{2}, \frac{1}{2}\right] = \left[0, \frac{1}{2}\right] + \left[-\frac{1}{2}, 0\right] \\ = T_{x^*}\left(-\frac{1}{2}\right) + T_{x^*}\left(\frac{1}{2}\right) = T_{x^*}(h_1) + T_{x^*}(h_2). \end{aligned}$$

On the other hand, it is easy to prove that if an interval-valued function F is gH-differentiable at a point $x^* \in S \subseteq \mathbb{R}$, then the interval-valued function $T_{x^*} : \mathbb{R} \to I(\mathbb{R})$ given by $T_{x^*}(h) = hF'_{gH}(x^*)$ is a quasilinear interval-valued function in the following sense.

Definition 4.3 (Assev 1986; Rojas-Medar et al. 2005) Considering the inclusion order relation $\subseteq$, an interval-valued function $\Gamma : \mathbb{R}^n \to (I(\mathbb{R}), \subseteq)$ is quasilinear if it satisfies the following conditions:

(C1) $\Gamma(\lambda x) = \lambda\Gamma(x)$ for all $\lambda \in \mathbb{R}, x \in \mathbb{R}^n$,
(C2) $\Gamma(x + y) \subseteq \Gamma(x) + \Gamma(y)$ for all $x, y \in \mathbb{R}^n$.
(C3) $\{x\} \leq \{y\} \Rightarrow \Gamma(x) \subseteq \Gamma(y)$.

Clearly, the condition (C3) holds true for every $x, y \in \mathbb{R}^n$. Thus, an interval-valued function $\Gamma : \mathbb{R}^n \to (I(\mathbb{R}), \subseteq)$ is quasilinear if and only if (C1) and (C2) hold.

Based on Theorem 3.2, Remark 4.2, and on the fact that a real-valued function $f : S \to \mathbb{R}$, defined on an open set $S \subseteq \mathbb{R}^n$, is differentiable at $x^* \in S$ if there exists a continuous and linear real-valued function $L_{x^*} : \mathbb{R}^n \to \mathbb{R}$ and a function $p : B(\mathbf{0}, \epsilon) \to \mathbb{R}$, where $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$ and $B(\mathbf{0}, \epsilon) := \{x \in \mathbb{R}^n : \|x\| < \epsilon\}$, with $\lim_{v \to \mathbf{0}} p(v) = 0$, such that

$$f(x^* + v) - f(v) = L_{x^*}(v) + \|v\| p(v),$$

our gH-differentiability definition for interval-valued functions of several real variables is given as follows.

Definition 4.4 Given $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n$ an open and nonempty set, let $F : S \to I(\mathbb{R})$ be an interval-valued function. We say that F is gH-differentiable at $x^* \in S$ if there exist a continuous and quasilinear interval-valued function $T_{x^*} : \mathbb{R}^n \to I(\mathbb{R})$ and an interval-valued function $P : B(\mathbf{0}, \epsilon) \subseteq \mathbb{R}^n \to I(\mathbb{R})$ with $\lim_{v \to \mathbf{0}} P(v) = [0, 0]$, such that

$$(F(x^* + v) \ominus_{gH} F(x^*)) \ominus_{gH} T_{x^*}(v) = ||v|| P(v), \tag{9}$$

for all $v \in B(\mathbf{0}, \epsilon) := \{x \in \mathbb{R}^n : \|x\| < \epsilon\}$ with $(x^* + v) \in S$, and in this case, $T_{x^*}(v)$ is called the gH-derivative of F at x^*. If F is gH-differentiable at all $x \in S$, then F is said to be gH-differentiable.

Theorem 4.1 *Given $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n$ an open and nonempty set, let $F : S \to I(\mathbb{R})$ be an interval-valued function. Then, F is gH-differentiable at $x^* \in S$ if and only if*

$$\lim_{v \to \mathbf{0}} \frac{\big(F(x^* + v) \ominus_{gH} F(x^*)\big) \ominus_{gH} T_{x^*}(v)}{\|v\|} = [0, 0], \tag{10}$$

where $T_{x^*} : \mathbb{R}^n \to I(\mathbb{R})$ is a continuous and quasilinear interval-valued function that satisfies (9).

Proof If F is gH-differentiable at $x^* \in S$, then directly from Definition 4.4 and Proposition 2.1, it follows that (10) holds. Reciprocally, if (10) holds, then setting $P : B(\mathbf{0}, \epsilon) \to I(\mathbb{R})$ by

$$P(v) = \begin{cases} \frac{(F(x^*+v) \ominus_{gH} F(x^*)) \ominus_{gH} T_{x^*}(v)}{\|v\|} & \text{if } v \neq \mathbf{0} \\ [0, 0] & \text{if } v = \mathbf{0}, \end{cases}$$

it follows that (9) holds with $\lim_{v \to \mathbf{0}} P(v) = [0, 0]$, and consequently, F is gH-differentiable at x^*. □

When $n = 1$, Definition 4.4 coincides with Definition 2.1 (see Stefanini and Bede 2009).

On the other hand, recently, Stefanini and Arana-Jiménez (2019) presented the following definition and result, respectively.

Definition 4.5 Let $F : S \subseteq \mathbb{R}^n \to I(\mathbb{R})$, $F(x) = [\underline{f}(x), \overline{f}(x)]$ and let $x^* \in S$, such that $(x^* + v) \in S$ for all $v \in \mathbb{R}^n$ with $\|v\| < \delta$ for some given $\delta > 0$. We say that F is gH-differentiable at $x^* \in S$ if and only if there exist two vectors $\hat{w} = (\hat{w}_1, \ldots, \hat{w}_n)$ and $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_n)$ in $\mathbb{R}^n$ and two functions $\hat{\epsilon}(v), \tilde{\epsilon}(v)$ with $\lim_{v \to \mathbf{0}} \hat{\epsilon}(v) = \lim_{v \to \mathbf{0}} \tilde{\epsilon}(v) = 0$, such that for all $v \neq 0$,

$$\left(\frac{\underline{f} + \overline{f}}{2}\right)(x^* + v) - \left(\frac{\underline{f} + \overline{f}}{2}\right)(x^*) = \sum_{i=1}^{n} v_i \hat{w}_i + \|v\| \hat{\epsilon}(v), \tag{11}$$

$$\left|\left(\frac{\overline{f} - \underline{f}}{2}\right)(x^* + v) - \left(\frac{\overline{f} - \underline{f}}{2}\right)(x^*)\right| = \left|\sum_{i=1}^{n} v_i \tilde{w}_i + \|v\| \tilde{\epsilon}(v)\right|. \tag{12}$$

The interval-valued function $D_{gH}F(x^*) : \mathbb{R}^n \to I(\mathbb{R})$ defined, for $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, by

$$D_{gH}F(x^*)(v) = \left[\sum_{i=1}^{n} v_i \hat{w}_i - \left| \sum_{i=1}^{n} v_i \tilde{w}_i \right|, \ \sum_{i=1}^{n} v_i \hat{w}_i + \left| \sum_{i=1}^{n} v_i \tilde{w}_i \right| \right]$$

is called the gH-differential (or total gH-derivative) of F at x^* and $D_{gH}F(x^*)(v)$ is the interval-valued differential of F at x^* with respect to v.

Proposition 4.1 *Let $F : S \subseteq \mathbb{R}^n \to I(\mathbb{R})$, $F(x) = [\underline{f}(x), \overline{f}(x)]$ and let $x^* \in S$ such that $(x^* + v) \in S$ for all $v \in \mathbb{R}^n$ with $\|v\| < \delta$ for some given $\delta > 0$. Then, F is gH-differentiable at $x^* \in S$ if and only if there exist two vectors $\hat{w} \in \mathbb{R}^n$ and $\tilde{w} \in \mathbb{R}^n$, such that the following limit condition is true*

$$\lim_{v \to \mathbf{0}} \frac{\big(F(x^* + v) \ominus_{gH} F(x^*)\big) \ominus_{gH} W(v)}{\|v\|} = [0, 0],$$

where $W(v)$ is the interval-valued function defined by

$$W(v) = \left[\sum_{i=1}^{n} v_i \hat{w}_i - \left| \sum_{i=1}^{n} v_i \tilde{w}_i \right|, \ \sum_{i=1}^{n} v_i \hat{w}_i + \left| \sum_{i=1}^{n} v_i \tilde{w}_i \right| \right].$$

In this case, for the differential function, we have that $D_{gH}F(x^*)(v) = W(v)$.

The following result allows us to know exactly the continuous and quasilinear interval-valued function T_{x^*} given in Definition 4.4 and allows us to see that the gH-differentiability definition for interval-valued functions of several real variables given in Stefanini and Arana-Jiménez (2019) is consistent with Theorem 3.2.

Theorem 4.2 *Let $F : S \to I(\mathbb{R})$, $F(x) = [\underline{f}(x), \overline{f}(x)]$. Then, F is gH-differentiable at $x^* \in S$ in the sense of Definition 4.4 if and only if F is gH-differentiable at $x^* \in S$ in the sense of Definition 4.5*

Proof From the unicity of the limit, Theorem 4.1, and from Proposition 4.1, it follows that for proving this result is sufficient to prove that $D_{gH}F(x^*) : \mathbb{R}^n \to I(\mathbb{R})$ is quasilinear, since from its definition, it is easy to see that it is continuous.

Given $v \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ with $\lambda < 0$, from definition of $D_{gH}F(x^*)$ and from (1), it follows that:

$$\begin{aligned}
D_{gH}F(x^*)(\lambda v) &= \left[\sum_{i=1}^{n} (\lambda v_i) \hat{w}_i - \left| \sum_{i=1}^{n} (\lambda v_i) \tilde{w}_i \right|, \ \sum_{i=1}^{n} (\lambda v_i) \hat{w}_i + \left| \sum_{i=1}^{n} (\lambda v_i) \tilde{w}_i \right| \right] \\
&= \left[\lambda \left(\sum_{i=1}^{n} v_i \hat{w}_i + \left| \sum_{i=1}^{n} v_i \tilde{w}_i \right) \right|, \ \lambda \left(\sum_{i=1}^{n} v_i \hat{w}_i - \left| \sum_{i=1}^{n} v_i \tilde{w}_i \right) \right| \right] \\
&= \lambda \left[\sum_{i=1}^{n} v_i \hat{w}_i - \left| \sum_{i=1}^{n} v_i \tilde{w}_i \right|, \ \sum_{i=1}^{n} v_i \hat{w}_i + \left| \sum_{i=1}^{n} v_i \tilde{w}_i \right| \right] \\
&= \lambda D_{gH}F(x^*)(v).
\end{aligned}$$

Now, given $\lambda \in \mathbb{R}$ with $\lambda > 0$, from definition of $D_{gH}F(x^*)$ and from (1), it follows that:

$$\begin{aligned} D_{gH}F(x^*)(\lambda v) &= \left[\sum_{i=1}^{n}(\lambda v_i)\hat{w}_i - \left|\sum_{i=1}^{n}(\lambda v_i)\tilde{w}_i\right|,\ \sum_{i=1}^{n}(\lambda v_i)\hat{w}_i + \left|\sum_{i=1}^{n}(\lambda v_i)\tilde{w}_i\right| \right] \\ &= \left[\lambda\left(\sum_{i=1}^{n} v_i\hat{w}_i - \left|\sum_{i=1}^{n} v_i\tilde{w}_i\right|\right),\ \lambda\left(\sum_{i=1}^{n} v_i\hat{w}_i + \left|\sum_{i=1}^{n} v_i\tilde{w}_i\right|\right) \right] \\ &= \lambda\left[\sum_{i=1}^{n} v_i\hat{w}_i - \left|\sum_{i=1}^{n} v_i\tilde{w}_i\right|,\ \sum_{i=1}^{n} v_i\hat{w}_i + \left|\sum_{i=1}^{n} v_i\tilde{w}_i\right| \right] \\ &= \lambda D_{gH}F(x^*)(v). \end{aligned}$$

Thus, $D_{gH}F(\lambda x^*)(v) = \lambda D_{gH}F(x^*)(v)$ for all $\lambda \in \mathbb{R}$.

Given $u, v \in \mathbb{R}^n$, it follows that:

$$\begin{aligned} &D_{gH}F(x^*)(u+v) \\ &= \left[\sum_{i=1}^{n}(u_i+v_i)\hat{w}_i - \left|\sum_{i=1}^{n}(u_i+v_i)\tilde{w}_i\right|,\ \sum_{i=1}^{n}(u_i+v_i)\hat{w}_i + \left|\sum_{i=1}^{n}(u_i+v_i)\tilde{w}_i\right| \right] \\ &= \left[\sum_{i=1}^{n} u_i\hat{w}_i + \sum_{i=1}^{n} v_i\hat{w}_i - \left|\sum_{i=1}^{n} u_i\tilde{w}_i + \sum_{i=1}^{n} v_i\tilde{w}_i\right|,\ \sum_{i=1}^{n} u_i\hat{w}_i + \sum_{i=1}^{n} v_i\hat{w}_i \right. \\ &\quad \left. + \left|\sum_{i=1}^{n} u_i\tilde{w}_i + \sum_{i=1}^{n} v_i\tilde{w}_i\right| \right] \end{aligned} \tag{13}$$

and

$$\begin{aligned} &D_{gH}F(x^*)(u) + D_{gH}F(x^*)(v) \\ &= \left[\sum_{i=1}^{n} u_i\hat{w}_i - \left|\sum_{i=1}^{n} u_i\tilde{w}_i\right|,\ \sum_{i=1}^{n} u_i\hat{w}_i + \left|\sum_{i=1}^{n} u_i\tilde{w}_i\right| \right], \\ &\quad \left[\sum_{i=1}^{n} v_i\hat{w}_i - \left|\sum_{i=1}^{n} v_i\tilde{w}_i\right| + \sum_{i=1}^{n} v_i\hat{w}_i + \left|\sum_{i=1}^{n} v_i\tilde{w}_i\right| \right] \\ &= \left[\sum_{i=1}^{n} u_i\hat{w}_i + \sum_{i=1}^{n} v_i\hat{w}_i - \left(\left|\sum_{i=1}^{n} u_i\tilde{w}_i\right| + \left|\sum_{i=1}^{n} v_i\tilde{w}_i\right|\right), \sum_{i=1}^{n} u_i\hat{w}_i + \sum_{i=1}^{n} v_i\hat{w}_i \right. \\ &\quad \left. + \left(\left|\sum_{i=1}^{n} u_i\tilde{w}_i\right| + \left|\sum_{i=1}^{n} v_i\tilde{w}_i\right|\right) \right]. \end{aligned} \tag{14}$$

Since $\left|\sum_{i=1}^{n} u_i\tilde{w}_i + \sum_{i=1}^{n} v_i\tilde{w}_i\right| \leq \left|\sum_{i=1}^{n} u_i\tilde{w}_i\right| + \left|\sum_{i=1}^{n} v_i\tilde{w}_i\right|$, from (13) and (14), it follows that:

$$D_{gH}F(x^*)(u+v) \subseteq D_{gH}F(x^*)(u) + D_{gH}F(x^*)(v).$$

Therefore, $D_{gH}F(x^*) : \mathbb{R}^n \to I(\mathbb{R})$ is quasilinear and continuous, and consequently, $D_{gH}F(x^*)(v) = T_{x^*}(v)$ for all $v \in \mathbb{R}^n$. □

5 An application of gH-differentiability for interval-valued functions: extension of gH-differentiability notion to fuzzy functions with several real variables

Let us recall some basic notions about the fuzzy environment to understand the extension to fuzzy context.

A fuzzy set on $\mathbb{R}^n$ is a mapping defined as $u : \mathbb{R}^n \to [0, 1]$. The α-level set of a fuzzy set, $0 \le \alpha \le 1$, is defined as

$$[u]^\alpha = \begin{cases} \{x \in \mathbb{R}^n | u(x) \ge \alpha\} & \text{if } \alpha \in (0, 1], \\ cl(supp \;\; u) & \text{if } \alpha = 0, \end{cases}$$

where $cl(supp \;\; u)$ denotes the closure of the support of u, $supp(u) = \{x \in \mathbb{R}^n | \, u(x) > 0\}$.

Definition 5.1 A fuzzy number is a fuzzy set u on $\mathbb{R}$ with the following properties:

1. u is normal, that is, there exists $x_0 \in \mathbb{R}$, such that $u(x_0) = 1$;
2. u is an upper semi-continuous function;
3. $u(\lambda x + (1 - \lambda)y) \ge min\{u(x), u(y)\}, \;\; x, y \in \mathbb{R}, \;\; \lambda \in [0, 1]$;
4. $[u]^0$ is compact.

Let $\mathcal{F}_C$ be the set of all fuzzy numbers on $\mathbb{R}$.

Obviously, if $u \in \mathcal{F}_C$, then $[u]^\alpha \in \mathcal{K}_C$ for all $\alpha \in [0, 1]$, and thus, the α-level sets of a fuzzy number are given by $[u]^\alpha = \left[\underline{u}_\alpha, \overline{u}_\alpha\right]$, $\underline{u}_\alpha, \overline{u}_\alpha \in \mathbb{R}$ for all $\alpha \in [0, 1]$.

For fuzzy numbers, $u, v \in \mathcal{F}_C$, represented by $\left[\underline{u}_\alpha, \overline{u}_\alpha\right]$ and $\left[\underline{v}_\alpha, \overline{v}_\alpha\right]$ respectively, and for any real number θ, we define the following operations between fuzzy numbers:

$$(u + v)(x) = \sup_{y+z=x} \min\{u(y), v(z)\}, \quad (\theta u)(x) = \begin{cases} u\left(\frac{x}{\theta}\right), & \text{if } \;\; \theta \ne 0, \\ 0, & \text{if } \;\; \theta = 0, \end{cases}$$

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} & (i) \;\; u = v + w, \\ \text{or} & (ii) \;\; v = u + (-1)w. \end{cases}$$

It is known that for every $\alpha \in [0, 1]$

$$[u + v]^\alpha = \left[(\underline{u + v})_\alpha, (\overline{u + v})_\alpha\right] = \left[\underline{u}_\alpha + \underline{v}_\alpha \;,\; \overline{u}_\alpha + \overline{v}_\alpha\right], \tag{15}$$

$$[\theta u]^\alpha = \left[(\underline{\theta u})_\alpha, (\overline{\theta u})_\alpha\right] = \theta[u]^\alpha = \theta\left[\underline{u}_\alpha, \overline{u}_\alpha\right] = \left[\min\{\theta\underline{u}_\alpha, \theta\overline{u}_\alpha\}, \max\{\theta\underline{u}_\alpha, \theta\overline{u}_\alpha\}\right], \tag{16}$$

and if $u \ominus_{gH} v$ exists, then, in terms of α-level sets, we can deduce that (see Stefanini and Bede (2009); Stefanini (2010))

$$[u \ominus_{gH} v]^\alpha = [u]^\alpha \ominus_{gH} [v]^\alpha = \left[\min\{\underline{u}_\alpha - \underline{v}_\alpha, \overline{u}_\alpha - \overline{v}_\alpha\}, \max\{\underline{u}_\alpha - \underline{v}_\alpha, \overline{u}_\alpha - \overline{v}_\alpha\}\right]. \tag{17}$$

Given $u, v \in \mathcal{F}_C$, we define the distance between u and v as

$$D(u, v) = \sup_{\alpha\in[0,1]} H\left([u]^\alpha, [v]^\alpha\right) = \sup_{\alpha\in[0,1]} \max\left\{\left|\underline{u}_\alpha - \underline{v}_\alpha\right|, |\overline{u}_\alpha - \overline{v}_\alpha|\right\}.$$

Therefore, $(\mathcal{F}_C, D)$ is a complete metric space.

We recall the usual order relations between fuzzy numbers (Osuna-Gómez et al. 2016):

Definition 5.2 For $u, v \in \mathcal{F}_C$, it is said that

(1) $u \preceq v$ if $[u]^\alpha \preceq [v]^\alpha$ for every $\alpha \in [0, 1]$.

(2) $u \preceq v$ if $u \underline{\preceq} v$ and $u \neq v$, i. e. $[u]^\alpha \underline{\preceq} [v]^\alpha$ for every $\alpha \in [0, 1]$, and $\exists \alpha_0 \in [0, 1]$, such that $[u]^{\alpha_0} \preceq [v]^{\alpha_0}$.
(3) $u \prec v$ if $u \underline{\preceq} v$ and $\exists \alpha_0 \in [0, 1]$, such that $[u]^{\alpha_0} \prec [v]^{\alpha_0}$.

Note that $\preceq$ is a partial order relation on $\mathcal{F}_C$. Hence, $v \underline{\succeq} u$ can be written instead of $u \underline{\preceq} v$. We observe that if $u \prec v$, then $u \preceq v$ and, therefore, $u \underline{\preceq} v$.

Let us consider $\tilde{f} : S \subseteq \mathbb{R}^n \to \mathcal{F}_C$ a fuzzy function or fuzzy mapping where S is an open and nonempty subset of $\mathbb{R}^n$. We associate with $\tilde{f}$ the family of interval-valued functions $f_\alpha : S \to I(\mathbb{R})$ where, for each $\alpha \in [0, 1]$, $f_\alpha(x) = [\tilde{f}(x)]^\alpha = \left[\underline{f}_\alpha(x), \overline{f}_\alpha(x)\right] = \left[\underline{f}(x; \alpha), \overline{f}(x; \alpha,)\right]$, are the α-level sets of $\tilde{f}$.

As a particular case of fuzzy quasilinear operators (Rojas-Medar et al. 2005), a fuzzy function is quasilinear if $\forall \lambda \in \mathbb{R}$, $x, y \in \mathbb{R}^n$

1. $\tilde{f}(\lambda x) = \lambda \tilde{f}(x)$.
2. $\tilde{f}(x + y) \preceq \tilde{f}(x) + \tilde{f}(y)$.

In Bede and Stefanini (2013), we find the gH-differentiability and level-wise gH-differentiability notions for fuzzy functions of a single variable. The gH-differentiability for fuzzy functions on $\mathbb{R}$ is a more general concept than the H-differentiability and G-differentiability, but less general than level-wise gH-differentiability, that is based on the gH-differentiability of every interval-valued function f_α.

Let us consider the definition of level-wise gH-differentiable fuzzy function when $n \geq 1$.

Definition 5.3 Let us consider $\tilde{f} : S \subseteq \mathbb{R}^n \to \mathcal{F}_C$, $f_\alpha(x) = \left[\underline{f}_\alpha(x), \overline{f}_\alpha(x)\right]$ for each $\alpha \in [0, 1]$ and let $x^* \in S$ such that $x^* + v \in S$ for all $v \in \mathbb{R}^n$ with $||v|| < \epsilon$ ($\epsilon > 0$). We say that $\tilde{f}$ is level-wise gH-differentiable at $x^* \in S$ if and only if for each $\alpha \in [0, 1]$, there exist two interval-valued functions, $T_{x^*}^\alpha$ continuous and quasilinear and P^α with $\lim_{v \to \mathbf{0}} P^\alpha(v) = [0, 0]$, such that

$$\left(f_\alpha(x^* + v) \ominus_{gH} f_\alpha(x^*)\right) \ominus_{gH} T_{x^*}^\alpha(v) = ||v|| P^\alpha(v).$$

The family of interval-valued functions $\{T_{x^*}^\alpha\}_{\alpha \in [0,1]}$ is called the level-wise gH-derivative of $\tilde{f}$ at x^*.

When $S \subseteq \mathbb{R}$ ($n = 1$), from Theorem 3.2, this definition coincides with Definition 23 in Bede and Stefanini (2013). In the general case, $S \subseteq \mathbb{R}^n$ with $n > 1$, from Theorem 4.2, it coincides with Definition 9-A in Stefanini and Arana-Jiménez (2019).

Example 5.1 Let us consider the fuzzy function $\tilde{f} : S \subseteq \mathbb{R}^2 \to \mathcal{F}_C$, defined by

$$\tilde{f}(x_1, x_2) = (1, 1, 1) \cdot x_1^2 + (0, 0, 1) \cdot x_2^2,$$

where its α-level sets for all $\alpha \in [0, 1]$ are $\left[\tilde{f}(x_1, x_2)\right]^\alpha = [\underline{f}(x; \alpha), \bar{f}(x; \alpha)] = \left[x_1^2\ ,\ x_1^2 + (1 - \alpha)x_2^2\right]$. From Theorem 4.2 and Definition 4.5, the level-wise gH-derivatives of the α-level sets functions satisfying the conditions of Definition 5.3 are

$$\left.\begin{array}{l} \frac{\partial \underline{f}}{\partial x_1}(x; \alpha) = 2x_1 \\ \frac{\partial \overline{f}}{\partial x_1}(x; \alpha) = 2x_1 \end{array}\right\} \Rightarrow \begin{array}{l} \hat{w_1} = 2x_1 \\ \tilde{w_1} = \frac{|2x_1 - 2x_1|}{2} = 0 \end{array}$$

$$\left.\begin{array}{l} \frac{\partial \underline{f}}{\partial x_2}(x; \alpha) = 0 \\ \frac{\partial \overline{f}}{\partial x_2}(x; \alpha) = 2(1 - \alpha)x_2 \end{array}\right\} \Rightarrow \begin{array}{l} \hat{w_2} = (1 - \alpha)x_2 \\ \tilde{w_2} = \frac{|0 - 2(1 - \alpha)x_2|}{2} = (1 - \alpha)|x_2|. \end{array}$$

Then, for each $\alpha \in [0, 1]$

$$[T_{x^*}(v)]^\alpha = \big[2x_1^* v_1 + (1-\alpha)x_2^* v_2 - (1-\alpha)|x_2^* v_2| \vee 2x_1^* v_1 + (1-\alpha)x_2^* v_2 + (1-\alpha)|x_2^* v_2|\big].$$

If $x_2^* v_2 \geq 0$

$$[T_{x^*}(v)]^\alpha = \big[2x_1^* v_1,\ 2x_1^* v_1 + 2(1-\alpha)x_2^* v_2\big] = \big[\underline{t}(x^*, \alpha)(v),\ \bar{t}(x^*, \alpha)(v)\big].$$

If $x_2^* v_2 \leq 0$

$$[T_{x^*}(v)]^\alpha = \big[2x_1^* v_1 + 2(1-\alpha)x_2^* v_2,\ 2x_1^* v_1\big] = \big[\underline{t}(x^*, \alpha)(v),\ \bar{t}(x^*, \alpha)(v)\big].$$

In any case, for each $\alpha \in [0, 1]$, $T_{x^*}^\alpha$ is an interval-valued function.

Fixed x^* and $v \in \mathbb{R}^n$, the family of interval-valued functions given by $\{[T_x^*(v)]^\alpha\}_{\alpha \in [0,1]}$ defines a fuzzy number.

Then, the fuzzy function $\tilde{f}$ is level-wise gH-differentiable in $\mathbb{R}^2$ and the level-wise gH-derivatives are given by

$$[T_{x^*}(v)]^\alpha = \big[2x_1^* v_1 \ \vee\ 2x_1^* v_1 + 2(1-\alpha)x_2^* v_2\big]$$

that define a fuzzy number. Therefore, the fuzzy function $\tilde{f}$ is gH-differentiable in $\mathbb{R}^2$ according to Definition 9-B in Stefanini and Arana-Jiménez (2019).

Example 5.2 Let us consider the fuzzy function $\tilde{f} : S \subseteq \mathbb{R}^2 \to \mathcal{F}_C$, defined by

$$\tilde{f}(x_1, x_2) = \begin{cases} [(1+\alpha)x_2, x_1^2 + (\alpha+4)x_2] & \text{if } x_2 \geq 0, \\ [(\alpha+4)x_2, x_1^2 + (1+\alpha)x_2] & \text{if } x_2 < 0. \end{cases}$$

For all $\alpha \in [0, 1]$, the end-point functions of the family of interval-valued functions associated with $\tilde{f}$ are: $\underline{f}(x; \alpha) = \begin{cases} (1+\alpha)x_2 & \text{if } x_2 \geq 0, \\ (\alpha+4)x_2 & \text{if } x_2 < 0 \end{cases}$ and $\overline{f}(x; \alpha) = \begin{cases} x_1^2 + (\alpha+4)x_2 & \text{if } x_2 \geq 0, \\ x_1^2 + (1+\alpha)x_2 & \text{if } x_2 < 0 \end{cases}$

Let us consider $x^* = (0, 0)$, and then, $f(\mathbf{0}; \alpha) = f_\alpha(\mathbf{0}) = [0, 0]$, $\forall \alpha \in [0, 1]$, but $\underline{f}(x; \alpha)$ and $\overline{f}(x; \alpha)$ are not differentiable at $x^* = (0, 0)$, since their partial derivatives with respect to x_2 do not exist. However, for each $\alpha \in [0, 1]$, there exists $T_{x^*}^\alpha(v)$, a quasilinear and continuous function which verifying Definition 4.4

$$T_{x^*}^\alpha(v) = [(1+\alpha)v_2 \vee (\alpha+4)v_2].$$

$$f_\alpha(x^* + v) \ominus_{gH} f_\alpha(x^*) = \begin{cases} [(1+\alpha)v_2, v_1^2 + (\alpha+4)v_2] & \text{if } v_2 \geq 0, \\ [\alpha+4)v_2, v_1^2 + (1+\alpha)v_2] & \text{if } v_2 < 0 \end{cases}$$

and

$$(f_\alpha(x^* + v) \ominus_{gH} f_\alpha(x^*)) \ominus_{gH} T_{x^*}^\alpha(v) = [0, v_1^2].$$

Therefore, for each $\alpha \in [0, 1]$, f_α is gH-differentiable at $x^* = (0, 0)$, and so $\tilde{f}$ is level-wise gH-differentiable at $x^* = (0, 0)$ if there exists an interval-valued function $P^\alpha : (-\epsilon, \epsilon) \to \mathbb{I}(\mathbb{R})$ such that $[0, v_1^2] = \|v\| P^\alpha(v)$ with $\lim_{v \to \mathbf{0}} P^\alpha(v) = [0, 0]$.

Given $\epsilon > 0$, let us define $P^\alpha(v) = \begin{cases} [\frac{1}{\|v\|}[0, v_1^2] & \text{if } v \neq (0, 0), \\ [0, 0] & \text{if } v = (0, 0). \end{cases}$ and it is easy to prove that $\lim_{v \to \mathbf{0}} P^\alpha(v) = [0, 0]$.

Then, $\tilde{f}$ is level-wise gH-differentiable at $x^* = (0, 0)$.

Let us notice that the family of level-wise gH-derivatives $\{T^{\alpha}_{x^*}\}_{\alpha\in[0,1]}$ does not have to determine a fuzzy number. Therefore, although it is a more restrictive notion, it makes sense to define the fuzzy gH-differentiability.

As we have said, when $S \subseteq \mathbb{R}$ ($n = 1$), from Theorem 3.2, Definition 5.3 coincides with Definition 23 in Bede and Stefanini (2013). In Bede and Stefanini (2013), the fuzzy gH-differentiability is defined directly, without the use of the interval-valued functions associated (Definition 20).

Now, we develop a necessary and sufficient condition for fuzzy gH- differentiability (Definition 20 given in Bede and Stefanini (2013) with $n = 1$) that can serve as the basis for further developments of fuzzy differentiability.

Theorem 5.1 *Let $\tilde{f} : S \subseteq \mathbb{R} \to \mathcal{F}_C$ be a fuzzy function and $v \in S$, such that $x^* + v \in S$. Then, there exists $\tilde{f}'(x^*) \in \mathcal{F}_C$ with*

$$\tilde{f}'(x^*) = \lim_{v\to 0} \frac{\tilde{f}(x^* + v) \ominus_{gH} \tilde{f}(x^*)}{v} \tag{18}$$

if and only if there exists $\tilde{u} \in \mathcal{F}_C$ and $\tilde{P} : (-\epsilon, \epsilon) \to \mathcal{F}_C$ with $\lim_{v\to 0} \tilde{P}(v) = \tilde{0}$, such that

$$(\tilde{f}(x^* + v) \ominus_{gH} \tilde{f}(x^*)) \ominus_{gH} v\tilde{u}(x^*) = v\tilde{P}(v). \tag{19}$$

Proof If there exists $\tilde{f}'(x^*) \in \mathcal{F}_C$ satisfying (18), then

$$\lim_{v\to 0} \frac{\tilde{f}(x^* + v) \ominus_{gH} \tilde{f}(x^*)}{v} \ominus_{gH} \tilde{f}'(x^*) = \tilde{0}.$$

Given $\epsilon > 0$, let us consider the fuzzy function $\tilde{P} : (-\epsilon, \epsilon) \to \mathcal{F}_C$ given by

$$\tilde{P}(v) = \begin{cases} \frac{\tilde{f}(x^*+v)\ominus_{gH}\tilde{f}(x^*)}{v} \ominus_{gH} \tilde{f}'(x^*) & \text{if } v \neq 0, \\ \tilde{0} & \text{if } v = 0. \end{cases}$$

It verifies that $\lim_{v\to 0} \tilde{P}(v) = \tilde{0}$, and furthermore, $(\tilde{f}(x^* + v) \ominus_{gH} \tilde{f}(x^*)) \ominus_{gH} v\tilde{f}'(x^*) = v\tilde{P}(v)$ that coincides with (19).

If (19) is verified, then $\frac{\tilde{f}(x^*+v)\ominus_{gH}\tilde{f}(x^*)}{v} \ominus_{gH} \tilde{u}(x^*) = \tilde{P}(v)$. Taking limit when $v \to 0$

$$\lim_{v\to 0} \frac{\tilde{f}(x^* + v) \ominus_{gH} \tilde{f}(x^*)}{v} \ominus_{gH} \tilde{u}(x^*) = \lim_{v\to 0} \tilde{P}(v) = \tilde{0},$$

and so $\lim_{v\to 0} \frac{\tilde{f}(x^*+v)\ominus_{gH}\tilde{f}(x^*)}{v} = \tilde{u}(x^*)$ that coincides with (18) □

Now, we extend the notion of gH-differentiability to a fuzzy function with several variables.

Definition 5.4 Given $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n$ an open and nonempty set, let $\tilde{f} : S \to \mathcal{F}_C$ be a fuzzy function. We say that $\tilde{f}$ is gH-differentiable at $x^* \in S$ if there exist a continuous and quasilinear fuzzy function $\tilde{T}_{x^*} : S \subseteq \mathbb{R}^n \to \mathcal{F}_C$ and a fuzzy function $\tilde{P} : B(\mathbf{0}, \epsilon) \subseteq \mathbb{R}^n \to \mathcal{F}_C$ with $\lim_{v\to\mathbf{0}} \tilde{P}(v) = \tilde{0}$, such that

$$(\tilde{f}(x^* + v) \ominus_{gH} \tilde{f}(x^*)) \ominus_{gH} \tilde{T}_{x^*}(v) = ||v||\tilde{P}(v), \tag{20}$$

for all $v \in B(\mathbf{0}, \epsilon) := \{x \in \mathbb{R}^n : \|x\| < \epsilon\}$, $\epsilon > 0$ with $(x^* + v) \in S$, and in this case, $\tilde{T}_{x^*}(v)$ is called the gH-derivative of $\tilde{f}$ at $x^* \in S$. If $\tilde{f}$ is gH-differentiable at all $x \in S$, then $\tilde{f}$ is said to be gH-differentiable.

6 Conclusions

In this paper, we present and study a gH-differentiability definition for interval-valued functions of several real variables, which extends the classical Fréchet differentiability notion.

This concept herein introduced overcomes some drawbacks of the gH-differentiability definition extended for interval-valued functions of several real variables in the previous literature, generating a gH-differential that is a quasilinear interval-valued function.

The gH-differentiability concept given by Markov (1979), Stefanini and Bede (2009), and Stefanini and Arana-Jiménez (2019) is extension of the classical Gâteaux differentiability to interval-valued functions. We prove that these concepts and the new definition, which is equivalent to an interval version of the Fréchet differentiability, are equivalents.

The extension of the new concept of differentiability to the fuzzy environment is presented.

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