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Some inverse problems for the Burgers equation and related systems

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ABSTRACT

In this article we deal with one-dimensional inverse problems concerning the Burgers equation and some related nonlinear systems (involving heat effects and/or variable density). In these problems, the goal is to find the size of the spatial interval from some appropriate boundary observations of the solution. Depending on the properties of the initial and boundary data, we prove uniqueness and non-uniqueness results. In addition, we also solve some of these inverse problems numerically and compute approximations of the interval sizes.

1. Introduction

This paper deals with some inverse problems for nonlinear time-dependent PDEs in one spatial dimension.

The analysis and solution of inverse problems of many kinds has recently increased a lot because of their relevance in many applications: elastography and medical imaging, seismology, potential theory, ion transport problems or chromatography, finances, etc.; see for instance [3,9,15]. The variety of inverse problems is huge in comparison with their direct analogs and many inverse problems coming from very classical and basic direct problems wait for theoretical and numerical research. Let us mention the monographs by Bellassoued and Yamamoto [2], Isakov [13], Romanov [16] and Hasanov and Romanov [10], where many theoretical and numerical aspects of inverse problems for partial differential equations are depicted.

In this paper, we consider problems related to the identification of the size of the spatial interval where a time-dependent governing nonlinear equation must be satisfied. We will focus on the Burgers equation and some variants, satisfied for $(x, t) \in (0, \ell)\times(0, T)$. We will assume that the equation is complemented with boundary and initial conditions corresponding to known data, respectively for $x \in \{0, \ell\}$ and $t = 0$. Then, we will try to determine the width ℓ of the spatial interval from some extra information, for instance given at $x = 0$. The main goals will be to establish or discard uniqueness

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and to compute approximations of the solutions to the inverse problems. Related questions have been analyzed recently for the linear heat and wave equations in [1].

The plan is the following. In Section 2, we consider the viscous Burgers equation under several different circumstances. Sections 3 and 4 respectively deal with the Burgers equation coupled to a heat equation and the variable density Burgers system. Finally, we present the results of some numerical experiments in Section 5.

Throughout this paper, $\|\cdot\|$ and $(\cdot\,,\cdot)$ will stand for the usual L^2 norm and scalar product, respectively. In the particular case of the space $L^2(0,\ell)$, we will sometimes write $(\cdot\,,\cdot)_\ell$ in order to make the length ℓ explicit. The symbol C will denote a generic positive constant.

2. Some positive and negative results for the viscous Burgers equation

Let us consider the following system for the Burgers equation:

$$\begin{cases} u_t - u_{xx} + uu_x = 0, & 0<x<\ell,\ 0<t<T, \\ u(0,t)=\eta(t), \quad u(\ell,t)=0, & 0<t<T, \\ u(x,0)=u_0(x), & 0<x<\ell. \end{cases} \tag{1}$$

The unknown $u=u(x,t)$ can be interpreted (for example) as the velocity of the particles of a homogeneous viscous fluid in a tube where the flow is allowed only lengthwise. It can also be viewed as the car traffic density in a road in a simplified model, see for instance [14].

The main inverse problem for (1) is the following:

IP-1: *Fix $u_0=u_0(x)$ and $\eta=\eta(t)$ in (1) in appropriate spaces and assume that $\beta := u_x|_{x=0}$ is known. Then, find ℓ.*

We are first interested in proving uniqueness. More precisely, the following question is in order:

Uniqueness for IP-1: *Let u^ℓ and u^L be the solutions to (1) respectively associated to the spatial intervals $(0,\ell)$ and $(0,L)$. Assume that the corresponding observations $u^\ell_x(0,\cdot)$ and $u^L_x(0,\cdot)$ coincide, that is,*

$$u^\ell_x(0,t) = u^L_x(0,t) \ \text{ in } \ (0,T). \tag{2}$$

Then, do we have $\ell = L$?

In the sequel, we will provide some positive and negative answers to this question, depending on the kind of imposed boundary or initial data.

2.1. The simplest cases: zero initial and/or boundary data

2.1.1. Case I: $\eta \not\equiv 0$ and $u_0 \equiv 0$

If $u_0 \equiv 0$, we get uniqueness:

Theorem 2.1. *Assume that $0<\ell\le L$, $\eta\in L^\infty(0,T)$ satisfies $\eta\not\equiv 0$ and $u_0\equiv 0$. Let u^ℓ and u^L be the solutions to (1) respectively corresponding to ℓ and L and let us assume that, for some $M>0$,*

$$|u^\ell_x(x,t)| \le M \ \text{ in } \ (0,\ell)\times(0,T) \ \text{ and } \ |u^L_x(x,t)| \le M \ \text{ in } \ (0,L)\times(0,T) \tag{3}$$

and (2) holds. Then, $\ell = L$.

Proof. The proof is standard. It can be achieved by contradiction, assuming that $\ell<L$. Indeed, note that $u^\ell\in L^\infty((0,\ell)\times(0,T))$ and $u^L\in L^\infty((0,L)\times(0,T))$. If we set $v:=u^\ell-u^L$, one has

$$v_t - v_{xx} + vu^\ell_x + u^L v_x = 0 \ \text{ in } \ (0,\ell)\times(0,T)$$

and also $v(0,t)=0$ and $v_x(0,t)=0$ in $(0,T)$. Consequently, from the unique continuation property of the heat equation (see [17]), we have $v=0$ in $(0,\ell)\times(0,T)$. This yields $u^L(x,t)=0$ in $(\ell,L)\times(0,T)$ and then (again from unique continuation) $u^L\equiv 0$, which is an absurd. □

2.1.2. Case II: $\eta \equiv 0$ and $u_0 \not\equiv 0$

Let us show that, as in the case of the linear heat equation (see [1]), non-uniqueness holds in general. More precisely, a counter-example to uniqueness can be found. We will follow three steps:

1- Using the Cole–Hopf transformation (named after J.D. Cole and E. Hopf's works [6,11], respectively), we will rewrite (1) as a system for the heat equation.
2- Then, we will prove a result similar to [1, Proposition 2.1] and we will deduce non-uniqueness for the inverse problem corresponding to the heat equation with Neumann boundary conditions.
3- Finally, coming back to the original variables, we will be able to conclude.

The Cole–Hopf transformation is given by

$$\varphi(x,t) = Me^{-\frac{1}{2}\int_0^x u(\xi,t)\,d\xi}$$

or, equivalently,

$$u(x,t) = -2\frac{\varphi_x(x,t)}{\varphi(x,t)}, \quad \varphi(0,t) \equiv M, \tag{4}$$

where M is a positive constant. Using (4), the Burgers system (1) can be rewritten in the form

$$\begin{cases} \varphi_t - \varphi_{xx} = 0, & 0 < x < \ell,\ 0 < t < T, \\ \varphi_x(0,t) = 0, \quad \varphi_x(\ell,t) = 0, & 0 < t < T, \\ \varphi(x,0) = \varphi_0(x), & 0 < x < \ell, \end{cases} \tag{5}$$

where we have introduced $\varphi_0(x) := Me^{-\frac{1}{2}\int_0^x u_0(\xi)\,d\xi}$.

Let us denote by λ_n and $\tilde{\varphi}_n$ (resp. μ_n and $\tilde{\psi}_n$) the eigenvalues and eigenfunctions of the Neumann Laplacian in $(0,\ell)$ (resp. $(0,L)$). Then,

$$\begin{cases} \lambda_n := \dfrac{n^2\pi^2}{\ell^2}, \quad n \in \mathbb{N} \cup \{0\}, \\ \tilde{\varphi}_n(x) := \begin{cases} \sqrt{\dfrac{2}{\ell}}\cos\left(\dfrac{n\pi x}{\ell}\right), & n \in \mathbb{N}, \\ \dfrac{1}{\sqrt{\ell}}, \quad n = 0, & 0 < x < \ell, \end{cases} \end{cases}$$

and

$$\begin{cases} \mu_n := \dfrac{n^2\pi^2}{L^2}, \quad n \in \mathbb{N} \cup \{0\}, \\ \tilde{\psi}_n(x) := \begin{cases} \sqrt{\dfrac{2}{L}}\cos\left(\dfrac{n\pi x}{L}\right), & n \in \mathbb{N}, \\ \dfrac{1}{\sqrt{L}}, \quad n = 0, & 0 < x < L. \end{cases} \end{cases}$$

The solutions to (5) corresponding to ℓ and L can be defined for all $t > 0$. They are respectively given by

$$\varphi^\ell(x,t) = \sum_{n=0}^{\infty} (\varphi_0, \tilde{\varphi}_n)_\ell\, e^{-\lambda_n t}\, \tilde{\varphi}_n(x), \quad 0 < x < \ell,\ t > 0 \tag{6}$$

and

$$\varphi^L(x,t) = \sum_{n=0}^{\infty} (\varphi_0, \tilde{\psi}_n)_L\, e^{-\mu_n t}\, \tilde{\psi}_n(x), \quad 0 < x < L,\ t > 0. \tag{7}$$

Recall that these scalar products are respectively given by

$$(f,g)_\ell := \int_0^\ell f(x)g(x)\,dx \quad \text{and} \quad (f,g)_L := \int_0^L f(x)g(x)\,dx.$$

For any set K, let us denote by $\#K$ the cardinal of K. Then, the following holds:

Proposition 2.2. *If $L/\ell \in \mathbb{Q}$, then there exist initial data φ_0 verifying*

$$\#\{n : (\varphi_0, \tilde{\varphi}_n)_\ell \neq 0\} = \#\{n : (\varphi_0, \tilde{\psi}_n)_L \neq 0\} = 1, \tag{8}$$

such that $\varphi^\ell_{xx}(0,t) = \varphi^L_{xx}(0,t)$ for all $t > 0$. Thus, we can have non-uniqueness with initial data φ_0 satisfying (8) even if $|L-\ell|$ is arbitrarily small.

Proof. Let $m_0, n_0 \in \mathbb{N}$ be given such that $n_0 < m_0$ and $\ell = n_0 L/m_0$, that is, $m_0/L = n_0/\ell$. Let us choose $k_1, n_1 \in \mathbb{N}$ such that $n_1 = k_1 m_0/n_0$. Note that

$$\lambda_{k_1} = \frac{k_1^2\pi^2}{\ell^2} = \frac{n_1^2\pi^2}{L^2} = \mu_{n_1}$$

and set

$$\varphi_0(x) := \cos\left(\frac{k_1\pi x}{\ell}\right) + a = \cos\left(\frac{n_1\pi x}{L}\right) + a, \quad x \in \mathbb{R}, \tag{9}$$

where a is a real constant.

The functions in (6) and (7) corresponding to this φ_0 are the following:

$$\varphi^\ell(x,t) = a + e^{-\frac{k_1^2\pi^2}{\ell^2}t}\cos\left(\frac{k_1\pi}{\ell}x\right) \tag{10}$$

and

$$\varphi^L(x,t) = a + e^{-\frac{n_1^2\pi^2}{L^2}t}\cos\left(\frac{n_1\pi}{L}x\right). \tag{11}$$

Consequently,

$$\varphi_x^\ell(0,t) = \varphi_x^L(0,t) = 0. \quad \Box$$

From (4), (10) and (11), we get

$$u^\ell(x,t) = \frac{2k_1\pi}{\ell}\,\frac{e^{-\frac{k_1^2\pi^2}{\ell^2}t}\sin\left(\frac{k_1\pi}{\ell}x\right)}{e^{-\frac{k_1^2\pi^2}{\ell^2}t}\cos\left(\frac{k_1\pi}{\ell}x\right)+a} \quad \text{and} \quad u^L(x,t) = \frac{2n_1\pi}{L}\,\frac{e^{-\frac{n_1^2\pi^2}{L^2}t}\sin\left(\frac{n_1\pi}{L}x\right)}{e^{-\frac{n_1^2\pi^2}{L^2}t}\cos\left(\frac{n_1\pi}{L}x\right)+a}.$$

If a is sufficiently large, these functions are well defined, solve the Burgers systems respectively in $(0,\ell)\times(0,T)$ and $(0,L)\times(0,T)$ for

$$u_0(x) = \frac{2k_1\pi}{\ell}\,\frac{\sin\left(\frac{k_1\pi}{\ell}x\right)}{\cos\left(\frac{k_1\pi}{\ell}x\right)+a} = \frac{2n_1\pi}{L}\,\frac{\sin\left(\frac{n_1\pi}{L}x\right)}{\cos\left(\frac{n_1\pi}{L}x\right)+a}$$

and, moreover, satisfy (3).

This ends the proof of non-uniqueness in this case. $\Box$

2.2. Results where $\eta(t)\not\equiv 0$ and $u_0(x)\not\equiv 0$

In order to prove uniqueness when both η and u_0 are nonzero (and η is sufficiently large), we need an auxiliary result concerning traces of functions in $H^2(0,\ell)$:

Lemma 2.3. *Let $L_* > 0$ be given. Then*

$$\left|\frac{df}{dx}(0)\right| \le \frac{C(L_*)}{\ell^{3/2}}\,\|f\|_{H^2(0,\ell)}$$

for any $f\in H^2(0,\ell)$ and any ℓ with $0<\ell\le L_*$.

The proof is elementary. It can be found in [1].

Theorem 2.4. *Assume that $0<\ell\le L\le L_*$, $0<T_0<T$,*

$$u_x^\ell(0,t) = u_x^L(0,t) \ \ in \ \ (0,T), \quad \|u_0\|_{L^2(0,L)} \le M_0,$$
$$|u_x^\ell(x,t)| \le M \ \ in \ \ (0,\ell)\times(T_0,T) \ \ and \ \ |u_x^L(x,t)| \le M \ \ in \ \ (0,L)\times(T_0,T),$$

where L_*, M_0 and M are some positive constants. There exists $\delta_0>0$ (only depending on L_*, T_0, T, M_0 and M) such that, if

$$\int_{T_0}^T |\eta(t)|^2\,dt \ge \delta_0, \tag{12}$$

one necessarily has $\ell = L$.

Proof. In this proof, we will denote by A the one-dimensional Dirichlet Laplacian in (ℓ,L), with the associated eigenvalues $0<\zeta_1<\zeta_2<\cdots$.

Let us assume that $\ell<L$. Then, arguing as in the proof of Theorem 2.1, we deduce that

$$u^L(\ell,t) = u^L(L,t) = 0 \ \text{ in } \ (0,T). \tag{13}$$

Therefore, from well known energy estimates, one has

$$\|u^L(\cdot\,,t)\|_{L^2(\ell,L)} = \|e^{-tA}u^L(\cdot\,,0)\|_{L^2(\ell,L)} \le M_0 e^{-\zeta_1 t} \quad \forall t\in(T_0,T),$$

where ζ_1 is the first eigenvalue of A, that is, $\zeta_1 = \pi^2(L-\ell)^{-2}$.

Let us put $u^L = u^h + z$ for $t \in (T_0, T)$, with $u^h(\cdot\,,t) := e^{-(t-T_0)A}u^L(\cdot\,,T_0)$. Then, we have:

$$\|u^h(\cdot\,,t)\|_{H^2(\ell,L)} \le \frac{M_0}{T_0}e^{-\zeta_1 T_0} \quad \text{in} \quad (T_0,T).$$

On the other hand,

$$z(\cdot\,,t) = \int_{T_0}^{t} e^{-(t-s)A}(u^L u^L_x)(\cdot\,,s)\,ds$$

and the standard parabolic regularity estimates and the fact that $|u^L_x| \le M$ yield:

$$\|z\|_{L^2(T_0,T;H^2(\ell,L))} \le C(M)\|u^L\|_{L^2(T_0,T;L^2(\ell,L))} \le C(T,M_0,M)e^{-\zeta_1 T_0}.$$

Therefore,

$$\|u^L\|_{L^2(T_0,T;H^2(\ell,L))} \le C(T,M_0,M)\left(1+\frac{1}{T_0}\right)e^{-\zeta_1 T_0}$$

and, from Lemma 2.3, we get:

$$\|u^L_x(\ell,\cdot)\|_{L^2(T_0,T)} \le \frac{C(L_*,T,M_0,M)}{(L-\ell)^{3/2}}\left(1+\frac{1}{T_0}\right)\exp\left(-\frac{\pi^2 T_0}{(L-\ell)^2}\right). \tag{14}$$

Maximizing the right hand side with respect to $L-\ell$, we obtain:

$$\|u^L_x(\ell,\cdot)\|_{L^2(T_0,T)} \le \frac{1}{T_0^{3/4}}\left(1+\frac{1}{T_0}\right)C(L_*,T,M_0,M).$$

Now, we can continue exactly as in the proof of Theorem 2.7 in [1] and deduce that, if δ_0 is large enough, we get a contradiction. □

3. The Burgers equation with heat effects

The system is now

$$\begin{cases} u_t - u_{xx} + uu_x = k\theta, & 0<x<\ell,\ t>0,\\ \theta_t - \theta_{xx} + u\theta_x = 0, & 0<x<\ell,\ t>0,\\ u(0,t)=\eta(t), \quad u(\ell,t)=0, & t>0,\\ \theta(0,t)=\lambda(t), \quad \theta(\ell,t)=0, & t>0,\\ u(x,0)=u_0(x), \quad \theta(x,0)=\theta_0(x), & 0<x<\ell. \end{cases} \tag{15}$$

Here, $k \in \mathbb{R}$ is given.

As before, u can be interpreted as the velocity of the fluid particles in a one-direction flow. This time, we assume that heat effects are important and, consequently, the evolution of a temperature $\theta = \theta(x,t)$ must also be taken into account.

We will deal with the following inverse problem:

IP-2: *Fix (u_0,θ_0) and (η,λ) in (15) in appropriate spaces and assume that $\beta := u_x|_{x=0}$ and $\alpha := \theta_x|_{x=0}$ are known. Then, find ℓ.*

The uniqueness property to analyze is as follows:

Uniqueness for IP-2: *Let (u^ℓ,θ^ℓ) and (u^L,θ^L) be the solutions to (15) associated to the spatial intervals $(0,\ell)$ and $(0,L)$, respectively. Assume that the corresponding observations $(u^\ell_x(0,\cdot),\theta^\ell_x(0,\cdot))$ and $(u^L_x(0,\cdot),\theta^L_x(0,\cdot))$ coincide, that is,*

$$u^\ell_x(0,t) = u^L_x(0,t) \ \text{ and } \ \theta^\ell_x(0,t) = \theta^L_x(0,t) \ \text{ in } \ (0,T). \tag{16}$$

Then, do we have $\ell = L$?

If $(u_0,\theta_0) \equiv (0,0)$, we have again uniqueness:

Theorem 3.1. *Assume that $0<\ell\le L<T$, $\eta,\lambda \in L^\infty(0,T)$ satisfy $(\eta,\lambda) \not\equiv (0,0)$ and $(u_0,\theta_0) \equiv (0,0)$. Let (u^ℓ,θ^ℓ) and (u^L,θ^L) be the solutions to (15) respectively corresponding to ℓ and L and let us assume that, for some $M>0$, $|u^\ell_x| + |\theta^\ell_x| \le M$ and $|u^L_x| + |\theta^L_x| \le M$ respectively in $(0,\ell)\times(0,T)$ and $(0,L)\times(0,T)$, furthermore, (16) is satisfied. Then, $\ell = L$.*

The proof is very similar to the proof of Theorem 2.1. Thus, if we assume that $\ell < L$ and we set $v := u^\ell - u^L$ and $\psi := \theta^\ell - \theta^L$, it is clear from unique continuation that $(v,\psi) = (0,0)$ in $(0,\ell)\times(0,T)$. From energy estimates, we deduce that $(u^L,\theta^L) = (0,0)$ in $(\ell,L)\times(0,T)$ and finally, again from unique continuation, $(u^L,\theta^L) \equiv (0,0)$, which is impossible.

On the other hand, it is obvious that any solution to (1) is a particular solution to (15), corresponding to $\theta_0 \equiv 0$ and $\lambda \equiv 0$. Consequently, the counter-example considered in Section 2.1.2 is also a counter-example to uniqueness for **IP-2** when we allow u_0 to be nonzero.

To our knowledge, it is unknown if a counter-example to uniqueness can also be found with $\theta_0 \not\equiv 0$.

As before, we can deduce a uniqueness result for (15) for large η. More precisely, the following holds:

Theorem 3.2. *Assume that $0 < \ell \le L \le L_*$, $0 < T_0 < T$, $\|(u_0, \theta_0)\|_{L^2(0,L)} \le M_0$, $|u_x^\ell(x,t)| + |\theta_x^\ell(x,t)| \le M$ in $(0,\ell)\times(T_0,T)$, $|u_x^L(x,t)| + |\theta_x^L(x,t)| \le M$ in $(0,L)\times(T_0,T)$ and (16) holds. There exists $\delta_1 > 0$ (only depending on L_*, T_0, T, M_0 and M) such that, if*

$$\int_{T_0}^{T} |\eta(t)|^2\, dt \ge \delta_1, \tag{17}$$

one necessarily has $\ell = L$.

Proof. It is similar to the proof of Theorem 2.4.

Thus, let us assume that $\ell < L$. As before, this implies

$$u^L(\ell,t) = u^L(L,t) = 0 \;\text{ and }\; \theta^L(\ell,t) = \theta^L(L,t) = 0 \;\text{ in }\; (0,T).$$

The following estimates for (u^L, θ^L) hold:

$$\|u^L(\cdot\,,T_0)\|_{L^2(\ell,L)} = M_0 e^{-\zeta_1 T_0} \text{ and } \|\theta^L(\cdot\,,T_0)\|_{L^2(\ell,L)} = M_0 e^{-\zeta_1 T_0},$$

$$\|u^L\|_{L^2(T_0,T;L^2(\ell,L))} = C(T,M_0)e^{-\zeta_1 T_0} \;\text{ and the same hold for } \theta^L.$$

Let us put $u^L = w + z$, with $w(\cdot\,,t) := e^{-(t-T_0)A}u^L(\cdot\,,T_0)$. Then

$$\|w(\cdot\,,t)\|_{H^2(\ell,L)} \le \frac{C}{T_0}\, e^{-\zeta_1 T_0} \;\text{ and }\; z(\cdot\,,t) = \int_{T_0}^{t} e^{(t-s)A}\left(-u^L u_x^L + k\theta^L\right)(\cdot\,,s)\, ds \;\text{ in }\; (T_0,T),$$

whence

$$\|z\|_{L^2(T_0,T;L^2(\ell,L))} \le C\left[M\|u^L\|_{L^2(T_0,T;L^2(\ell,L))} + k\|\theta^L\|_{L^2(T_0,T;L^2(\ell,L))}\right].$$

Consequently,

$$\|u^L\|_{L^2(T_0,T;H^2(\ell,L))} \le C(T,L_*,M,M_0)\left(1 + \frac{1}{T_0}\right)e^{-\zeta_1 T_0}$$

and

$$\|u_x^L(\ell,\cdot)\|_{L^2(T_0,T)} \le \frac{C(T,L_*,M,M_0)}{(L-\ell)^{3/2}}\left(1 + \frac{1}{T_0}\right)e^{-\frac{\pi^2 T_0}{(L-\ell)^2}}.$$

At this point, we can continue as in the proof of Theorem 2.4 and deduce that, for δ_1 large enough, (17) leads to a contradiction. □

It is interesting to note that, in this result, the size of λ (that is, $\theta|_{x=0}$) is not relevant at all.

Remark 3.3. A simplified version of (15) can be obtained if we skip the transport terms. We find the linear system

$$\begin{cases} u_t - u_{xx} = k\theta, & 0 < x < \ell,\ t > 0,\\ \theta_t - \theta_{xx} = 0, & 0 < x < \ell,\ t > 0,\\ u(0,t) = \eta(t),\quad u(\ell,t) = 0, & t > 0,\\ \theta(0,t) = \lambda(t),\quad \theta(\ell,t) = 0, & t > 0,\\ u(x,0) = u_0(x),\quad \theta(x,0) = \theta_0(x), & 0 < x < \ell \end{cases} \tag{18}$$

It is not difficult to check that the assertions on uniqueness/nonuniqueness in Section 2 can be extended to this system with very similar (and in fact simpler) arguments. □

Similar inverse problems can be considered for coupled Burgers-heat systems where the heat flux is given and the temperature is observed at $x = 0$. These are the following:

$$\begin{cases} u_t - u_{xx} + uu_x = k\theta, & 0 < x < \ell,\ t > 0,\\ \theta_t - \theta_{xx} + u\theta_x = 0, & 0 < x < \ell,\ t > 0,\\ u(0,t) = \overline{u}(t),\quad u(\ell,t) = 0, & t > 0,\\ \theta_x(0,t) = \chi(t),\quad \theta_x(\ell,t) = 0, & t > 0,\\ u(x,0) = u_0(x),\quad \theta(x,0) = \theta_0(x), & 0 < x < \ell \end{cases} \tag{19}$$

and

$$\begin{cases} u_t - u_{xx} = k\theta, & 0 < x < \ell,\ t > 0, \\ \theta_t - \theta_{xx} = 0, & 0 < x < \ell,\ t > 0, \\ u(0,t) = u(\ell,t) = 0, & t > 0, \\ \theta_x(0,t) = \chi(t), \quad \theta_x(\ell,t) = 0, & t > 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), & 0 < x < \ell. \end{cases} \tag{20}$$

Now, the problems for (19) and (20) are as follows: (u_0, θ_0), $\overline{u}$, χ and the additional observations $\beta := u_x|_{x=0}$ and $\zeta := \theta|_{x=0}$ are known and, again, we try to find ℓ.

The same questions above are in order. Results similar to Theorems 3.1 and 3.2 can be proved in this context.

4. The case of the variable density Burgers equation

This is more interesting, but also more difficult. We consider a non-homogeneous (or variable density) one-dimensional fluid, modelled as follows:

$$\begin{cases} \rho(u_t + uu_x) - u_{xx} = 0, & 0 < x < \ell,\ t > 0, \\ \rho_t + u\rho_x = 0, & 0 < x < \ell,\ t > 0, \\ u(0,t) = \overline{u}(t), \quad u(\ell,t) = 0, & t > 0, \\ \rho(0,t) = \overline{\rho}(t), & t \in \mathbb{R}_+ \cap \{t : \overline{u}(t) > 0\}, \\ u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x), & 0 < x < \ell. \end{cases} \tag{21}$$

Of course, this can be viewed as a toy model for the variable density Navier–Stokes system. The corresponding inverse problem is the following:

IP-3: *Fix (u_0, ρ_0) and $(\overline{u}, \overline{\rho})$ in (21) in appropriate spaces and assume that $\beta := u_x|_{x=0}$ and $\gamma := \rho|_{x=0} 1_{\{t:\overline{u}(t)\leq 0\}}$ are known. Then, find ℓ.*

This is the uniqueness question we are interested in:

Uniqueness for IP-3: *Let (u^ℓ, ρ^ℓ) and (u^L, ρ^L) be the solutions to (21) respectively associated to $(0,\ell)$ and $(0,L)$. Assume that the corresponding $(u_x^\ell(0,\cdot), \rho^\ell(0,\cdot))$ and $(u_x^L(0,\cdot), \rho^L(0,\cdot))$ coincide. Then, do we have $\ell = L$?*

4.1. A result for zero initial data

When the initial data vanish, we have a positive uniqueness result for this problem:

Theorem 4.1. *Assume that $0 < \ell \leq L$, $T > 0$ and (u_0, ρ_0) and $(\overline{u}, \overline{\rho})$ satisfy*

$$\begin{cases} \overline{u}, \overline{\rho} \in L^\infty(0,T), \quad \overline{u} \not\equiv 0, \quad \overline{\rho} \geq 0, \\ u_0 \equiv 0, \quad \rho_0 \in L^\infty(0,L), \quad \rho_0 \geq a_0 > 0. \end{cases}$$

Let (u^ℓ, ρ^ℓ) and (u^L, ρ^L) be the solutions to (21) for $0 < t < T$ respectively corresponding to ℓ and L. Let us assume that $|u_t^\ell| + |u_x^\ell| + |\rho_x^\ell| \leq M$ and $|u_t^L| + |u_x^L| + |\rho_x^L| \leq M$ respectively in $(0,\ell)\times(0,T)$ and $(0,L)\times(0,T)$ and $u_x^\ell(0,\cdot) = u_x^L(0,\cdot)$ and $\rho^\ell(0,\cdot) = \rho^L(0,\cdot)$. Then, $\ell = L$.

For the proof, we will use a unique continuation property satisfied by the solutions to systems of the form

$$\begin{cases} a(x,t)v_t - v_{xx} + b(x,t)v_x + c(x,t)v + d(x,t)p = 0, & (x,t) \in Q, \\ p_t + m(x,t)p_x + r(x,t)v = 0, & (x,t) \in Q, \end{cases} \tag{22}$$

where we assume that $Q := (0,\ell)\times(0,T)$,

$$b, c, d, m, r \in C^0(\overline{Q}),\ a \in C^1(\overline{Q}) \ \text{ and } \ a \geq a_0 > 0 \ \text{ in } \ Q. \tag{23}$$

More precisely, we have the following:

Proposition 4.2. *Assume that (23) is satisfied and (v,p) solves (22), with $v, v_x, v_{xx}, p, p_x \in C^0(\overline{Q})$. Also, assume that*

$$\begin{cases} v(0,t) = 0, \quad v_x(0,t) = 0, \quad p(0,t) = 0, & 0 < t < T, \\ v(x,0) = 0, \quad p(x,0) = 0, & 0 < x < \ell. \end{cases} \tag{24}$$

Then, one has $v \equiv 0$ and $p \equiv 0$.

The proof of this Proposition relies on appropriate local Carleman estimates for the solutions to (22) and is postponed to Section 4.2.

Proof of Theorem 4.1. Note that $u^\ell \in L^\infty((0,\ell)\times(0,T))$ and $u^L \in L^\infty((0,L)\times(0,T))$. If we set $v := u^\ell - u^L$ and $p := \rho^\ell - \rho^L$, one has

$$\begin{cases} \rho^\ell v_t - v_{xx} + \rho^\ell v u^\ell_x + \rho^\ell u^L v_x + (u^L_t + u^L u^L_x)p = 0, & 0<x<\ell,\ t>0,\\ p_t + u^L p_x + v\rho^\ell_x = 0, & 0<x<\ell,\ t>0,\\ v(0,t)=0,\quad v_x(0,t)=0,\quad p(0,t)=0, & t>0,\\ v(x,0)=0,\quad p(x,0)=0, & 0<x<\ell. \end{cases}$$

Consequently, v and p satisfies (22) with $a=\rho^\ell$, $b=\rho^\ell u^L$, $c=\rho^\ell u^\ell_x$, $d=u^L_t+u^L u^L_x$, $m=u^L$ and $r=\rho^\ell_x$ and (24).

In view of Proposition 4.2, one has $v=0$ and $p=0$ in $(0,\ell)\times(0,T)$. This yields $u^L(x,t)=0$ in $(\ell,L)\times(0,T)$. Since the equations satisfied by u^L and ρ^L also possess the unique continuation property, we find that $u^L\equiv 0$, which is impossible. □

It would be interesting to find nonzero initial data (u_0,ρ_0) such that uniqueness fails. On the other hand, it would also be interesting to prove a result similar to Theorem 3.2 asserting that, if the boundary data are large enough (with respect to the other data in the system), uniqueness is satisfied. However, to our knowledge these questions are open.

A still more complex situation is found when we deal with a variable density fluid where thermal effects are relevant. For example, we can consider the variable density Boussinesq-like system

$$\begin{cases} \rho(u_t+uu_x)-u_{xx}=\theta, & 0<x<\ell,\ t>0,\\ \rho(\theta_t+u\theta_x)-\theta_{xx}=0, & 0<x<\ell,\ t>0,\\ \rho_t+u\rho_x=0, & 0<x<\ell,\ t>0,\\ u(0,t)=\overline{u}(t),\quad u(\ell,t)=0, & t>0,\\ \rho(0,t)=\overline{\rho}(t), & t\in\mathbb{R}_+\cap\{t:\overline{u}(t)>0\},\\ \theta_x(0,t)=\theta_x(\ell,t)=0, & t>0,\\ \rho(x,0)=\rho_0(x),\quad u(x,0)=u_0(x),\quad \theta(x,0)=\theta_0(x), & 0<x<\ell. \end{cases} \tag{25}$$

This is the related inverse problem: (u_0,θ_0,ρ_0) and $(\overline{u},\overline{\rho})$ are given and the additional observations $\beta := u_x|_{x=0}$ and $\zeta := \theta|_{x=0}$ are known for $t\in(0,T)$ and we try to find ℓ.

A result similar to Theorem 4.1 can also be established in this case.

4.2. Proof of Proposition 4.2

The proof of Proposition 4.2 can be obtained by combining two Carleman inequalities that can be deduced for the solutions to the first and the second equation in (22). The main steps are the following:

- To choose a suitable weight function (the same in both inequalities);
- To argue as in [18] and [12] deduce appropriate estimates for v and p.
- Finally, to add and eliminate all undesirable terms on the right hand sides.

Step 1: Let us first recall some known Carleman estimates for the solutions to equations like those in (22).

Thus, assume that a, b and c are as in Proposition 4.2 and set $Lv := av_t - v_{xx} + bv_x + cv$ for any suitable v. For any $\lambda>0$, $\beta>0$, $x_0>\ell$, $\delta>0$ and $T>0$ (to be definitively fixed below), we take

$$\varphi(x,t) := e^{\lambda\psi(x,t)},\quad \text{with}\quad \psi(x,t) := |x-x_0|^2 - \frac{4\delta^2\beta}{T^2}|t-T/2|^2. \tag{26}$$

Note that φ can be used in the proof of the Carleman inequality in [18]. Consequently, the following holds:

Theorem 4.3. *There exists $\lambda_0>0$ with the following property: for any $\lambda\ge\lambda_0$, there exist constants $s_0=s_0(\lambda)>0$ and $C_0=C_0(\lambda)$ such that*

$$\begin{aligned} &\iint_Q \left(\frac{1}{s\varphi}(|v_t|^2+|v_{xx}|^2) + s\lambda^2\varphi|v_x|^2 + s^3\lambda^4\varphi^3|v|^2\right)e^{2s\varphi}\,dx\,dt\\ &\quad\le C_0\left(\iint_Q |Lv|^2 e^{2s\varphi}\,dx\,dt + \int_0^T \left(s^3\lambda^3\varphi^3|v|^2 + s\lambda\varphi|v_x|^2 + |v_t|^2\right)e^{2s\varphi}\,dt\Big|_{x=0,\ell}\right.\\ &\quad\left. + s^2\lambda^2 e^{C_0\lambda}\int_0^\ell \left(|v|^2+|v_x|^2\right)e^{2s\varphi}\,dx\Big|_{t=0,T}\right) \end{aligned} \tag{27}$$

for all $s\ge s_0$ and any $v\in H^{2,1}(Q)$.

Now, let m be as in (23) and let us set $B := \varphi_t + m\varphi_x$ and $Ep := p_t + mp_x$ for any p. We can also adapt the proof of the Carleman estimate for transport equations in [4] and deduce the following result:

Theorem 4.4. *Assume that* $\min_{(x,t)\in\overline{Q}} |B(x,t)| \geq B_0 > 0$. *Then, there exist constants* $s_0 > 0$ *and* $C > 0$ *such that*

$$
\begin{aligned}
s^2 \iint_Q |p|^2 e^{2s\varphi}\,dx\,dt \leq C \iint_Q |Ep|^2 e^{2s\varphi}\,dx\,dt \\
+ s\int_0^T mB|p|^2 e^{2s\varphi}\,dt\Big|_{x=0}^{x=\ell} + s\int_0^\ell B|p|^2 e^{2s\varphi}\,dx\Big|_{t=0}^{t=T}
\end{aligned}
\tag{28}
$$

for all $s \geq s_0$ *and any* $p \in H^1(Q)$.

Step 2: Let us assume that $t_0 \in (0,T)$ and $\delta > 0$ is such that $0 < t_0 - \delta < t_0 + \delta < T$ and let us set

$$Q_\delta := (0,\ell) \times (t_0 - \delta, t_0 + \delta).$$

Let us introduce the new variable $\tilde{t}$ with $\tilde{t} = t_0 - \delta + \dfrac{2\delta}{T} t$ and the new function $\tilde{\varphi}$ with

$$\tilde{\varphi}(x,\tilde{t}) := e^{\lambda\tilde{\psi}(x,\tilde{t})} \quad \text{and} \quad \tilde{\psi}(x,\tilde{t}) := \psi(x,\tilde{t}) \equiv |x-x_0|^2 - \beta|\tilde{t}-t_0|^2.$$

Then, (27) can be rewritten as an estimate in Q_δ. By denoting $\tilde{t}$ (resp. $\tilde{\varphi}$) again by t (resp. φ), the following is found:

$$
\begin{aligned}
\iint_{Q_\delta} \Big(\frac{1}{s\varphi}(|v_t|^2 + |v_{xx}|^2) + s\lambda^2\varphi|v_x|^2 + s^3\lambda^4\varphi^3|v|^2\Big) e^{2s\varphi}\,dx\,dt \\
\leq C\left(\iint_{Q_\delta} |p|^2 e^{2s\varphi}\,dx\,dt + K_1 + K_2\right),
\end{aligned}
\tag{29}
$$

where

$$
\begin{aligned}
K_1 \ &:= \int_{t_0-\delta}^{t_0+\delta} \Big(s^3\lambda^3\varphi^3|v|^2 + s\lambda\varphi|v_x|^2 + |v_t|^2\Big) e^{2s\varphi}\,dt\Big|_{x=0,\ell} \\
&\leq Cs^3\lambda^3 e^{C\lambda}\int_{t_0-\delta}^{t_0+\delta}\Big(|v(0,t)|^2 + |v_x(0,t)|^2 + |v_t(0,t)|^2\Big) e^{2s\varphi(0,t)}\,dt \\
&\quad + Cs^3\lambda^3 e^{C\lambda} M^2 \int_{t_0-\delta}^{t_0+\delta} e^{2s\varphi(\ell,t)}\,dt
\end{aligned}
\tag{30}
$$

and

$$
K_2 := Cs^2\lambda^2 e^{C\lambda}\int_0^\ell \Big(|v|^2 + |v_x|^2\Big) e^{2s\varphi}\,dx\Big|_{t=t_0-\delta,t_0+\delta} \leq Cs^2\lambda^2 e^{C\lambda} M^2 e^{2se^{\lambda(|x_0|^2-\beta\delta^2)}}.
\tag{31}
$$

On the other hand, the estimate (28) applied to the second equation of (22) in Q_δ gives:

$$
\begin{aligned}
s^2 \iint_{Q_\delta} |p|^2 e^{2s\varphi}\,dx\,dt \leq C \iint_{Q_\delta} |v|^2 e^{2s\varphi}\,dx\,dt \\
+ s\int_{t_0-\delta}^{t_0+\delta} mB|p|^2 e^{2s\varphi}\Big|_{x=0}^{x=\ell}\,dt + s\int_0^\ell B|p|^2 e^{2s\varphi}\,dx\Big|_{t=t_0-\delta}^{t=t_0+\delta^2}
\end{aligned}
$$

and we find that

$$
s^2 \iint_{Q_\delta} |p|^2 e^{2s\varphi}\,dx\,dt \leq C \iint_{Q_\delta} |v|^2 e^{2s\varphi}\,dx\,dt + R_1 + R_2,
\tag{32}
$$

where

$$
\begin{aligned}
R_1 := Cse^{C\lambda}M^2 \int_{t_0-\delta}^{t_0+\delta} |p|^2 e^{2s\varphi}\,dt\Big|_{x=0}^{x=\ell} \\
\leq Cse^{C\lambda}M^2 \int_{t_0-\delta}^{t_0+\delta} |p(0,t)|^2 e^{2s\varphi(0,t)}\,dt + Cse^{C\lambda}M^4 \int_{t_0-\delta}^{t_0+\delta} e^{2s\varphi(\ell,t)}\,dt
\end{aligned}
\tag{33}
$$

and

$$
R_2 := Cse^{C\lambda}M \int_0^\ell |p|^2 e^{2s\varphi}\,dx\Big|_{t=t_0-\delta}^{t=t_0+\delta} \leq Cse^{C\lambda}M^3 e^{2se^{\lambda(|x_0|^2-\beta\delta^2)}}.
\tag{34}
$$

In (30), (31), (33) and (34), we have used that $|v| + |v_x| + |v_t| + |p| \leq M$ in $\overline{Q}$. It is not restrictive to assume that $M \geq 1$.

Step 3: After adding (29) and (32), if we take into account the estimates of the K_i and R_i and the data and observations, assuming that s and λ are sufficiently large, we find:

$$\begin{aligned}
&\iint_{Q_\delta}\Big(\frac{1}{s\varphi}(|v_t|^2+|v_{xx}|^2)+s\lambda^2\varphi|v_x|^2+s^3\lambda^4\varphi^3|v|^2\Big)e^{2s\varphi}\,dx\,dt+s^2\iint_{Q_\delta}|p|^2e^{2s\varphi}\,dx\,dt\\
&\quad\le Cs^3\lambda^3e^{C\lambda}M^2\int_{t_0-\delta}^{t_0+\delta}\Big(|v(0,t)|^2+|v_x(0,t)|^2+|v_t(0,t)|^2+|p(0,t)|^2\Big)e^{2s\varphi(0,t)}\,dt\\
&\qquad+Cs^3\lambda^3e^{C\lambda}M^4\int_{t_0-\delta}^{t_0+\delta}e^{2s\varphi(\ell,t)}\,dt+Cs^2\lambda^2e^{C\lambda}M^3e^{2se^{\lambda(|x_0|^2-\beta\delta^2)}}\\
&\quad=Cs^3\lambda^3e^{C\lambda}M^4\int_{t_0-\delta}^{t_0+\delta}e^{2s\varphi(\ell,t)}\,dt+Cs^2\lambda^2e^{C\lambda}M^3e^{2se^{\lambda(|x_0|^2-\beta\delta^2)}}.
\end{aligned}\tag{35}$$

Now, we argue as follows:

- First, we fix $\lambda > 0$ such that (35) holds and choose x_0, t_0 and δ as before and $\varepsilon \in (0,\ell)$.
- Then, we take $\beta > 0$ large enough such that $|x_0|^2 - |x_0-\ell+\epsilon|^2 < 3\beta\frac{\delta^2}{4}$.
- Finally, we choose $\kappa \in (0,\delta/2)$ such that $\beta\kappa^2 < 2\varepsilon\,(x_0-\ell)+\varepsilon^2$.

With these constants ε and κ, one has

$$|x-x_0|^2-\beta|t-t_0|^2 \ge \mu := |x_0-\ell+\varepsilon|^2-\beta\kappa^2 > \max(|x_0-\ell|^2, |x_0|^2-\beta\delta^2) \tag{36}$$

for all $(x,t)\in(0,\ell-\varepsilon)\times(t_0-\kappa,t_0+\kappa)$. Taking into account (24), we deduce from (35) that

$$\begin{aligned}
&\iint_{(0,\ell-\varepsilon)\times(t_0-\kappa,t_0+\kappa)}\Big(s\lambda^4|v|^2+|p|^2\Big)\,dx\,dt\\
&\quad\le 2\delta Cs\lambda^3e^{C\lambda}M^4e^{2s(e^{\lambda|x_0-\ell|^2}-e^{\lambda\mu})}+Cs\lambda^2e^{C\lambda}M^3e^{2s(e^{\lambda(|x_0|^2-\beta\delta^2)}-e^{\lambda\mu})}\\
&\quad\le C_*s\left(e^{2s(e^{\lambda|x_0-\ell|^2}-e^{\lambda\mu})}+e^{2s(e^{\lambda(|x_0|^2-\beta\delta^2)}-e^{\lambda\mu})}\right),
\end{aligned}\tag{37}$$

where C_* depends on M, δ and λ but is independent of s. But, in view of (36), this right hand side goes to zero as $s\to+\infty$. Consequently, $v(x,t)=0$ and $p(x,t)=0$ in $(0,\ell-\varepsilon)\times(t_0-\kappa,t_0+\kappa)$.

Since ε and κ are arbitrarily small and t_0 is arbitrary in $(0,T)$, $v\equiv 0$ and $p\equiv 0$ and the proof is achieved. □

5. Some numerical results

In this section, we will perform some numerical experiments for the previous inverse problems. We will carry out the reconstruction of the unknown length through the resolution of some appropriate extremal problems. This strategy has been applied in some previous papers of the authors for other similar problems, see [5,7,8]. The results of the numerical tests that follow will serve to illustrate the theoretical results in the previous sections.

5.1. Inverse problems for the Burgers equation

We deal with the following

Reformulation of IP-1: *Given $T>0$, $\eta=\eta(t)$, $u_0=u_0(x)$ and $\beta=\beta(t)$, find $\ell\in(\ell_0,\ell_1)$ such that*

$$J_1(\ell)\le J_1(\ell')\quad \forall\,\ell'\in(\ell_0,\ell_1), \tag{38}$$

where J is given by

$$J_1(\ell) := \frac{1}{2}\int_0^T |\beta(t)-u_x^\ell(0,t)|^2\,dt. \tag{39}$$

Here, u^ℓ is the state, i.e. the solution to (1) corresponding to the length ℓ.

Three different situations will be analyzed for the Burgers equation. In the first two cases, we will check that uniqueness holds: zero initial data and nonzero initial data and sufficiently large η. In the third case we will consider a non-uniqueness situation corresponding to some nonzero initial data and "small" η and we will study the behavior of the numerical algorithm. To this purpose (and also in the experiences in the following sections), we will implement the `fmincon` function from the MatLab Optimization Toolbox using the `active-set` minimization algorithm.

Case 1.1: Burgers equation with $u_0=0$ and $\eta\ne 0$.

We take $T=5$, $\eta(t)=5\sin^3 t$ in $(0,T)$ and $u_0(x)\equiv 0$. Starting from $L_i=3$, our goal is to recover the desired value of the length $L_d=2$.

The results of this numerical experiments can be seen in Table 1, where the effect of random noise on the target are shown. The computed length is denoted by L_c. The corresponding solution to (38)–(39) is displayed in Fig. 1. The

Table 1
Burgers equation, $u_0 = 0$ and $\eta \neq 0$. Results with random noise in the target (desired length: $L_d = 2$).

% noise	Cost	Iterates	Computed L_c
1%	1.e−3	12	1.997140631
0.1%	1.e−5	15	1.999169558
0.01%	1.e−7	11	1.999912907
0.001%	1.e−9	10	2.000021375
0%	1.e−17	9	1.999999985

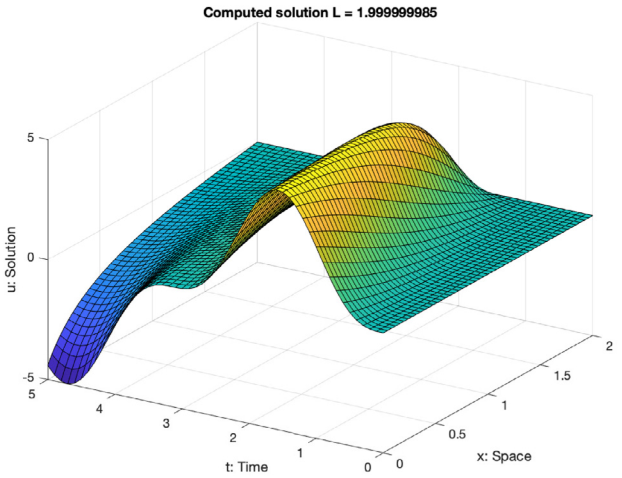

Fig. 1. Burgers equation with $u_0 = 0$ and $\eta \neq 0$. The computed solution.

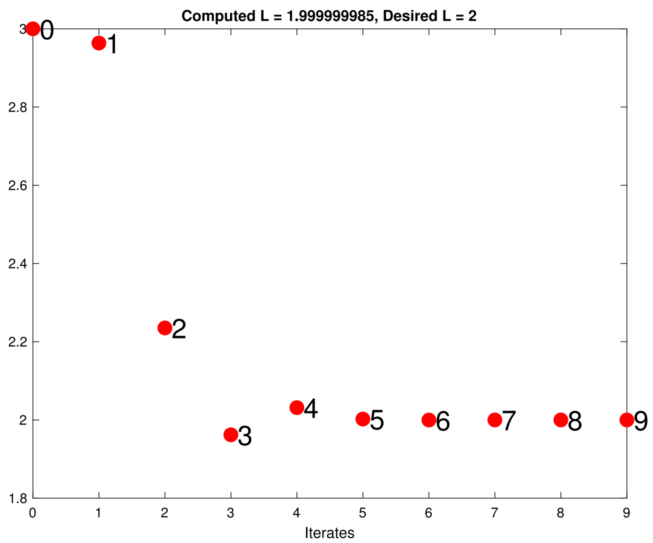

Fig. 2. Burgers equation, $u_0 = 0$ and $\eta \neq 0$. The iterates in `active-set` algorithm.

evolution of the iterates and the cost in the minimization process in the absence of random noise appear in Figs. 2 and 3, respectively.

Case 1.2: Burgers equation with $u_0 \neq 0$ **and** large η.

We take $T = 5$, $\eta(t) = 5(\sin t)^3$ in $(0, T)$ and $u_0(x) \equiv 3x(2 - x)$. Now, starting from $L_i = 2.4$, the target value that we want to recover is $L_d = 2$.

The results of the numerical implementation are shown in Table 2, where again random noise was incorporated. The contents of Figs. 4–6 are similar to those above.

Case 1.3: Burgers equation with $u_0 \neq 0$ **and "small"** η.

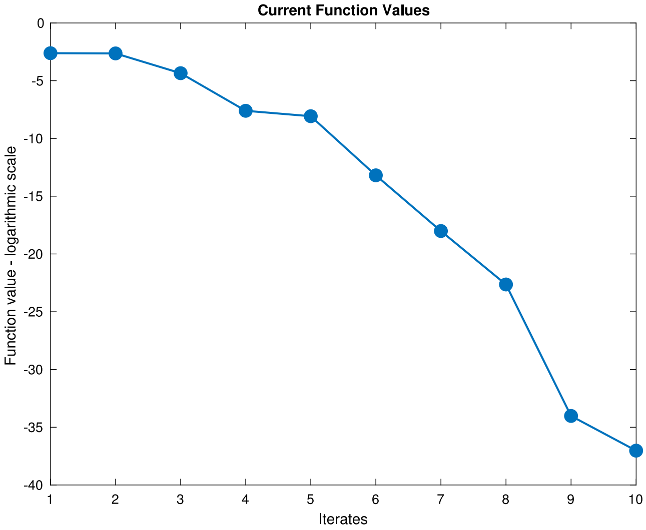

Fig. 3. Burgers equation, $u_0 = 0$ and $\eta \neq 0$. Evolution of the cost.

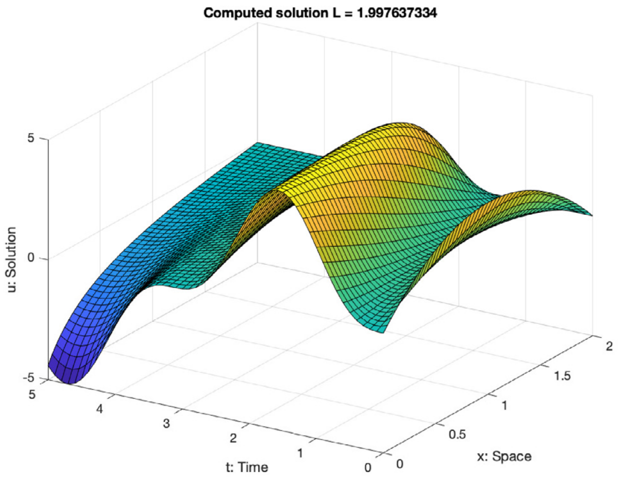

Fig. 4. Burgers equation, $u_0 \neq 0$ and large η. The computed solution.

Table 2
Burgers equation, fixed u_0 and large η. Results with random noise in the target (desired length: $L_d = 2$).

% noise	Cost	Iterates	Computed L_c
1%	1.e−2	6	2.032815856
0.1%	1.e−5	11	2.012510004
0.01%	1.e−5	9	1.985859861
0.001%	1.e−6	9	1.994836103
0%	1.e−6	9	1.997637334

Here, we deal with a non-uniqueness situation. Our aim is to investigate the behavior of the algorithm in a situation of this kind.

We take $T = 6$, $\eta = 0$ in $(0, T)$ and $u_0(x) \equiv \pi \sin(\pi x/2)/(2 + \cos(\pi x/2))$. Note that we have $u_0(x) \equiv \sin(3\pi x/L_d^1)/(2 + \cos(3\pi x/L_d^1)) \equiv \sin(2\pi x/L_d^2)/(2 + \cos(2\pi x/L_d^2))$, with $L_d^1 = 6$ and $L_d^2 = 4$; consequently, this initial data can be used as in Section 2.1.2 to prove non-uniqueness.

We will consider the following experiments:

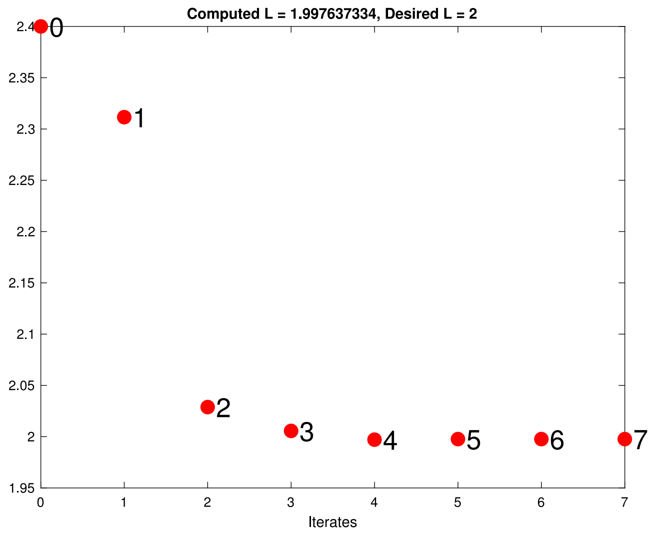

Fig. 5. Burgers equation, fixed u_0 and large η. The iterates in `active-set` algorithm.

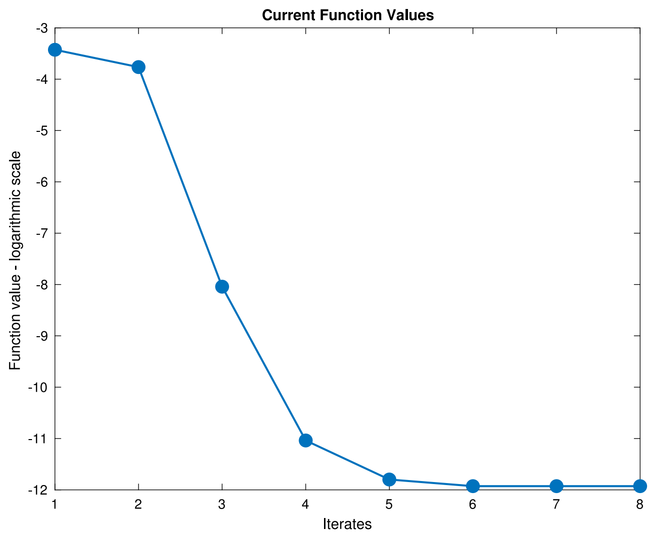

Fig. 6. Burgers equation, fixed u_0 and large η. Evolution of the cost.

- First, we start from $L_i = 5.6$, and we obtain the results exhibited in Figs. 7 and 8. The computed value is $L_c^1 = 5.998083259$ and the associated cost is $J(L_c^1) < 10^{-8}$.
- Then, we start from $L_i = 4.6$, and we obtain the results exhibited in Figs. 9 and 10. The computed value is $L_c^2 = 4.000601673$ and the associated cost is again $J(L_c^2) < 10^{-9}$.

The corresponding computed boundary observations are displayed in Figs. 11 and 12, respectively. Thus, we confirm that these identical observations correspond, as we already knew, two different solutions. (See Figs. 13 and 14.)

5.2. Inverse problems for the Burgers-heat system

This section is concerned with **IP-2** and other related problems. We will consider several choices of boundary conditions and also several different observations.

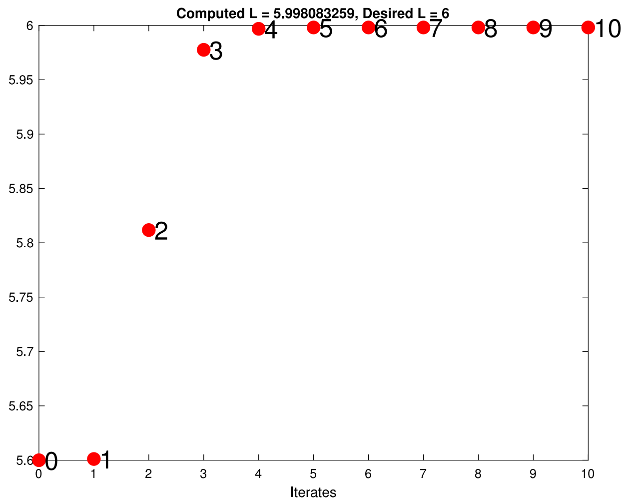

Fig. 7. Burgers equation, $\eta = 0$, fixed $u_0(x)$. Iterates in `active-set` algorithm with $L_d^1 = 6$.

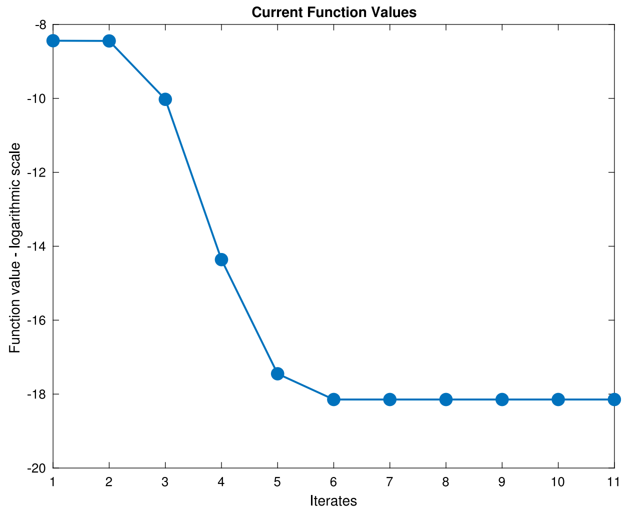

Fig. 8. Burgers equation, $\eta = 0$, fixed $u_0(x)$. Evolution of the cost for $L_d^1 = 6$, $J(L_c^1) < 10^{-8}$.

5.2.1. Dirichlet boundary conditions for u and θ and stress and flux observations

We consider the system (15). A reformulation of **IP-2** is the following:

$$\begin{cases} \text{Minimize } J_2(\ell) := \dfrac{1}{2}\displaystyle\int_0^T \left|\beta(t) - u_x^\ell(0,t)\right|^2 dt + \dfrac{1}{2}\displaystyle\int_0^T \left|\alpha(t) - \theta_x^\ell(0,t)\right|^2 dt \\ \text{Subject to: } \ell \in (\ell_0, \ell_1), \quad (u^\ell, \theta^\ell) \text{ satisfies (15).} \end{cases}$$

Case 2.1: Burgers-heat system with $(u_0, \theta_0) = (0, 0)$ and $\eta \neq 0$ **and** $\lambda \neq 0$.

We take $T = 5$, $\eta(t) \equiv 5\sin^3 t$, $\lambda(t) \equiv 0.2\cos(t)\sin(t)$ and $(u_0(x), \theta_0(x)) \equiv (0, 0)$. Starting from $L_i = 1$, our goal is to recover the desired value of the length $L_d = 2$.

The computed length is $L_c = 1.999999534$, the cost is $J(L_c) < 10^{-14}$ is reached at the iterate 8 of the optimization algorithm. The corresponding solution to (15) is displayed in Figs. 15 and 16. The evolution of the iterates and the cost in the minimization process in the absence of the random noise appear in Figs. 17 and 18, respectively.

Case 2.2: Burgers-heat system with $(u_0, \theta_0) \neq (0, 0)$ **and large** η.

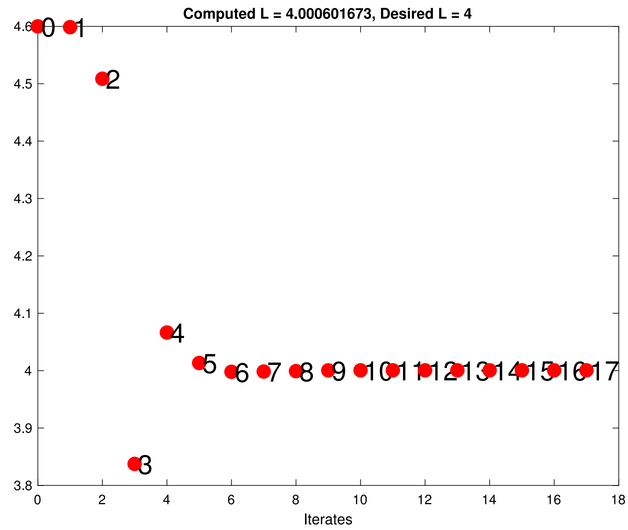

Fig. 9. Burgers equation, $\eta = 0$, fixed $u_0(x)$. Iterates in `active-set` algorithm with $L_d^2 = 4$.

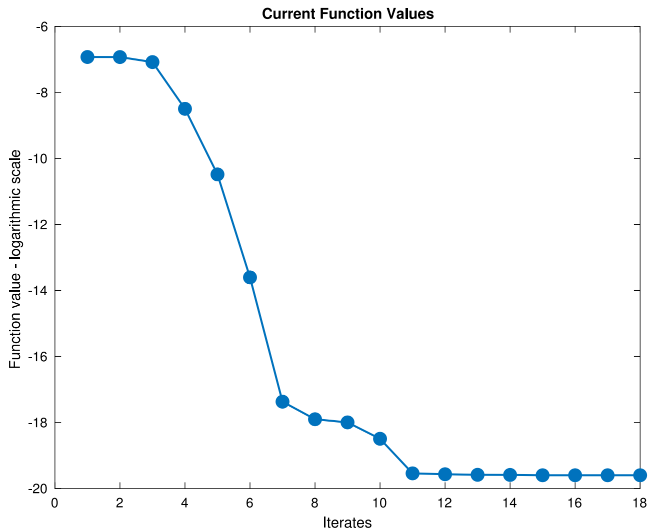

Fig. 10. Burgers equation, $\eta = 0$, fixed $u_0(x)$. Evolution of the cost for $L_d^2 = 4$, $J(L_c^2) < 10^{-9}$.

We take $T = 5$, $\eta(t) = 5\sin^3 t$ and $\lambda(t) = 6\sin(t)\cos(t)$ in $(0, T)$, $u_0(x) \equiv 0.1x(2-x)$ and $\theta_0(x) \equiv 0.1x^2(x-3)$. Starting from $L_i = 1$, our goal is to recover the desired value of the length $L_d = 2$.

The computed length is $L_c = 2.000000005$, the cost is $J(L_c) < 10^{-17}$ is reached at the iterate 9 of the optimization algorithm. The corresponding solution to (15) is displayed in Figs. 19 and 20. The evolution of the iterates and the cost in the minimization process in the absence of the random noise appear in Figs. 21 and 22, respectively.

5.2.2. Dirichlet boundary conditions for u, Neumann boundary conditions for θ and stress observation

In this section, the system under study is (19). The inverse problem is similar to **IP-2** and a suitable reformulation is:

$$\begin{cases} \text{Minimize } J_3(\ell) := \dfrac{1}{2}\displaystyle\int_0^T |\beta(t) - u_x^\ell(0,t)|^2\,dt \\ \text{Subject to: } \ell \in (\ell_0, \ell_1), \quad (u^\ell, \theta^\ell) \text{ satisfies (19).} \end{cases}$$

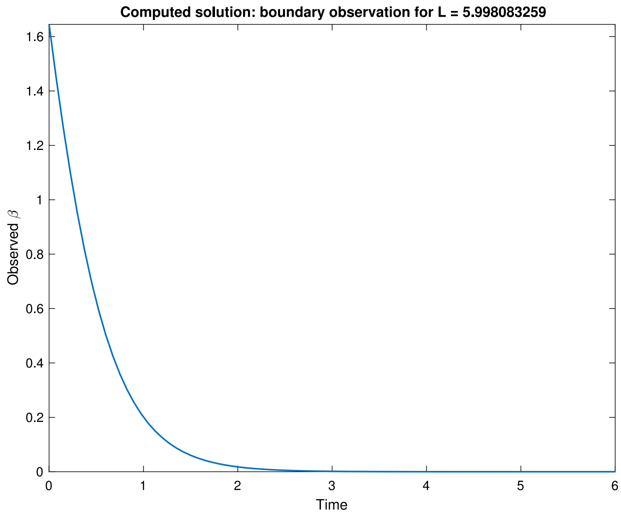

Fig. 11. Burgers equation, $\eta = 0$, fixed $u_0(x)$. The computed boundary observation $u_x(0, \cdot)$ for $L_c^1 = 5.996562049$.

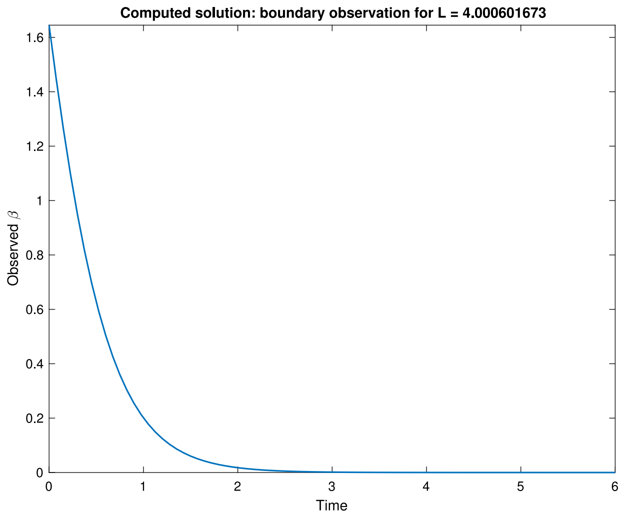

Fig. 12. Burgers equation, $\eta = 0$, fixed $u_0(x)$. The computed boundary observation $u_x(0, \cdot)$ for $L_c^2 = 4.007345905$.

As before, two different situations will be analyzed for this problem. In both cases, respectively corresponding to zero initial data and nonzero initial data and sufficiently large η, we will check that uniqueness holds.

Case 2.3: Burgers-heat system with $(u_0, \theta_0) = (0, 0)$ **and** $\eta \neq 0$.

We observe that this case is reduced to the Burgers single equation.

We take $T = 5$, $\eta(t) = 5\sin^3 t$ in $(0, T)$ and $(u_0(x), \theta_0(x)) \equiv (0, 0)$. Starting from $L_i = 1$, our goal is to recover the desired value of the length $L_d = 2$.

The computed length is $L_c = 1.999999964$, the cost is $J(L_c) < 10^{-16}$ is reached in the iterate 10 of the optimization algorithm. The corresponding solution to (15) is displayed in Figs. 23 and 24. The evolution of the iterates and the cost in the minimization process in the absence of the random noise appear in Figs. 25 and 26, respectively.

Case 2.4: Burgers-heat system with $(u_0, \theta_0) \neq (0, 0)$ **and large** η.

We take $T = 5$, $\eta(t) = 5\sin^3 t$ in $(0, T)$ and $u_0(x) = 0.1x(2 - x)$, $\theta_0(x) = 0.1(1 + x^2(x - 3))$. Starting from $L_i = 1.4$, our goal is to recover the desired value of the length $L_d = 2$.

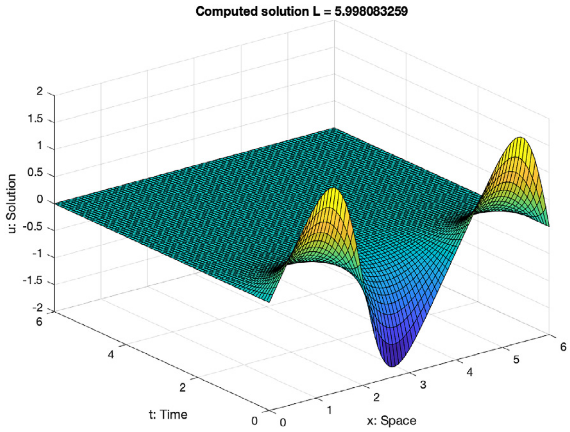

Fig. 13. Burgers equation, $\eta = 0$, fixed $u_0(x)$. The computed solution corresponding to $L_c^1 = 5.998083259$.

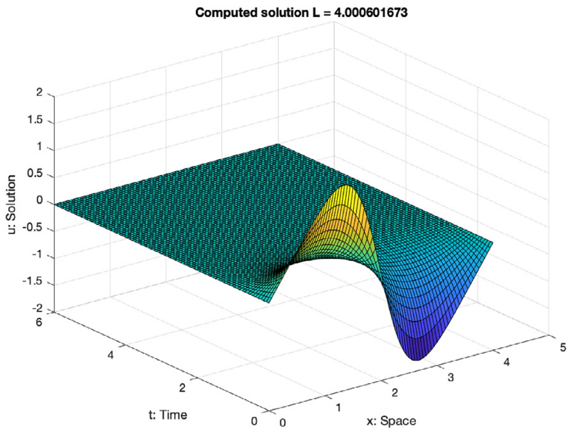

Fig. 14. Burgers equation, $\eta = 0$, fixed $u_0(x)$. The computed solution corresponding to $L_c^2 = 4.000601673$.

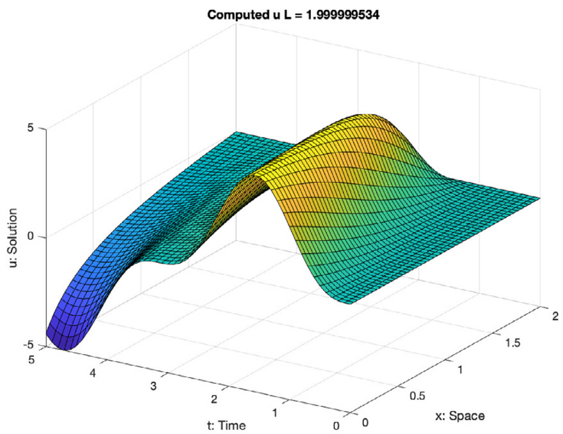

Fig. 15. Burgers equation with heat effect with $(u_0, \theta_0) = (0, 0)$ and $(\eta, \lambda) \neq (0, 0)$ with two observations $u_x(0, t)$ and $\theta_x(0, t)$. The computed solution u.

The computed length is $L_c = 2.001874913$, the cost is $J(L_c) < 10^{-6}$ is reached in the iterate 9 of the optimization algorithm. The corresponding solution to (15) is displayed in Figs. 27 and 28. The evolution of the iterates and the cost in the minimization process in the absence of the random noise appear in Figs. 29 and 30, respectively.

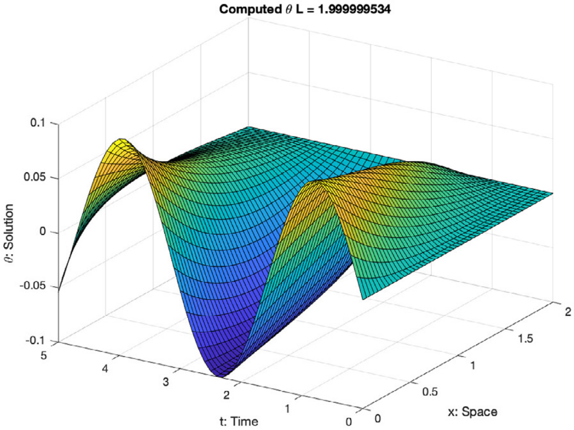

Fig. 16. Burgers equation with heat effect with $(u_0, \theta_0) = (0, 0)$ and $(\eta, \lambda) \neq (0, 0)$ with two observations $u_x(0, t)$ and $\theta_x(0, t)$. The computed solution θ.

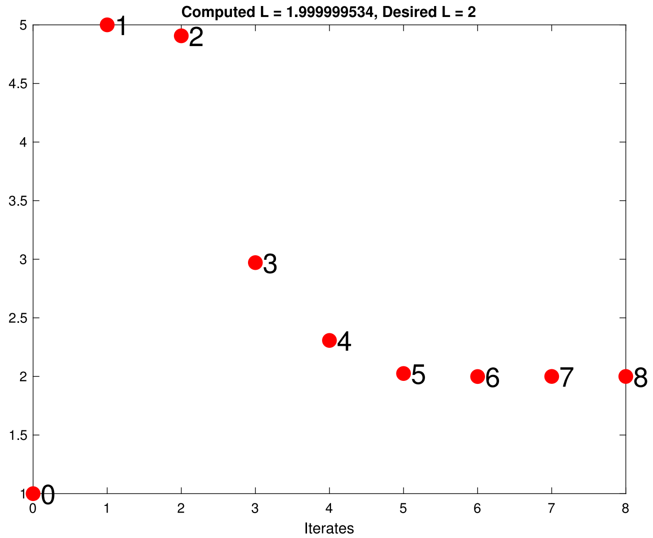

Fig. 17. Burgers equation with heat effect with $(u_0, \theta_0) = (0, 0)$ and $(\eta, \lambda) \neq (0, 0)$ with two observations $u_x(0, t)$ and $\theta_x(0, t)$. The iterates in `active-set` algorithm.

Declaration of competing interest

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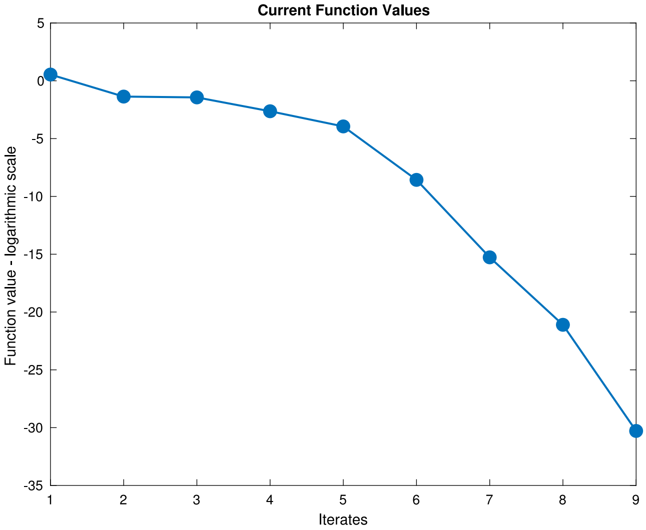

Fig. 18. Burgers equation with heat effect with $(u_0, \theta_0) = (0, 0)$ and $(\eta, \lambda) \neq (0, 0)$ with two observations $u_x(0, t)$ and $\theta_x(0, t)$. Evolution of the cost.

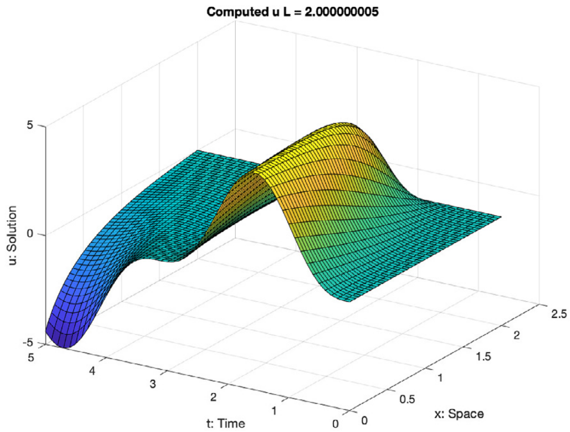

Fig. 19. Burgers equation with heat effect with $(u_0, \theta_0) \neq (0, 0)$ and large (η, λ) with two observations $u_x(0, t)$ and $\theta_x(0, t)$. The computed solution u.

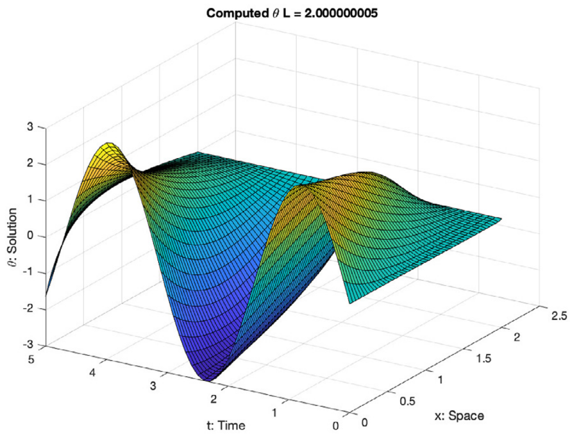

Fig. 20. Burgers equation with heat effect with $(u_0, \theta_0) \neq (0, 0)$ and large (η, λ) with two observations $u_x(0, t)$ and $\theta_x(0, t)$. The computed solution θ.

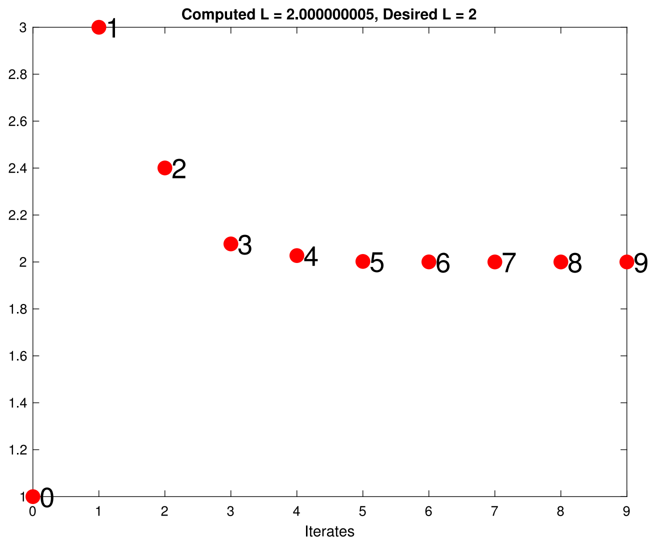

Fig. 21. Burgers equation with heat effect with $(u_0, \theta_0) \neq (0, 0)$ and large (η, λ) with two observations $u_x(0, t)$ and $\theta_x(0, t)$. The iterates in `active-set` algorithm.

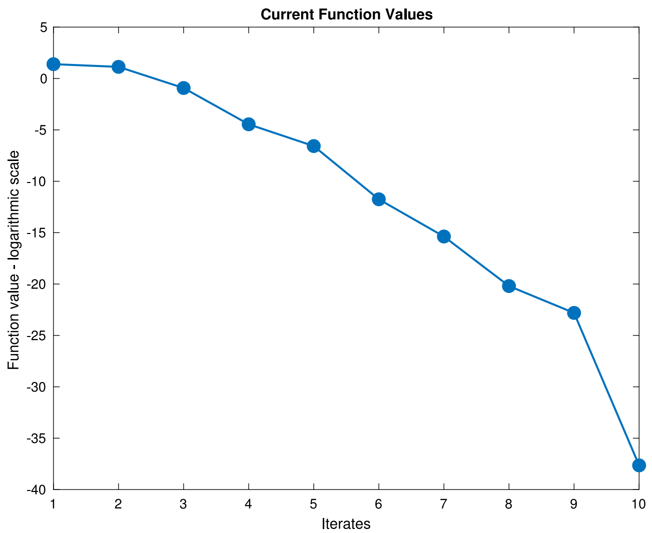

Fig. 22. Burgers equation with heat effect with $(u_0, \theta_0) \neq (0, 0)$ and large (η, λ) with two observations $u_x(0, t)$ and $\theta_x(0, t)$. Evolution of the cost.

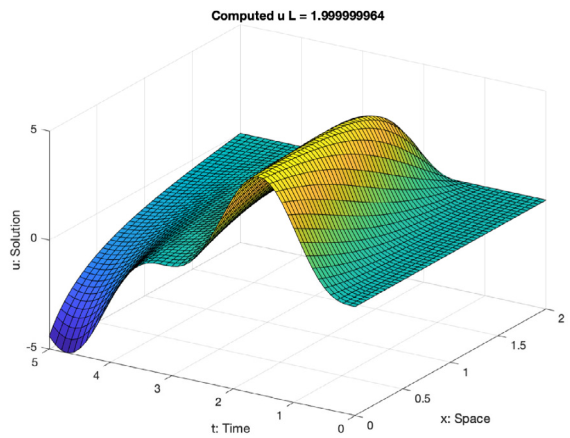

Fig. 23. Burgers equation with heat effect with $(u_0, \theta_0) = (0, 0)$ and $\eta \neq 0$ with one observation $u_x(0, t)$. The computed solution u.

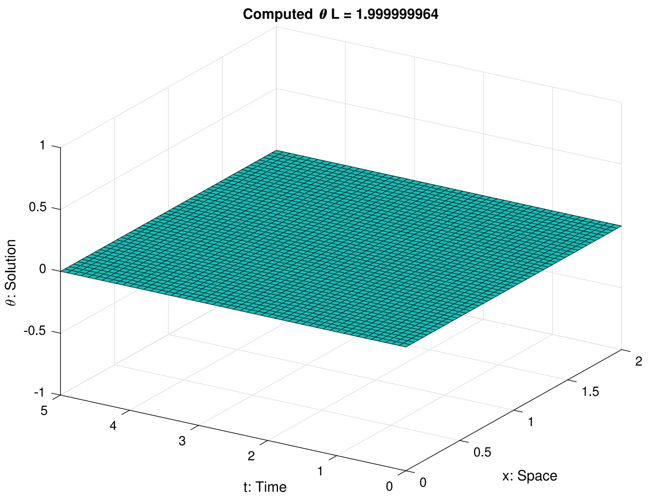

Fig. 24. Burgers equation with heat effect with $(u_0, \theta_0) = (0, 0)$ and $\eta \neq 0$ with one observation $u_x(0, t)$. The computed solution θ.

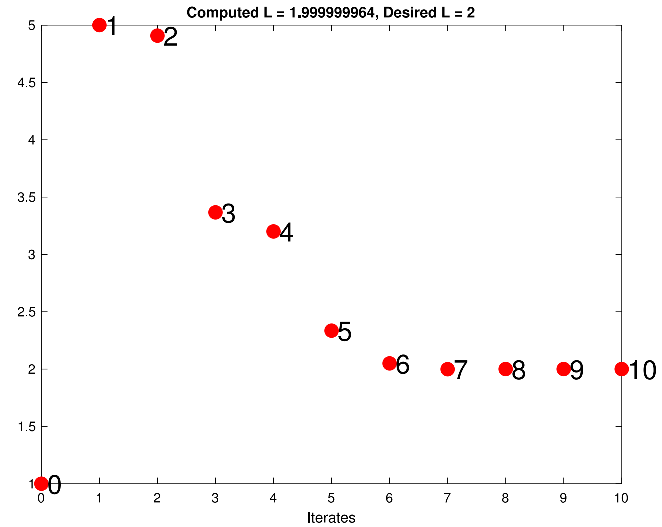

Fig. 25. Burgers equation with heat effect with $(u_0, \theta_0) = (0, 0)$ and $\eta \neq 0$ with one observation $u_x(0, t)$. The iterates in `active-set` algorithm.

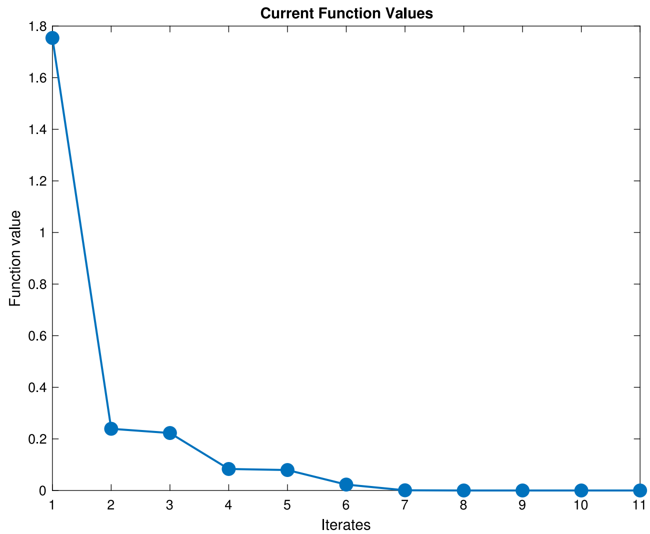

Fig. 26. Burgers equation with heat effect with $(u_0, \theta_0) = (0, 0)$ and $\eta \neq 0$ with one observation $u_x(0, t)$. Evolution of the cost.

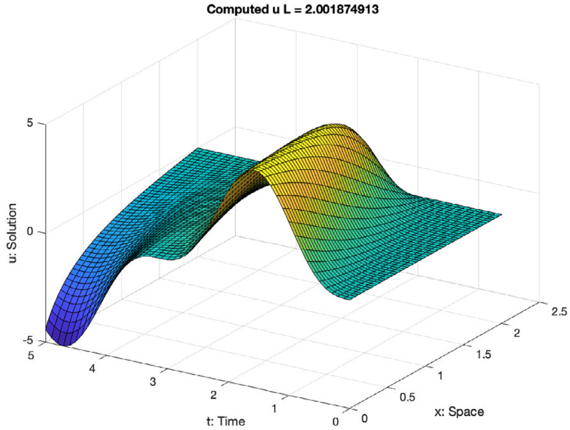

Fig. 27. Burgers equation with heat effect with $(u_0, \theta_0) = (0, 0)$ and $\eta \neq 0$ with one observation $u_x(0, t)$. The computed solution u.

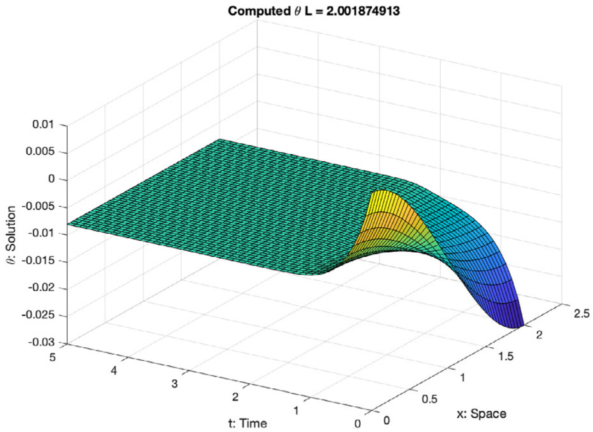

Fig. 28. Burgers equation with heat effect with $(u_0, \theta_0) = (0, 0)$ and $\eta \neq 0$ with one observation $u_x(0, t)$. The computed solution θ.

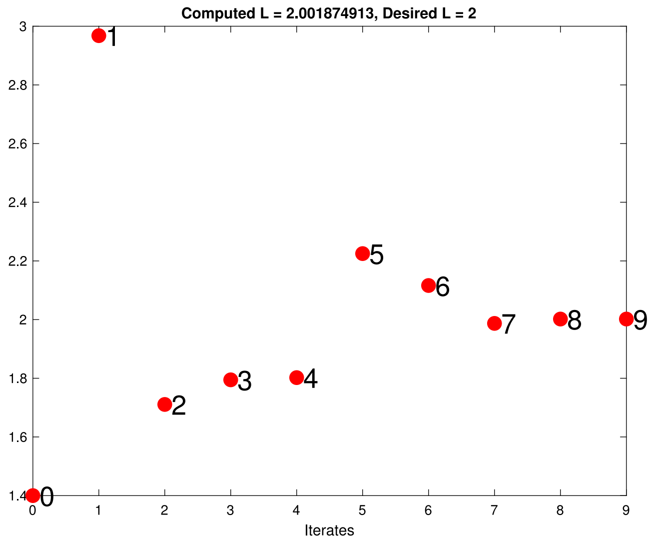

Fig. 29. Burgers equation with heat effect with $(u_0, \theta_0) = (0, 0)$ and $\eta \neq 0$ with one observation $u_x(0, t)$. The iterates in `active-set` algorithm.

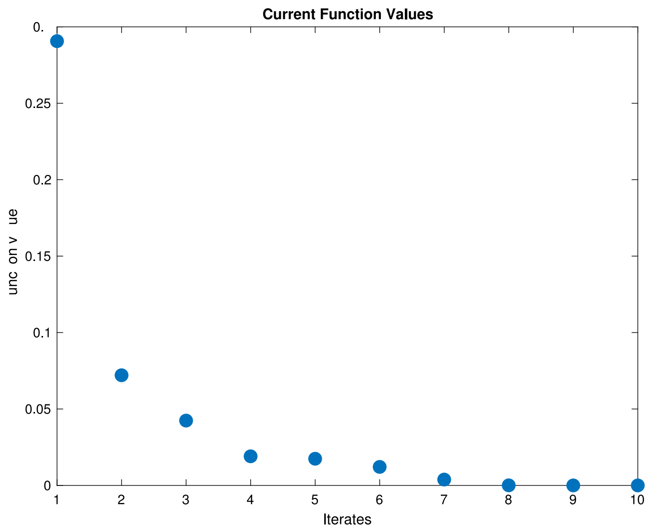

Fig. 30. Burgers equation with heat effect with $(u_0, \theta_0) = (0, 0)$ and $\eta \neq 0$ with one observation $u_x(0, t)$. Evolution of the cost.

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